TOPOLOGY AND TIME REVERSAL

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ABSTRACT

In this lecture we address some topological questions connected with the existence on a general spacetime manifold of diffeomorphisms connected to the identity which reverse the time-orientation.

1 Introduction

If one regards Quantum Gravity as an attempt to unify two distinct but equally fundamental physical theories; quantum mechanics on the one hand and general relativity on the other, one can ask what elements of either theory is it most likely that one will have to sacrifice in the eventual unification. Perhaps the most fundamental innovations of general relativity relate to its treatment of the notion of time. One of most striking features of quantum mechanics is its use of complex amplitudes. One may argue that the introduction of complex numbers into the basic structure of quantum mechanics is closely connected to the treatment in that theory of the notion of change and of time evolution. It therefore seems reasonable to regard the use of complex numbers in conventional quantum mechanics as a potential casuality. More precisely, one may argue that if, as is commonly supposed in quantum cosmology, the classical idea of time is an emergent concept, valid only at late times, low energies and large distances, then so too is our usual idea of a quantum mechanical Hilbert space with its attendant complex structure. *In other words, the complex numbers in quantum mechanics should be thought of as having an essentially historical origin.* Some ideas along these lines were discussed within the context of the semi-classical approach to quantum cosmology in [16–17].

A related question is to ask: how in a theory in which one assumes that spacetime has an everywhere well-defined Lorentzian metric are the properties of quantum fields in those spacetimes affected by such global properties of the spacetime as the existence of closed timelike curves ('CTCs '), a lack of time-orientability or some other pathology which would normally be excluded in a globally hyperbolic spacetime? Are there restrictions on the possible spacetimes for example? One possible restriction comes about by demanding that spacetime admit a spin or pin structure [6]. Another possible restriction arises by demanding that the spacetime has a time-orientation. If it does not, one may argue that one may not be able to construct a quantum mechanical Hilbert space endowed with a complex structure. This suggestion was made some time ago [25] and it has received further support from the work of Bernard Kay [7].

One motivation for asking this question is to try to extend the range of applicability of quantum field theory in a fixed background. Another motivation might be to answer questions about what possibilities the laws of physics in principle allow. This has provided much of the impetus behind recent work on CTCs . Another, and possibly more cogent, reason for considering non-globally-hyperbolic spacetimes is that in the path integral approach to quantum gravity in which one sums over all possible Lorentzian metrics there is *a priori* no good reason for excluding them. One might attempt to perform the functional integral by first freezing the metric and integrating over all matter fields on that spacetime, and then summing over all spacetimes. The first part of the integral is then tantamount to quantizing matter fields on a fixed background. It is customary in the Euclidean formulation to replace the sum over Lorentzian metrics by a sum over Riemannian metrics but one may ask what happens if one tries to avoid this step.

In the Euclidean version one is often concerned with anomalies that may arise when functional determinants fail to be well-defined, for example they may not be invariant under spacetime diffeomorphisms. The diffeomorphisms in question may either be continuously connected to the identity or not. The latter type of global anomalies are closely related to discrete symmetries, or lack of them, such as parity or orientation. They may also be investigated from an Hamiltonian point of view . However, this does not address the possibility of anomalies of a purely Lorentzian kind which manifest themselves only in non-globally-hyperbolic spacetimes. An example is a breakdown of spin structure. If one assumes that spacetime is both time and space-orientable, this can *only* occur in a spacetime which is not globally hyperbolic . If one drops the requirement of space-orientability, however, there may exist pin structure even though the spacetime *is* globally hyperbolic. An example is provided by $\mathbb{RP}^2 \times \mathbb{R}^2$, endowed with the product metric formed from the standard 'round' metric on $\mathbb{RP}^2$ and the Minkowski metric on $\mathbb{R}^2$ (with either signature) [6].

One possible viewpoint on the difficulties experienced with non-time-orientable spacetimes is precisely that there is some sort of anomaly. Roughly speaking, for each complex amplitude in the functional sum one must, if there is no global time-orientation, include its complex conjugate which is associated to the time-reversed amplitude. The result must then necessarily be real and so no true quantum interference is possible. It is interesting to note that this sort of problem would also arise in some attempts to generalize the usual quantum formalism being made by Gell-Mann and Hartle [22] since they also make use of *complex* amplitudes and they incorporate a rule relating complex conjugation to time reversal of a sequence of observables.

The purpose of this lecture is to explore some of these issues in more depth. In particular, we will discuss the relation between the topology of a time-orientable spacetime $\{\mathcal{M}, g\}$ and the existence and properties of various kinds of time-reversing diffeomorphisms. We shall, for the sake of mathematical precision, mainly concentrate on spacetime manifolds $\mathcal{M}$ which are compact and without boundary, but we will comment on the case of non-compact spacetimes and spacetimes with boundaries.

As well as the motivations given above, our results are also relevant to suggestions like that of Sakharov [18] that the early universe may simply be a time reflection of the late universe. Such a viewpoint is essentially a Lorentzian version of the (historically later) no-boundary proposal or the idea of a universe born from nothing [23].

2 Compact Spacetimes

An assumption of compactness in spatial directions is quite natural when discussing topological questions because one has in mind a situation where the non-trivial topology can be localized, at least to the extent that it is not allowed to escape from the spacetime altogether. Compactness in the time direction is less easy to justify (unless there are spacelike boundaries) because it necessarily implies the existence of closed timelike curves . Formerly this was thought to rule out consideration of such spacetimes but more recently, with the advent of studies of the properties of time machines, this view has been abandoned and so we shall not be put off by this feature.

In fact the Euler number $\chi(\mathcal{M})$ of a compact spacetime of arbitrary dimension must vanish:

$$\chi(\mathcal{M}) = 0, \tag{1}$$

and in four dimensions:

$$\chi = 2 - 2b_1 + b_2, \tag{2}$$

where b_i are the Betti numbers. Thus a compact spacetime must have an even second Betti number and infinite fundamental group, and so its universal covering space is non-compact. In this sense it may be thought of as a non-compact spacetime which has been periodically identified and this is indeed typically how examples of time machines are constructed in the literature. However, the reader is cautioned that there is, as we shall see later, no logical connection between whether or not a curve is closed and timelike and whether or not it is homotopically trivial. In general, one expects the fundamantal group $\pi_1(\mathcal{M})$ to be non-Abelian. This is what one expects in the case of two or more time machines, for example if the spacetime has in a connected sum decomposition two summands of the form $S^1 \times S^3$ with time running around the S^1 factors.

In the exceptional case that the fundamental group *is* Abelian, it may be shown [8–11] that the possible Betti numbers (b_1, b_2) must belong to the set: {(1,0), (2,2), (3,4), (4,6)} . This is because for any closed orientable manifold of any dimension which has an Abelian fundamental group one has the inequality:

$$\frac{1}{2}\, b_1(b_1 - 1) \leq b_2 \tag{3}$$

The result follows from (1) which holds for any spacetime dimension and (2) which holds in four dimensions.

The significance of this non-Abelian-ness in the case that homotopically non-trivial time machines are present is presumably that some physical effects may depend upon the order in which one enters the time machines. It would be interesting to explore this point further. In that connection, it is perhaps worth recalling why it is that non-simply-connected four-manifolds are not classifiable [26]. The point is that by taking the connected sum $\#_k S^1 \times S^3$ of k copies of $S^1 \times S^3$ one obtains a four-manifold whose fundamental group is the free group on k generators (which of course is maximally non-Abelian). One may now perform surgery on this manifold to obtain a new manifold whose fundamemtal group has k generators and r arbitrarily chosen relations. Since there is no algorithm for deciding whether two different presentations give an isomorphic group there can be no algorithm for deciding whether two four-manifolds are homeomorphic .

The process of surgery can be described as follows. Given an element $g \in \pi_1(\mathcal{M}')$ of a four-manifold $\mathcal{M}'$ one can represent it by a closed curve $\gamma \in \mathcal{M}'$. Now surround this closed curve γ by a tube or collared neighbourhood $\mathcal{N}$ of the form $\mathcal{N} = D^3 \times \gamma \equiv D^3 \times S^1$ where D^3 is a closed 3-dimensional disc. The boundary $\partial\mathcal{N}$ of this tube has topology $\partial\mathcal{N} \equiv S^1 \times S^2$. One now removes the tube $\mathcal{N}$ from $\mathcal{M}'$ and replaces it with the simply connected manifold $D^2 \times S^2$ which has the same boundary. The result is a new manifold $\mathcal{M}''$ whose fundamental group differs from that of $\mathcal{M}'$ only by the imposition of the relation $g = 0$. This process is called 'killing an element of the fundamental group'. It may be shown that by a succesion of such killings one may obtain from $\#_k S^1 \times S^3$ a manifold with any desired finitely generated fundamental group.

From a physical point of view it is interesting to note two things. Firstly that the undecidability problem reviewed above may give rise to limitations on what is 'in principle' allowed by the laws of physics when it comes to the sort of wormhole and time machine engineering envisaged by Thorne and others. The possibility arises of having two sets of instructions for building a multiple time machine but having no algorithm for deciding whether the two spacetimes have the same topology. Whether or not this is true is not obvious from the general result quoted above because a compact spacetime must have vanishing Euler number. We do not know whether such manifolds are classifiable or not.

The second point is that the process of surgery gives rise to a manifold which physically looks rather like one containing the creation and annihilation of an extra Einstein-Rosen throat. If the 2-disc D^2 has coordinates $X + iT = r \exp\left(it - \frac{i\pi}{2}\right)$ where the cyclic 'time' coordinate t which parameterizes the original curve γ runs between 0 and 2π then 'half-way round', i.e. on the real axis $T = 0$, the interior of the tube $\mathcal{N}$ has been replaced by a manifold which has the same topology as the Kruskal manifold of a black hole and therefore it has embedded in it a three-manifold which has the topology of a bridge, i.e. of $\mathbb{R} \times S^2$. If these sorts of manifolds do arise in a Lorentzian form of quantum gravity it seems reasonable to think of them as containing 'virtual black holes'.

This interpretation receives some support from the observation that the Riemannian manifolds used as instantons or real tunnelling geometries in the Euclidean approach to vacuum instability and black hole pair creation may be obtained by surgery on a circle , which we would like to associate with the world line of a virtual black hole, from the corresponding false vacuum spacetime. Thus the Euclidean Schwarzschild manifold ($\mathbb{R}^2 \times S^2$) may be obtained from the hot flat space manifold $S^1 \times \mathbb{R}^3$, the Ernst instanton manifold ($S^2 \times S^2 - \{pt\}$) for the creation of pairs of oppositely charged non-extreme black holes from a constant electromagnetic field (topology $\mathbb{R}^4$), and the Nariai and Mellor-Moss Instantons (both with topology $S^2 \times S^2$) are obtained from the De Sitter manifold (S^4). In Kaluza-Klein theory, Witten [27] has argued that the five-dimensional Schwarzschild solution (topology $\mathbb{R}^2 \times S^3$) is the bounce solution which mediates the decay of the Kaluza-Klein vacuum (topology $S^1 \times \mathbb{R}^4$). The five-dimensional manifold corresponding to a magnetic field also has topology $S^1 \times \mathbb{R}^4$. This may decay via Witten's instability but it may also decay into a monopole-anti-monopole pair. The instanton for this process has topology $S^5 - S^1 \equiv \mathbb{R}^2 \times S^3$ and so may also be obtained by surgery on a circle from the false vacuum spacetime manifold.

3 Time Reversal in a General Spacetime

Let $\{\mathcal{M}, g^L\}$ be a time-orientable spacetime. Thus the bundle of time-oriented frames $SO_\uparrow(n-1,1)(\mathcal{M}, g^L)$ falls into two connected components. One typically thinks of time reversal Θ as a diffeomorphism:

$$\Theta : \mathcal{M} \to \mathcal{M}$$

which reverses time-orientation, whose lift to $SO_\uparrow(n-1,1)(\mathcal{M}, g^L)$ exchanges the two connected components and is an involution of order two:

$$\Theta^2 = \text{id}.$$

It need not necessarily be an isometry (in general the spacetime will not admit any isometries). One could imagine considering a more general finite group action but presumably one could always find a $\mathbb{Z}_2$ subgroup and we shall assume that this can be done.

In a general non-globally-hyperbolic spacetime it is not obvious whether Θ should reverse space-orientation, or total orientation (assuming $\{\mathcal{M}, g^L\}$ to be space or time-orientable respectively) , whether it should act freely on $\mathcal{M}$ or fix a three-surface for example, or whether it should belong to the identity component $\mathrm{Diff}_0(\mathcal{M})$. The existence and uniqueness and other properties of Θ depends both on the topology of the manifold $\mathcal{M}$ and on the Lorentz metric g^L.

To illustrate these subtleties, consider even the simplest globally hyperbolic spacetime $\mathcal{M} \equiv \mathbb{R} \times \Sigma$ with coordinates $t, \mathbf{x}$, t being timelike and Σ being an orientable $(n-1)$-manifold. Naively we might take

$$\Theta^T : (t, \mathbf{x}) \to (-t, \mathbf{x})$$

but nothing prevents us from considering

$$\Theta^J : (t, \mathbf{x}) \to (-t, \mathbf{x}^*)$$

where

$$J : \mathbf{x} \to \mathbf{x}^*$$

is an involution on the $(n-1)$-manifold Σ. Clearly Θ^T fixes the three-manifold Σ and reverses total orientation. It therefore lies outside the identity component $\mathrm{Diff}_0(\mathcal{M})$. On the other hand, we might arrange for J to act freely on Σ, possibly reversing or not reversing space-orientation.

These seemingly rather artificial examples actually arise in some applications. In quantum field theory in De Sitter spacetime, dS_n, Σ is the $(n-1)$-sphere and J its antipodal map. This preserves space-orientation if the spacetime dimension n is even. The map Θ^J is an isometry and is the centre of the isometry group $O(n, 1)$. One may identify points under the action of Θ^J to obtain the 'elliptic interpretation'. This then provides a possible non-singular realisation of Sakharov's ideas of a Lorentzian model of a universe born from nothing. The idea immediately generalizes to a Friedman model whose scale factor is an even function of time. Sakharov's idea was in fact to impose some sort of time-reflection symmetry about a singular big bang at which the scale factor vanishes. He did not use the involution J. In spatially closed models the scale factor often starts from a zero value at the big bang, $t = 0$, rises to a maximum at $t = t_{\max}$ say, and then symmetrically decreases to a vanishing value at the big crunch at $t = 2t_{\max}$. This has led Gold [2] to conjecture that the 'arrow of time reverses' in the contracting phase. In effect he proposed that the entire quantum state is invariant under a time-reversing involution whose action on spacetime is given by:

$$\Theta^G : \ t \to t_{\max} - t.$$

By contrast Davies [1] (see also Albrow [4]) prefers to continue through the Big Bang and Big Crunch to get a model in which the arrow of time reverses in sucessive cycles. In other words, one imposes invariance under the action of semi-direct product $\mathbb{Z} \odot \mathbb{Z}_2$ given by

$$t \to t + 2t_{\max}$$

and

$$t \to -t.$$

It is clear that similar options are available for non-singular periodic models in which there is neither a Big Bang nor a Big Crunch. Thus for example, in the case of Anti-De Sitter spacetime AdeS_n, the scale factor is a sinusoidal function of cosmic time but the vanishing of the scale factor is an artefact of a poor choice of coordinates. In fact $\mathcal{M} \equiv S^1 \times H^{n-1}$ where $H^{n-1} \equiv \mathbb{R}^{n-1}$ is hyperbolic space and time t runs around the circle, $0 \le t < 2\pi$. The center of the isometry group $O(n-1, 2)$ does not reverse time (it sends $(t, \mathbf{x})$ to $(t + \pi, -\mathbf{x})$). Intuitively, it seems clear that time reversal must have fixed points since we must reverse t and compose with an involution J which may be thought to act on Euclidean space.

In the examples so far (at least if we wish to maintain the boundary conditions) there was no natural choice of Θ in the identity component $\mathrm{Diff}_0(\mathcal{M})$. However in more exotic situations, as we shall see in detail shortly, this seemingly paradoxical situation can occur. Now if no possible Θ lies in the identity component $\mathrm{Diff}_0(\mathcal{M})$ it is reasonable to say that the spacetime $\{\mathcal{M}, g^L\}$ has an intrinsic sense of the passage of time (even though time itself may not be defined!). If however there exists a Θ which does lie in the identity component this is not reasonable. The general situation with respect to $\mathrm{Diff}(\mathcal{M})$

appears to be quite difficult to analyse and so we shall restrict attention here to a simpler question. Is there a homotopy rather than a diffeomorphism carrying the metric g^L with one time-orientation to the same metric with the opposite time-orientation? If there does exist a suitable Θ in the identity component $\mathrm{Diff}_0(\mathcal{M})$ then a homotopy will certainly exist (simply pull back g^L by a curve f_s, $0 \leq s \leq 1$ of diffeomorphisms joining $f_1 = \Theta$ to the identity $f_0 = \mathrm{id}$). However the converse is not necessarily true. Given a homotopy g^L_t of Lorentz metrics there may exist no diffeomorphism producing it. Now from the point of view of homotopy theory, a closed time-oriented Lorentzian spacetime $\{\mathcal{M}, g^L\}$ contains no more information than a Riemannian manifold $\mathcal{M}$ equipped with a unit vector field $\mathbf{V}$. The spacetime with the opposite time-orientation corresponds homotopically to the same manifold equipped with the negative unit vector field $-\mathbf{V}$.

4 Mathematical Interlude

This following mathematical interlude follows some conversations with Graeme Segal.

4.1 Linear and General Homotopies

We suppose that M is a closed, n-dimensional time-orientable Lorentzian manifold. We may, in the standard way, endow M with a Riemannian metric and hence deduce that M admits a global section $\mathbf{V}$ of the bundle $S(M)$ of unit vectors over M. At each point x in M the fibre S_x of $S(M)$ is an $n-1$ sphere.

Pulling the Lorenzian metric back under the action of diffeomorphisms induces an action on $\mathbf{V}$ and we would like to know whether there exists a diffeomorphism $f : M \rightarrow M$ which takes $\mathbf{V}$ to its negative, i.e. which reverses the direction of time. In particular we would like to know whether there exits such a diffeomorphism f contained in the identity component $\mathrm{Diff}_0(M)$ of the diffeomorphism group $\mathrm{Diff}(M)$. An easier question to ask is whether there exists a homotopy taking $\mathbf{V}$ to $-\mathbf{V}$ since if there exists a diffomorphism in the identity component a homotopy is given by a curve f_t in $\mathrm{Diff}_0(M)$ joining

f to the identity. The converse is however not sufficient because, as we shall see, if one considers $M = S^1 \times S^{2n-1}$ with the vector field running around the S^1 factor one finds that this cannot be reversed by a diffeomorphism but it may be reversed by a homotopy

A homotopy $\mathbf{V}_t$ between $\mathbf{V}$ and $-\mathbf{V}$ thus gives at each point x in M a continuous path $\gamma_x(t)$ from the north pole to the south pole of S^{n-1}. In other words a *general homotopy* $\mathbf{V}_t$ provides a global section s_Z of a bundle $Z(M)$ whose fibres Z_x are the space of paths from the north to the south pole of S^{n-1}. Since any path from the north pole to the south pole of S^{n-1} is homotopic to a closed path on S^{n-1} one sees that from the point of view of homotopy the fibre Z_x is equivalent to the loop space $\Omega(S^{n-1})$ of based loops on S^{n-1} .

Consider now a *special* or *linear* homotopy from $\mathbf{V}$ to $-\mathbf{V}$. By definition this is one for which, at each point x in M, $\mathbf{V}(\mathbf{x})_t$ lies in a an oriented two plane $\pi(x)$ spanned say by the vectors $\mathbf{V}_0$ and $\mathbf{V}_{t_1}$ where $0 < t_1 < 1$. A linear homotopy gives a particular kind of path $\gamma_x(t)$ from the north to the south pole of S^{n-1}, one which is along a great circle in the 2-plane defined by by the vectors $\mathbf{V}_0$ and $\mathbf{V}_{t_1}$. The set of such great circles is parameterized by where the great circle intersects the equatorial S^{n-1}.

The existence of a linear homotopy is thus equivalent to the existence of a global section s_Y of the S^{n-2} bundle $Y(M)$ of unit vectors orthogonal to the vector field $V(x)$. One may think of this S^{n-2} fibre Y_x as the equatorial S^{n-2} in the S^{n-1} fibre S_x of the bundle $S(M)$. It follows that the bundle $Y(M)$ is a sub-bundle of the bundle $Z(M)$. The question of whether every homotopy can be deformed into a linear homotopy then reduces to the question whether every section s_Z may be deformed to a section s_Y.

It should also be clear that the existence of the vector field and a linear homotopy is equivalent to a non-vanishing section of the bundle $V_{n,2}(M)$ of dyads, i.e of ordered pairs of linearly independent vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ say. The fibre of the dyad bundle $V_{n.2}(M)$ is the Stiefel manifold $V_{n,2}$ of dyads. In addition a linear homotopy provides a global section s_G of the bundle $G_{n,2}(M)$ of oriented 2-planes whose fibre is the Grassman manifold $G_{n,2}$. The existence of a section s_G is, in fact, the necessary and suffient condition that a manifold admit a metric of signature $(n-2,2)$.

We note *en passant* the following

Lemma *If M is even dimensional a sufficient condition for M to admit a linear homotopy is that it admit an almost complex structure J. In four* *dimensions this condition is also neccessary.*

The point is that one may then take

$$\mathbf{V}_t = e^{t\pi J}\mathbf{V_0}$$

In four dimensions the existence of an almost complex structure is also necessary since given the dyad field one obtains an almost complex structure by extending the rotation through $\frac{\pi}{2}$ in the two-plane spanned by the two vectors to the unique orthogonal two-plane. The sign ambiguity may be fixed by the convention that the associated two-form is anti-self-dual.

A simple example is provided by the manifold mentioned earlier: $S^1 \times S^{2n-1}$. As is well known this is a complex manifold and hence it certainly admits a complex structure. Thus the vector field which just winds around the S^1 factor can certainly be reversed by a homotopy but it is clear, by using a metric to convert the vector to a one-form and considering the line integral of the one-from around the circle, that it cannnot be reversed by a diffeomorphism. To see that $S^1 \times S^{2n-1}$ is a complex manifold one notes that $S^1 \times S^{2n-1} \equiv \mathbb{C}^{2n}/\mathbb{Z}$ where the integers $\mathbb{Z}$ act on $\mathbb{C}^{2n} \equiv \mathbb{R}^{4n}$ by $(z^1, z^2, \dots, z^n) \rightarrow (\lambda^m z^1, \lambda^m z^2, \dots \lambda^m z^n,)$ where $m \in \mathbb{Z}$ and λ is a real number not equal to zero or unity.

Now it is known, eg. from Morse theory, that the homotopy type of the fibre Y_x of loops on S^{n-1} is that of a cell-complex corresponding to the geodesic paths. Thus there is a cell corresponding to going once around the sphere, the descending directions parmetrizerized by the equatorial S^{n-1}, and next comes a cell $S^{2(n-2)}$ and so-forth. If $n > 4$ this second cell is higher in dimension than the dimension of the base M of the bundle $Y(M)$. It follows from obstruction theory that there is no obstruction to pushing points of any section s_Y in the fibre Y_x down onto the S^{n-2} of the fibre Y_x of the dyad bundle $Y(M)$. In other words we have the following

Proposition: *In dimensions greater than* 4 *a general homotopy is deformable to a linear homotopy and thus the necessary and sufficient condition for a general homotopy is the existence a global section of the* S^{n-2} *bundle* $Y(M)$, *or equivalently a global section of the dyad bundle* $V_{n,2}(M)$.

Atiyah [15] (see also Thomas [14]) has obtained some necessary conditions for the existence of a non-singular dyad field. From their work one has one has the following

Proposition *A necessary condition that a* $4k$ *dimensional manifold,* $k > 1$ *admit a time revesing homotopy is that the signature* $\tau(M)$ *be divisible* *by 4. A necessary condition that a $4k+1$ dimensional manifold admit a time-reversing homotopy is that the real Kervaire semi-characteristic $k(M)$ vanish.*

The real Kervaire semi-characteristic is defined by

$$k(M) = \sum b_{2p} \bmod 2$$

where b_{2p} are the Betti numbers, $b_{2p} = \dim H^{2p}(M;\mathbb{R})$.

4.2 Four-dimensional case

In four dimensions the situation is more delicate because the cell-decomposition of the fibre Y_x contains a 4-sphere which has the same dimension as the base. We now turn to a more detailed discussion of the four dimensional case. We begin with some general facts about S^2 and S^3 bundles over oriented four manifolds.

Firstly note that oriented 3-plane, or equivalently S^2 bundles $Q \to M$ have characteristic classes $w_2 \in H^2(M;\mathbb{Z}/2)$ and $p_1 \in H^4(M;\mathbb{Z})$ which satisfy:

$$p_1 = w_2^2 \bmod 4$$

The characteristic classes w_2 and p_1 subject to this condition determine and are determined by the bundle Q. Moreover, the bundle Q admits a cross section if and only if there exists an element $\xi \in H^2(M;\mathbb{Z})$ such that

$$\xi = w_2 \bmod 2$$

and

$$\xi^2 = p_1.$$

The class ξ may be thought of as follows. A non-zero section s of an oriented three-plane bundle gives rise to an oriented 2-plane bundle whose fibres consist of vectors orthogonal to the section s. This oriented two-plane bundle may be thought of as a complex line bundle and ξ is its first Chern class c_1.

Similarly, in four dimensions, real four-dimensional oriented vector bundles $E \to M$ determine and are determined by classes $w_2 \in H^2(M;\mathbb{Z}/2)$ and $p_1, e \in H^4(M;\mathbb{Z})$ such that

$$w_2^2 = p_1 + 2e \bmod 4.$$

Given E one may pass to the bundle of two forms $\Lambda(E)$. Giving the fibres a positive definite metric we obtain two 3-plane bundles, $\Lambda^{\pm}$ of self-dual or anti-self-dual two forms. We have

$$w_2(E) = w_2(\Lambda^+) = w_2(\Lambda^-)$$

and

$$p_1(\Lambda^{\pm}) = p_1(E) \pm 2e(E)$$

We are of course interested in the case when E is the tangent bundle of the manifold M. Then w_2 is the second Steifel-Whitney class $w_2(M)$, e its Euler class $e(M)$, and p_1 its Pontryagin class $p_1(M)$. The Pontryagin class is related to the signature τ by

$$p_1(M) = 3\tau(M),$$

moreover

$$\tau = e \bmod 2$$

Now if E admits a global section s_E (which can happen if and only if the euler class e vanishes) then the bundles $\Lambda^+(E)$ and $\Lambda^-(E)$ are isomorphic. This is because given any vector u orthogonal to the section s_E we get a self or anti-self dual two form, i.e.

$$u \wedge s_E \pm \star u \wedge s_E.$$

Thus both Λ^+ and Λ^- are isomorphic to the bundle of vectors orthogonal to s_E, $E^{\perp}$. The set of such unit vectors corresponds to the equatorial two-sphere in the three-sphere in our general discussion above.

An almost complex structure or equivalently a linear homotopy therefore exists if and only if there exists a section $u_{E^{\perp}}$. Such a section exits if and only if there exists an an element $\xi \in H^2(M;\mathbb{Z})$ such that

$$\xi = w_2 \bmod 2$$

and

$$\xi^2 = p_1 = 3\tau.$$

Now in four dimensions $F^2(M) = H^2(M;\mathbb{Z})/\mathrm{Tor}(M)$ is an integral lattice since it is equipped, via the cup product, with an integral valued bilinear product: the intersection form $Q(\ ,\ )$. By Wu's formula the second Stiefel Whitney class satisfies

$$Q(w_2, x) = Q(x, x)$$

for all $x \in F^2(M)$ and thus

$$Q(\xi, x) = Q(x, x) \bmod 2$$

for all $x \in F^2(M)$, in other words the element ξ is a so-called *characteristic* element of the integral lattice $F^2(M)$. It follows on purely arithmetic grounds, by a lemma of Van der Blij [24], that for such an element

$$Q(\xi, \xi) = \tau \bmod 8.$$

The other condition on ξ becomes

$$\xi^2 = p_1 = 3\tau$$

that is, eliminating ξ^2

$$2\tau = 0 \bmod 8$$

or

$$\tau = 0 \bmod 4.$$

Lemma *A neccessary condition that a closed Lorentz 4-manifold admit a linear homotopy is that the signature is divisible by 4*

Moreover we have also shown that

Proposition *A Lorentz 4-manifold admits a linear homotopy if and only if it admits an almost complex structure*

These conditions are non-trivial because, while for a spin manifold $\tau = 0 \bmod 16$ [14], in general one only knows that if the Euler characteristic nanishes then $\tau = 0 \bmod 2$. In fact to obtain an example of a Lorentz 4-manifold which does not admit a linear homotopy consider the connected sum of $2n$ copies of $\mathbb{CP}^2$ with $n+1$ copies of $S^1 \times S^3$. This has $\tau = 2n$, and does not admit a spin structure, even if n is a multiple of 8. Unless n is divisible by 4 it cannot admit a linear homotopy.

We may relate this discussion to the question of the existence of global sections of the bundle of dyads $V_{4,2}(M)$, a subject studied by Hirzebruch and Hopf [28]. Generically a section will have singularities isolated at points in the manifold M. Surounding each point by small 3-sphere we get a map from $S^3 \to V_{4,2}$. Since $\pi_3(V_4, 2) \equiv \mathbb{Z} \oplus \mathbb{Z}$ one has an index consisting of two integers (a, b) associated with each singularity. If $\alpha = Q(\xi, \xi)$ where ξ is a characteristic element of $H^2(M; \mathbb{Z})$ then the allowed values are given by

$$(a, b) = \frac{1}{4}(\alpha - 3\tau - 2e, \alpha - 3\tau + 2e).$$

In terms of Betti numbers one has

$$\frac{1}{4}(3\tau + 2e) = \frac{1}{4}(b^+ - b^-) + 1 - b_1$$

and

$$\frac{1}{4}(3\tau - 2e) = \frac{5}{4}(b^+ - b^-) - 1 + b_1$$

So the integrality of (a, b) is automatic. In the present case $e = 0$ and if we have a global section then there exists a characteristic element $\xi \in H^2(M; \mathbb{Z})$ such that

$$\xi^2 = 3\tau$$

This is of course the same condition that we used above.

Our necessary condition for the existence of an almost complex structure may also be obtained by considering the index of the associated Dolbeault complex. This is called the arithmetic genus, $ag(M)$. In dimension 4

$$ag(M) = \frac{1}{4}(\chi + \tau) = \frac{1}{2}(b^+ + 1 - b_1)$$

and thus under our assumption that $\chi = 0$ this again leads to the necessary condition for the existence of an almost complex structure is that the signature τ be divisible by 4.

Consider the more general problem of whether a general homotopy exists. This requires the existence of a section of the bundle $Z(M)$. As always, the potential obstructions lie in $H^i(M; \pi_{i-1}(Z_x))$. Since M is four dimensional $H^i(M; \pi_{i-1}(Z_x))$ for $i > 4$, so we need only consider $\pi_i(Z_x)$ for $i \leq 3$. Now Z_x is the space of based loops on S^3 and so

$$\pi_i(Z_x) = \pi_{i+1}(S^3).$$

The possible obstructions are therefore in $H^i(M;\pi_i(S^3))$. Thus there are two potential obstructions: the primary one, which is an element of $H^3(M;\mathbb{Z})$ and a secondary one which is an element of $H^4(M;\mathbb{Z}/2)$.

The primary obstruction coincides with the obstruction for the bundle $Y(M)$ and is the third integral Stiefel Whitney class $W_3(M)$ which is the obstruction to lifting the second $\mathbb{Z}/2$-Stiefel Whitney class $w_2 \in H^2(M;\mathbb{Z}/2)$ to an integral class ξ. It is the obstruction to the introduction of a Spin_c structure, ξ being the Chern class of the circle bundle.. This is well known to vanish for an orientable four-manifold. There remains the secondary obstruction. In the case of the bundle $Y(M)$ this vanishes if $\xi^2 = p_1$. In the case of the bundle $Z(M)$ it vanishes under the weaker condition that

$$\xi^2 = p_1 = 3\tau \bmod 8$$

But as before ξ^2 is congruent to τ mod 8 and thus we have the following **Proposition** *The necessary and sufficient condition for general homotopy is*

$$\tau = 0 \bmod 4.$$

The necessary and sufficent condition for a general homotopy is the same as the necessary condition for a linear homotopy obtained above. It is a non-trivial requirement as the examples constructed above illustrate. The remaining question, whose answer is not known at present, is whether the necessary condition is sufficient. This boils down to a purely arithmetic question about the possible intersection forms Q.

5 Some Examples

Every odd dimensional sphere S^{2r+1} admits a time-orientable Lorentz metric g^L. One takes:

$$g^L = g^R - 2\mathbf{V}^\flat \otimes \mathbf{V}^\flat,$$

where g^R is the standard round metric, $\mathbf{V}^\flat$ is the one-form dual to the vector field $\mathbf{V}$ obtained by using the musical isomorphism (i.e. lowering the index with the metric g^R) and the unit vector field $\mathbf{V}$ is tangent to the Hopf fibration. If Z^a, $a = 1, \dots 2r+2$ are complex coordinates for $\mathbb{R}^{2r+2} \equiv \mathbb{C}^{r+1}$ then the Hopf fibration corresponds to the $SO(2) \subset SO(2r+2)$ action :

$$Z^a \to \exp(it) Z^a$$

and

$$\mathbf{V} = \frac{\partial}{\partial t}.$$

The case $r = 1$ should be familiar because it is encountered in the Taub-NUT solutions of Einstein's equations. The general case also arises in higher dimensions as we shall describe later.

Atiyah's result tells us that if r is even, $r = 2k$, then the Lorentz structure described above cannot be obtained by a diffeomorphism which is connected to the identity to the Lorentz structure whose light cones differ merely by being upside down. On the other hand, we may trivially reverse the light cones by using the diffeomorphism Γ consisting of r reflections $\in O(2r+2)$, i.e. by complex conjugation:

$$\Gamma : \qquad Z^a \to \bar{Z}^{\dot{a}}.$$

Now if r is odd then Γ lies in the identity component $SO(2r+2)$ of $O(2r+2)$ and hence in the identity component $\mathrm{Diff}_0(S^{2r+1})$ of $\mathrm{Diff}(S^{2r+1})$. If however r is even, $r = 2k$, then Γ is not in the identity component of $O(4k+2)$ and, by Atiyah's result, not in $\mathrm{Diff}_0(S^{4k+2})$ either.

Thus Lorentz metrics on S^{4k+1} of the type we have been considering fall into two classes with opposite time-orientation. This is similar to the situation with respect to orientation (i.e. combined space and time-orientation). An oriented manifold may or may not be diffeomorphic to the same manifold with the opposite orientation. Manifolds which are, are called *reversible*. Manifolds which are not, are called *irreversible* or sometimes *chiral*. Of course in this latter case the diffeomorphism must lie outside the identity component $\mathrm{Diff}_0(\mathcal{M})$.

Chiral manifolds are analogous to enantiomorphic crystal forms, such as seen in quartz for example. In that case, they arise because the point group of the crystal is contained in $SO(3)$ and thus includes no orientation reversing isometries of Euclidean space. In our case, however, we are *not* requiring our diffeomorphism to be an isometry of any metric.

The spheres S^n are obviously reversible because they admit reflections. By contrast, some of the three-dimensional lens spaces, $L_{p,q}$ (with p and q co-prime) are chiral, as first noticed by Kneser. They are obtained from S^3 by identifying points under the action of the cyclic group C_p given by [12–13]:

$$Z^1 \to \exp\left(\frac{2\pi i}{p}\right) Z^1$$

$$Z^2 \to \exp\left(\frac{2\pi q i}{p}\right) Z^2$$

Because the action of the cyclic group commutes with the Hopf fibration the time-orientable Lorentz metric described above descends to all of the lens spaces.

The topological classification of the lens spaces depends on the bi-linear map:

$$\lambda : H_1(\mathcal{M}; \mathbb{Z}) \times H_1(\mathcal{M}; \mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

called the linking form defined on the first homology group $H_1(\mathcal{M}; \mathbb{Z}) \equiv \mathbb{Z}/p\mathbb{Z}$. The linking form λ changes sign under reversal of orientation.

Now the integral curve of the timelike vector field $\mathbf{V}$ gives a generator γ of $H_1(\mathcal{M}, \mathbb{Z})$ with linking invariant:

$$\lambda(\gamma, \gamma) = \frac{q}{p}.$$

The remaining elements α of $H_1(\mathcal{M}, \mathbb{Z})$ are of the form $\alpha = x\gamma$, $x = 0, 1, \ldots p-1$. The bilinearity of $\lambda(,)$ implies that

$$\lambda(\alpha, \alpha) = x^2 \frac{q}{p}.$$

To exhibit a chiral lens space it suffices to find a pair of co-prime natural numbers (p, q) such that for no $x = 0, 1, \ldots p-1$ is it true that:

$$x^2 \frac{q}{p} = -\frac{q}{p} \qquad \pmod{p}.$$

Thus $L_{3,1}$ is an example of a chiral three-dimensional spacetime. On the other hand, the action of the cyclic group C_p may or may not commute with the reflection Γ. If it does, then the action of Γ will descend to the quotient and then we can still reverse time.

We remark here that, consistent with our general idea, the partition function $Z(\mathcal{M}_3)$ for Witten's topological field theory is invariant under all diffeomorphisms whether or not they are in the identity component $\mathrm{Diff}_0(\mathcal{M})$, and obeys:

$$Z(\overline{\mathcal{M}}_3) = \overline{Z(\mathcal{M}_3)}$$

where $\overline{\mathcal{M}}$ is the same manifold as $\mathcal{M}$ but with the opposite orientation. Thus for reversible manifolds it is real, and conversely if it is complex, then the manifold must be chiral.

Turning to four-dimensional manifolds: a standard example of an irreversible four-manifold is $\mathbb{CP}^2$. Notationally one distinguishes between $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$. The Euler characters χ are the same, but the Hirzebruch signatures $\tau = b_2^+ - b_2^-$ are opposite in sign:

$$\tau(\mathbb{CP}^2) = 1 = -\tau(\overline{\mathbb{C}}\mathbb{P}^2).$$

Quite generally, a four-manifold with non-vanishing Hirzebruch signature cannot admit an orientation-reversing diffeomorphism. Now consider, for example the connected sum of $K3$ with 12 copies of $S^1 \times S^3$. This has vanishing Euler characteristic and signature 16. It therefore admits no total orientation-reversing diffeomorphism but the Lorentz structure g^L is homotopic to the time-reversed Lorentz structure. We do not know, however, whether there exist diffeomorphisms (connected to the identity or not) which will produce this time-reversal.

6 Generalized Taub-NUT Spacetimes

The four-dimensional Taub-NUT solution of Einstein's vacuum equations has provided many examples of the possible exotic behaviour of Lorentzian metrics. In this section we provide a family of higher-dimensional examples, based on some work by Bais and Batenberg [3] on the associated Riemannian metrics, which serve to illustrate our general results.

Suppose $\{\mathcal{B}, g^{\mathcal{B}}, \omega^{\mathcal{B}}\}$ is a $2p$-dimensional Einstein-Kähler manifold with Kähler form $\omega^{\mathcal{B}}$ which obeys the Dirac quantization condition, i.e., it represents an integral class

$$\left[\frac{1}{2\pi}\omega^{\mathcal{B}}\right] \in H_2(\mathcal{B};\mathbb{Z})$$

Then $\omega^{\mathcal{B}}$ may be thought of as the curvature of an S^1 bundle over $\mathcal{B}$. Let

$$e^0 = dt + A$$

where $0 \leq t < 2\pi$ be a coordinate on the S^1 fibre and A the connection such that:

$$dA = \omega^{\mathcal{B}}$$

Then the $(2p+2)$-dimensional time-orientable Lorentzian metric

$$F^{-1}(r)dr^2 + (r^2+N^2)g^{\mathcal{B}} - 4N^2F(r)e^0 \otimes e^0$$

is Ricci flat, provided

$$F(r) = \frac{r}{(r^2+N^2)^p}\int^r (s^2+N^2)^p \frac{ds}{s^2}$$

The function $F(r)$ contains two arbitrary constants, the generalized 'NUT ' charge N and an arbitary constant of integration. If $p=1$ then $\{\mathcal{B}, g^{\mathcal{B}}, \omega^{\mathcal{B}}\}$ is $\mathbb{CP}^1 \equiv S^2$, the S^1 bundle is S^3 and we recover the usual Taub-NUT solution. Indeed, when $p=1$ we have

$$F(r) = \frac{r}{(r^2+N^2)}(r - \frac{N^2}{r} - 2m)$$

where m is a constant of integration. One now recovers the usual Taub-NUT metric [5] with A = $\cos\theta \mathrm{d}\phi$, t = ψ and r = t.

For higher values of p one finds that

$$\begin{aligned} F(r) &= \frac{1}{(r^2+N^2)}(\frac{r^2p}{(2p-1)} + \frac{pN^2r^2p-2}{(2p-3)} + ... - N^2p - 2mr) \\ &= \frac{1}{(r^2+N^2)}(\frac{r^2}{(2p-1)}P(r) - N^2p - 2mr) \end{aligned}$$

where $P(r)$ is a polynomial of degree $2(p-1)$ containing only even powers of r, all of whose coefficients are positive. It follows that the numerator of F(r) has just two real roots.

In these higher p generalisations we can choose $\{\mathcal{B}, g^{\mathcal{B}}, \omega^{\mathcal{B}}\}$ to be $\mathbb{CP}^p$ and then the S^1 bundle becomes S^{2p+1} with its standard Hopf fibration. In this case, the isometry group of the spacetime is $U(p+1)$ which acts transitively on S^{2p+1} and contains a $U(1)$ factor acting as time translations. The group acting on the base $\mathcal{B}$ is $SU(2p)/\mathbb{Z}_{2p}$.

The resulting $(2p+2)$-dimensional spacetime may be thought of as a time-orientable two-plane bundle over $\mathcal{B}$ carrying an $SO(2)$-invariant Lorentzian metric on the fibres with local coordinates t, r. Its structure is independent of the particular metric on the base $\mathcal{B}$. Because the numerator of $F(\mathrm{r})$ has only two real roots the structure is qualitatively the same as that of the usual four-dimensional Taub-NUT case. In particular, the Penrose diagram is the same as that shown on page 177 of [5]. Note that if the constant of integration m is chosen to vanish, the metric has an additional discrete isometry $\mathrm{r} \to -\mathrm{r}$, interchanging different asymptotic regions.

From the point of view of this paper, we are interested in whether one can find a diffeomorphism which reverses the time-orientation. If we consider the case when $\mathcal{B} \equiv \mathbb{CP}^p$, and we confine ourselves to diffeomorphisms keeping the coordinate r fixed, then we are in the same position as above in our discussion of the odd-dimensional spheres S^{2p+1}. Thus if p is odd, we can and if p is even, we cannot reverse the sense of time by means of a diffeomorphism in the identity component $\mathrm{Diff}_0(\mathcal{M})$. Presumably this means that there is no invariant significance in the sign of the NUT charge N if p is odd but there may be if p is even. *Presumably therefore if p is odd, as it is in the usual four-dimensional case, then a Taub-NUT solution should be considered as its own anti-particle.*

7 Time-Reversal for Dynamical Systems

The topological ideas about time reversal discussed in this lecture may be applied in a different but related context. Suppose we have a finite-dimensional autonomous dynamical system with a compact phase space. That is, we have a symplectic manifold $\{\mathcal{M}_{2r}, \omega\}$ with symplectic form ω and Hamiltonian vector field

$$\mathbf{H} = \omega^{-1} dH.$$

Now time reversal is an *anti-symplectic involution*

$$f : \mathcal{M}_{2r} \to \mathcal{M}_{2r};\; f^2 = \mathrm{id}$$

such that

$$f^*\omega = -\omega.$$

and

$$f^*H = H$$

and therefore

$$f_*\mathbf{H} = -\mathbf{H}$$

The standard (non-compact) example is of course $\mathbb{R}^{2r} \equiv T^*(\mathbb{R}^r)$ for which $f : (\mathbf{q}, \mathbf{p}) \to (\mathbf{q}, -\mathbf{p})$. To get a compact example, one may replace $\mathbb{R}^r$ by any configuration space manifold $\mathcal{Q}$. If $\mathcal{Q}$ were compact and we had a Hamiltonian action of some symmetry group G, we might pass to the symplectic quotient which might be compact.

Since the r-th power:

$$\omega \wedge \ldots \wedge \omega$$

defines a volume form on $\mathcal{M}_{2r}$, time reversal is orientation-reversing if r is odd and orientation- preserving if r is even.

Thus if r is odd, it cannot live in the identity component $\mathrm{Diff}_0(\mathcal{M}_{2r})$. Therefore if r is odd and $\mathcal{M}_{2r}$ is irrevesible, then no such f can exist. *Thus if such an $\{\mathcal{M}_{2r}, \omega\}$ exists, it would mean that no dynamical sytem on this phase space, whatever its Hamiltonian function, could be invariant under time reversal!*

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