TOPOLOGICAL INVARIANCE OF WHITEHEAD TORSION.

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1. Introduction. In this paper we use infinite-dimensional topology to prove that Whitehead torsion is a topological invariant for compact connected CW-complexes. This answers affirmatively a question raised by Whitehead in [12]. Our techniques are motivated by the work of Kirby-Siebenmann [7], where handle straightening was used to prove the invariance of torsion for compact connected PL manifolds. This type of approach very strongly uses the fact that PL manifolds have nice neighborhoods of each point, a property which is not generally satisfied for CW-complexes.

Our proof of torsion invariance uses some recent results concerning Hilbert cube manifolds (or Q-manifolds), where a *Q-manifold* is a separable metric manifold modeled on the Hilbert cube Q. The first key idea involved in the proof is the following result of West [10]: *If X is any compact CW-complex, then $X \times Q$ is a Q-manifold. This has the effect of converting a CW-complex into a space which has nice neighborhoods of each point. Indeed any point in a Q-manifold lies in an open set which is homeomorphic to $Q \times [0,1)$* [2].

The second key idea in the proof is an infinite-dimensional version of the finite-dimensional handle straightening idea which was used in [7]. This infinite-dimensional handle straightening technique is the main result of [4] and is summarized in Section 2 of this paper. Combining these ideas we obtain the following characterization of simple homotopy equivalences in terms of homeomorphisms of Q-manifolds. We remark that the "only if" part of this characterization is essentially Corollary 3 of [10] (see our Lemma 2.2).

Main Theorem. *Let X and Y be compact connected CW-complexes and let $f: X \to Y$ be a map* (i.e. *a continuous function*). *Then f is a simple homotopy*

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equivalence if and only if the map

$$X \times Q \overset{f \times id}{\rightarrow} Y \times Q$$

is homotopic to a homeomorphism of $X \times Q$ *onto* $Y \times Q$.

We now obtain invariance of torsion as a straightforward consequence of the Main Theorem.

THEOREM 1 (Invariance of torsion). *Any homeomorphism between compact connected CW-complexes is a simple homotopy equivalence.*

We can also use the Main Theorem to characterize simple homotopy types in terms of homeomorphisms of Q-manifolds.

THEOREM 2. *If* X *and* Y *are compact connected CW-complexes, then* X *and* Y *have the same simple homotopy type if and only if* $X \times Q \cong Y \times Q$ (i.e. $X \times Q$ *is homeomorphic to* $Y \times Q$).

A topological space X is said to be a *Q-manifold factor* provided that $X \times Q$ is a Q-manifold. In particular X could be an n-manifold, or more generally any space which is locally a CW-complex (by the aforementioned result of West). In [5] it was shown that every compact Q-manifold Y can be triangulated, i.e. $Y \cong |K| \times Q$, for some finite simplicial complex K. Using this result and Theorem 2 it follows that each compact connected Q-manifold factor can be given a well-defined simple homotopy type. In [7] a method was given for assigning to each compact connected n-manifold a well-defined simple homotopy type; namely the type of any of its normal disc bundles triangulated as a PL manifold.

THEOREM 3. *Each compact connected Q-manifold factor* X *can be given a well-defined simple homotopy type; namely the type of any simplicial complex* K *for which* $X \times Q \cong |K| \times Q$. *In case* X *is an* n*-manifold, this assignment agrees with a triangulation of its normal disc bundle.*

In connection with Theorem 3 it should be remarked that the following question is open [1]: *Is every compact ANR (metric) a Q-manifold factor*?

In [1] the following question was raised: *If* X *and* Y *are compact Q-manifolds of the same homotopy type, then are* X *and* Y *homeomorphic*? We can easily use Theorem 2 to answer this question negatively. We remark that by using different techniques Philip Martens has also claimed a negative answer to this question.

THEOREM 4. *There exist compact Q-manifolds of the same homotopy type which are not homeomorphic.*

In Section 2 we give a brief summary of all the infinite-dimensional results which we will need. Thus no prior experience with infinite-dimensional topology is necessary in order to read this paper. In Section 3 we summarize the results on Whitehead torsion which we will need, but for the most part we assume familiarity with the standard results in this area (see [8] and [12]). The proof of the Main Theorem is by induction and in Section 4 we prove the inductive step. In Section 5 we prove the Main Theorem and finally in Section 6 we prove Theorems 1–4.

2. Infinite-Dimensional Preliminaries. We use the notation $Q=\prod_{i=1}^{\infty}I_i$, where each I_i is the closed interval $[-1,1]$. Let $0=(0,0,\ldots)\in Q$ and for any integer $k>0$ let

$$Q_k=I_k\times I_{k+1}\times\cdots.$$

R^n denotes Euclidean n-space, B_r^n denotes the standard n-ball of radius r, S_r^{n-1} denotes the boundary of B_r^n, and $\mathrm{Int}(B_r^n)=B_r^n\setminus S_r^{n-1}$. In general $\mathrm{Bd}(A)$ and $\mathrm{Int}(A)$ denote the topological boundary and interior, respectively, of a subset A of a space X.

Let M be a PL manifold (i.e., a finite-dimensional manifold equipped with a PL structure). A *handle* in $M\times Q$ is an open embedding $h:R^n\times Q\to M\times Q$. We say that a handle $h:R^n\times Q\to M\times Q$ can be *straightened* provided that there exists a homeomorphism $f:B_2^n\times Q\to B_2^n\times Q$, an integer $k>0$, and a compact PL submanifold N of $M\times I^k$ such that

1. $f|S_2^{n-1}\times Q=id$ (the identity),
2. $hf(B_1^n\times Q)=N\times Q_{k+1}$,
3. $\mathrm{Bd}(N)$ is a PL submanifold of $M\times I^k$ which is PL bicollared.*

The following is the main result of [4].

Lemma 2.1. (Handle straightening theorem). *If $h:R^n\times Q\to M\times Q$ is a handle, for $n\geq 2$, then h can be straightened.*

The case $n=1$ can also be done using the techniques of [4], but we won't need it here. We remark that the proof of Lemma 2.1 requires infinite-dimensional surgery and an infinite-dimensional version of the torus homeomorphism technique involved in handle straightening in finite-dimensional manifolds.

*$\mathrm{Bd}(N)$ is PL bicollared means that there exists a PL open embedding $\alpha:\mathrm{Bd}(N)\times(-1,1)\to M\times I^k$ which satisfies $\alpha(n,0)=n$, for all $n\in\mathrm{Bd}(N)$.

We will also need the following result which is essentially due to West [10]. This result is not explicitly given in [10], but a simple modification of the proof of Corollary 3 of that paper easily gives it (see [4], Lemma 4.1, for details of the simplicial case).

LEMMA 2.2. *If X and Y are compact connected CW-complexes and $f:X\to Y$ is a simple homotopy equivalence, then $f\times id:X\times Q\to Y\times Q$ is homotopic to a homeomorphism of $X\times Q$ onto $Y\times Q$.*

In the proof of Theorem 3 we will need the following bundle-trivialization result.

LEMMA 2.3. *If $\xi=(E,p,B)$ is a locally-trivial bundle with paracompact base space B and fiber $F\cong Q$, then ξ is trivial.*

Proof. In [3] it was shown that if $\xi'=(E',p',B')$ is any locally-trivial bundle with paracompact base space B' and fiber $F'\cong l_2$ (separable infinite-dimensional Hilbert space), then ξ' is trivial. The same proof goes through if we replace l_2 by Q.

3. Torsion preliminaries. If X and Y are compact connected CW-complexes* and $f:X\to Y$ is a cellular homotopy equivalence, then the Whitehead torsion $\tau(f)$ is defined as an element of the Whitehead group $\mathrm{Wh}(\pi_1(Y))$. Some of the more well-known properties of torsion which we will need are listed below. We refer the reader to [8] and [12] for details. For (1), (2), and (3) let X, Y, and Z be CW-complexes.

1. If $f:X\to Y$ is any cellular homotopy equivalence, then $\tau(f)=0$ if and only if f is a simple homotopy equivalence (in the geometric sense).
2. If $f,g:X\to Y$ are homotopic cellular homotopy equivalences, then $\tau(f)=\tau(g)$.
3. If $f:X\to Y$ and $g:Y\to Z$ are cellular homotopy equivalences, then

$$\tau(gf)=\tau(g)+g_*\tau(f),$$

where $g_*:\mathrm{Wh}(\pi_1(Y))\to\mathrm{Wh}(\pi_1(Z))$ is the isomorphism induced by g.

Since any map between CW-complexes is homotopic to a cellular map, it follows from (2) that we can define $\tau(f)$ for any homotopy equivalence $f:X\to Y$.

We will need a sum theorem for simple homotopy equivalences.

*In the sequel all of our spaces will be compact and connected (with the possibility of obvious exceptions).

LEMMA 3.1. *Let X, Y be CW-complexes which are the union of subcomplexes $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. Assume that $X_0 = X_1 \cap X_2$ and $Y_0 = Y_1 \cap Y_2$ are connected and let $f: X \to Y$ be a cellular map such that $f_i = f|X_i : X_i \to Y_i$ is a simple homotopy equivalence, for $i = 0, 1, 2$. Then f is also a simple homotopy equivalence.*

Proof. In [6] it was shown that

$$\tau(f) = (j_1)_* \tau(f_1) + (j_2)_* \tau(f_2) - (j_0)_* \tau(f_0),$$

where $j_i : Y_i \subset Y$ is inclusion, for $i = 0, 1, 2$. Since each $\tau(f_i) = 0$ it follows that $\tau(f) = 0$.

We will also need the following result which can be found on page 51 of [12].

LEMMA 3.2. *If X is any CW-complex, then X has the simple homotopy type of a simplicial complex.*

Finally we will need the following result of Stallings [9].

LEMMA 3.3. $\mathrm{Wh}(Z * Z * \cdots * Z) = 0$, *where $*$ denotes free product.*

4. The Inductive Step. Our proof of the Main Theorem is by induction on the n-skeleta of one of the given CW-complexes. The inductive step is given in Lemma 4.2 below. For convenience we use the term *PL space* for a topological space equipped with a *PL* structure. We also use $f \cong g$ to indicate that maps f and g are homotopic.

For spaces X and Y let $f: X \times Q \to Y \times Q$ be a homeomorphism. For any integer $k \geqslant 0$ we use $f_k : X \to Y \times I^k$ to denote the composition

$$X \xrightarrow{\times 0} X \times Q \xrightarrow{f} Y \times Q \xrightarrow{p_k} Y \times I^k$$

where $Y \times I^0 = Y$, $\times 0$ is defined by $\times 0\ (x) = (x, 0)$, and p_k is just projection. A *PL* space X is said to have *Property P* provided that for any *PL* space Y and homeomorphism $f: X \times Q \to Y \times Q$, $\tau(f_0) = 0$. If $p: Y \times I^k \to Y$ denotes projection, then $\tau(p) = 0$ and $f_0 = p f_k$. Since

$$\tau(f_0) = \tau(p) + p_* \tau(f_k),$$

it follows that $\tau(f_0) = 0$ if and only if $\tau(f_k) = 0$, for any fixed $k > 0$.

Our first result establishes the invariance of Property P.

LEMMA 4.1. *Let X_1 and X_2 be PL spaces such that $X_1 \times Q \cong X_2 \times Q$. If X_1 has Property P, then X_2 also has Property P.*

Proof. Let Y be a PL space and let $f: X_2 \times Q \to Y \times Q$ be a homeomorphism. We want to show that $\tau(f_0)=0$. Let $g: X_1 \times Q \to X_2 \times Q$ be the given homeomorphism. We have the following diagrams.

$$\begin{array}{ccccc} X_1 \times Q & \xrightarrow{g} & X_2 \times Q & \xrightarrow{f} & Y \times Q \\ \uparrow & & \downarrow\uparrow & & \downarrow \\ X_1 & \xrightarrow[g_0]{} & X_2 & \xrightarrow[f_0]{} & Y \end{array}$$

$$\begin{array}{ccc} X_1 \times Q & \xrightarrow{fg} & Y \times Q \\ \uparrow & & \downarrow \\ X_1 & \xrightarrow[(fg)_0]{} & Y, \end{array}$$

where the vertical arrows are either $\times 0$ or projection. It is easy to check that $(fg)_0 \cong f_0 g_0$. Since X_1 has Property P it follows that $\tau((fg)_0)=0$, hence $\tau(f_0 g_0) = 0$. Once more using the fact that X_1 has Property P it follows that $\tau(g_0)=0$. It then follows from the formula

$$\tau(f_0 g_0) = \tau(f_0) + (f_0)_* \tau(g_0)$$

that $\tau(f_0)=0$.

We now turn our attention to the inductive step in the proof of the Main Theorem. First we make some remarks to simplify the proof. Let X be a PL space and assume that given any PL manifold M and homeomorphism $f: X \times Q \to M \times Q$, $f_0: X \to M$ satisfies $\tau(f_0)=0$. It then follows that X has Property P. To see this let Y be a PL space and let $g: X \times Q \to Y \times Q$ be a homeomorphism. It is clear that there exists a PL manifold M and a simple homotopy equivalence $\alpha: Y \to M$. [Just take a PL embedding of Y into some Euclidean space and let M be a regular neighborhood of the image.] Using Lemma 2.2 there exists a homeomorphism $\beta: Y \times Q \to M \times Q$ such that $\beta \cong \alpha \times id$. Thus $\beta g: X \times Q \to M \times Q$ is a homeomorphism. By our assumption it follows that $(\beta g)_0: X \to M$ satisfies $\tau((\beta g)_0)=0$, and since $(\beta g)_0 \cong \beta_0 g_0$ it follows that $\tau(\beta_0 g_0)=0$. We chose β so that $\beta \cong \alpha \times id$, thus $\beta_0 \cong \alpha$. This implies that $\tau(\beta_0)=0$, and as

$$\tau(\beta_0 g_0) = \tau(\beta_0) + (\beta_0)_* \tau(g_0)$$

it follows that $\tau(g_0)=0$.

LEMMA 4.2. *Let X be a PL space and let $h: R^n \to X$ be a PL open embedding, for $n \geqslant 2$. If $X_0 = X \setminus h(\mathrm{Int}(B_1^n))$ has Property P, then X also has Property P.*

Proof. Let M be a PL manifold and let $f: X \times Q \to M \times Q$ be a homeomorphism. We will show that $\tau(f_0) = 0$, and by the above remarks it will then follow that X has Property P.

Using Lemma 2.1 we can find an integer $k > 0$, a PL submanifold N of $M \times I^k$, and a homeomorphism $g: X \times Q \to M \times Q$ such that

1. $g|(X \setminus h(B_2^n)) \times Q = f|\,(X \setminus h(B_2^n)) \times Q$,
2. $g(h(B_1^n) \times Q) = N \times Q_{k+1}$,
3. $\mathrm{Bd}(N)$ is a PL submanifold of $M \times I^k$ which is PL bicollared.

Note that $g \cong f$. Thus $g_k \cong f_k$ and $\tau(g_k) = 0$ if and only if $\tau(f_k) = 0$. Therefore all we need to do is prove that $\tau(g_k) = 0$.

Using the fact that $\mathrm{Bd}(N)$ is PL bicollared it easily follows that there exists a PL map $g': X \to M \times I^k$ such that

1. $g' \cong g_k$,
2. $g'(X_0) \subset M_0 = (M \times I^k) \setminus \mathrm{Int}(N)$ and $g'|X_0 \cong g_k|X_0$(in M_0),
3. $g'h(B_1^n) \subset N$ and $g'|h(B_1^n) \cong g_k|h(B_1^n)$ (in N),
4. $g'h(S_1^{n-1}) \subset \mathrm{Bd}(N)$ and $g'|h(S_1^{n-1}) \cong g_k|h(S_1^{n-1})$ (in $\mathrm{Bd}(N)$).

Since X_0 has Property P it follows that the map $g_k|X_0: X_0 \to M_0$ satisfies $\tau(g_k|X_0) = 0$. By Lemma 3.3 we have $\mathrm{Wh}(\pi_1(S_1^1)) = 0$ and for any $n > 2$ we have $\mathrm{Wh}(\pi_1(S_1^{n-1})) = 0$. Thus the map $g_k|h(S_1^{n-1}): h(S_1^{n-1}) \to \mathrm{Bd}(N)$ satisfies $\tau(g_k|h(S_1^{n-1})) = 0$. Since $h(B_1^n)$ is contractible it follows that the map $g_k|h(B_1^n): h(B_1^n) \to N$ also satisfies $\tau(g_k|h(B_1^n)) = 0$. Using (2), (3), and (4) above and Lemma 3.1 it follows that $\tau(g') = 0$. Then by (1) above we have $\tau(g_k) = 0$.

5. Proof of the Main Theorem. We have already observed that the "only if" part (which is stated in Lemma 2.2) is due to West. For the proof of the other half we will first consider the PL case. Thus let X and Y be PL spaces and let $f: X \to Y$ be a map such that $f \times id: X \times Q \to Y \times Q$ is homotopic to a homeomorphism $g: X \times Q \to Y \times Q$. We must prove that $\tau(f) = 0$. Since $f \cong g_0$ all we need to do is prove that $\tau(g_0) = 0$. In the language of Section 4 this entails proving that X has Property P. For this we will regard X as $|K|$, the underlying space of a simplicial complex K. Let $n = \dim|K|$ and let K_i denote the i-skeleton of K. We will inductively prove that each $|K_i|$ has Property P. This will fulfill our requirements.

For $i=1$ note that $\pi_1(|K_1|)$ is a free product $Z*Z*\cdots*Z$ (as $|K_1|$ has the homotopy type of a finite bouquet of 1-spheres). It follows from Lemma 3.3 that $\mathrm{Wh}(\pi_1(|K_1|))=0$. Thus $|K_1|$ has Property P.

Now assume that $|K_i|$ has Property P, for some i satisfying $1 \leqslant i < n$. We want to prove that $|K_{i+1}|$ has Property P. Let $\{\sigma_j\}_{j=1}^r$ be the collection of $(i+1)$-simplices of K_{i+1}. Then let A be the union of all closed simplices of K''_{i+1} which meet $|K_i|$ (where K''_{i+1} is the second barycentric subdivision of K_{i+1}). Note that A and $|K_i|$ have the same simple homotopy type. Thus Lemma 2.2 implies that $A \times Q \cong |K_i| \times Q$, and by Lemma 4.1 it follows that A has Property P. For each j, $1 \leqslant j \leqslant r$, let σ_j' be the closure of $\sigma_j \backslash A$. Then we can apply Lemma 4.2 to inductively add the cells σ_j' to A to conclude that $A \cup (\cup_{j=1}^r \sigma_j') = |K_{i+1}|$ has Property P. This completes the inductive step and therefore the proof of the PL case.

Now assume that X and Y are CW-complexes and let $f: X \times Q \to Y \times Q$ be a homeomorphism. We must prove that $\tau(f_0)=0$. It follows from Lemma 3.2 that there exist PL spaces X_1, Y_1 and simple homotopy equivalences $\alpha: X_1 \to X, \beta: Y \to Y_1$. Using Lemma 2.2 it follows that $\alpha \times id: X_1 \times Q \to X \times Q$ is homotopic to a homeomorphism $\alpha': X_1 \times Q \to X \times Q$ and $\beta \times id: Y \times Q \to Y_1 \times Q$ is homotopic to a homeomorphism $\beta': Y \times Q \to Y_1 \times Q$. We have the diagrams

$$\begin{array}{ccccccc}
X_1 \times Q & \xrightarrow{\alpha'} & X \times Q & \xrightarrow{f} & Y \times Q & \xrightarrow{\beta'} & Y_1 \times Q \\
\uparrow & & \downarrow\uparrow & & \downarrow\uparrow & & \downarrow \\
X_1 & \xrightarrow[(\alpha')_0]{} & X & \xrightarrow[f_0]{} & Y & \xrightarrow[(\beta')_0]{} & Y_1
\end{array}$$

$$\begin{array}{ccc}
X_1 \times Q & \xrightarrow{\beta' f \alpha'} & Y_1 \times Q \\
\uparrow & & \downarrow \\
X_1 & \xrightarrow[(\beta' f \alpha')_0]{} & Y_1
\end{array}$$

We have just shown that $\tau((\beta' f \alpha')_0)=0$. But $(\beta' f \alpha')_0 \cong (\beta')_0 f_0 (\alpha')_0 \cong \beta f_0 \alpha$. Since $\tau(\alpha)=0$ and $\tau(\beta)=0$ we can use the formula

$$\tau(\beta f_0 \alpha) = \tau(\beta) + \beta_* \tau(f_0) + \beta_*(f_0)_* \tau(\alpha)$$

to conclude that $\tau(f_0)=0$.

6. Proofs of Theorems 1–4. We have already observed that Theorem 1 is an immediate corollary of the Main Theorem. The "only if" part of Theorem 2 is a consequence of West's theorem. For the other half we note that if $f: X \times Q \to Y \times Q$ is a homeomorphism, then we have shown that $\tau(f_0) = 0$. Thus X and Y have the same simple homotopy type.

For the proof of Theorem 3 let M be a finite-dimensional manifold and let E be the total space of a triangulated normal disc bundle of M. Then Kirby-Siebenmann assign to M the simple homotopy type of E. Note that $M \times Q$ is a Q-manifold, and by the triangulation result of [5] it follows that $M \times Q \cong X \times Q$, for some PL space X. In Section 1 we assigned to M the simple homotopy type of X. We must prove that E and X have the same simple homotopy type. We are given a bundle projection $p: E \to M$ so that $\xi = (E, p, M)$ is a locally-trivial bundle with fiber $F = B_1^n$, for some $n > 0$. Consider now the associated bundle $\xi^* = (E \times Q, p^*, M)$, where $p^*: E \times Q \to M$ is defined by $p^*(e, q) = p(e)$. Then ξ^* is a locally-trivial bundle with fiber $B_1^n \times Q \cong Q$. It follows from Lemma 2.3 that ξ^* is trivial. Thus $E \times Q \cong M \times Q$, and by Theorem 2 it follows that E and X have the same simple homotopy type.

For the proof of Theorem 4 let X and Y be PL spaces which have the same homotopy type but not the same simple homotopy type [11]. Then Theorem 2 implies that $X \times Q$ and $Y \times Q$ are nonhomeomorphic Q-manifolds of the same homotopy type.

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