APPROXIMATING HOMOTOPY EQUIVALENCES BY HOMEOMORPHISMS.

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1. Introduction.

Let M^n, N^n be topological n-manifolds, $n \geq 5$, and let $f: M \to N$ be a proper map (i.e., a map such that inverse images of compacta are compact). The purpose of this paper is to answer the following question: *When is f close to a homeomorphism?* Our answer is phrased in terms of local homotopy restrictions on f which give us necessary and sufficient conditions for f to be close to a homeomorphism.

Here is the basic definition. If α is an open cover of N, then the proper map $f: M \to N$ is said to be an *α-equivalence* provided that for some map $g: N \to M$ there are homotopies θ_t from fg to the identity on N, and φ_t from gf to the identity on M, such that

(1) for each $m \in M$, there is a $U \in \alpha$ containing $\{f\varphi_t(m) | 0 \leq t \leq 1\}$,
(2) for each $n \in N$, there is a $U \in \alpha$ containing $\{\theta_t(n) | 0 \leq t \leq 1\}$.

Thus an α-equivalence is a special type of homotopy equivalence in which we place restrictions on the size of the homotopies $fg \simeq \text{id}$ and $gf \simeq \text{id}$. It is easy to show that if f is close to a homeomorphism, then f must be an α-equivalence (for α a fine open cover). Here is our main result.

α-Approximation Theorem. *Let N^n be an n-manifold $n \geq 5$. For every open cover α of N there is an open cover β of N such that any β-equivalence $f: M^n \to N^n$ which is already a homeomorphism from ∂M to ∂N is α-close to a homeomorphism $h: M \to N$ (i.e., for each $m \in M$, there is a $U \in \alpha$ containing $f(m)$ and $h(m)$).*

An α-approximation theorem was first proved for Q-manifolds (Q = Hilbert cube). In [5] the second author proved that if N^Q is a Q-manifold, then there is an open cover α of N^Q such that any α-equivalence $f: M^Q \to N^Q$ is close to a homeomorphism $h: M \to N$. The proof given there involved global mapping cylinder constructions which do not seem to have analogs in finite-dimensional topology. By means of these mapping cylinder constructions it was shown in [5]

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that the homeomorphism $h: M \to N$ can be chosen to depend continuously on f, thus leading to a proof that the homeomorphism group of a compact Q-manifold is an ANR. The proof we give here is a handle-by-handle approach in which we lose the continuous dependence of h on f.

In [13] Siebenmann proved that if $f: M^n \to N^n$ is a proper surjection of n-manifolds, $n \geqslant 5$, which is already a homeomorphism from ∂M to ∂N, and for which each point inverse $f^{-1}(n)$ has the Borsuk shape of a point, then f can be approximated arbitrarily closely by homeomorphisms. (Such maps f are called CE.) It follows immediately from Lacher [10] that any CE map $f: M \to N$ is an α-equivalence, for any α. Thus our α-Approximation Theorem generalizes the CE approximation theorem of [13].

Our proof of the α-Approximation Theorem is similar to Siebenmann's proof of the CE approximation theorem. In Section 3 we formulate a handle problem, which is solved by a torus trick. In Section 4 we use Siebenmann's inversion idea of [13] to solve a second handle problem. This is used in Section 5 to prove the α-Approximation Theorem by induction on handles. Of crucial importance in the handle problems of Sections 3, 4 is the Splitting Theorem, which is established in Section 6. The proof given there was suggested to the authors by R. D. Edwards and L. C. Siebenmann. It is an improvement over an earlier proof that the authors had, which was modeled on the splitting theorem of [12]. The earlier proof used handlebody theory and was limited to dimensions $\geqslant 6$.

The α-Approximation Theorem will be generalized in a subsequent paper by the second author [6]. In that paper it will be proven that the α-Approximation Theorem is true if the map f is merely assumed to be a β-domination rather than a β-equivalence. This, in turn, is used to show that maps between manifolds of dimension $\geqslant 5$ which have small point inverses are homotopic to homeomorphisms. This verifies a conjecture of Kirby and Siebenmann [13].

As an application of the α-Approximation Theorem we establish the following result, which gives us a way to homotopically detect locally trivial bundles with compact n-manifold fibers. It is a finite-dimensional version of a corresponding Q-manifold result [2].

BUNDLE THEOREM (Section 7). *Let $p: E \to B$ be a Hurewicz fibration such that E and B are locally compact metric spaces, B is locally path connected and locally finite-dimensional, and the fibers $p^{-1}(b)$ are compact n-manifolds, n fixed and $\geqslant 5$. Define $\partial E = \bigcup \{\partial p^{-1}(b) \mid b \in B\}$, and assume that $p|\partial E: \partial E \to B$ is a locally trivial bundle. Then p is also a locally trivial bundle.*

The idea of the proof given in Section 7 is to use lifting functions to establish complete regularity of the map $p: E \to B$, and then apply the main result of [7].

We also mention that the Bundle Theorem is related to a question raised by Raymond in [11]. He conjectured that a Hurewicz fibering of a compact n-manifold without boundary over an ANR must be locally trivial. Husch [8] gave an example of a Hurewicz fibering of $S^3 \times S^1$ over S^1 in which all the fibers but one are S^3, the singular fiber being S^3/α, where $\alpha \subset S^3$ is a non-cellular arc. Our theorem shows that Raymond's conjecture is true (even without assuming the total space to be a manifold) if the fibers are assumed to be manifolds of dimension ≥ 5.

2. Some Preliminaries.

The purpose of this section is to introduce some more notation and establish some results which will be needed in the sequel. All spaces will be locally compact, separable, and metric.

If $f, g: X \to Y$ are maps and α is an open cover of Y, then we say that f is *α-homotopic* to g (written $f \overset{\alpha}{\simeq} g$) if there is a homotopy $F_t: f \simeq g$, $t \in I = [0,1]$, such that the *track* of each point, $\{F_t(x) | 0 \leq t \leq 1\}$, lies in some element of α. We call F_t an *α-homotopy*. If $h: Y \to Z$ is a map and β is an open cover of Z, then we write $f \overset{h^{-1}(\beta)}{\simeq} g$ to indicate that f is $h^{-1}(\beta)$-homotopic to g, where $h^{-1}(\beta)$ is the open cover of Y defined by $h^{-1}(\beta) = \{h^{-1}(U) | U \in \beta\}$. If a metric is specified for Y and $\epsilon > 0$ is given, then we write $f \overset{\epsilon}{\simeq} g$ to indicate that there is a homotopy $F_t: f \simeq g$ such that the diameter of the track of each point is less than ϵ. If $h: Y \to Z$ is as above and a metric for Z is specified, then we write $f \overset{h^{-1}(\epsilon)}{\simeq} g$ to indicate that there is a homotopy $F_t: f \simeq g$ such that the diameter of each $h(\{F_t(x) | 0 \leq t \leq 1\})$ is less than ϵ.

Our first result is just an estimated version of the usual homotopy extension theorem.

PROPOSITION 2.1 (Estimated homotopy extension theorem). *Let $A \subset X$ be closed, and let $f_t: A \to Y$ be an α-homotopy such that f_0 extends to $\tilde{f}_0: X \to Y$. Assume that* (1) *both A and X are ANRs or* (2) *Y is an ANR. Then $\tilde{f}_0$ is α-homotopic to a map $\tilde{f}_1: X \to Y$ which extends f_1.*

Proof. (1) We proceed in the usual manner. Choose an open set U in X containing A for which there is a retraction

$$r: (X \times \{0\}) \cup (U \times I) \to (X \times \{0\}) \cup (A \times I).$$

If U is close to A, then r does not move points very far. By moving points in the I-direction we can construct a map $g: X \times I \to (X \times \{0\}) \cup (U \times I)$ which is the identity on $(X \times \{0\}) \cup (A \times I)$. Let $h: (X \times \{0\}) \cup (A \times I) \to Y$ be defined by

$h(x,0)=\tilde{f}_0(x)$ and $h(x,t)=f_t(x)$. Then we define $\tilde{f}_t:X\to Y$ by $f_t(x)=hrg(x,t)$. Note that each track $\{\tilde{f}_t(x)|0\leqslant t\leqslant 1\}$ is a single point for $x\notin U$. For $x\in U$ we may choose r and U so that the track $\{\tilde{f}_t(x)|0\leqslant t\leqslant 1\}$ is close to some $\{f_t(x')|0\leqslant t\leqslant 1\}$, where $x'\in A$. Thus $\tilde{f}_t$ is an α-homotopy.

(2) For the case in which Y is an ANR, let $h:(X\times\{0\})\cup(A\times I)\to Y$ be defined as above, and use the fact that Y is an ANR to extend h to $\tilde{h}:G\to Y$, where G is an open set in $X\times I$ containing $(X\times\{0\})\cup(A\times I)$. If the map g above is constructed appropriately, then it will carry $X\times I$ into G. Thus we can define $\tilde{f}_t:X\to Y$ by $\tilde{f}_t(x)=hg(x,t)$. Q.E.D.

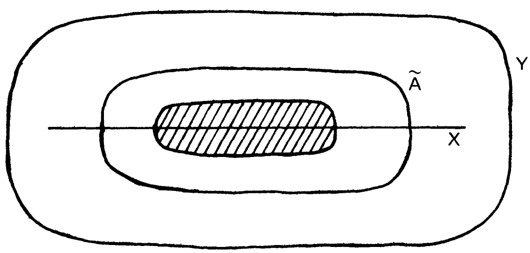
If α is an open cover of X and $A\subset X$, define

$$\mathrm{St}(A,\alpha) = \bigcup\{A\cup U|A\cap U\neq\varnothing \text{ and } U\in\alpha\}.$$

Using this, we can inductively define open covers $\mathrm{St}^n(\alpha)$ of X by $\mathrm{St}^0(\alpha)=\alpha$ and $\mathrm{St}^{n+1}(\alpha)=\{\mathrm{St}(A,\alpha)|A\in\mathrm{St}^n(\alpha)\}$. The symbol id will be overworked. It will sometimes mean the identity map of a space to itself, and other times it will be used to denote inclusion maps.

The following is a local version of the well-inown result that the notions of weak deformation retraction and strong deformation retraction are equivalent for ANR pairs. Our proof is a slight modification of the standard proof of this fact (see [15, p. 31]).

LEMMA 2.2. *Let Y be an ANR, let $A\subset Y$ be closed with closed neighborhood $\tilde{A}$, and let $X\subset Y$ be a closed ANR. For some open cover α of Y let $g:\tilde{A}\to X$ be a map such that $g\overset{\alpha}{\simeq}\mathrm{id}$ (in Y) and $g|\tilde{A}\cap X\overset{\alpha}{\simeq}\mathrm{id}$ (in X). If $\mathrm{St}(A,\mathrm{St}(\alpha))\subset\tilde{A}$, then there exists a map $\tilde{g}:A\to X$ such that $\tilde{g}|A\cap X=\mathrm{id}$ and $\tilde{g}\overset{\mathrm{St}^4(\alpha)}{\simeq}\mathrm{id}\ \mathrm{rel}A\cap X$ (in Y). (See Figure 1.)*



A = shaded region

FIGURE 1.

Proof. By Proposition 2.1 we conclude that g is α-homotopic to $g_1:\tilde{A}\to X$ such that $g_1|\tilde{A}\cap X=\text{id}$. Thus $g_1 \overset{\text{St}(\alpha)}{\simeq} \text{id}$ (in Y). Let $F:\tilde{A}\times I\to Y$ be such a homotopy, where $F_0=\text{id}$ and $F_1=g_1$. Define a homotopy

$$G:[(A\times\{0\})\cup((A\cap X)\times I)\cup(A\times\{1\})]\times J\to Y \qquad (J=[0,1])$$

by the equations

$$\begin{aligned} &G((x,0),t)=x && \text{for } x\in A,\\ &G((x,s),t)=F(x,(1-t)s) && \text{for } x\in A\cap X,\\ &G((x,1),t)=F(g_1(x),1-t) && \text{for } x\in A. \end{aligned}$$

Observe that in order for the third equation to make sense we must have $g_1(A)\subset\tilde{A}$. This is the reason for requiring that $\text{St}(A,\text{St}(\alpha))$ be contained in $\tilde{A}$. Also note that $g_1\circ g_1|A=g_1|A$, because $g_1(A)\subset\tilde{A}\cap X$.

Note that G_0 can be extended to $F|A\times I$ and G is a $\text{St}(\alpha)$-homotopy. By Proposition 2.1 we can extend G_1 to a map $H:A\times I\to Y$ which is $\text{St}(\alpha)$-homotopic to $F|A\times I$. This implies that H is a $\text{St}^4(\alpha)$-homotopy. Then $\tilde{g}=g_1|A$ fulfills our requirements. Q.E.D.

Let $f:X\to Y$ be a proper map, let $A\subset Y$, and let α be an open cover of Y. We say that f is an *α-equivalence over A* if there exists a map $g:A\to X$ such that

$$fg\overset{\alpha}{\simeq}\text{id} \quad\text{and}\quad gf|f^{-1}(A)\overset{f^{-1}(\alpha)}{\simeq}\text{id}.$$

We call g an *α-inverse for f over A*. If $h:Y\to Z$ and a metric is specified for Z, then f is an *$h^{-1}(\epsilon)$-equivalence over A* if there is a $g:A\to X$ such that $fg\overset{h^{-1}(\epsilon)}{\simeq}\text{id}$ and $gf|f^{-1}(A)\overset{(hf)^{-1}(\epsilon)}{\simeq}\text{id}$. Note that if $f:X\to Y$ is an α-equivalence over Y, then f must be an α-equivalence in the sense of Section 1. Also note that if $f:X\to Y$ is an α-equivalence, $U\subset Y$ is open, and $A\subset U$, then the restriction $f|f^{-1}(U):f^{-1}(U)\to U$ is an α-equivalence over A provided α is chosen so that no element of $\text{St}(A,\alpha)$ meets $Y-U$. For example, if A is closed, α can always be chosen fine enough so that this is the case.

If $f:X\to Y$ is a proper map, then the *mapping cylinder* of $f, M(f)$, is the decomposition space $(X\times I)\cup Y/\sim$, where $\sim$ is the minimal equivalence relation generated by $(x,1)\sim f(x)$. We represent $M(f)$ by $X\times[0,1)\cup Y$, so that $\lim_{t\to 1}(x,t)=f(x)$. Thus Y is naturally identified with a subset of $M(f)$, called the *base*, and we identify X with $X\times\{0\}$ in $M(f)$, called the *top*. There is a natural retraction $c:M(f)\to Y$ defined by $c|Y=\mathrm{id}$ and $c(x,t)=f(x)$. It is called the *collapse to the base*. The intervals $\{x\}\times[0,1)\cup\{f(x)\}$ in $M(f)$ are called the *rays* of $M(f)$.

The following result will be needed several times in the sequel. It says that we can sew together α-equivalences.

PROPOSITION 2.3. *Let Y be an ANR with open cover α, and let $A_1,A_2\subset Y$ be closed with closed neighborhoods $\tilde{A}_1,\tilde{A}_2$, respectively. Then there exists an open cover β of Y such that if X is any ANR and $f:X\to Y$ is a proper map which is a β-equivalence over $\tilde{A}_1$ and $\tilde{A}_2$, then f is an α-equivalence over $A_1\cup A_2$.*

Proof. Form the mapping cylinder, $M(f)$, which is an ANR, and let $c:M(f)\to Y$ denote the collapse to the base. Since f is a β-equivalence over $\tilde{A}_i$, we have a map $g_i:\tilde{A}_i\to X$ such that

$$fg_i\overset{\beta}{\simeq}\mathrm{id}\quad\text{and}\quad g_if|f^{-1}(\tilde{A}_i)\overset{f^{-1}(\beta)}{\simeq}\mathrm{id}.$$

This gives us a map $g_i':c^{-1}(\tilde{A}_i)\to X$ defined by $g_i'=g_ic|c^{-1}(\tilde{A}_i)$. Certainly

$$g_i'|c^{-1}(\tilde{A}_i)\cap X\overset{c^{-1}(\beta)}{\simeq}\mathrm{id}\qquad(\text{in }X),$$

and we have a homotopy $g_i'\overset{c^{-1}(\beta)}{\simeq}\mathrm{id}$ (in $M(f)$ given by $g_i'=g_ic\simeq fg_ic\simeq c\simeq\mathrm{id}$, where the first homotopy arises by deforming down the rays of $M(f)$, the second arises from $fg_i\simeq\mathrm{id}$, and the third arises by coming back up the rays of $M(f)$.

Using Lemma 2.2 we can choose β such that there is a closed neighborhood $\bar{A}_i$ of A_i, $\bar{A}_i\subset\tilde{A}_i$, and a map $\bar{g}_i:c^{-1}(\bar{A}_i)\to X$ such that $\bar{g}_i|c^{-1}(\bar{A}_i)\cap X=\mathrm{id}$ and $\bar{g}_i\simeq\mathrm{id}\ \mathrm{rel}\,c^{-1}(\bar{A}_i)\cap X$, where this is a $\mathrm{St}^4c^{-1}(\beta)$-homotopy. Let $F^i:c^{-1}(\bar{A}_i)\times I\to M(f)$ be such a homotopy for which $F_0^i=\bar{g}_i$ and $F_1^i=\mathrm{id}$.

Let $\varphi:A_1\cup A_2\to I$ be a map such that $\varphi(A_1)=\{0\}$ and $\varphi(A_2-\bar{A}_1)=\{1\}$. Define $h_1:c^{-1}(A_1\cup A_2)\to M(f)$ by

$$h_1(x)=\begin{cases}x & \text{for}\quad x\in c^{-1}(A_2-\bar{A}_1),\\ F^1(x,\varphi c(x)) & \text{for}\quad x\in c^{-1}\big(\bar{A}_1\cap(A_1\cup A_2)\big).\end{cases}$$

The effect of h_1 is to take $c^{-1}(A_1)$ into X. Clearly h_1 is $c^{-1}\mathrm{St}^4(\beta)$-homotopic to id $\mathrm{rel}\,f^{-1}(A_1\cup A_2)$. If β is chosen so that $h_1c^{-1}(A_2)\subset c^{-1}(\bar{A}_2)$, then we define $h_2:c^{-1}(A_1\cup A_2)\to X$ by

$$h_2(x)=\begin{cases}h_1(x) & \text{for}\quad x\in c^{-1}(A_1),\\ F_1^2h_1(x) & \text{for}\quad x\in c^{-1}(A_2).\end{cases}$$

Since F^1 and F^2 are $c^{-1}\mathrm{St}^4(\beta)$-homotopies, we conclude that h_2 is $c^{-1}\mathrm{St}^9(\beta)$-homotopic to id rel $f^{-1}(A_1 \cup A_2)$.

Now define $g: A_1 \cup A_2 \to X$ by $g = h_2 | A_1 \cup A_2$. We have $fg = ch_2 | A_1 \cup A_2$, and this is $\mathrm{St}^9(\beta)$-homotopic to id. Also $gf | f^{-1}(A_1 \cup A_2) = h_2 c | f^{-1}(A_1 \cup A_2)$, which is $f^{-1}\mathrm{St}^9(\beta)$-homotopic to id. Q.E.D.

We use R^n to denote euclidean n-space and $rB^n = [-r, r]^n \subset R^n$, with $1B^n = B^n$. In the usual manner, ∂B^n denotes the boundary of B^n (also written S^{n-1}), and $\mathring{B}^n$ denotes its interior. The *standard norm* on R^n is given by

$$\|x - y\| = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2},$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. In general, if M is a topological manifold, ∂M will denote its boundary. (We will not need a symbol for topological boundary.) If $A \subset X$, then $\mathring{A}$ will denote the topological interior of A in X. If X has a specified metric and $x \in X$, then $B_\epsilon(X)$ will denote the open ϵ-ball in X around x.

If $f, g: X \to Y$ are maps and $A \subset Y$, then the statement $f = g$ *over* A means that $f^{-1}(A) = g^{-1}(A)$ and $f | f^{-1}(A) = g | f^{-1}(A)$. The statement f *is a homeomorphism over* A means that $f | f^{-1}(A): f^{-1}(A) \to A$ is a homeomorphism.

3. The Handle Lemma.

The purpose of this section is to establish the Handle Lemma, which is the main step in the proof of the Handle Theorem of Section 4. Our proof is based on the proof of an analogous result from [13] (see Main Lemma 2.2 of [13]), in which all the maps are CE. Some care must be taken to avoid this restriction to CE maps. In particular, we will need the Splitting Theorem of Section 6, but it is not needed until steps III and IV of the proof given below, so there is no need to burden the reader with a statement at this point.

For notation, let V^n be a topological manifold, $n = m + k \geqslant 5$, and let $f: V \to B^k \times R^m$ be a proper map such that $\partial V = f^{-1}(\partial B^k \times R^m)$ and f is a homeomorphism over $(B^k - \frac{1}{2}\mathring{B}^k) \times R^m$.

HANDLE LEMMA. *For every* $\epsilon > 0$ *there exists a* $\delta > 0$ *so that if* f *is a* δ*-equivalence over* $B^k \times 3B^m$ *and* $m \geqslant 1$, *then*

(1) *there exists an* ϵ*-equivalence* $F: B^k \times R^m \to B^k \times R^m$ *such that* $F = \mathrm{id}$ *over* $(B^k - \frac{5}{6}\mathring{B}^k) \times R^m \cup B^k \times (R^m - 4\mathring{B}^m)$,

(2) *there exists a homeomorphism* $\varphi: f^{-1}(U) \to F^{-1}(U)$ *such that* $F\varphi = f | f^{-1}(U)$, *where* $U = (B^k - \frac{5}{6}\mathring{B}^k) \times R^m \cup B^k \times 2B^m$.

Remarks. (1) The δ in the above statement depends only on n and ϵ.

(2) The ϵ and δ in the above statement are calculated with respect to the standard metric on $B^k \times R^m$ (see Section 2).

Proof of the Handle Lemma. We first set up the notation necessary to apply the now standard torus wrapping and unwrapping trick. Let $S^1 \subset R^2$ be the set of complex numbers of absolute value 1, and let $e: R \to S^1$ be the covering projection defined by $e(x) = e^{\pi i x/4}$. The product covering projection of R^m onto the m-torus $T^m = S^1 \times \cdots \times S^1$ is given by $e^m = e \times \cdots \times e$. Choose a point $x_0 \in T^m - e^m(2B^m)$, and let $T_0^m = T^m - \{x_0\}$, the punctured torus. A map is an *immersion* if it is locally an open embedding, and it follows from [4] that there is an immersion $i: T_0^m \to 2.5\mathring{B}^m$. By the Schoenflies theorem we can adjust i so that $ie^m|2B^m: 2B^m \to 2B^m$ is the identity (see [9, p. 48] for more details). We will work our way through the following diagram of spaces and maps. The ϵ-equivalence F that we are looking for appears at the top of the diagram in Figure 2.

$$
\begin{array}{ccccc}
B^k \times R^m & & \xrightarrow{\quad F \quad} & & B^k \times R^m \\
\uparrow{\scriptstyle j} & & & & \uparrow{\scriptstyle j} \\
B^k \times R^m & & \xrightarrow{\quad F' \quad} & & B^k \times R^m \\
\downarrow{\scriptstyle \mathrm{id} \times e^m} & & & & \downarrow{\scriptstyle \mathrm{id} \times e^m} \\
B^k \times T^m & \xleftarrow{h} & W_3 & \xrightarrow{f_3} & B^k \times T^m \\
 & & \cup & & \cup \\
 & & W_2 & \xrightarrow{f_2} & B^k \times T^m - \mathring{D}^n \\
 & & \cap & & \cap \\
 & & W_1 & \xrightarrow{f_1} & (B^k \times T^m) - (\tfrac{2}{3}B^k \times \{x_0\}) \\
 & & \cup & & \cup \\
 & & W_0 & \xrightarrow{f_0} & B^k \times T_0^m \\
 & & \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle \mathrm{id} \times i} \\
 & & V & \xrightarrow{f} & B^k \times R^m
\end{array}
$$

FIGURE 2.

I. Construction of W_0. W_0 is formed by taking the pullback,

$$W_0 = \{(x,y) \in V \times (B^k \times T_0^m) | f(x) = (\mathrm{id} \times i)(y)\}.$$

Here i_0 and f_0 are restrictions of projection maps. It is easy to see that i_0 is an immersion, so W_0 is a topological n-manifold. Also $f_0^{-1}(\partial B^k \times T_0^m) = \partial W_0$, and f_0 is a homeomorphism over $(B^k - \frac{1}{2}\mathring{B}^k) \times T_0^m$.

Now write $T_0^m = Y_0 \cup (S^{m-1} \times [0,\infty))$ and let $Y_t = Y_0 \cup (S^{m-1} \times [0,t])$. We arrange notation so that $Y_0 \cap (S^{m-1} \times [0,\infty)) = S^{m-1} \times \{0\}$ and $S^{m-1} \times \{0\}$ is collared in Y_0. Note that $T^m - \mathring{Y}_t$ is an m-ball containing x_0.

ASSERTION. *For any $\delta_0 > 0$ we can choose $\delta > 0$ small enough so that f_0 is a δ_0-equivalence over $B^k \times Y_3$ (where δ_0 is calculated with respect to any convenient choice of a metric on $B^k \times T_0^m$.)*

Proof. By definition we have a map $g: B^k \times 3B^m \to V$ such that

$$fg \overset{\delta}{\simeq} \mathrm{id} \quad \text{and} \quad gf | f^{-1}(B^k \times 3B^m) \overset{f^{-1}(\delta)}{\simeq} \mathrm{id}.$$

We need a map $g_0: B^k \times Y_3 \to W_0$ such that $f_0 g_0 \overset{\delta_0}{\simeq} \mathrm{id}$ and

$$g_0 f_0 | f_0^{-1}(B^k \times Y_3) \overset{f_0^{-1}(\delta_0)}{\simeq} \mathrm{id}.$$

Since Y_3 is compact, we can choose $\mu > 0$ so that $\mathrm{id} \times i | B_\mu(y)$ is 1-1 for each $y \in B^k \times Y_3$. Choose δ small enough so that $B_\delta((\mathrm{id} \times i)(y)) \subset (\mathrm{id} \times i) B_\mu(y)$ for all $y \in B^k \times Y_3$. This enables us to define $g_0: B^k \times Y_3 \to W_0$ by

$$g_0(y) = \Big(g(\mathrm{id} \times i)(y), \big[(\mathrm{id} \times i) | B_\mu(y)\big]^{-1}(fg(\mathrm{id} \times i)(y))\Big).$$

We need to check that g_0 fulfills our requirements.

By definition we have $f_0 g_0(y) \in B_\mu(y)$, which implies that $f_0 g_0 \overset{\delta_0}{\simeq} \mathrm{id}$ for μ small. Let $\varphi_t: f^{-1}(B^k \times 3B^m) \to V$ be a $f^{-1}(\delta)$-homotopy such that $\varphi_0 = gf | f^{-1}(B^k \times 3B^m)$ and $\varphi_1 = \mathrm{id}$. Consider the homotopy $\theta_t: f_0^{-1}(B^k \times Y_3) \to W_0$ defined by

$$\theta_t(x,y) = \Big(\varphi_t(x), \big[(\mathrm{id} \times i) | B_\mu(y)\big]^{-1}(f\varphi_t(x))\Big).$$

Then θ_t is an $f_0^{-1}(\mu)$-homotopy of $g_0 f_0 | f_0^{-1}(B^k \times Y_3)$ to id. Q.E.D.

II. Construction of W_1. By adding a copy of $(B^k - \frac{2}{3}\mathring{B}^k) \times \{x_0\}$ to W_0 we can form a manifold W_1 containing W_0 so that f_0 extends to a proper map

$f_1: W_1 \to (B^k \times T^m) - (\frac{2}{3}B^k \times \{x_0\})$ which is a homeomorphism over $(B^k - \frac{2}{3}B^k) \times T^m$. Using Proposition 2.3, it follows that f_1 is a δ_1-equivalence over

$$B^k \times T^m - \left[\tfrac{3}{4}\mathring{B}^k \times (T^m - Y_2)\right],$$

for a small choice of δ_0.

III. Construction of W_2. Consider the open set

$$G = \left[\tfrac{4}{5}\mathring{B}^k \times (T^m - Y_1)\right] - \left[\tfrac{3}{4}B^k \times \left(T^m - \mathring{Y}_2\right)\right],$$

which is a copy of $S^{n-1} \times R$. For the moment let us identify G with $S^{n-1} \times R$. Then we see that if δ_1 is sufficiently small, f_1 restricts to a proper map $f_1 | f_1^{-1}(S^{n-1} \times R): f_1^{-1}(S^{n-1} \times R) \to S^{n-1} \times R$ which is a δ_1-equivalence over $S^{n-1} \times [-2,2]$. If δ_1 is sufficiently small, then it is possible to find an $(n-1)$-sphere $S \subset f_1^{-1}(S^{n-1} \times (-1,1))$ which is bicollared, which separates $f_1^{-1}(S^{n-1} \times \{-1\})$ from $f_1^{-1}(S^{n-1} \times \{1\})$, and for which $f_1 | S: S \to S^{n-1} \times R$ is a homotopy equivalence. The existence of S follows immediately from the Splitting Theorem of Section 6. Observe that we do not require the full strength of the Splitting Theorem, for the Splitting Theorem only requires δ_1-control in the R-factor. Since we are given that $f_1 | f_1^{-1}(S^{n-1} \times R)$ is a δ_1-equivalence over $S^{n-1} \times [-2,2]$, we have δ_1-control in both the S^{n-1} and R-factors.

Define an n-ball by

$$D^n = \tfrac{3}{4}B^k \times \left(T^m - \mathring{Y}_2\right),$$

and let W_2 be the closure of the component of $W_1 - S$ containing $f_1^{-1}(Y_0)$. Our map $f_2: W_2 \to B^k \times T^m - \mathring{D}^n$ is defined by $f_2 = f_1 | W_2$. This makes sense, because for δ_1 small we must have $f_1(W_2) \subset B^k \times T^m - \mathring{D}^n$.

IV. Construction of W_3. W_3 is constructed from W_2 by attaching to W_2 the cone over S. W_3 is a compact n-manifold which is homotopy equivalent to $B^k \times T^m$. In fact, we will show that for δ_1 small, there is a δ_3-equivalence $f_3: W_3 \to B^k \times T^m$ which agrees with f_1 over $(B^k - \frac{5}{6}\mathring{B}^k) \times T^m \cup B^k \times Y_0$. We start by extending $f_2: W_2 \to B^k \times T^m - \mathring{D}^n$ to $f_3': W_3 \to B^k \times T^m$ so that f_3' takes $W_3 - \mathring{W}_2$ into $\frac{4}{5}\mathring{B}^k \times (T^m - Y_1)$. The remainder of this step is concerned with showing that f_3' can be modified to get our desired δ_3-equivalence $f_3: W_3 \to B^k \times T^m$. The argument is straightforward but tedious. We suggest that on first reading the reader should accept this and go on to step V.

Choose evenly spaced balls D_i^n, $0 \leqslant i \leqslant 5$, such that $D_0^n = D^n$, $D_5^n = \frac{4}{5}B^k \times (T^m - \mathring{Y}_1)$, and $D_i^n \subset \mathring{D}_{i+1}^n$. We may assume that $S \subset f_1^{-1}(\mathring{D}_1^n - D^n)$ and $f_3'(W_3 -$

$\mathring{W}_2) \subset \mathring{D}_1^n$. It follows from the Addendum to the Splitting Theorem that for δ_1 sufficiently small, $f_1^{-1}(D_i^n) \cap W_2$ deforms into S rel S, with the deformation taking place in $f_1^{-1}(D_{i+1}^n) \cap W_2$, for $1 \leqslant i \leqslant 4$. This implies that $(f_3')^{-1}(D_i^n)$ contracts to a point in $(f_3')^{-1}(D_{i+1}^n)$ for $1 \leqslant i \leqslant 4$. To see this we just note that

$$(f_3')^{-1}(D_i^n) = \left(f_1^{-1}(D_i^n) \cap W_2\right) \cup \left(W_3 - \mathring{W}_2\right),$$

which is easily checked.

We know that $f_1 : W_1 \to B^k \times T^m - (\frac{2}{3} B^k \times \{x_0\})$ is a δ_1-equivalence over $B^k \times T^m - \mathring{D}^n$, so we can choose a δ_1-inverse $g_1 : B^k \times T^m - \mathring{D}^n \to W_1$. Let $\bar{g}_3 : B^k \times T^m - \mathring{D}_2^n \to W_2$ be defined by $\bar{g}_3 = g_1 | B^k \times T^m - \mathring{D}_2^n$. [This is certainly defined, because for δ_1 small we must have $g_1(B^k \times T^m - \mathring{D}_2^n) \subset f_1^{-1}(B^k \times T^m - \mathring{D}_1^n)$, which lies in W_2.] Note that for δ_1 small we must have $\bar{g}_3(\partial D_2^n) \subset f_1^{-1}(D_3^n) \cap W_2$, which lies in $(f_3')^{-1}(D_3^n)$. Since $(f_3')^{-1}(D_3^n)$ contracts to a point in $(f_3')^{-1}(D_4^n)$, we can extend $\bar{g}_3$ to $g_3' : B^k \times T^m \to W_3$ so that $g_3'(D_2^n) \subset (f_3')^{-1}(D_4^n)$.

ASSERTION 1. *$f_3' g_3' \simeq \mathrm{id}$, with a homotopy that is a δ_1-homotopy on $B^k \times T^m - \mathring{D}_2^n$, and on D_2^n it takes place in D_3^n.*

Proof. We have $f_3' g_3' = f_1 g_1$ on $B^k \times T^m - \mathring{D}_2^n$, and since g_1 is a δ_1-inverse of f_1, we have

$$f_3' g_3' | B^k \times T^m - \mathring{D}_2^n \overset{\delta_1}{\simeq} \mathrm{id}.$$

If δ_1 is small, then the restriction of this homotopy to ∂D_2^n takes place in D_3^n. Thus $f_3' g_3' | \partial D_2^n \simeq \mathrm{id}$ extends to a homotopy $f_3' g_3' | D_2^n \simeq \mathrm{id}$ which takes place in D_3^n. Q.E.D.

ASSERTION 2. *$g_3' f_3' \simeq \mathrm{id}$, with a homotopy that is an $(f_3')^{-1}(\delta_1)$-homotopy on $(f_3')^{-1}(B^k \times T^m - \mathring{D}_3^n)$, and on $(f_3')^{-1}(D_3^n)$ it takes place in $(f_3')^{-1}(D_5^n)$.*

Proof. We have $g_3' f_3' = g_1 f_1$ on $(f_3')^{-1}(B^k \times T^m - \mathring{D}_3^n) = f_1^{-1}(B^k \times T^m - \mathring{D}_3^n)$, and thus have an $f_1^{-1}(\delta_1)$-homotopy $g_3' f_3' | (f_3')^{-1}(B^k \times T^m - \mathring{D}_3^n) \simeq \mathrm{id}$. This homotopy takes place in $f_1^{-1}(B^k \times T^m - \mathring{D}_2^n) = (f_3')^{-1}(B^k \times T^m - \mathring{D}_2^n)$, so it is an $(f_3')^{-1}(\delta_1)$-homotopy. The restriction of this homotopy to $(f_3')^{-1}(\partial D_3^n)$ takes place in $(f_3')^{-1}(D_4^n)$. But $(f_3')^{-1}(D_4^n)$ contracts to a point in $(f_3')^{-1}(D_5^n)$, so $g_3' f_3' | (f_3')^{-1}(\partial D_3^n) \simeq \mathrm{id}$ extends to a homotopy $g_3' f_3' | (f_3')^{-1}(D_3^n) \simeq \mathrm{id}$ which takes place in $(f_3')^{-1}(D_5^n)$. Q.E.D.

We note that if D_5^n were a small ball in $B^k \times T^m$, then f_3' would be a δ_1-equivalence and we would be done. To remedy this let $\theta : B^k \times T^m \to B^k \times T^m$ be a homeomorphism such that $\theta(D_5^n)$ has small diameter $(<\delta_3)$ and which is supported on $\frac{5}{6} B^k \times (T^m - \mathring{Y}_0)$. Define $f_3 = \theta f_3' : W_3 \to B^k \times T^m$ and $g_3 =$

$g_3'\theta^{-1}: B^k \times T^m \to W_3$. We have

$$f_3 g_3 = \theta(f_3' g_3')\theta^{-1} \simeq \theta(\mathrm{id})\theta^{-1} = \mathrm{id}.$$

If δ_1 is small, then this must be a δ_3-homotopy. Moreover, if δ_1 is small, then $g_3 f_3 = g'_3\theta^{-1}\theta f_3' = g'_3 f_3' \simeq \mathrm{id}$ is an $f_3^{-1}(\delta_3)$-homotopy.

V. Construction of h. We want $h: W_3 \to B^k \times T^m$ to be a homeomorphism which agrees with f_3 over $(B^k - \frac{5}{6}\mathring{B}^k) \times T^m$ and which is homotopic to f_3. For details see [13, p. 280], where three proofs are given.

VI. Construction of F'. $F': B^k \times R^m \to B^k \times R^m$ is the covering of $f_3 h^{-1}$ which is the identity on $(B^k - \frac{5}{6}\mathring{B}^k) \times T^m$. Since $f_3 h^{-1} \simeq \mathrm{id}$, it follows from elementary covering-space theory that F' is *bounded*, i.e.,

$$\{\|F'(x) - x\| \mid x \in B^k \times R^m\}$$

is bounded above, where $\| \ \|$ is the standard norm on R^{k+m} (see Section 2). If δ_3 is small, then we can prove that F' is an ϵ-equivalence (just as we proved that f_0 is a δ_0-equivalence).

VII. Construction of j. Define $J: R^n \to 4\mathring{B}^k \times 4\mathring{B}^m$ to be the radial homeomorphism which is fixed on $2B^k \times 2B^m$. Then $j: B^k \times R^m \to B^k \times R^m$ is defined by restricting J. It follows that j is an open embedding.

VIII. Construction of F. We define $F: B^k \times R^m \to B^k \times R^m$ as follows:

$$F(x) = \begin{cases} jF'j^{-1}(x) & \text{for } x \in j(B^k \times R^m), \\ x & \text{for } x \notin j(B^k \times R^m). \end{cases}$$

We observe that $F = \mathrm{id}$ on $[(B^k - \frac{5}{6}\mathring{B}^k) \times R^m] \cup [B^k \times (R^m - 4\mathring{B}^m)]$, $F = F'j^{-1}$ over $B^k \times 2B^m$, and F is still an ϵ-equivalence.

IX. Construction of φ. It is easy to check that we have a commutative diagram,

$$\begin{array}{ccc} F^{-1}(B^k \times 2B^m) & \xrightarrow{F} & B^k \times 2B^m \\ \downarrow{\scriptstyle h(\mathrm{id}\times e^m)j^{-1}} & & \downarrow{\scriptstyle \mathrm{id}\times e^m} \\ f_0^{-1}(\mathrm{id}\times e^m)(B^k \times 2B^m) & \xrightarrow{f_0} & (\mathrm{id}\times e^m)(B^k \times 2B^m) \\ \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle \mathrm{id}\times i} \\ f^{-1}(B^k \times 2B^m) & \xrightarrow{f} & B^k \times 2B^m \end{array}$$

The vertical arrows are homeomorphisms, and by composing the inverses of the two on the left we get a homeomorphism

$$\psi : f^{-1}(B^k \times 2B^m) \to F^{-1}(B^k \times 2B^m)$$

which satisfies $F\psi = f | f^{-1}(B^k \times 2B^m)$ (recall that $ie^m | 2B^m = \mathrm{id}$). Note that $\psi = f$ over $(B^k - \frac{5}{6}\mathring{B}^k) \times 2B^m$. Thus ψ extends to a homeomorphism $\varphi : f^{-1}(U) \to F^{-1}(U)$ by defining $\varphi = f$ on $f^{-1}\big((B^k - \frac{5}{6}\mathring{B}^k) \times R^m\big)$. Q.E.D.

4. The Handle Theorem

We now use the Handle Lemma to prove the main result needed in the proof of the α-Approximation Theorem. Our strategy is to use the inversion trick of [13] to switch the roles of 0 and ∞ in the Handle Lemma.

For notation let V^n be a topological manifold, $n = m + k \geqslant 5$, and let $f : V \to B^k \times R^m$ be a proper map such that $\partial V = f^{-1}(\partial B^k \times R^m)$ and f is a homeomorphism over $(B^k - \frac{1}{2}\mathring{B}^k) \times R^m$.

MAIN THEOREM. *For every $\epsilon > 0$ there exists a $\delta > 0$ so that if f is a δ-equivalence over $B^k \times 3B^m$, then there exists a proper map $\tilde{f} : V \to B^k \times R^m$ such that*

(1) *$\tilde{f}$ is an ϵ-equivalence over $B^k \times 2.5B^m$,*
(2) *$\tilde{f} = f$ over $[(B^k - \frac{2}{3}\mathring{B}^k) \times R^m] \cup [B^k \times (R^m - 2\mathring{B}^m)]$,*
(3) *$\tilde{f}$ is a homeomorphism over $B^k \times B^m$.*

Remark. As in the Handle Lemma, δ depends only on n and ϵ.

Proof of the Handle Theorem. We first treat the somewhat easier case $m = 0$. In this case V is compact, and $f : V^n \to B^n$ is a δ-equivalence which is a homeomorphism over $B^n - \frac{1}{2}\mathring{B}^n$. We want an ϵ-equivalence $\tilde{f} : V \to B^n$ such that $\tilde{f} = f$ over $B^n - \frac{2}{3}\mathring{B}^n$ and $\tilde{f}$ is a homeomorphism (and therefore $\tilde{f}$ will automatically be an ϵ-equivalence). To see this we only need to note that $f^{-1}(\frac{2}{3}B^n)$ is a contractible n-manifold bounded by an $(n-1)$-sphere, and therefore it must be an n-ball. Clearly $f | f^{-1}(B^n - \frac{2}{3}\mathring{B}^n)$ extends to our desired homeomorphism $\tilde{f}$.

We now treat the cases $m \geqslant 1$. For any $\delta_1 > 0$ we can choose δ small enough so that if f is a δ-equivalence over $B^k \times 3B^m$, then there exists a δ_1-equivalence $F : B^k \times R^m \to B^k \times R^m$ and a homeomorphism $\varphi : f^{-1}(U) \to F^{-1}(U)$ as described in the Handle Lemma. Consider the restriction

$$B^k \times R^m - F^{-1}(B^k \times \{0\}) \xrightarrow{F|} B^k \times (R^m - \{0\}).$$

For any compactum in $B^k \times (R^m - \{0\})$, δ_1 can be chosen small enough so that $F|$ is a δ_1-equivalence over this compactum. Since $F = \mathrm{id}$ over a neighborhood of ∞, we can identify S^m with $R^m \cup \{\infty\}$ and extend $F|$ to $F_1 : V_1 \to B^k \times (S^m - \{0\})$, where V_1 is an n-manifold. This extension can be carried out so that F_1 is a homeomorphism over $(B^k - \frac{5}{6}\mathring{B}^k) \times (S^m - \{0\})$. Also F_1 will be a δ_1-equivalence over any conveniently chosen compactum in $B^k \times (S^m - \{0\})$.

Again using the Handle Lemma, there exists (for δ_1 small) a δ_2-equivalence $F_2 : B^k \times (S^m - \{0\}) \to B^k \times (S^m - \{0\})$ such that $F_2 = \mathrm{id}$ on

$$\left[\left(B^k - \tfrac{6}{7}\mathring{B}^k\right) \times (S^m - \{0\})\right] \cup \left[B^k \times (B^m - \{0\})\right],$$

and there exists a homeomorphism $\varphi_1 : F_1^{-1}(U_1) \to F_2^{-1}(U_1)$ such that $F_2\varphi_1 = F_1 | F_1^{-1}(U_1)$, where

$$U_1 = \left[\left(B^k - \tfrac{6}{7}\mathring{B}^k\right) \times (S^m - \{0\})\right] \cup \left[B^k \times \left(S^m - \tfrac{3}{2}\mathring{B}^m\right)\right].$$

Note that $F_2\varphi_1\varphi = f$ over

$$\left[\left(B^k - \tfrac{6}{7}\mathring{B}^k\right) \times (R^m - \{0\})\right] \cup \left[B^k \times \left(2B^m - \tfrac{3}{2}\mathring{B}^m\right)\right].$$

Extend F_2 to $\tilde{F}_2 : B^k \times S^m \to B^k \times S^m$ by defining $\tilde{F}_2 | B^k \times \{0\} = \mathrm{id}$. Then $\tilde{F}_2$ is still a δ_2-equivalence.

Now consider the open set

$$G = \left(\tfrac{7}{8}\mathring{B}^k \times 2\mathring{B}^m\right) - \left(\tfrac{6}{7}B^k \times \tfrac{3}{2}B^m\right),$$

which is homeomorphic to $S^{n-1} \times R$. Using the Splitting Theorem we can construct a bicollared submanifold S of $f^{-1}(G)$ such that $f|S : S \to G$ is a homotopy equivalence and S is an $(n-1)$-sphere.

Assertion. *S bounds an n-ball B in $f^{-1}(B^k \times 2\mathring{B}^m)$ which contains $f^{-1}(\frac{6}{7}B^k \times \frac{3}{2}B^m)$.*

Proof. (Compare with step IV in the proof of the Handle Lemma.) Choose evenly spaced n-balls D_i^n, $0 \leqslant i \leqslant 2$, such that $D_0^n = \frac{6}{7}B^k \times \frac{3}{2}B^m$, $D_2^n = \frac{7}{8}B^k \times 2B^m$, and $D_i^n \subset \mathring{D}_{i+1}^n$. We may assume that $S \subset f^{-1}(\mathring{D}_1^n - D_0^n)$. Let B be the compact connected n-manifold in $f^{-1}(\mathring{D}_2^n)$ bounded by S. We will prove that B is contractible. For δ small it is clear that B contracts to a point in $f^{-1}(\mathring{D}_2^n)$. By the Addendum to the Splitting Theorem there exists a retraction of $f^{-1}(D_2^n) - \mathring{B}$ onto S. This extends to a retraction $r : f^{-1}(D_2^n) \to B$. Then the composition of r with the contraction of B to a point in $f^{-1}(D_2^n)$ yields a contraction of B to a point in B. Thus B is contractible, and it must be an n-ball. Q.E.D.

Using the Schoenflies theorem, let B' be the n-ball in $\tilde{F}_2^{-1}(B^k \times 2\mathring{B}^m)$ bounded by $S' = \varphi_1\varphi(S)$. Choose a homeomorphism $h: B \to B'$ so that $h|S: S \to S'$ is given by $\varphi_1\varphi$. We define $\tilde{f}: V \to B^k \times R^m$ by

$$\tilde{f} = \begin{cases} f & \text{on } V - \mathring{B}, \\ \tilde{F}_2 h & \text{on } B. \end{cases}$$

Clearly $\tilde{f} = f$ over $[(B^k - \frac{7}{8}\mathring{B}^k) \times R^m] \cup [B^k \times (R^m - 2\mathring{B}^m)]$, and $\tilde{f}$ is a homeomorphism over $B^k \times B^m$. To see that $\tilde{f}$ is an ϵ-equivalence over $B^k \times 2.5B^m$ we just apply Proposition 2.3. Finally we note that $\frac{7}{8}$ may be replaced by $\frac{2}{3}$ to give condition (2) in the statement of the Main Theorem. Q.E.D.

5. The Approximation Theorem

We now use the Handle Theorem to prove the main result of this paper. The idea is to pass from the Handle Theorem to a global theorem. This requires an examination of the process of adding on a single handle, which is carried out in the following Lemma 5.1.

For notation let M^n, N^n be topological manifolds, $n \geq 5$ and $\partial M = \partial N = \varnothing$, and let $f: M \to N$ be a proper map. Let $P^n \subset N$ be a submanifold which is compact, and let $Q^n \subset N$ be obtained from P by adding on a handle. Assume that f is a homeomorphism over a neighborhood $\tilde{P}$ of P, and let $U \subset N$ be any neighborhood of $Q - \mathring{P}$. Finally, let $C \subset N$ be a compactum containing $Q \cup U$ in its interior, and let $\tilde{C}$ be a compactum containing C in its interior.

LEMMA 5.1. *For every open cover α of N there is an open cover β such that if the map f mentioned above is a β-equivalence over $\tilde{C}$, then there is a proper map $g: M \to N$ such that g is a homeomorphism over Q, $g = f$ over $N - U$, and g is an α-equivalence over C.*

Proof. By slight abuse of notation we may assume that $B^k \times R^m$ is an open subset of $U - \mathring{P}$ such that

$$(B^k \times R^m) \cap P = \partial B^k \times R^m \subset \partial P$$

and such that $Q = P \cup (B^k \times B^m)$. We may also assume that $(B^k - \frac{1}{2}\mathring{B}^k) \times R^m \subset \tilde{P}$. The restriction

$$f| : f^{-1}(B^k \times R^m) \to B^k \times R^m$$

is a homeomorphism over $(B^k - \frac{1}{2}\mathring{B}^k) \times R^m$ and a δ-equivalence over $B^k \times 3B^m$, where δ is small corresponding to a small choice of β. The Handle Theorem gives us a proper map $\tilde{f}: f^{-1}(B^k \times R^m) \to B^k \times R^m$ such that

(1) $\tilde{f}$ is an ϵ-equivalence over $B^k \times 2.5B^m$,
(2) $\tilde{f} = f$ over $[(B^k - \frac{2}{3}\mathring{B}^k) \times R^m] \cup [B^k \times (R^m - 2\mathring{B}^m)]$,
(3) $\tilde{f}$ is a homeomorphism over $B^k \times B^m$.

Define $g: M \to N$ by $g = f$ over $N - (\mathring{B}^k \times R^m)$ and $g = \tilde{f}$ over $B^k \times R^m$. By Proposition 2.3 we conclude that g is an α-equivalence over C. Q.E.D.

The following is now a straightforward consequence of Lemma 5.1 in a standard way.

LEMMA 5.2. *Let $f: M^n \to N^n$ be a proper map, where $\partial M = \varnothing$, $n \geqslant 5$, and N is an open subset of R^n. Choose a compactum $C \subset N$ with compact neighborhood $\tilde{C}$. Then for every α there exists a β such that if f is a β-equivalence over $\tilde{C}$, then there is a proper map $g: M \to N$ which is α-close to f and which is a homeomorphism over C.*

Remarks on the Proof. Choose a linear triangulation of N so fine that each handle in the standard handle decomposition of this triangulation which meets C must lie in $(\tilde{C})^0$. Then inductively work through the handles which meet C by using Lemma 5.1 to deform f to a homeomorphism over C. The g that we get must be close to f, because all of the k-handles can be taken care of simultaneously. For more details see [13, Section 3.2]. Q.E.D.

Proof of the α-Approximation Theorem. With Lemma 5.2 we are now in a position to complete the proof of our Approximation Theorem. For the case $\partial N = \varnothing$ write $N = \bigcup_{i=1}^{\infty} N_i$, where each N_i is openly embeddable in R^n, and where $\{N_i\}_{i=1}^{\infty}$ is a star-finite cover of N. Moreover, let $C_i \subset \tilde{C}_i \subset N_i$ be compact, so that $C_i \subset (\tilde{C}_i)^0$ and $\bigcup_{i=1}^{\infty} C_i = N$. By Lemma 5.2 there is a proper map $g_i: f^{-1}(N_i) \to N_i$ approximating $f | f^{-1}(N_i)$ which is a homeomorphism over $\tilde{C}_i$. By the Deformation Theorem of [3] we can glue the embeddings $g_i | C_i$ together to obtain a homeomorphism $g: M \to N$ approximating f.

If $\partial N \neq \varnothing$, consider the restriction $f | M - \partial M: M - \partial M \to N - \partial N$. [We may assume that $f^{-1}(N - \partial N) = M - \partial M$] Let $\partial N \times [0,1] \subset N$ be a boundary collar, $\partial N \equiv \partial N \times \{0\}$. By a simple extension of the above case we can find a proper map $g': M - \partial M \to N - \partial N$ which approximates $f | M - \partial M$ and which is a homeomorphism over $N - \left(\partial N \times \left[0, \frac{1}{2}\right)\right)$. If f is already a homeomorphism over $\partial N \times [0,1]$, then

$$f | f^{-1}\left(\partial N \times \left(\tfrac{4}{6}, \tfrac{5}{6}\right)\right): f^{-1}\left(\partial N \times \left(\tfrac{4}{6}, \tfrac{5}{6}\right)\right) \to N,$$
$$g' | f^{-1}\left(\partial N \times \left(\tfrac{4}{6}, \tfrac{5}{6}\right)\right): f^{-1}\left(\partial N \times \left(\tfrac{4}{6}, \tfrac{5}{6}\right)\right) \to N$$

are embeddings which are close in the majorant topology. By the Deformation Theorem of [3] we can therefore piece together these embeddings to obtain a homeomorphism $g: M \to N$ approximating f. For more details see [13, Section 3].

Q.E.D.

6. The Splitting Theorem

In this section we will prove the following Splitting Theorem, which was of crucial importance in the proof of the Handle Theorem.

For notation let W^n be an n-manifold, $n \geqslant 5$ and $\partial W \neq \varnothing$, and let $f: W \to S^{n-1} \times R$ be a proper map which is a $p^{-1}(\epsilon)$-equivalence over $S^{n-1} \times [-2,2]$, where $p: S^{n-1} \times R \to R$ is the projection map.

Splitting Theorem. *If $\epsilon = \epsilon(n)$ is sufficiently small, then there is an $(n-1)$-sphere $S \subset (pf)^{-1}(-1,1)$ such that $f|S: S \to S^{n-1} \times R$ is a homotopy equivalence, S is bicollared, and S separates the component of W containing $(pf)^{-1}([-1,1])$ into two components, one containing $(pf)^{-1}(-1)$ and the other containing $(pf)^{-1}(1)$.*

Addendum. *It also follows that if C_0 is the closure of the component of $(pf)^{-1}(-1,\frac{4}{3}) - S$ containing $(pf)^{-1}(1)$, and C_1 is the closure of the component of $(pf)^{-1}(-1,\frac{5}{3}) - S$ containing $(pf)^{-1}(1)$, then C_0 deforms into S rel S, with the deformation taking place in C_1.*

Our method is to find an open subset $X \subset (pf)^{-1}(-1,1)$ which is homeomorphic with $S^{n-1} \times R$. Here is a proposition, due to Siebenmann [13], which allows us to recognize such an X.

A manifold X is said to be 1-LC at ∞ if for each compactum $C \subset X$ there is a compactum $D \subset X$ such that $C \subset D$ and every map $f: S^1 \to X - D$ is homotopic to a constant map in $X - C$.

Proposition 6.1 [13, Lemma 2.5, p. 288]. *If X^n is an n-manifold, $n \geqslant 5$ and $\partial X = \varnothing$, which is homotopy equivalent to $S^{n-1} \times R$ and 1-LC at ∞, then X is homeomorphic to $S^{n-1} \times R$.* Q.E.D.

Throughout this section we will retain the notation preceding the statement of the Splitting Theorem. Thus, W will denote an n-manifold, $f: W \to S^{n-1} \times R$ will denote a proper map which is a $p^{-1}(\epsilon)$equivalence over $S^{n-1} \times [-2,2]$, and $g: S^{n-1} \times [-2,2] \to W$ will denote a $p^{-1}(\epsilon)$-inverse for f. We will be concerned with the construction of the splitting $(n-1)$-sphere $S \subset (pf)^{-1}(-1,1)$. The Addendum follows immediately from the proof of the theorem.

Definition 6.2. By a *collection of chambers* in W we will mean a collection $\{U_i\}_{i=1}^m$ of subsets obtained as follows. Let $-1<t_1<\cdots<t_{m+1}<1$ be a subdivision of $(-1,1)$, and let $U_i=(pf)^{-1}[t_i,t_{i+1}]$. We will denote $(pf)^{-1}(t_i)$ by V_i. $\{U_i\}_{i=1}^m$ is said to be *nice* if $|t_{i+1}-t_i|>3\epsilon$ for all i. (See Figure 3.)

We are interested in nice collections of chambers because of the next proposition. If $A\subset B\subset C\subset X$, we say that C *deforms into* B rel$A$ *in* X if there is a homotopy $H_t: C\to X$ such that H_0 is inclusion, $H_1(C)\subset B$, and $H_t(a)=a$ for all $a\in A$ and $0\leqslant t\leqslant 1$.

Proposition 6.3. *If* $\{U_i\}_{i=1}^m$ *is a nice collection of chambers in* W, *then*

(i) *there is a deformation of* $U_i\cup U_{i+1}$ *into* U_i relV_i, *with the deformation taking place in* $U_i\cup U_{i+1}\cup U_{i+2}$;

(ii) *there is a deformation of* $U_{i+1}\cup U_{i+2}$ *into* U_{i+2} relV_{i+3}, *with the deformation taking place in* $U_i\cup U_{i+1}\cup U_{i+2}$.

Proof. We will only construct the deformation of (i). The deformation of (ii) is entirely similar. Let $a_i=\frac{2}{3}t_i+\frac{1}{3}t_{i+1}$ and $b_i=\frac{1}{3}t_i+\frac{2}{3}t_{i+1}$. Also let $H_t:(pf)^{-1}[-2,2]\to W$ be a $(pf)^{-1}(\epsilon)$-homotopy from the inclusion map to $gf|(pf)^{-1}[-2,2]$. Define a homotopy $s_t: S^{n-1}\times[-2,2]\to S^{n-1}\times[-2,2]$ by

$$s_t(x,u) = (x,(1-t)u+tb_i).$$

Putting all of this together, we then define a homotopy h_t of $(pf)^{-1}[b_i,t_{i+2}]$ into $(pf)^{-1}([a_i,t_{i+3}])$ by

$$h_t(x) = \begin{cases} H_{2t}(x) & \text{for } 0\leqslant t\leqslant \frac{1}{2}, \\ gs_{2t-1}f(x) & \text{for } \frac{1}{2}\leqslant t\leqslant 1. \end{cases}$$

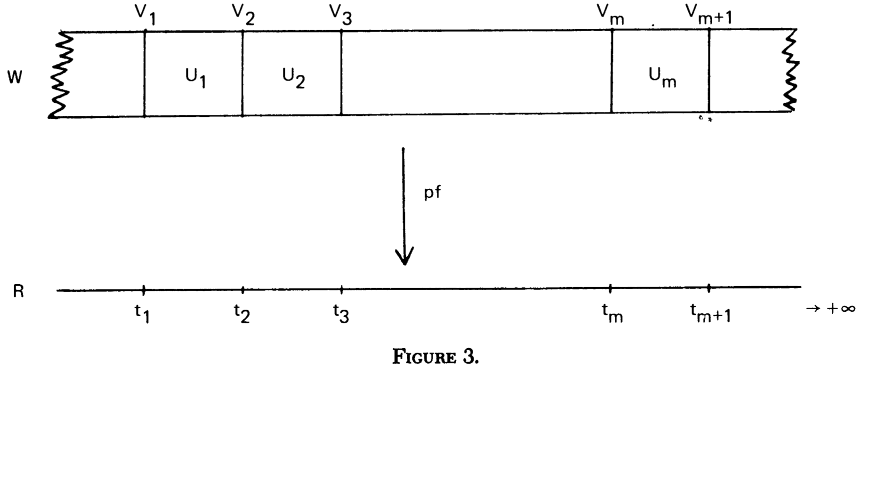


Figure 3.

h_t is a homotopy from the inclusion to $gs_1 f$ such that

(1) $gs_1 f(pf)^{-1}[b_i, t_{i+2}] \subset (pf)^{-1}[a_i, t_{i+1}]$,
(2) $h_t|(pf)^{-1}(b_i)$ is a $(pf)^{-1}(\epsilon)$-homotopy.

By Lemma 2.1 we can extend the homotopy $h_t|(pf)^{-1}(b_i)$ to a homotopy $h_t': (pf)^{-1}[t_i, b_i] \to (pf)^{-1}[t_i, t_{i+1}]$ such that $h_t' = \mathrm{id}$ on $(pf)^{-1}[t_i, a_i]$ and $h_t' = h_t$ on $(pf)^{-1}(b_i)$. Piecing h_t' and h_t together, we get our desired deformation of $(pf)^{-1}[t_i, t_{i+2}]$ into $(pf)^{-1}[t_i, t_{i+1}]$. Q.E.D.

Here is a crucial result which is the key ingredient in the proof of our Splitting Theorem.

PROPOSITION 6.4 [14, Sections 2, 3]. *There exists an integer $N = N(n)$ such that if $\{U_i\}_{i=1}^m$ is a nice collection of chambers in W and k is an integer, $2N + k \leq m$, then there is an isotopy $h_t: W \to W$, supported on $U_1 \cup \cdots \cup U_{2N+k}$, such that $h_0 = \mathrm{id}$ and $h_1(U_1 \cup \cdots \cup U_N) \supset U_1 \cup \cdots \cup U_{N+k}$.*

Remarks on the Proof. In case W is a PL n-manifold, the desired isotopy is constructed by engulfing a skeleton from the left and a dual skeleton from the right (as in Stallings [16]). The strategy in [14] is to carry out a topological version of this idea.

We now use Proposition 6.4 to prove the next result, which will be used to produce our open subset of $(pf)^{-1}(-1, 1)$ which is homeomorphic to $S^{n-1} \times R$.

PROPOSITION 6.5. *There exists an integer $N = N(n)$ such that if $\{U_i\}_{i=1}^{8N}$ is a nice collection of chambers in W, then there is a homeomorphism $h: W \to W$, supported on $U_1 \cup \cdots \cup U_{8N}$, such that*

(i) $U_1 \cup \cdots \cup U_{3N} \subset h(U_1 \cup \cdots \cup U_N) \subset U_1 \cup \cdots \cup U_{4N}$,
(ii) $U_1 \cup \cdots \cup U_{5N} \subset h^2(U_1 \cup \cdots \cup U_N) \subset U_1 \cup \cdots \cup U_{6N}$,
(iii) $U_1 \cup \cdots \cup U_{7N} \subset h^3(U_1 \cup \cdots \cup U_N) \subset U_1 \cup \cdots \cup U_{8N}$.

Proof. By Proposition 6.4 there is an integer N such that there are homeomorphisms $h_0, h_1, h_2: W \to W$ satisfying

(1) h_0 is supported on $U_1 \cup \cdots \cup U_{4N}$ and $h_0(U_1 \cup \cdots \cup U_N) \supset U_1 \cup \cdots \cup U_{3N}$,
(2) h_1 is supported on $U_{2N+1} \cup \cdots \cup U_{6N}$ and $h_1(U_{2N+1} \cup \cdots \cup U_{3N}) \supset U_{2N+1} \cup \cdots \cup U_{5N}$,
(3) h_2 is supported on $U_{4N+1} \cup \cdots \cup U_{8N}$ and $h_2(U_{4N+1} \cup \cdots \cup U_{5N}) \supset U_{4N+1} \cup \cdots \cup U_{7N}$.

(See Figure 4.) Define h to be the homeomorphism $h = h_0h_1h_2$. Since h_0 and h_2 have disjoint supports, they commute, and one easily verifies that

$$h(U_1 \cup \cdots \cup U_N) = h_0(U_1 \cup \cdots \cup U_N),$$
$$h^2(U_1 \cup \cdots \cup U_N) = h_1h_0(U_1 \cup \cdots \cup U_N),$$
$$h^3(U_1 \cup \cdots \cup U_N) = h_2h_1h_0(U_1 \cup \cdots \cup U_N).$$

The desired properties follow immediately. Q.E.D.

Definition 6.6. We are now ready to define our open subset of W which will be homeomorphic to $S^{n-1} \times R$. Let

$$Y = h(U_1 \cup \cdots \cup U_N) - (U_1 \cup \cdots \cup U_N)^0,$$
$$X = \bigcup_{i=-\infty}^{\infty} h^i(Y),$$

where h is the homoemorphism of Proposition 6.5, and h^i means the composition of h with itself i times, for $i \geqslant 0$, and for $i < 0$, $h^i = (h^{-i})^{-1}$. To finish the proof of the Splitting Theorem all we have to do is prove that X is open, X is 1-LC at ∞, and $f|X : X \to S^{n-1} \times R$ is a homotopy equivalence, for then X will be homeomorphic to $S^{n-1} \times R$ by Propositon 6.1 and therefore X will contain a splitting $(n-1)$-sphere S.

Proposition 6.7. *X is an open subset of W.*

Proof. It certainly suffices to prove that $Y \cup h(Y)$ contains $h(V_{N+1})$ in its interior. Using properties (i) and (ii) of Proposition 6.5 it is easy to argue that $h(V_{N+1})$ lies in $U_{3N+1} \cup \cdots \cup U_{4N}$ and $U_{2N+1} \cup \cdots \cup U_{5N}$ lies in $Y \cup h(Y)$. Thus $(U_{2N+1} \cup \cdots \cup U_{5N})^0$ is an open set in $Y \cup h(Y)$ containing $h(V_{N+1})$. (See Figure 5.) Q.E.D.

The following deformation result enables us to prove that X is 1-LC at ∞ and $f|X : X \to S^{n-1} \times R$ is a homotopy equivalence.

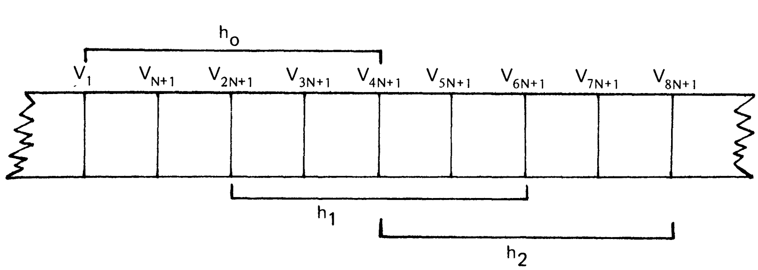


Figure 4.

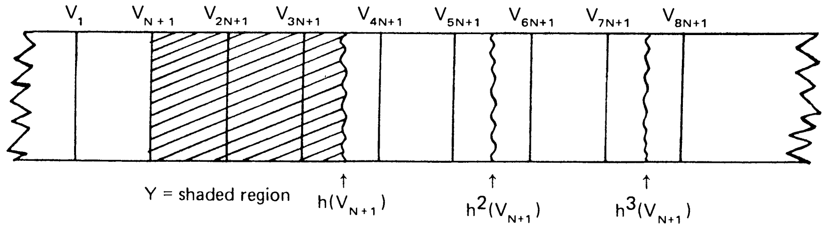


FIGURE 5.

PROPOSITION 6.8. *With Y as above,*

(i) *there is a deformation of $Y \cup h(Y)$ into Y $\operatorname{rel} V_{N+1}$, with a deformation that takes place in $Y \cup h(Y) \cup h^2(Y)$;*
(ii) *there is a deformation of $h(Y) \cup h^2(Y)$ into $h^2(Y)$ $\operatorname{rel} h^3(V_{N+1})$, with a deformation that takes place in $Y \cup h(Y) \cup h^2(Y)$.*

Proof. We refer the reader again to Figure 5. Using Proposition 6.5, one readily verifies the following four properties of h:

(1) $Y \cup h(Y) \subset U_{N+1} \cup \cdots \cup U_{6N}$,
(2) $Y \cup h(Y) \cup h^2(Y) \supset U_{N+1} \cup \cdots \cup U_{7N}$,
(3) $h^2(Y) \supset U_{6N+1} \cup \cdots \cup U_{7N}$,
(4) $[h(Y) \cup h^2(Y)] \cap (U_1 \cup \cdots \cup U_{7N}) \subset U_{3N+1} \cup \cdots \cup U_{7N}$.

By Proposition 6.3 (i) there is a deformation of $U_{N+1} \cup \cdots \cup _{6N}$ into U_{N+1} $\operatorname{rel} V_{N+1}$, with a deformation which takes place in $U_{N+1} \cup \cdots \cup _{6N+1}$. Properties (1) and (2) imply that the restriction of such a deformation to $Y \cup h(Y)$ suffices to establish (i). Similarly, one uses Properties (2), (3), and (4) and Proposition 6.3 (ii) to establish (ii). Q.E.D.

PROPOSITION 6.9. *X is 1-LC at ∞, and $f|X : X \to S^{n-1} \times R$ is a homotopy equivalence.*

Proof. We will first prove that $f|X$ is a homotopy equivalence. Recall the map $g : S^{n-1} \times [-2,2] \to W$. We will use it to prove that X is simply connected. If $\alpha : S^1 \to X$ is a map, then repeated use of Proposition 6.8 allows us to homotop α to a map $\bar{\alpha} : S^1 \to U_{4N+1}$. $\bar{\alpha}$ is easily seen to be homotopic in $U_{4N} \cup U_{4N+1} \cup U_{4N+2}$ to a constant map. To see this we first use the homotopy from $gf|(pf)^{-1}[-2,2]$ to the inclusion to homotop $\bar{\alpha}$ to $gf\bar{\alpha}$, and the fact that S^{n-1} is simply connected allows us to homotop $f\bar{\alpha}$ to a constant map. Then $gf\bar{\alpha}$ is homotopic to a constant map. The $(pf)^{-1}(\epsilon)$-control on the homotopy $gf|(pf)^{-1}[-2,2]$ shows that this homotopy of $\bar{\alpha}$ to a constant map takes place in $U_{4N} \cup U_{4N+1} \cup U_{4N+2}$.

It now suffices to show that f induces isomorphisms on homology groups. This is roughly the same as the argument on π_1. Choose $t \in R$ so that $g(S^{n-1} \times \{t\}) \subset U_{4N+1}$. Then $g|S^{n-1} \times \{t\}: S^{n-1} \times (t\} \to X$ is a map for which $fg|S^{n-1} \times \{t\} \simeq \mathrm{id}$. Thus f induces surjections on homology. On the other hand, Proposition 6.8 implies thaat the inclusion induced map $i_*: H_*(h(Y)) \to H_*(X)$ is onto. Just as in the π_1 argument above, $i_* \alpha = g_*(f|h(Y))_* \alpha$ for each $\alpha \in H_*(h(Y))$. Thus f_* is 1-1, and we are done.

To prove that X is 1-LC at ∞ it suffices to show that the inclusion-induced maps

$$\pi_1\left(\bigcup_{i=2}^{\infty} h^i(Y)\right) \to \pi_1\left(\bigcup_{i=1}^{\infty} h^i(Y)\right),$$
$$\pi_1\left(\bigcup_{i=-\infty}^{1} h^i(Y)\right) \to \pi_1\left(\bigcup_{i=-\infty}^{2} h^i(Y)\right)$$

are zero. This is done exactly as in the π_1 argument given above. Q.E.D.

Proof of the Splitting Theorem (conclusion). If W satisfies the hypotheses of the Splitting Theorem, choose $\epsilon(n)$ small enough so that there is a nice collection of chambers $\{U_i\}_{i=1}^{8N}$ in $(pf)^{-1}(-1,1)$, and construct X as above. By Proposition 6.1 there is a homeomorphism $k: S^{n-1} \times R \to X$. Then $S = k(S^{n-1} \times \{0\})$ is our bicollared $(n-1)$-sphere in $(pf)^{-1}(-1,1)$ which separates $(pf)^{-1}(-1)$ from $(pf)^{-1}(1)$ by construction. Q.E.D.

Finally we remark that in dimensions $\geqslant 6$ one can prove a much stronger theorem.

THEOREM 6.10. *Let W^n be a CAT manifold (CAT = TOP, PL, or Diff), $n \geqslant 6$, and $\partial W = \varnothing$, and let $f: W \to K \times R$ be a proper map, where K is a finite simplicial complex. Moreover, assume that f is a $p^{-1}(\epsilon)$-equivalence over $K \times [-2,2]$, where $p: K \times R \to R$ is the projection map. If $\epsilon = \epsilon(n)$ is sufficiently small, then there is a well-defined obstruction in the projective class group $\tilde{K}_0 Z[\pi_1(K)]$, which vanishes iff there is a codimension 1, bicollared, compact, CAT submanifold S of $(pf)^{-1}(-1,1)$ such that S separates $(pf)^{-1}(-1)$ from $(pf)^{-1}(1)$ and $f|S$ is a homotopy equivalence.*

Remarks on Proof. The idea is to proceed as above to construct an open set $X \subset (pf)^{-1}(-1,1)$ such that X is proper homotopy equivalent to $K \times R$. The desired obstruction in $\tilde{K}_0 Z[\pi_1(K)]$ is then Siebenmann's obstruction to putting a boundary on X [12]. The obstruction is well defined, since Siebenmann's obstruction is well defined, and different choices of X can be forced to overlap. Q.E.D.

7. Proof of the Bundle Theorem

Before giving the proof of the Bundle Theorem we will need to introduce some more terminology and preliminary results. Recall from [7] that a proper surjection $p:E\to B$ of spaces is *completely regular* if for each $b_0\in B$ and $\epsilon>0$, there is a $\delta>0$ such that $d(b_0,b)<\delta$ implies that there is a homeomorphism $h:p^{-1}(b)\to p^{-1}(b_0)$ such that $d(h,\mathrm{id})<\epsilon$. We will need the following result from [7].

PROPOSITION 7.1. *Let $p:E\to B$ be completely regular. If B is locally finite-dimensional, and the homeomorphism group of each point-inverse is locally contractible, then $p:E\to B$ is a locally trivial bundle.*

Recall from [1] that a proper surjection $\mathrm{p}:\mathrm{E}\to\mathrm{B}$ is *strongly regular* if for each $b_0\in B$ and $\epsilon>0$, there is a $\delta>0$ so that $d(b_0,b)<\delta$ implies that there are ϵ-maps

$$p^{-1}(b)\underset{g}{\overset{f}{\rightleftarrows}}p^{-1}(b_0)$$

and ϵ-homotopies

$$fg\overset{\epsilon}{\simeq}\mathrm{id},\qquad gf\overset{\epsilon}{\simeq}\mathrm{id}.$$

(The given homotopies must take place in their respective point inverses.) We will need the following result from [1].

PROPOSITION 7.2. *Let $p:E\to B$ be a Hurewicz fibration such that p is proper and surjective, and B is locally path connected. Then p is strongly regular.*

Proof of the Bundle Theorem. We are given a Hurewicz fibration $p:E\to B$ such that B is locally finie-dimensional and locally path connected, and each fiber $p^{-1}(b)$ is a compact n-manifold. Moreover, $p|\partial E:\partial E\to B$ is a locally trivial bundle map. Without loss of generality we may assume that p is a surjection, and it is easy to argue that p is proper. To show that $p:E\to B$ is a locally trivial bundle it suffices, by Proposition 7.1, to show that p is completely regular. (Recall from [3] that the homeomorphism group of any compact n-manifold is locally contractible.)

Choose any $b_0\in B$. By Proposition 7.2 it follows that if b is close to b_0, then there are small maps

$$p^{-1}(b)\underset{g}{\overset{f}{\rightleftarrows}}p^{-1}(b_0)$$

and small homotopies $\theta_t : fg \simeq \mathrm{id}$, $\varphi_t : gf \simeq \mathrm{id}$. Since f is a small map, $f\varphi_t : p^{-1}(b) \to p^{-1}(b_0)$ is a small homotopy. All this implies that if $\epsilon > 0$ is given, then there is a $\delta > 0$ such that $d(b_0, b) < \delta$ implies that there is an ϵ-equivalence $f : p^{-1}(b) \to p^{-1}(b_0)$ for which $d(f(x), x) < \epsilon$.

Since $p|\partial E : \partial E \to B$ is a locally trivial bundle, we may assume that the δ chosen above is small enough so that $d(b_0, b) < \delta$ implies that there is a homeomorphism $h : \partial p^{-1}(b) \to \partial p^{-1}(b_0)$ for which $d(h, \mathrm{id}) < \epsilon$. Now

$$f|\partial p^{-1}(b) : \partial p^{-1}(b) \to p^{-1}(b_0),$$
$$h : \partial p^{-1}(b) \to p^{-1}(b_0)$$

are maps into $p^{-1}(b_0)$ which are within 2ϵ of each other. Thus there is a small homotopy $f|\partial p^{-1}(b) \simeq h$. By Proposition 2.1 we can therefore adjust f slightly to get $f' : p^{-1}(b) \to p^{-1}(b_0)$ which agrees with h on $p^{-1}(b)$. Since f' is close to f, f' must be a small equivalence, so that by the α-Approximation Theorem there is a small homeomorphism of $p^{-1}(b) \to p^{-1}(b_0)$. This establishes complete regularity. Q.E.D.

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REFERENCES.

[1] David F. Addis, Strongly regular mappings: A survey, in *Proc. of First Conference on Monotone Mappings and Open Mappings*, Binghamton, N.Y., 1970, pp. 261–269.

[2] T. A. Chapman and Steve Ferry, Hurewicz fiber maps with ANR fibers, *Topology*, **16** (1977), pp. 131–143.

[3] R. Edwards and R. Kirby, Deformations of spaces of imbeddings, *Ann. of Math.* **93** (1971), pp. 63–88.

[4] Steve Ferry, An immersion of $T^n - C^n$ into R^n, *Enseignement Math.* **20** (1974), pp. 12–13.

[5] ———, The homeomorphism group of a compact Q-manifold is an ANR, *Ann. of Math.* **106** (1977), pp. 101–119.

[6] ———, Homotoping ϵ-maps to homeomorphisms, *Am. Jour. Math.* **101** (1979), pp. 567–582.

[7] M. E. Hamstrom and Eldon Dyer, Completely regular mappings, *Fund. Math.* **45** (1957), pp. 103–118.

[8] L. Husch, Hurewicz fibrations need not be locally trivial, *Proc. Amer. Math. Soc.*, to appear.

[9] R. Kirby, Lectures on triangulation of manifolds, U.C.L.A. Lecture Notes, Los Angeles, 1969.

[10] R. Lacher, Cell-like mappings I, *Pacific J. Math.* **30** (1969), pp. 717–731.

[11] F. Raymond, Local triviality for Hurewicz fiberings of manifolds, *Topology* **3** (1965), pp. 43–57.

[12] L. C. Siebenmann, The obstruction to finding a boundary for an open manifold of dimension greater than five, thesis, Princeton University, 1965.

[13] ———, Approximating cellular maps by homeomorphisms, *Topology* **11** (1972), pp. 271–294.

[14] L. Siebenmann, L. Guillou, and H. Hahl, Les voisinages ouverts réguliers: critères homotopiques d'existence, *Ann. Sci. École Norm. Sup.* 7 (1974), pp. 431–462.

[15] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1969.

[16] J. Stallings, The piecewise linear structure of euclidean space, *Proc. Cambridge Philos. Soc.* **58** (1962), pp. 481–488.