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SPECTRAL GEOMETRY OF SINGULAR RIEMANNIAN SPACES

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0. Introduction

In [6], [8], [9] we announed an extension of the theory of the Laplace operator on smooth manifolds to certain riemannian spaces with singularities.. The details for parts of our program were given in [9], [10], [15], [16]. The purpose of the present paper is to give further details, especially those concerning the trace of the heat kernel and the application of the heat equation method to the index theorem for the Euler characteristic and signature complexes.

Recall that the simplest geometric singularity is that of a metric cone. If N^m is a riemannian manifold, then the *metric cone* $C(N^m)$ on N^m is the space $R^+ \times N^m$ with the metric $dr^2 + r^2 g$. The completed cone is denoted by $C^*(N^m) = C(N^m) \cup p$.

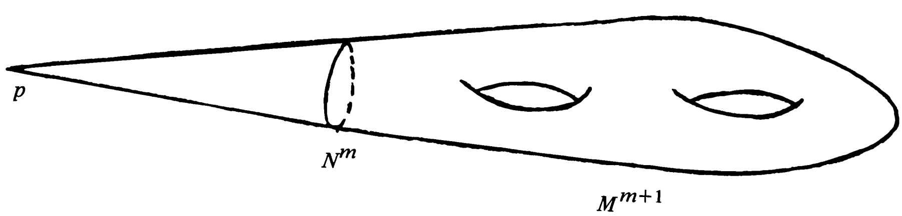

FIG. 0.1

X^{m+1} is called a space with an *isolated metrically conical singularity* if $X^{m+1} = \overline{C_{0,u}(N^m)} \cup M^{m+1}$. Here $C_{0,u}(N^m) = \{(r, x) \in C(N^m) \mid 0 < r < u\}$, $\partial M^{m+1} = N^m$, and the union is along the boundary. As in [8] we can also consider the case in which $\partial N^m \neq \varnothing$.[1] By definition, analysis on X^{m+1} means analysis on the incomplete riemannnian manifold $X^{m+1} \setminus p$.

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[1] Some times we will assume for simplicity that $\partial M^n = \varnothing$, but all our results generalize to the case $\partial M^n \neq \varnothing$.

In [9] we proved the Hodge theorem in the more general context of pseudomanifolds with metrics which are inductively of the above type (see also the work of Teleman [52] for Hodge theory on Lipschitz manifolds). Recall that a (closed) n-dimensional pseudomanifold X^n is a simplicial complex such that every point is contained in a closed n-simplex, and every $(n-1)$-simplex is contained in precisely two n-simplicies. The metrics we considered were (somewhat more general than) quasi-isometrically piecewise flat, i.e., $X^n \setminus \Sigma^{n-2}$ is a flat riemannian manifold and an open dense subset, and each closed simplex $\bar{\sigma}^n$ is isometric to a linear simplex in $\mathbf{R}^n$. Here Σ^i denotes the i-skeleton of X. Analysis on X^n means, by definition, analysis on $X^n \setminus \Sigma^{n-2}$ (one shows that this definition is independent of triangulation). Insofar as L^2-cohomology and Hodge theory are concerned, in fact, it is only the quasi-isometry class of the metric which plays a role. Analysis on pseudomanifolds of the above type reduces inductively to the conical case because each point $x \in \sigma^i \subset \Sigma^i$ has a neighborhood which is isometrically of the form $U^i \times C_{0,\epsilon}(L(\sigma^i))$ where $U^i \subset \sigma^i$ is flat ($L(\sigma^i)$ denotes the link of σ^i and the neighborhood has the product metric).

An important feature of the discussion of [8], [9], was the fact that (if a certain local topological condition is satisfied) the space of L^2 forms which are closed and coclosed forms is isomorphic to the L^2-cohomology, which is in turn isomorphic to the dual of the "middle intersection homology" introduced by Goresky and MacPherson [29], [30]. The connection between our work and that of Goresky-MacPherson was pointed out by Dennis Sullivan in 1976 (see [12] for some "historical remarks" on the evolution of these ideas). For spaces with isolated conical singularities the local topological condition in question is: $m = \dim N = 2k+1$, or if $m = 2k$, then $H^k(N^{2k}, R) = 0$. If this condition is not satisfied, then $\bar{d}_k$ and $\bar{\delta}_{k+1}$ are not adjoint operators, and "ideal boundary conditions" must be introduced.

We want to emphasize that our discovery that Poincaré duality can be restored in the context of pseudomanifolds was completely independent of that of Goresky-MacPherson; from our analytic viewpoint, this was reflected by the action of $*$-operator on harmonic forms.[2] In [7], [8], [9], we indicated how a local formula for the resulting signature can be treated by the heat equation method; see §9 for details.

As further background we recall that the interplay between L^2-cohomology and intersection homology theory led to certain natural conjectures concerning

[2] In the most general case one chooses ideal boundary conditions which are invariant under the $*$-operator. This is closely related to the extension of the Goresky-MacPherson theory to more general spaces, due to Morgan [40]; see also [48].

the topology of complex algebraic varieties; see [9], [10], [12] (recently most of these have been proved by other methods). Also in joint work with M. Taylor [15], [16] we gave a description of the phenomenon diffraction of waves by cones of arbitrary cross section, for which the starting point is the fundamental solution of the wave equation on a cone constructed in [7]. Finally in his thesis, State University of New York, 1982, Arthur Chou studied the Dirac operator on spaces with isolated conical-singularities.

The paper has nine remaining sections:

1. Parametrices for the heat kernel
2. Conformal homogeneity of $C(N^m)$ and the trace of the heat kernel
3. Functional calculus on cones
4. The asymptotics of the trace of the heat kernel
5. The Euler characteristic
6. The η-invariant and signature
7. Pseudomanifolds
8. The Chern Gauss Bonnet formula for pseudomanifolds
9. The η-invariant and combinatorial formulas for Pontrjagin classes

§§2–6 deal with the case of isolated conical singularities, while §§7–9 are concerned with the inductive generalization to polyhedra. The main result of the first six sections is the explicit calculation of the asymptotic expansion of the trace of the heat kernel. This is carried out in §§1–4. In §5 we find a formula for the Euler characteristic by the heat equation method. In so doing, we obtain new expressions for the Lipschitz-Killing curvatures as spectral invariants on a smooth manifold. In §6, the η-invariant formula of Atiyah-Patodi-Singer [2] is derived by applying the heat equation method to the signature complex of a space with isolated conical singularities. The main result of §7 is the derivation of the asymptotic expansion for a piecewise flat pseudomanifold, X^n. We justify the simple heuristic arguments which suggest that for all i, the expansion for i-forms contains only nonpositive (half) powers of t, and that the coefficient of $t^{l-n/2+j/2}$ is locally computable on the $(n-j)$-skeleton of X^n. As an application, in §8 we derive a formula for the Euler characteristic in which spectral invariants drop out, and what remains are *interior* dihedral angles between simplices of X^n (in [14], it is shown that the limit of this expression under fat subdivision is the Chern-Gauss-Bonnet formula in the smooth case). In §9 the analogous procedure is applied to derive a local formula for the signature of a certain class of pseudomanifolds in terms of η-invariants of links. This yields a canonical local combinatorial formula for the generalization of L-classes to such spaces, and in particular for the L-classes of piecewise linear manifolds. Unfortunately the η-invariants are difficult to compute explicitly and may well not be rational numbers in general. However, it seems possible that approximate calculations might be done on a computer in simple cases. Finally, we consider the invariant $\rho_E(Y)$, $\sigma_1(\tilde{Y})$ where E is a flat vector bundle and $\tilde{Y}$ is a finite covering which was introduced in [2] for smooth closed manifolds. We show that these have analogs $\hat{\rho}_E(Y)$, $\hat{\sigma}_1(\tilde{Y})$ for a general class of pseudomanifolds. The $\hat{\rho}_E(Y)$ are piecewise linear invariants, but the $\rho_E(Y)$ are only known to be diffeomorphism invariants in general. We conjecture that for Y smooth, $\rho_E(Y) = \hat{\rho}_E(Y)$.

We point out that in recent work, Werner Müller [41] has shown that most of our results for isolated conical singularities, have close analogs, for "metric cups." Moreover, in this work, Müller investigates further highly interesting aspects of the spectral geometry of cusps, and gives important applications.

We also wish to mention some earlier works of Fedosov [19], [20]. Although these works do not employ the Hankel transform, nor do they pursue matters in detail in dimensions other than 2, they do espouse a point of view which has much in common with that of the present paper (see also [21] for further references to the 2-dimensional case).

We are grateful to Arthur Chou and Michael Taylor for several very helpful conversations concerning this work.

Most of the results of this paper were obtained in 1977–78 during which time the author was a visiting member of the Institute for Advanced Study. He wishes to thank the Institute for its hospitality.

1. Parametrices for the heat kernel

Let Y be a riemannian manifold (possibly incomplete) with empty boundary. Let Δ_0 denote the Laplacian restricted to the space Λ^i_0 of i-forms of compact support. If we fix a particular selfadjoint extension Δ of Δ_0, then the heat kernel $e^{-\Delta t}$ can be defined as a bounded selfadjoint operator, via the spectral theorem. As in [17] for example, it can then be shown that for $t > 0$, the action of $e^{-\Delta t}$ on L^2 is given by the action of a smooth symmetric kernel $E(x, y, t)$. If $P(d_x, \delta_x, d_y, \delta_y)$ is any polynomial, then $P(d_x, \delta_x, d_y, \delta_y)E(x, y, t)$ is in the domain of all powers of the Laplacian when considered as a function of either space variable. Moreover, given open sets U, V with compact closure such that $y \in V$, $\overline{V} \subset U$, we have for all $N > 0$,

$$(1.1) \qquad \|P(d_x, \delta_x, d_y, \delta_y)E(x, y, t)\|_{M \setminus U} \leq K_N t^N, \quad \text{as } t \to 0,$$

where the norm on the left-hand side is the L^2 norm of $E(x, y, t) \mid M$, regarded as a function of x.

Now let M_1, M_2 be arbitrary riemannian manifolds. For simplicity assume $\partial M_1 = \partial M_2 = \varnothing$. Assume there are open manifolds $Z_j \subset M_j$ with compact

FIG. 1.1

smooth boundary $\partial Z_j \subset M_j$ and an isometry $I: Z_1 \to Z_2$. From now on we just write $Z \subset M_j$, $j = 1, 2$. Let Δ_j be selfadjoint extensions of the Laplacians on $\Lambda^i_0(M_j)$, $j = 1, 2$. Assume that the restrictions of the Δ_j to Z agree in the following sense. If $w_j \in \Lambda^i_0(M_j)$, then

$$(1.2)\qquad \int_Z \Delta_1 w_1 \wedge * w_2 - \int_Z w_1 \wedge * \Delta_2 w_2 = \int_{\partial Z} \delta w_1 \wedge * w_2 + \cdots,$$

where the dots denote the standard boundary terms. If we let E_j denote the heat kernel on M_j and apply Duhamel's principle, we obtain

$$(1.3)\qquad \begin{aligned} &E_2(x, y, t) - E_1(x, y, t) \\ &\quad = \int_0^t \int_{\partial Z} E_1(x, z, t-s) \wedge * dE_2(z, y, s) \\ &\qquad + (-1)^{ni+1} \int_0^t \int_{\partial Z} * dE_1(x, z, t-s) \wedge E_2(z, y, s) \\ &\qquad + (-1)^{n(i-1)+1} \int_0^t \int_{\partial Z} * E_1(x, z, t-s) \wedge \delta E_2(z, y, s) \\ &\qquad + \int_0^t \int_{\partial Z} \delta E_1(x, z, t-s) \wedge * E_2(z, y, s), \end{aligned}$$

where all operations are applied to the variable z. Let $\overline{W} \subset Z$. Then in view of (1.1), equation (1.3) exhibits $E_1 - E_2 \,|\, W \times W$ as an integral over $[0, t] \times \partial Z$ of a family of finite rank (rank 1) operators whose trace norm is uniformly bounded. Thus $E_1 - E_2$ is trace class, and moreover, by (1.1), as $t \to 0$, for all N,

$$(1.4)\qquad \left| \int_W E_1(x, x, t) - \int_W E_2(x, x, t) \right| < K_N t^N.$$

At this point we specialize the discussion to the case of spaces with isolated conical singularities. Let N^m be compact, $\partial N^m = \varnothing$ and suppose

$$(1.5)\qquad \begin{aligned} &M_1 = C(N^m), \quad M_2 = X^{m+1} = C_{0,1}(N^m) \cup M^{m+1}, \\ &Z = C_{0,1}(N^m), \quad W = C_{0,1/2}(N^m). \end{aligned}$$

In case $m = 2k - 1$ or $m = 2k$ and $H^k(N^{2k}, R) = 0$, let $\bar{\Delta}_j$ be the selfadjoint extensions $\bar{d}\bar{\delta} + \bar{\delta}\bar{d}$ (see [7]). More generally, if $H^k(N^{2k}, R) \neq 0$, choose ideal boundary conditions as in [7]. Recall (see [7]–[9]) that this means the following: Let α be a smooth k-form such that α, $d\alpha \in L^2$. Let $\tilde{\mathcal{H}}^k$ denote the space of harmonic k-forms on N^{2k}, and choose an orthogonal direct sum decomposition

$$(1.6)\qquad \tilde{\mathcal{H}}^k = V_a \oplus V_r.$$

On $C_{0,1}(N^{2k})$, write $\alpha = \phi + dr \wedge \omega$ and

$$(1.7)\qquad \phi = \phi_{\tilde{\mathcal{H}}} + \phi_{c\tilde{\varepsilon}} + \phi_{\tilde{\varepsilon}},$$

where

$$(1.8)\qquad \phi_{\tilde{\mathcal{H}}} = \phi_{V_a} + \phi_{V_r}.$$

Let $\{h_j\}$ be an orthonormal basis for $\tilde{\mathcal{H}}^k$ such that $\{h_j\}, j = 1 \cdots p$, and $\{h_j\}$, $j = p + 1 \cdots q$, are orthonormal bases for V_a, V_r respectively. If

$$(1.9)\qquad \phi_{V_a} = \sum_{j=1}^{p} f_j(r) h_j, \quad \phi_{V_r} = \sum_{p+1}^{q} g_j(r) h_j,$$

then we say that α *is in* $\operatorname{dom} d$ if f_j, g_j satisfy Neumann and Dirichlet conditions respectively at $r = 0$. Similarly, $\beta \in \operatorname{dom} \delta_{2k+1}$ for the decomposition $V_a \oplus V_r$, if $*\beta \in \operatorname{dom} d$ for the decomposition $*V_r \oplus *V_a$ (Neumann conditions on $*V_r$, Dirichlet on $*V_a$). Then it is not difficult to check that $\bar{d}^*_{2k} = \bar{\delta}_{2k+1}$. As above we define $\Delta_j = \delta_{j+1} d_j + d_{j-1}\delta_j$. Then in general, if we wish to guarantee that Poincaré duality

$$(1.10)\qquad * \ker \Delta_{2k} = \ker \Delta_{2k+1}$$

holds, we must assume that

$$(1.11)\qquad * V_a = V_r, \quad * V_r = V_a,$$

i.e., that the boundary conditions are $*$-invariant. Since also $V_a^{\perp} = V_r$, this implies that V_a, V_r are maximal self-annihilating subspaces for the cup product pairing on N^{2k}.

We now apply (1.4) with M_1, M_2, Z, W as in (1.5). A standard argument shows that

$$(1.12)\qquad \int_{M^{m+1}} E_2(x, x, t) < \infty,$$

and that the integral in (1.12) has the usual asymptotic expansion. Moreover in §4 we show that for $t > 0$,

$$\int_{C_{0,1}(N^m)} E_1(x, x, t) < \infty. \tag{1.13}$$

Thus as $t \to 0$, for all N,

$$\int_{X^{m+1}} E_2(x, x, t) = \int_{C_{0,1}(N^m)} E_1(x, x, t) + \int_{M^{m+1}} E_2(x, x, t) + O(t^N). \tag{1.14}$$

If we use the semigroup property

$$E_2(t) = E_2(t/2) \cdot E_2(t/2), \tag{1.15}$$

then (1.12)–(1.14) imply that $E_2(t/2)$ is Hilbert-Schmidt. Hence $E_2(t)$ is a product of Hilbert-Schmidt operators and thus is trace class with trace given by (1.14). We will use similar arguments ((1.14), (1.15) imply trace class) elsewhere in the paper without further comment.

In case N^m is the interior of a compact manifold with boundary, the discussion is entirely similar. One can choose either Dirichlet or Neumann boundary conditions for N^m and the corresponding Laplacians for generalized Dirichlet or Neumann conditions $\overline{\Delta}_D = \bar{\delta}\bar{d}_0 + \bar{d}_0\bar{\delta}$, $\overline{\Delta}_N = \bar{\delta}_0\bar{d} + \bar{d}_0$ on X^{m+1}. If $H^k(N^{2k}, \partial N^k, R) \neq 0$ (respectively $H^k(N^{2k}, R) \neq 0$) one can also introduce ideal boundary conditions. Away from the singularity, the generalized boundary conditions reduce to ordinary boundary conditions, and (1.1)–(1.4) and (1.12), (1.13) still hold.

We close this section by mentioning that the techniques of this section can be used to give a simple proof of a quite general *Relative index theorem* of the type recently formulated by Gromov and Lawson in [31]. They consider a pair of generalized Dirac operators $\mathfrak{D}_1$, $\mathfrak{D}_2$ on complete manifolds M_1, M_2 such that $\mathfrak{D}_1$, $\mathfrak{D}_2$, M_1, M_2 are identified outside a compact set. They assume that the scalar curvatures of M_1, M_2 are positive at ∞; this implies in particular that all kernels and cokernels are finite. They then prove that the appropriate characteristic class (giving the topological index) in $M_1 \setminus C \cup M_2 \setminus C$ gives the difference of the indices. In the generalization referred to above, the kernels and cokernels could be *infinite dimensional*. The individual indices, index $\mathfrak{D}_j$, need not be defined, but we show index $\mathfrak{D}_1$-index $\mathfrak{D}_2$ is well defined and equal to the appropriate characteristic class. For this we need an assumption on the behavior of the Green's operators for $\mathfrak{D}_j^*\mathfrak{D}_j$, $\mathfrak{D}_j\mathfrak{D}_j^*$. That some such assumption is necessary even in the complete case, is clear from considering the example of half line with Dirichlet and Neumann boundary conditions, and the operator $d\colon \Lambda^0 \to \Lambda^1$. Here $\text{index}_D - \text{index}_N = 1$, but all kernels and cokernels are zero dimensional; see [11] for further details.

2. Conformal homogeneity of $C(N^m)$ and the trace of the heat kernel

In the previous section we saw that the study of the trace tr $E_i(t)$ of the heat kernel on i-forms of a manifold $X = C_{0,1}(N^m) \cup M^{m+1}$, with conical singularities, reduces to studying

$$\int_{C_{0,1}(N^m)} \operatorname{tr} \mathcal{E}_i(t), \tag{2.1}$$

where $\mathcal{E}_i(t)$ is the heat kernel on $C(N^m)$. In the present section, by using the conformal homogeneity of $C(N^m)$ we exhibit the form of the asymptotic expansion of (2.1) as $t \to 0$. Further, we reduce the explicit calculation of the coefficients to calculation of the pointwise coefficients of $\operatorname{tr}(\mathcal{E}_i(t))$ at $r = 1$, and to the calculation of what turns out to be a certain global spectral invariant of N^m, giving the contribution to the constant term coming from the singularity at p. These calculations are carried out in §§3 and 4.

Let $\mathcal{E}_i(r_1, x_1, r_2, x_2, t)$ denote the heat kernel on $C(N^m)$. Let $\omega(r, x)$ denote the volume form, and let $T_k : C(N^m) \to C(N^m)$ be the homothetic transformation defined by $T_k((r, x)) = (kr, x)$. Set

$$\operatorname{tr}(\mathcal{E}_i(r, x, r, x, t)) = f(r, x, t)\omega, \tag{2.2}$$

where ω is the volume form. Then, as a consequence of the conformal homogeneity of $C(N^m)$,

$$T_r^*(\omega(r, x)) = r^{m+1}\omega(1, x), \tag{2.3}$$

$$f(r, x, t) = r^{-(m+1)}f(1, x, t/r^2). \tag{2.4}$$

Let $dr \wedge \beta$ be the volume form on $R^+ \times N^m$ with respect to the product metric, and $S : R^+ \times N^m \to C(N^m)$ be the polar coordinate map. Then

$$S^*(\omega) = r^m \, dr \wedge \beta. \tag{2.5}$$

If the pointwise asymptotic expansion of tr $\mathcal{E}_i(r, x, t)$ is given by[3]

$$\sum \bar{a}_{j/2}(r, x)\omega t^{-(m+1)/2+j/2}, \tag{2.6}$$

then (2.3) and (2.4) imply

$$\bar{a}_{j/2}(r, x) = r^{-j}\bar{a}_{j/2}(1, x). \tag{2.7}$$

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[3] If we assume that $\partial N^m = \varnothing$, we actually could write j instead of $j/2$. The bar in $\bar{a}_{j/2}$ indicates that $\bar{a}_{j/2}$ is a function. We will usually write $\bar{a}_{j/2}\omega = a_{j/2}$.

In particular, if $X_u = X \backslash C_{0,u}(N)$ then

$$\int_{X_u} \bar{a}_{j/2}(r, x)\omega = c_{j/2} + \begin{cases} -\int_N \bar{a}_{j/2}(1, x)\beta \dfrac{u^{m+1-j}}{m+1-j}, & j \neq m+1, \\ -\int_N \bar{a}_{(m+1)/2}(1, x)\beta \cdot \log u, & j = m+1, \end{cases} \tag{2.8}$$

for some constant $c_{j/2}$. Thus, although the integral on the right-hand side of (2.8) diverges as $u \to 0$ if $j \geqslant m + 1$, its *finite part* may be defined in all cases as

$$\text{p.f.} \int_X \bar{a}_{j/2}\omega = c_{j/2}. \tag{2.9}$$

If $j < m + 1$, then

$$\text{p.f.} \int_X \bar{a}_{j/2}\omega = \int_X \bar{a}_{j/2}\omega. \tag{2.10}$$

The above calculation makes it apparent that for $j \geqslant m + 1$, the coefficient in $\operatorname{tr}(\mathcal{E}_i(t))$ cannot possibly be obtained by integrating the pointwise coefficient over X, since the integral does not converge. It also suggests that the correct answer might be obtained by taking the finite part of the integral. This turns out to be the essentially correct. However, there is an additional contribution to the constant term coming from the singular point, and a logarithmic term also enters. Set

$$\int_N \bar{a}_{j/2}(1, x)\beta = a_{j/2}(1). \tag{2.11}$$

Define $\mu_k(u)$ by

$$\mu_K(u) = \int_N f(1, x, u)\beta - \sum_{j=0}^{K} a_{j/2}(1) u^{-(m+1)/2+j/2}. \tag{2.12}$$

Theorem 2.1.

(i)
$$\int_{C_{0,1}(N^m)} \operatorname{tr} \mathcal{E}_i(t) = \tfrac{1}{2}\int_t^\infty u^{-1} \int_N f_i(1, x, u)\beta \, du.$$

(ii) *For all* $K > m + 1$,

$$\begin{aligned} \int_X \operatorname{tr} E(t) \sim \sum_{j=0}^{K} \left(p.f. \int_X \bar{a}_{j/2} \right) \cdot t^{-(m+1)/2+j/2} - \tfrac{1}{2} a_{(m+1)/2}(1) \log t \\ + \tfrac{1}{2}\int_1^\infty \int_N u^{-1} f(1, x, u)\beta \, du + \tfrac{1}{2}\int_0^1 u^{-1} u_K(u) \, du \\ + \tfrac{1}{2} \sum_{j \neq m+1} \frac{a_{j/2}(1)}{-(m+1)/2 + j/2}. \end{aligned} \tag{2.13}$$

Proof. It follows from (2.2), (2.3), (2.4) that

$$\begin{aligned}\int_{C_{0,1}(N^m)} \operatorname{tr} \mathfrak{S}(r, x, t) &= \int_0^1 \int_{N^m} f(r, x, t) r^m \beta \, dr \\ &= \int_0^1 \int_{N^m} f(1, x, t/r^2) r^{-1} \beta \cdot dr.\end{aligned} \tag{2.14}$$

Set $t/r^2 = u$. Then

$$-\frac{t}{u^2} du = 2r\, dr, \quad \frac{u}{t} = r^{-2},$$

so that $-\frac{1}{2}u^{-1}du = r^{-1}dr$. Thus the right-hand side of (2.14) becomes

$$\tfrac{1}{2}\int_t^\infty u^{-1} \int_N f(1, x, u)\beta \, du, \tag{2.15}$$

which gives (i).

To prove (ii), first write

$$\int_X \operatorname{tr} E(t) = \int_{C_{0,1}(N)} \operatorname{tr} E(t) + \int_{X \setminus C_{0,1}(N)} \operatorname{tr} E(t).$$

By §1, as $t \to 0$,

$$\int_X \operatorname{tr} E(t) \sim \int_{C_{0,1}(N)} \operatorname{tr} \mathfrak{S}(t) + \int_{X \setminus C_{0,1}(N)} \operatorname{tr} E(t), \tag{2.16}$$

and it clearly suffices to prove the formula for $\int_{C_{0,1}(N)} \operatorname{tr} \mathfrak{S}(t)$. Take $K > m + 1$. Add and subtract $\int_0^t u^{-1}\mu_K(u)\, du$. Then (2.15) can be rewritten as

$$\begin{aligned}&\tfrac{1}{2}\int_1^\infty u^{-1} \int_N f(1, x, u)\beta \, du + \tfrac{1}{2}\int_0^1 u^{-1}\mu_K(u) \\ &+ \tfrac{1}{2}\sum_{j=0}^{K} \int_t^1 a_{j/2}(1) u^{-(m+3)/2 + j/2} - \tfrac{1}{2}\int_0^t u^{-1}\mu_K(u)\, du.\end{aligned} \tag{2.17}$$

Since

$$\begin{aligned}\frac{1}{2}\sum \int_t^1 a_{j/2}(1) u^{-(m+3)/2+j/2} = \frac{1}{2}\sum_{j \neq m+1} &\left[\frac{a_{j/2}(1)}{-(m+1)/2 + j/2}\right. \\ &\left. - \frac{a_{j/2}(1)}{-(m+1)/2 + j/2} t^{-(m+1)/2 + j/2}\right] \\ &- \frac{1}{2} a_{(m+1)/2}(1) \log t,\end{aligned} \tag{2.18}$$

the second term on the right-hand side of (2.18) is just

$$\sum_{j=0}^{K}\left(\text{p.f}\int_{C_{0,1}(N^m)}\bar{a}_{j/2}\right)t^{-(m+1)/2+j/2}. \tag{2.19}$$

The last term of (2.17) is $O(t^{(K+1)/2-(m+1)/2})$. Thus by inserting (2.18) into (2.19), (ii) follows. q.e.d.

The last three terms in (2.13) constitute the contribution to the constant term coming from the singular point p. If we think of the connection between the heat kernel and the zeta function which holds in the compact case, then we see that formally this contribution is the constant term in the Laurent expansion at $s = 0$ of

$$\Gamma(s)\zeta(s) = \Gamma(s)\int_{(1,N^m)}\Delta^{-s}(1,x). \tag{2.20}$$

The explicit calculations of §4 will exploit such a relationship. For the case of the pointwise coefficients of $\operatorname{tr} E_i(t)$, some small modifications are necessary, because in fact, $C(N^m)$ is not compact. In §3 we will derive the formulas for $\mathcal{E}_i(t)$ and $\Gamma(s)\Delta^{-s}$ on $C(N^m)$.

3. Functional calculus on cones

In [8] we described a functional calculus for the Laplacian Δ on the cone, which, by employing the evaluation of various classical integrals, allowed the explicit calculation of the kernels representing the most important functions $f(\Delta)$. We now recall this calculus and give further details on the functions $e^{-\Delta t}$ and $\Gamma(s)\Delta^{-s}$ which are of particular interest for the present paper.

As in [8] we introduce the following notation. Operations on the cross section are indicated by a tilda. The coclosed eigenforms of $\tilde{\Delta}$ in dimension 1 are denoted by ϕ_j, and the corresponding eigenvalules by μ_j. We set

$$\alpha(i) = \tfrac{1}{2}(1 + 2i - m), \tag{3.1}$$

$$\nu_j(i) = \sqrt{\mu_j + \alpha^2(i)}\,, \tag{3.2}$$

$$a_j^{\pm}(i) = \alpha(i) \pm \nu_j(i), \tag{3.3}$$

$$\nu(i) = \sqrt{\delta_{i+1}d_i + \alpha^2(i)} - |\alpha(i)|\, P_{\mathcal{K}} = P_{\tilde{c}e}\sqrt{\tilde{\Delta}_i + \alpha^2(i)}\,. \tag{3.4}$$

Thus $\nu(i)$ is a pseudodifferential operator; P_{cc}, P_{ce}, P_c, P_e, $P_{\mathcal{H}}$ denote orthogonal projection on the subspaces of coclosed, coexact, closed, exact and harmonic forms respectively.

$$e^{\nu(i-1)} = P_e\sqrt{\tilde{\Delta}_i + \alpha^2(i-1)}\,. \tag{3.5}$$

If $\theta(r,x) = \beta(r,x) + dr \wedge \omega(r,x)$ is a form on $C(N^m)$, and $\partial\omega/\partial r = \omega'$, etc., then

$$*\,\theta = r^{m-2i+2}\tilde{*}\omega + (-1)^i r^{m-2i} dr \wedge \tilde{*}\beta, \tag{3.6}$$

$$\delta\theta = r^{-2}\tilde{\delta}\beta - r^{-2}dr \wedge \tilde{\delta}\omega - \left(\omega' + (m-2i+2)r^{-1}\omega\right), \tag{3.7}$$

$$\begin{aligned} \Delta\theta = {} & -\beta'' - (m-2i)r^{-1}\beta' + r^{-2}\tilde{\Delta}\beta - 2r^{-3}dr \wedge \tilde{\delta}\beta \\ & + dr \wedge \left[-\omega'' - (m-2i+2)r^{-1}\omega' + (m-2i+2)r^{-2}\omega + r^2\tilde{\Delta}\omega\right] \\ & -2r^{-1}\tilde{d}\omega. \end{aligned} \tag{3.8}$$

See [10] for details. Formula (3.8) corrects some misprints which appeared in the corresponding formulas of [6], [8].

Let $\phi^i(r,x)$ be an i-form such that for each r, $\phi^i(r,x)\,|\,(r,N)$ is coexact. We can then introduce i-forms of the following types, to be called 1, 2, 3, 4 respectively.

$$r^{\alpha(i)}\phi^i, \tag{3.9}$$

$$r^{\alpha(i-1)}\tilde{d}\phi^{i-1} + dr \wedge \left(r^{\alpha(i-1)}\phi^{i-1}\right)', \tag{3.10}$$

$$r^{2\alpha(i-1)+1}\left(r^{-\alpha(i-1)}\tilde{d}\phi^{i-1}\right)' + r^{\alpha(i-1)-1}dr \wedge \tilde{\delta}\tilde{d}\phi^i, \tag{3.11}$$

$$r^{\alpha(i-2)+1}dr \wedge \tilde{d}\phi^{i-2}. \tag{3.12}$$

The rational behind our convention concerning the powers of r will become apparent below. Note that types 1 and 3 are coexact, while types 2 and 4 are exact. The operator d carries types 1 and 3 to types 2 and 4, while δ carries types 2 and 4 to types 1 and 3. In addition to the above forms, we also introduce types E and O:

$$r^{\alpha(i)}h^i, \tag{3.13}$$

$$dr \wedge \left(r^{\alpha(i-1)}h^{i-1}\right)'. \tag{3.14}$$

Here for each r, $h(r,x)\,|\,(r,N)$ is harmonic.

Note that the forms in (3.13) and (3.14) would appear as special cases of the forms in (3.9) and (3.10) respectively if we allowed ϕ^i, ϕ^{i-1} to be coclosed with the convention which we have adopted, the $*$-operator interchanges types 1 and 3, 2 and 4, E and O.

The eigenforms of Δ with eigenvalue $\lambda^2 \neq 0$ can be decomposed into the types above. Let J_ν be the Bessel function of order ν. If ϕ_j^{i-2}, ϕ_j^{i-1}, ϕ_j^i are coexact eigenforms of $\tilde{\Delta}$ in dimensions $i-2$, $i-1$, i, respectively, then we have the eigenforms

$$r^{\alpha(i)} J_{\pm \nu_j(i)}(\lambda r)\phi_j^i, \tag{3.15}$$

$$r^{\alpha(i-1)} J_{\pm \nu_j(i-1)}(\lambda r) d\phi_j^{i-1} + \left(r^{\alpha(i-1)} J_{\pm \nu_j(i-1)}(\lambda r)\right)' dr \wedge \phi_j^{i-1}, \tag{3.16}$$

$$\begin{aligned} &r^{2\alpha(i-1)+1}\left(r^{-\alpha(i-1)} J_{\pm \nu_j(i-1)}(\lambda r)\right)' d\phi_j^{i-1} \\ &\quad + r^{\alpha(i-1)-1} J_{\pm \nu_j(i-1)}(\lambda r) dr \wedge \tilde{\delta}\tilde{d}\phi_j^{i-1}, \end{aligned} \tag{3.17}$$

$$r^{\alpha(i-2)+1} J_{\pm \nu_j(i-2)}(\lambda r) dr \wedge d\phi_j^{i-2}. \tag{3.18}$$

If ν_j is half integer, the $(+)$ and $(-)$ solutions above are not independent, and logarithmic $(-)$ solutions must be introduced (see [7]). Since (apart from the case of ideal boundary conditions) we are interested only in the () solutions, this will not concern us further here. In addition to (3.15)–(3.18), we have E and O solutions

$$r^{\alpha(i)} J_{\pm|\alpha(i)|}(\lambda r) h_j^i, \tag{3.19}$$

$$\left(r^{\alpha(i-1)} J_{\pm|\alpha(i-1)|}(\lambda r)\right)' dr \wedge h_j^{i-1} \tag{3.20}$$

corresponding to the kernel of $\tilde{\Delta}$. Using the identities

$$(z^{-\nu} J_\nu(z))' = -z^{-\nu} J_{\nu+1}(z), \tag{3.21}$$

$$(z^{-\nu} J_{-\nu}(z))' = z^{-\nu} J_{-\nu-1}(z), \tag{3.22}$$

(see [53, p. 66]), one sees that

$$\begin{aligned} &\lambda * \left(r^{\alpha(m+1-i)} J_{\nu(m+1-i)}(\lambda r)\right) * h_j^{m+1-i} \\ &\quad = (-1)^{(m+1-i)(i-1)}\left(r^{\alpha(i-1)} J_{\nu(i-1)}(\lambda r)\right)' dr \wedge h_j^{i-1}, \end{aligned} \tag{3.23}$$

from which it follows that the family of forms in (3.19), (3.20) is invariant under d, $\delta *$ (up to the appropriate factor of λ).

To obtain the harmonic forms ($\lambda^2 = 0$), we may proceed directly or examine the limiting behavior of (3.15)–(3.18) as $\lambda \to 0$. If $\mu_j \neq 0$, $d\phi_j = \psi_j$, then the limits of (3.16), (3.17) give rise to forms of the same type $(2 + 3)$ given in (3.25) below. These forms are closed and coclosed. However, by considering the difference of the forms in (3.16), (3.17), we obtain a type of harmonic form $(2 - 3)$ which does not fit into the above scheme and which was overlooked in

[8]. Thus the statements of [8] concerning harmonic forms must be taken as applying only to (3.24), (3.25), (3.27) below (the appropriate modifications for type (2 − 3) were given in [7] and [10]). The four types of harmonic forms are

$$r^{a_j^{\mp}(i)}\phi_j^i, \tag{3.24}$$

$$r^{a_j^{\pm}(i-1)}d\phi_j^{i-1} + a_j^{\pm}(i-1)r^{a_j^{\mp}(i-1)-1}dr \wedge \phi_j^{i-1}, \tag{3.25}$$

$$r^{a_j^{\pm}(i-1)+2}d\phi_j^{i-1} + a_j^{\mp}(i-1)r^{a_j^{\pm}(i-1)+1}dr \wedge \phi_j^{i-1}, \tag{3.26}$$

$$r^{a_j^{\pm}(i-2)+1}dr \wedge d\phi_j^{i-2}, \tag{3.27}$$

where (for economy of notation) ϕ_j is a coclosed eigenform of $\tilde{\Delta}$.

If $\tilde{\Delta}\phi_j = 0$, some of the forms corresponding to the above coalesce while others vanish; see [7] for a precise description. It follows by inspection from (3.24)–(3.27) (and [7] for $\mu_j = 0$) that there are no square integrable harmonic forms on $C(N^m)$. Thus the Hodge decomposition takes the form

$$L^2 = \overline{d\Lambda_0^{i-1}} + \overline{\delta\Lambda_0^{i+1}}\,; \tag{3.28}$$

see [9]. It then follows from the above that $\overline{\delta\Lambda_0^{i+1}}$ is the orthogonal direct sum of the closures of the type 1, 3 and E subspaces of $\delta\Lambda_0^{i+1}$, and that $\overline{d\Lambda_0^{i-1}}$ is the orthogonal direct sum of the closures of the type 2, 4 and O subspaces of $d\Lambda_0^{i-1}$. That the corresponding statements do not hold for harmonic forms is explained by the fact that these forms are not square integrable (of course, the eigenforms with $\lambda^2 \neq 0$ are also not square integrable but the decomposition does happen to apply in that case).

Suppose $m = 2k - 1$ or $H^k(N^{2k}, R) = 0$. If the $(+)$ solutions in (3.15)–(3.18) are multiplied by a cutoff function $h(r)$, with $h(r) \equiv 1$ near $r = 0$ and $h(r) \equiv 0$ for say $r \geqslant \frac{1}{2}$, forms which are in the domain of the Laplacian $\Delta = d\delta + \delta d$ are obtained. Multiplying (–) solutions by $h(r)$ gives rise to forms which are not in the domain of Δ. For the case of ideal boundary conditions corresponding to $\mathcal{H}^k = V_a + V_r$ as in [8], we use the corresponding $(+)$ and (–) solutions; see [7] for a more complete description of the properties of $(+)$ and (–) solutions. The facts that $h(r)$ times a $(+)$ solution is in dom Δ, and that d and δ are adjoints, lead immediately to a functional calulus for Δ on $C(N^m)$, based on the Hankel inversion formula. Let

$$H_{\nu_j}(g_j) = \int_0^{\infty} J_{\nu_j}(\lambda r)g(r)r\,dr \tag{3.29}$$

denote the Hankel transform. Then as explained in [15] for functions, the map

$$\theta = \sum_j g_j\phi_j \to \left(H_{\nu_0}\left(r^{-\alpha(i)}g_0\right), H_{\nu_1}\left(r^{-\alpha(i)}g_1\right), \cdots\right) \tag{3.30}$$

provides an isometry of the space of L^2 type 1 forms onto the Hilbert space $L^2(R^+, l^2, \lambda d\lambda)$ of square integrable functions on R^+ with values in l^2, with respect to the measure $\lambda\, d\lambda$. This isometry carries Δ into multiplication by λ^2, for $\theta \in \operatorname{dom} \Delta$, and thus provides the spectral representation of Δ on type 1 forms. However, to obtain a closer analog of the Fourier transform on $\mathbf{R}^{m+1} = C(S_1^m)$, we can replace (3.30) by

$$\theta = \sum_j g_j(r)\phi_j(x) \overset{\mathfrak{F}}{\to} \sum_j \lambda^{\alpha(i)} H_{\nu_j}\left(r^{-\alpha(i)}g_j\right)\phi_j(y), \tag{3.31}$$

which is an isometry of L^2 of the (r, x) cone onto L^2 of the (λ, y) cone, such that

$$\mathfrak{F}(\Delta\theta) = \lambda^2\mathfrak{F}(\theta), \tag{3.32}$$

for $\theta \in \operatorname{dom} \Delta$ (if $H^k(N^{2k}, R) \neq 0$, use $H_{-\nu_j}$ in (3.31) for those ν_j corresponding to V_a). Note that $\mathfrak{F}$ is given by the action of the kernel

$$(\lambda r)^{\alpha(i)} J_\nu(\lambda r) = \sum_j (\lambda r)^{\alpha(i)} J_{\nu_{j(i)}}(\lambda r)\phi_j(y)\phi_j(x), \tag{3.33}$$

and corresponds to

$$f \to \frac{1}{(2\pi)^{(m+1)/2}} \int (\cos \xi x f_e(x) - \sin \xi x f_0(x)) \tag{3.34}$$

on $R^{m+1} = C(S_1^m)$, where $f = f_e + f_o$ is the decomposition of f into its even and odd parts. $\mathfrak{F}$ extends to forms of types 2, 3, 4 by specifying that

$$\mathfrak{F}(d\theta) = \lambda d\lambda \wedge \mathfrak{F}(\theta), \tag{3.35}$$

$$\mathfrak{F}(*\theta) = *\mathfrak{F}(\theta) \tag{3.36}$$

(if N^m is not orientable, we identify forms on $C(N^m)$ with those forms of $C(\tilde{N}^m)$ which are invariant under the natural involution). So defined, $\mathfrak{F}$ provides an isometry of $L^2(C(N^m))$ to $L^2(C(N^m))$, such that $\mathfrak{F}(\Delta\theta) = \lambda^2\mathfrak{F}(\theta)$ for $\theta \in \operatorname{dom} \Delta$. It preserves types 1 and 4 and also the direct sum of the spaces of types 2 and 3, but does not preserve types 2 and 3 individually.

In view of the discussion of types E and O surrounding (3.19)–(3.23) it follows that we can take account of these forms (without counting them twice), by allowing coclosed eigenforms ϕ_j in the formulas for types 1 and 2, but not in types 3 and 4. This will be done below. With this convention, $*$ of a form of type 1 or 3 is no longer necessarily of type 2 or 4, but this will cause no problem.

By combining the above discussion with the spectral theorem and the Hankel inversion formula, [53], we find that the following formal relation holds for type 1 forms:

$$\begin{aligned} f(\Delta) &= (r_1 r_2)^{\alpha(i)} \\ &\cdot \sum_j \int_0^\infty f(\lambda^2) J_{\nu_j(i)}(\lambda r_1) J_{\nu_j(i)}(\lambda r_2)\lambda\, d\lambda \phi_j(x_1) \otimes \phi_j(x_2). \end{aligned} \tag{3.37}$$

In general, the Hankel transform in (3.37) may have to be understood in the distribution sense, but for our present purpose this is not necessary.

The formula for i-forms of type 2 corresponding to (3.37) is

$$\begin{aligned} f(\Delta) = &\sum_j d_1 d_2 (r_1 \cdot r_2)^{\alpha(i-1)} \\ &\cdot \int_0^\infty f(\lambda^2) J_{\nu_j(i-1)}(\lambda r_1) J_{\nu_j(i-1)}(\lambda r_2) \lambda^{-} d\lambda \phi_j^{i-1} \otimes \phi_j^{i-1} \\ = &\sum_j \int_0^\infty f(\lambda^2) \Big\{ \Big[\alpha(i-1) r_1^{\alpha(i-1)-1} J_{\nu_j(i-1)}(\lambda r_1) \\ &\qquad + r_1^{\alpha(i-1)} J'_{\nu_j(i-1)}(\lambda r_1)\lambda \Big] dr_1 \wedge \phi_j^{i-1} \\ &\qquad + r_1^{\alpha(i-1)} J_{\nu_j(i-1)}(\lambda r_1) d\phi_j^{i-1} \Big\} \\ &\otimes \Big\{ \Big[\alpha(i-1) r_2^{\alpha(i-1)-1} J_{\nu_j(i-1)}(\lambda r_2) \\ &\qquad + r_2^{\alpha(i-1)} J'_{\nu_j(i-1)}(\lambda r_2)\lambda \Big] dr_2 \wedge \phi_j^{i-1} \\ &\qquad + r_2^{\alpha(i-1)} J_{\nu_j(i-1)}(\lambda r_2) d\phi_j^{i-1} \Big\} \lambda^{-1} d\lambda. \end{aligned} \tag{3.8}$$

It is sometimes convenient to rewrite (3.38) using the identity

$$J'_{=} \frac{J_{\nu-1} - J_{\nu+1}}{z}. \tag{3.39}$$

The expression for forms of type 3 is

$$
\begin{aligned}
f(\Delta) = \sum_j \int_0^\infty f(\lambda^2) \Big\{ \Big[&-\alpha(i-1) r_1^{\alpha(i-1)} J_{\nu_j(i-1)}(\lambda r_1) \\
&+ r_1^{\alpha(i-1)+1} J'_{\nu_j(i-1)}(\lambda r_1)\lambda \Big] \frac{d\phi_j^{i-1}}{\sqrt{\mu_j}} \\
&+ r_1^{\alpha(i-1)-1} J_{\nu_j(i-1)} dr_1 \wedge \sqrt{\mu_j}\,\phi_j^{i-1} \Big\} \\
\otimes \Big\{ \Big[&-\alpha(i-1) r_2^{\alpha(i-1)} J_{\nu_j(i-1)}(\lambda r_1) \\
&+ r_2^{\alpha(i-1)+1} J'_{\nu_j(i-1)}(\lambda r_2)\lambda \Big] \frac{d\phi_j^{i-1}}{\sqrt{\mu_j}} \\
&+ r_2^{\alpha(i-1)-1} J_{\nu_j(i-1)} dr_2 \wedge \sqrt{\mu_j}\,\phi_j^{i-1} \Big\} \lambda^{-1}\, d\lambda .
\end{aligned}
\tag{3.40}
$$

For forms of type 4, we have

$$
\begin{aligned}
f(\Delta) &= (r_1 r_2)^{\alpha(i-2)} \\
&\cdot \sum_j \int_0^\infty f(\lambda^2) J_{\nu_j(i-2)}(\lambda_1 r_1) J_{\nu_j(i-2)}(\lambda r_2)\lambda\, d\lambda\, dr_1 \wedge \frac{d\phi_j^{i-2}}{\sqrt{\mu_j}} \otimes dr_2 \wedge \frac{d\phi_j^{i-2}}{\sqrt{\mu_j}} .
\end{aligned}
\tag{3.41}
$$

As was emphasized in [7], [8] and exploited in [15], it is crucial to recognize that the right-hand sides of (3.37), (3.38), (3.40), (3.41) define families of functions of the Laplacian $\tilde{\Delta}$ on N^m, parameterized by r_1, r_2, where the sum of the series may have to be understood in the distribution sense. This observation allows one to bring in the functional calculus on N^m, and thereby "sum the series".

Of course, to pass from (3.31) to the above representations, in effect one is required to reverse the order of integration in a double integral and then to interchange an integration and summation. The discussion of [15, §3] provides a rigorous justification for these operations in a context which is sufficiently general for our present purposes.

Example 3.1. (*The heat kernel* $e^{-\Delta t}$). For i-forms of type 1, this reduces to *Weber's second exponential integral* [53, p. 395]:

$$(3.42)\qquad \begin{aligned}&(r_1r_2)^{\alpha(i)}\int_0^\infty e^{-\lambda^2 t}J_{\nu_j}(\lambda r_1)J_{\nu_j}(\lambda r_2)\lambda\, d\lambda\phi_j\otimes\phi_j\\ &\quad=(r_1r_2)^{\alpha(i)}\sum_j\frac{1}{2t}e^{-(r_1^2+r_2^2)/4t}I_{\nu_j}\left(\frac{r_1r_2}{2t}\right)\phi_j\otimes\phi_j,\end{aligned}$$

where I_ν is the modified Bessel function.

In the case of a wedge, this formula is due to Sommerfeld [50]. For solid 3-dimensional cones of circular cross section, it was considered by Carslaw [5]; both derived it by other means. The elliptic estimate for $\tilde{\Delta}$, together with the elementary estimate

$$(3.43)\qquad I_\nu(z)\sim e^{-\pi i/2}\exp\left(\nu+\nu\log\left(\tfrac{1}{2}zi\right)-\left(\nu+\tfrac{1}{2}\right)\log\nu\right)\left[c_0+\frac{c_1}{\nu}+\cdots\right],$$

shows that the series converges uniformly on compact subsets of $R^+\times C(N^m)\times C(N^m)$. The rigorous justification for identifying the kernel in (3.42) with the heat kernel on forms of type 1 now follows immediately from [11, Lemma 3.5], and the discussion following that lemma.

Example 3.2 ($\Gamma(s)\Delta^{-s}$). For i-forms of type 1 we use the *Weber-Schaftheitlin integral* [53, p. 401]:

$$(3.44)\qquad \begin{aligned}&\Gamma(s)\int_0^\infty \lambda^{1-2s}J_\nu(\lambda r_1)J_\nu(\lambda r_2)d\lambda\\ &\quad=\frac{r_1^\nu r_2^{-\nu+2(s-1)}\Gamma(\nu-s+1)}{2^{2s-1}\Gamma(\nu+1)}F\left(1-s+\nu,s+1,\nu+1,\frac{r_1^2}{r_2^2}\right),\quad r_1<r_2,\end{aligned}$$

where F is the hypergeometric function and $\nu>s-1>-\frac{1}{2}$. In particular, for $\frac{1}{2}<s<1$, (3.44) holds for all ν_j. On the basis of [15, §3] and analytic continuation arguments it is possible to give a rigorous interpretation of the expression for $\Gamma(s)\Delta^{-s}$ corresponding to (3.44) for all s. However, in this paper we are primarily interested only in the trace of the corresponding kernel. Thus in §4 we will carry out an analogous argument in that context. For the moment we think of ν $(=\nu_j)$ as fixed in (3.44) and in the relations which follow. In case $r_1\to r_2$, the hypergeometric function reduces via Gauss' formula; see [34, p. 243] and also [53, p. 403]. This gives the following expression for the pointwise trace of the jth component of the kernel at $r_1=r_2=1$:

$$(3.45)\qquad \frac{1}{2\sqrt{\pi}}\frac{\Gamma(\nu_j(i)-s+1)}{\Gamma(\nu_j(i)+s)}\Gamma\left(s-\tfrac{1}{2}\right)dr\wedge\phi_j\wedge\tilde{*}\phi_j.$$

We now calculate the corresponding expressions for types 2, 3, 4. For i-forms of type 2, we get

$$(3.46)\quad \Gamma(s)\int_0^\infty \lambda^{-1-2s}\Big\{\Big[\alpha^2(i-1)J^2_{\nu_j(i-1)} + 2\alpha(i-1)J_{\nu_j(i-1)}J'_{\nu_j(i-1)}\lambda + \big(J'_{\nu_j(i-1)}\big)^2\lambda^2\Big]dr\wedge\phi_j\wedge\tilde{*}\phi_j + J^2_{\nu_j(i-1)}dr\wedge d\phi_j\wedge\tilde{*}d\phi_{j-1}\Big\}d\lambda.$$

For economy of notation we will write α, ν for $\alpha(i-1)$, $\nu_j(i-1)$ below. Using (3.44) and the Weber-Schaftheitlin formula [53, p. 403], (3.46) becomes

$$\begin{aligned}
&\Gamma(s)\Bigg\{\frac{\alpha^2\Gamma(2s+1)\Gamma(\nu-s)}{2^{2s+1}\Gamma(s+1)\Gamma(\nu+s+1)\Gamma(s+1)}\\
&\quad+\frac{\alpha\Gamma(2s)}{2^{2s}}\left[\frac{\Gamma(\nu-s)}{\Gamma(s+1)\Gamma(\nu+s)\Gamma(s)}-\frac{\Gamma(\nu-s+1)}{\Gamma(s)\Gamma(\nu+s+1)\Gamma(s+1)}\right]\\
(3.47)\quad&\quad+\frac{\Gamma(2s-1)}{2^{2s+1}}\Bigg[\frac{\Gamma(\nu-s)}{\Gamma(s)\Gamma(\nu+s-1)\Gamma(s)}-2\frac{\Gamma(\nu-s+1)}{\Gamma(s-1)\Gamma(\nu+s)\Gamma(s+1)}\\
&\qquad+\frac{\Gamma(\nu-s+2)}{\Gamma(s)\Gamma(\nu+s+1)\Gamma(s)}\Bigg]\Bigg\}dr\wedge\phi^{i-1}\wedge\tilde{*}\phi^{i-j}\\
&\quad+\Gamma(s)\frac{\Gamma(2s+1)\Gamma(\nu-s)}{2^{2s+1}\Gamma(s+1)\Gamma(\nu+s+1)\Gamma(s+1)}dr\wedge d\phi^{i-1}\wedge\tilde{*}d\phi^{i-1}.
\end{aligned}$$

By using Legendre's duplication formula [34, p. 3], this reduces to

$$\begin{aligned}
&\Gamma(s)\Bigg\{\alpha^2\frac{1}{2\sqrt{\pi}}\frac{\Gamma(\nu-s)}{\Gamma(\nu+s+1)}\frac{\Gamma(s+\frac12)}{\Gamma(s+1)}\\
&\quad+\alpha\frac{1}{2\sqrt{\pi}}\left[\frac{\Gamma(\nu-s)}{\Gamma(\nu+s)}-\frac{\Gamma(\nu-s+1)}{\Gamma(\nu+s+1)}\right]\frac{\Gamma(s+\frac12)}{\Gamma(s+1)}\\
(3.48)\quad&\quad+\frac{1}{8\sqrt{\pi}}\Bigg[\frac{\Gamma(\nu-s)}{\Gamma(\nu+s-1)}\frac{\Gamma(s-\frac12)}{\Gamma(s)}-2\frac{\Gamma(\nu-s+1)}{\Gamma(\nu+s)}\frac{\Gamma(s-\frac12)}{\Gamma(s+1)}(s-1)\\
&\qquad+\frac{\Gamma(\nu-s+2)}{\Gamma(\nu+s+1)}\frac{\Gamma(s-\frac12)}{\Gamma(s)}\Bigg]\Bigg\}dr\wedge\phi^{i-1}\wedge\tilde{*}\phi^{i-1}\\
&\quad+\Gamma(s)\frac{1}{2\sqrt{\pi}}\frac{\Gamma(\nu-s)}{\Gamma(\nu+s+1)}\frac{\Gamma(s+\frac12)}{\Gamma(s+1)}dr\wedge d\phi^{i-1}\wedge\tilde{*}d\phi^{i-1}.
\end{aligned}$$

For i-forms of type 3, a similar analysis leads to an expression which can be obtained form (3.48) by making the following changes: (1) replace ϕ^{i-1} by $d\phi^{i-1}/\sqrt{\mu_j}$ and $d\phi^{i-1}$ by $\sqrt{\mu_j}\phi^{i-1}$, (2) replace $\alpha(i-1)$ by $-\alpha(i-1)$. Call the resulting expression (3.48)′. We can then combine the last term of (3.48) with the first term of (3.48)′, and the last term of (3.48)′ with the first term of (3.48). In order to write the resulting expression concisely, it is convenient to introduce the notation

$$\psi(x,s) = \frac{1}{2\sqrt{\pi}}\frac{\Gamma(x-s+1)}{\Gamma(x+s)}\Gamma(s-\tfrac{1}{2}). \tag{3.49}$$

After some routine simplifications, the sum of (3.48) and (3.48)′ then becomes

$$\begin{aligned} &\{\psi(\nu_j(i-1),s) \\ &\quad +2[s+\alpha(i-1)]\psi(\nu_j(i-1),s+1)\}\,dr\wedge\phi_j^{i-1}\wedge\tilde{*}\phi_j^{i-1} \\ &\quad +\{\psi(e^{\nu_j(i-1)},s) \\ &\quad +2[s-\alpha(i-1)]\psi(e^{\nu_j(i-1)},s+1)\}\,dr\wedge\frac{d\phi_j^{i-1}}{\sqrt{\mu_j}}\wedge\tilde{*}\frac{d\phi_j^{i-1}}{\sqrt{\mu_j}}. \end{aligned} \tag{3.50}$$

In this notation, (3.45), the contribution from type 1 forms, is

$$\psi(\nu_j(i),s)\,dr\wedge\phi_j^i\wedge\tilde{*}\phi_j^i. \tag{3.51}$$

Similarly, the contribution from forms of type 4 is

$$\psi(\nu_j(i-2),s)\,dr\wedge\frac{d\phi_j^{i-2}}{\sqrt{\mu_j}}\wedge\tilde{*}\frac{d\phi_j^{i-2}}{\sqrt{\mu_j}}. \tag{3.52}$$

4. The asymptotic expansion of the trace of the heat kernel

Let ${}_1\mathcal{E}_i(t)$ denote the fundamental solution of the heat equation on the cone $C(N^m)$, given in (3.42). Recall that $I_\nu(s)\sim s^\nu$ as $s\to 0$, and $I_\nu(s)\sim s^{-1/2}e^s$ as $s\to\infty$ (see [53]). For i forms of type 2, the corresponding kernel is given by

$$ {}_s\mathcal{E}_i(t) = \int_t^\infty d_1 d_2\,{}_1\mathcal{E}_{i-1}(t)\,ds. \tag{4.1}$$

The elliptic estimate for $\tilde{\Delta}$, together with the asymptotic behavior of I_ν, easily implies that the integral in (4.1) converges. For types 3, 4 the corresponding expressions are obtained by applying $*_1*_2$ to the expressions for $m+1-i$ forms of types 1 and 2. In this section we will calculate the asymptotic expansion of $\operatorname{tr} E_i(t)$, where $E_i(t)$ is the heat kernel of the space with conical singularities, $X^{m+1} = \overline{C_{0,1}(N^m)} \cup M^{m+1}$. In view of the result of §§1 and 2, it will suffice to calculate the expansion for

$$\int_{C_{0,1}(N^m)} \operatorname{tr} \mathcal{E}_i(t).$$

In our calculation we would like to exploit the usual relationship between the heat kernel and the zeta function. That is, we would like to write

$$\Gamma(s)\zeta(s) = \int_0^\infty t^{s-1} \operatorname{tr}(\mathcal{E}(t)\, dt, \tag{4.2}$$

for some range of values of s, and to identify the coefficients in the asymptotic expansion with the residues at the poles of the analytic continuation of $\Gamma(s)\zeta(s)$. If, however, there exist $\nu_j < (n-1)/2$, then it turns out that there are no values of s for which the integral in (4.2) converges. So in order to carry out the procedure we must first split off the terms corresponding to the small eigenvalues. Write

$$\mathcal{E} = \mathcal{E}_{\leqslant b} + \mathcal{E}_{>b}, \tag{4.3}$$

where $\mathcal{E}_{\leqslant b}$ denotes the sum of those finitely many terms in (4.2) for which $\nu_j \leqslant b$, and $\mathcal{E}_{>b}$ the sum of the infinitely many remaining terms.

Let b_i denote the dimension of $\ker \tilde{\Delta}_i$, and let h_j^i be an orthonormal basis for the corresponding space of harmonic forms. In (4.4) below, ϕ_j denotes a *coexact* eigenform of $\tilde{\Delta}$. Let $\psi(x, s)$ be as in (3.49), and for convenience of notation, introduce the convention $\psi(\nu(i), s) \equiv 0$ if $i \notin 0 \cdots m-1$. In view of (3.50)–(3.52) we are led to define $\Gamma(s)\zeta(s)$ by

$$\begin{aligned}\Gamma(s)\zeta_i(s) \overset{\text{def}}{=} & \sum_{j=1}^{b_i} \psi(|\alpha(i)|, s) dr \wedge h_j^i \wedge \tilde{*} h_j^i + \sum_{j=1}^{b_{i-1}} \{\psi(|\alpha(i-1)|, s) \\ & \qquad + 2[s+\alpha(i-1)]\psi(|\alpha(i-1)|, s+1)\} dr \wedge h_j^{i-1} \wedge \tilde{*} h_j^{i-1} \\ & + \sum_j \big(\psi_j(i), s\big) dr \wedge \phi_j^i \wedge \tilde{*}\phi_j^i + \sum_j \big\{\psi\big(\nu_j(i-1), s\big) \\ & \quad + 2[s+\alpha(i-1)]\psi\big(\nu_j(i-1), s+1\big)\big\} dr \wedge \phi_j^{i-1} \wedge \tilde{*}\phi_j^{i-1} \\ & + \sum_j \big\{\psi\big(\nu_j(i-1), s\big) \\ & \quad + 2[s-\alpha(i-1)]\psi\big(\nu_j(i-1), s+1\big)\big\} dr \wedge \frac{d\phi_j^{i-1}}{\sqrt{\mu_j}} \wedge \frac{\tilde{*}\phi_j^{i-1}}{\sqrt{\mu_j}} \\ & + \sum_j \psi\big(\nu_j(i-2), s\big) dr \wedge \frac{d\phi_j^{i-2}}{\sqrt{\mu_j}} \wedge \frac{\tilde{*} d_j^{i-2}}{\sqrt{\mu_j}}\end{aligned} \tag{4.4}$$

(compare the discussion surrounding (3.19)–(3.23) for the explanation of the harmonic terms). For fixed s, the individual terms in the summation are finite provided that for no j is $\nu_j(\cdot) - s$ a nonpositive integer. For s bounded away from such values, with $\operatorname{Re} s > \frac{1}{2}(m+1)$, it follows from the elliptic estimate for $\tilde{\Delta}$ that the series converges uniformly to a holomorphic function of s. The functions $\Gamma(s)\zeta_{i,\leqslant b}(s)$, $\Gamma(s)\zeta_{i,\geqslant b}(s)$ are defined as in (4.3). Clearly $\zeta_{i,\leqslant b}(s)$ is meromorphic in the whole complex plane.

The connection between $\Gamma(s)\zeta(s)$ and $\operatorname{tr}(\mathfrak{S}(t))$ is established in the following theorem.

Theorem 4.1. *In the following* (i)–(v) *assume that* $\mu = 0$, *and that* $\alpha = \frac{1}{2}$ *does not occur.*

(i) *The pointwise relation*

$$\int_0^\infty t^{s-1} \operatorname{tr} \mathfrak{S}_{\leqslant b}(t) = \Gamma(S)\zeta_{\leqslant b}(s) \tag{4.5}$$

holds in the strip $\frac{1}{2} < \operatorname{Re} s \leqslant \nu_0 + 1$.

(ii) *The pointwise relation*

$$\int_0^\infty t^{s-1} \operatorname{tr} \mathfrak{S}_{>b}(t) = \Gamma(s)\zeta_{>b}(s) \tag{4.6}$$

holds in the strip $(m+1)/2 < \operatorname{Re} s < b + 1$.

(iii) $\Gamma(s)\zeta(s)$ *has an analytic continuation to a meromorphic function in all of* $\mathbf{C}$ *with possible poles at*

(a) $s = (m+1)/2 - j/2$.

(b) $s = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \cdots$.

(c) s *such that* $\nu_j - s = 0, -1, \cdots$ *for some* ν_j.

(iv) *For* $(m+1)/2 - j/2 \neq \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$,

$$a_{j/2}(x) = \operatorname*{Res}_{s=(m+1)/2-j/2} \Gamma(s)\zeta_{>(m+1)/2}(s).$$

(v) *For* $(m+1)/2 - j/2 = -\frac{1}{2}, -\frac{3}{2}, \cdots$,

$$a_{j/2}(x) = \operatorname*{Res}_{s=(m+1)/2-j/2} \Gamma(s)\zeta(s).$$

If $(m+1)/2 - j/2 = \frac{1}{2}$, *the result is the same except that those terms of* $\psi(\nu(i-1), s+1)$ *and* ${}_e\psi_i(s+1)$, *for which* $\nu = \frac{1}{2}$ *must be omitted from* $\Gamma(s)\zeta(s)$ *before the residue is taken*; *see* (3.4), (3.5).

(vi) *If there exist* $\mu = 0$, $\alpha = \frac{1}{2}$, *let* n_a *be the dimension of* V_a *in* (1.5). *The results are as above except that in dimension* $(m-1)/2$, $(m+1)/2$, *a contribution of* $\frac{1}{2}n_a/\sqrt{\pi}$ *must be added to the coefficient of* $t^{-1/2}$.

Proof. Consider first the case of 0-forms (functions). Since

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}, \tag{4.7}$$

it follows that as $t \to \infty$,

$$\operatorname{tr} \mathcal{E}_{\leqslant b}(t) \sim \frac{1}{2t} e^{-1/2t} \sum_{\nu_j \leqslant b} \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu_j + k + 1)} \left(\frac{1}{4t}\right)^{\nu_j + k}, \tag{4.8}$$

$$\operatorname{tr} \mathcal{E}_{>b}(t) \leqslant K t^{-(b+1)}. \tag{4.9}$$

Moreover, the asymptotic expansion for $I_\nu(z)$ as $z \to \infty$ (see [53, p. 203]) implies that for $r = 1$, as $t \to 0$

$$\operatorname{tr} \mathcal{E}_{\leqslant b}(t) \sim \frac{1}{2\sqrt{\pi}} \sum_k (-1)^k \sum_{\nu_j \leqslant b} \frac{\Gamma(\nu_j + k + 1/2) t^{-1/2+k}}{k!\Gamma(\nu_j - k + 1/2)} \phi_j \wedge \tilde{*}\phi_j. \tag{4.10}$$

Since

$$\operatorname{tr} \mathcal{E}_{>b}(t) = \operatorname{tr} \mathcal{E}(t) - \operatorname{tr} \mathcal{E}_{\leqslant b}(t),$$

we have for $r = 1$,

$$\begin{aligned} \operatorname{tr} \mathcal{E}_{>b}(t) &\sim \sum a_{j/2}(x) t^{-(m+1)/2 + j/2} \\ &\quad - \frac{1}{2\sqrt{\pi}} \sum_k (-1)^k \sum_{\nu_j \leqslant b} \frac{\Gamma(\nu_j + k + 1/2) t^{-1/2+k}}{k!\Gamma(\nu_j - k + 1/2)} \phi_j \wedge \tilde{*}\phi_j. \end{aligned} \tag{4.11}$$

It follows from (4.8)–(4.11) that for $\frac{1}{2} < \operatorname{Re} s \leqslant \nu_0 + 1$ and $(m+1)/2 < \operatorname{Re} s < b + 1$ respectively, the integrals

$$\int_0^\infty t^{s-1} \operatorname{tr} \mathcal{E}_{\leqslant b}(t)\, dt, \tag{4.12}$$

$$\int_0^\infty t^{s-1} \operatorname{tr} \mathcal{E}_{>b}(t)\, dt \tag{4.13}$$

converge. Moreover, the expression in (4.12) is equal to

$$\int_0^\infty t^{s-1} \sum_{\nu_j \leqslant b} \int_0^\infty e^{-\lambda^2 t} J_{\nu_j}^2 \lambda \, d\lambda\, dt. \tag{4.14}$$

We can reverse the order of integration to get

$$\Gamma(s) \sum_{\nu_j \leqslant b} \int_0^\infty \lambda^{1-2s} J_\nu^2 \, d\lambda = \Gamma(s) \zeta_{\leqslant b}(s).$$

This establishes (i). (ii) follows similarly. (i) makes the analytic continuation of $\Gamma(s)\zeta_{\leqslant b}(s)$ apparent. But the standard argument (see [6, p. 273]) based on the asymptotic expansions (4.8) and (4.10) also gives the analytic continuation. Further it shows that the poles in $\operatorname{Re} s \leqslant$ occur at $s = \frac{1}{2} - k$, and that residues are just the coefficients in (4.8). For $\operatorname{Re} s \geqslant \nu_0 + 1$, the poles are those in (iii)(c), and the residues are the negatives of the coefficients in (4.10). These assertions can also be verified directly by comparing (3.51) to (4.8), (4.10). The same argument shows that $\Gamma(s)\zeta_{>b}(s)$ has an analytic continuation to all of $\operatorname{Re} s < b+1$ for which the residues are the coefficients in (4.11). Since (4.11) is just the difference of the asymptotic expansions for $\operatorname{tr} \mathfrak{S}(t)$ and $\operatorname{tr} \mathfrak{S}_{\leqslant b}(t)$ it follows that the appropriate residues of

$$\Gamma(s)\zeta(s) = \Gamma(s)\zeta_{\leqslant b}(s) + \Gamma(s)\zeta_{>b}(s)$$

are just the coefficients for $\operatorname{tr} \mathfrak{S}(t)$. This gives (iii), (iv), (v).

The proof in the general case is essentially the same, once one has the expressions corresponding to (4.8)–(4.10). For type 1 forms, the formulas are again just (4.8)–(4.10). To express ${}_2\mathfrak{S}(t)$ in terms of I_ν one can write out (4.1) explicitly. The expressions for types 3, 4 are obtained by applying $*_1*_2$ to (3.42), (4.1). We omit further details, since they are similar to the above. q.e.d.

We are now going to obtain the analytic continuation of $\Gamma(s)\zeta_i(s)$ directly from the expression in (4.4). This will allow us to compute the residues at the poles explicitly, and thus, in view of (4.5) and Theorem 4.1, to calculate the pointwise coefficients of $\mathfrak{S}_i(t)$. By Theorem 2.1, this will suffice to calculate the coefficients of the integrated trace of $E_i(t)$, apart from the contribution coming from the singular point p. We then calculate this last contribution by a similar argument.

According to (4.4) it suffices to consider $\psi(\nu(\cdot), s) = \sum_j \psi(\nu_j(\cdot), s)$, where for the next few pages, for ecnomy of notation, we will omit writing $dr \wedge \phi_j \wedge \tilde{*}\phi_j$ or $dr \wedge d\phi_j/\sqrt{\mu_j} \wedge \tilde{*}d\phi_j/\sqrt{\mu_j}$. We will begin by considering the simpler function $\nu^{-2s} = \sum_j \nu_j^{-2s}$, and relate it to $\mu^{-s} = \sum \mu_j^{-s}$. We then reduce the calculation of the residues of $\sum \psi(\nu_j(\cdot), s)$ to those of $\sum_j \nu_j(\cdot)^{-2s}$. Since $\Gamma(s)\mu^{-s}$ is the zeta function on coexact i-forms, we have

$$\operatorname*{Res}_{m/2-j/2} \Gamma(s)\mu^{-s} = \tilde{a}^i_{j/2} - \tilde{a}^{i-1}_{j/2} + \cdots + (-1)^i \tilde{a}^0_{j/2}, \tag{4.15}$$

where $\tilde{a}^i_{j/2}$ is the coefficient of $t^{-m/2+j/2}$ on i-forms. Thus we will, in effect, have calculated the pointwise coefficients for $e^{-\Delta t}$ in terms of those for $e^{-\tilde{\Delta} t}$. In [36], P. C. Lue gives a generalization of this result to arbitrary warped products, for the case of 0-forms. His expression for the coefficients is not quite as explicit as the one given here.

Write

$$\begin{aligned} \nu_j^{-2s} &= \mu_j^{-s}\left(1 + \alpha^2/\mu_j\right)^{-s} \\ &= \mu_j^{-s}\left(1 - \frac{s\alpha^2}{\mu_j} + \cdots + (-1)^n \binom{s+n-1}{n} \frac{\alpha^{2n}}{\mu_j^n} + O\left(\mu_j^{-(n+1)}\right)\right). \end{aligned} \tag{4.16}$$

Summing over ν_j ($\neq 0$) gives

$$(4.17)\qquad \nu^{-2s} = \sum_{i=0}^{n} (-1)^i \binom{s+i-1}{i} \alpha^{2i} \zeta(s+i) + O\Big(\sum_j \mu_j^{-(s+n+1)}\Big).$$

The first term is meromorphic in all of **C**. By the elliptic estimate, the series in the second term converges uniformly for $\operatorname{Re} s > m/2 - (n+1)$, and thus represents a holomorphic function in that half plane. Clearly, the residues can only occur at $s = m/2 - j/2$ and are given by

$$(4.18)\qquad \operatorname*{Res}_{s+m/2-j/2} \nu^{-2s} = \sum_{i=0}^{i\leq j/2} (-1)^i \alpha^{2i} \operatorname*{Res}_{s=m/2-j/2} \binom{s}{i} \zeta(s+i).$$

Now let f be a function having an asymptotic expansion about ∞, of the form

$$f(x) \sim \sum_{k=1}^{n} a_k x^{-b_k} + O(x^{-m/2+\epsilon}),$$

where $b_0 < b_1 < \cdots$. Then we can define $f(\mu)$ by

$$(4.19)\qquad f(\mu) = nf(0) + \sum_{b_k \leq m} a_k \zeta(b_k) + \left[\sum_j \Big(f(\mu_j) - \sum_{b_k\leq m} a_k \mu^{-b_k}\Big)\right],$$

where n is the number of zero eigenvalues. Equivalently, for $\operatorname{Re} s$ large, the series

$$(4.20)\qquad \sum f(\mu_j)\mu_j^{-s}$$

converges to an analytic function of s. $f(\mu)$ is the analytic continuation of this function to $s = 0$. Thus

$$(4.21)\qquad f(\mu) \overset{\text{def}}{\equiv} f(\mu)\mu^{-s}\Big|_{s=0}.$$

Similarly, we set

$$(4.22)\qquad \operatorname{Res} f(\mu) \overset{\text{def}}{\equiv} \operatorname*{Res}_{s=0} f(\mu)\mu^{-s}.$$

Notice that although for example $\nu^0 = 1$, in general of course

$$(4.23)\qquad \nu^{-2s}|_{s=0} \neq I(\mu) = \mu^{-s}|_{s=0} = \zeta(0).$$

In fact, from (4.19) it is clear that

$$
\begin{aligned}
\nu^{-2s}\big|_{s=0} &= \zeta(0) + \sum_{j=1}^{n>m} (-1)^j \frac{s(s+1)\cdots(s+j-1)\alpha^{2j}}{j!}\zeta(s+j)\bigg|_{s=0} \\
&= \zeta(0) + \sum_{j=1}^{m} \operatorname*{Res}_{s=0} \frac{(-1)^j\alpha^{2j}}{j}\zeta(s+j).
\end{aligned}
\tag{4.24}
$$

We are now ready to treat the function $\Gamma(s)\zeta_{>b}(s)$. Recall the asymptotic expansion as $\nu \to \infty$,

$$
\frac{\Gamma(\nu - s + 1)}{(\nu + s)} \sim \frac{\nu^{1-2s}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} G_j(s)\nu^{-j},
\tag{4.25}
$$

which holds uniformly for s in any compact set; see [42]. Here the $G_j(s)$ are certain polynomials. In view of (4.25), an argument entirely similar to the one we gave for ν^{-2s} establishes the analytic continuation of $\Gamma(s)\zeta_{>b}(s)$. To compute the residues we need some explicit knowledge of the $G_j(s)$. This will be recalled in the following lemma.

Let B_j denote the jth Bernoulli number.

Lemma 4.2.

$$
\frac{\Gamma(\nu - s + 1)}{\Gamma(\nu + s)} \sim \nu^{1-2s}\left(1 + s\sum_j (-1)^{(j-1)}\frac{B_j}{j}\nu^{-2j}\right) + O(s^2).
\tag{4.26}
$$

Proof. According to [53, p. 252] we have

$$
\begin{aligned}
\log\Gamma(\nu - s) &\sim (\nu - s - \tfrac{1}{2})\log(\nu - s) - (\nu - s) + \tfrac{1}{2}\log 2\pi \\
&\quad + \sum \frac{(-1)^{(j-1)}}{2j(2j-1)} \frac{B_j}{(\nu - s)^{2j-1}},
\end{aligned}
\tag{4.27}
$$

$$
\begin{aligned}
\log\Gamma(\nu + s) &\sim (\nu + s - \tfrac{1}{2})\log(\nu + s) - (\nu + s) + \tfrac{1}{2}\log 2\pi \\
&\quad + \sum_{j=1}^{\infty} \frac{(-1)^{(j-1)}}{2j(2j-1)} \frac{B_j}{(\nu + s)^{2j-1}}.
\end{aligned}
\tag{4.28}
$$

Write

$$\log(\nu - s) = \log\nu + \log(1 - s/\nu), \qquad \log(\nu + s) = \log\nu + \log(1 + s/\nu). \tag{4.29}$$

Insert (4.29) in (4.27), (4.28) and subtract (4.28) from (4.27). This gives

$$\begin{aligned}\log\frac{\Gamma(\nu - s)}{\Gamma(\nu + s)} &\sim -2s\log\nu + \left(\nu - s - \frac{1}{2}\right)\log\left(1 - \frac{s}{\nu}\right)\\ &\quad - \left(\nu + s - \frac{1}{2}\right)\log\left(1 + \frac{s}{\nu}\right) + 2s\\ &\quad + \sum_{j=1}^{\infty}\frac{(-1)^{(j-1)}}{2j(2j-1)}B_j\left[\frac{1}{(\nu - s)^{2j-1}} - \frac{1}{(\nu + s)^{2j-1}}\right]\\ &\sim -2s\log\nu + \left(\nu - s - \frac{1}{2}\right)\left(-\frac{s}{\nu}\right) - \left(\nu + s - \frac{1}{2}\right)\left(\frac{s}{\nu}\right) + 2s\\ &\quad + \sum_{j=1}^{\infty}\frac{(-1)^{(j-1)}}{2j(2j-1)}B_j\left[\frac{(\nu + s)^{2j-1} - (\nu - s)^{2j-1}}{(\nu^2 - s^2)^{2j-1}}\right] + O(s^2)\\ &\sim -2s\log\nu + \frac{s}{\nu} + s\sum_{j=1}(-1)^{(j-1)}\frac{B_j}{j}\nu^{-2j} + O(s^2).\end{aligned} \tag{4.30}$$

Thus

$$\begin{aligned}\frac{\Gamma(\nu - s + 1)}{\Gamma(\nu + s)} &= (\nu - s)\frac{\Gamma(\nu - s)}{\Gamma(\nu + s)}\\ &\sim (\nu - s)\nu^{-2s}\left(1 + \frac{s}{\nu} + s\sum_{j=1}^{\infty}(-1)^{(j-1)}\frac{B_j}{j}\nu^{-2j}\right) + O(s^2)\\ &\sim \nu^{1-2s}\left(1 + s\sum_{j=1}^{\infty}(-1)^{(j-1)}\frac{B_j}{j}\nu^{-2j}\right) + O(s^2).\end{aligned} \tag{4.31}$$

We now have the following result which, in view of Theorem 4.1, suffices to compute the pointwise asymptotic expansion.

Theorem 4.3. *Let* $(m + 1)/2 - j/2 = a$.

(i) *If* $a \neq \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$, *we have the pointwise relation*

$$\operatorname*{Res}_{s=a}\psi_{\geqslant(m+1)/2}(\nu, s) = \frac{1}{2\sqrt{\pi}}\operatorname{Res}\frac{\Gamma_{\geqslant(m+1)/2}(\nu - a + 1)\Gamma(a - 1/2)}{\Gamma_{\geqslant(m+1)/2}(\nu + a)},$$

with obvious notation.

(ii) *If* $a = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \cdots$, *we have the pointwise relation*

$$\operatorname*{Res}_{s=a}\psi(\nu, s) = \frac{1}{2\sqrt{\pi}}\frac{(-1)^{1/2-a}}{(1/2-a)!}$$

$$\cdot\left\{\frac{\Gamma(\nu-a+1)}{\Gamma(\nu+a)} + \operatorname{Res}\frac{d}{ds}\left(\frac{\Gamma(\nu-a-s+1)}{\Gamma(\nu+a-s)}\right)\right|_{s=0}$$

$$+\operatorname{Res}\frac{\Gamma(\nu-a+1)}{\Gamma(\nu+a)}\left[-\log(1+\alpha^2/\mu) - 2\log(1+a/\nu)\right.$$

$$\left.\left. + \frac{1}{\nu+a} + \sum_j (-1)^{j-1}\frac{B_j}{j}\nu^{-2j}\right]\right\}.$$

(iii) *In particular*

$$\operatorname*{Res}_{s=0}\psi(\nu, s) = -\operatorname{Res}\nu.$$

Before giving the proof of Theorem 4.3 we make some remarks. When we write $\log(1 + \alpha^2/\mu)$, $\log(1 + a/\nu)$, we understand

$$\sum(-1)^j\frac{1}{j}\left(\frac{\alpha^2}{\mu}\right)^j, \qquad \sum(-1)^j\frac{1}{j}\left(\frac{a}{\nu}\right)^j.$$

In these sums, as in the sum involving Bernoulli numbers, only finitely many terms actually contribute to the residue. The expression $\Gamma(\nu - a + 1)/\Gamma(\nu + a)$ is a polynomial in ν since $a = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \cdots$.

Proof of Theorem 4.3. Note that $\Gamma(a - \frac{1}{2})$ is finite for $a \neq \frac{1}{2}, -\frac{1}{2}, \cdots$. Thus it suffices to compute

$$\text{(4.32)} \quad \operatorname*{Res}_{s=a}\frac{\Gamma(\nu-s+1)}{\Gamma(\nu+s)} = \operatorname*{Res}_{s'=0}\frac{\Gamma(\nu-a+1-s')}{\Gamma(\nu+a-s')}\frac{\Gamma(\nu+a-s')}{\Gamma(\nu+a+s')},$$

where $s' = s - a$. By (4.31) with ν replaced by $\nu + a$,

$$\text{(4.33)} \quad \frac{\Gamma(\nu+a-s')}{\Gamma(\nu+a+s')} \sim (\nu+a)^{-2s'} + O(s').$$

Now as in (4.16)

$$\text{(4.34)} \quad (\nu+a)^{-2s'} = \mu^{-s'} + O(s').$$

Since $\mu^{-s'}$ has at most simple poles, terms involving s' can be dropped since they cannot contribute to the residue at $s' = 0$. The first factor on the right-hand side of (4.32) is just a rational function of $(\nu - s')$, and again terms which are $O(s')$ can be ignored. Thus in view of (4.32) and (4.34), (i) follows.

For (ii) we note that for $a = \frac{1}{2}, -\frac{1}{2}, \cdots$, $\Gamma(s - \frac{1}{2})$ has a simple pole at $s = a$ with

$$\operatorname*{Res}_{s=a} \Gamma\left(s - \frac{1}{2}\right) = \frac{(-1)^{a-1/2}}{(1/2 - a)!}.$$

On the other hand, since $\Gamma(\nu - a + 1)/\Gamma(\nu + a)$ is just a polynomial in μ for the values under consideration, it follows that $\Gamma(\nu - a + 1)/\Gamma(\nu + a)$ (in the sense of (4.21)) is finite. However, as explained in (4.23), this time we must also take into account those terms which are $O(s')$. The first factor on the right-hand side of (4.32) is

$$\frac{\Gamma(\nu - a + 1)}{\Gamma(\nu + a)} + s'\frac{d}{ds}\left[\frac{\Gamma(\nu - a + 1 - s')}{\Gamma(\nu + a - s')}\right]\Bigg|_{s=0}. \tag{4.35}$$

By (4.31),

$$\begin{aligned}&\frac{\Gamma(\nu + a - s')}{\Gamma(\nu + a + s')}\\ &\sim (\nu + a)^{-2s'}\left(1 + \frac{s'}{\nu + a} + s'\sum(-1)^{j-1}\frac{B_j}{j}(\nu + a)^{-2j}\right) + O(s'^2),\end{aligned} \tag{4.36}$$

$$\begin{aligned}(\nu + a)^{-2s'} &= \nu^{-2s'}(1 + \tfrac{a}{\nu})^{-2s'}\\ &= \nu^{-2s'}(1 - 2s'\log(1 + \tfrac{a}{\nu}) + O(s'^2))\\ &= \mu^{-s'}(1 + \tfrac{a^2}{\mu})^{-s'}(1 - 2s'\log(1 + \tfrac{a}{\nu})) + O(s'^2)\\ &= \mu^{-s'}[1 - s'\log(1 + \tfrac{a^2}{\nu}) - 2s'\log(1 + \tfrac{a}{\nu})] + O(s'^2).\end{aligned} \tag{4.37}$$

(ii) follows by inserting (4.36), (4.37) into (4.32).

Since $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$, (iii) follows from (i).

We now evaluate the explicity of the contribution to the constant term in the asymptotic expansion of $\operatorname{tr} E_i(t)$ due to the singular point. Note that if $g(s)$ is a holomorphic function having at most a simple pole at $s = 0$, then $(d/ds)(s \cdot g(s))|_{s=0}$ is just a way of writing the constant term in the Laurent expansion of $g(s)$ at $s = 0$. In Theorem 4.4 below we continue to follow the notation of Theorem 2.1.

Theorem 4.4. *On i-forms, the constant term in the asymptotic expansion of* $\operatorname{tr} E_i(t)$ *due to the singular point p is given by*

$$\begin{aligned}
&\frac{1}{2}\int_1^\infty \int_N u^{-1} f(1, x, u)\beta\, du + \frac{1}{2}\int_0^1 u^{-1}\mu_K(u)\, du \\
&\quad + \frac{1}{2}\sum_{j \neq m+1}^{K} \frac{a_{j/2}(1)}{-(m+1)/2 + j/2} \\
&\quad = \frac{1}{2}\frac{d}{ds}(s \cdot \zeta(s))\bigg|_{s=0} \\
&\quad = -\frac{1}{2}(|\alpha(i)|b_i + |\alpha(i-1)|b_{i-1}) + \frac{1}{2}\frac{\alpha(i-1)}{|\alpha(i-1)|}b_{i-1} \\
&\qquad -\frac{1}{2}\frac{d}{ds}\left[s\{\nu^{1-2s}(i) + 2\nu^{1-2s}(i-1) + \nu^{1-2s}(i-2)\}\right]\bigg|_{s=0} \\
&\qquad -\frac{1}{2}\operatorname{Res}\left\{\sum_{j=1}^{\infty}(-1)^{j-1}\frac{B_j}{j}(\nu^{1-2j}(i) + 2\nu^{1-2j}(i-1) + \nu^{1-2j}(i-2))\right\} \\
&\qquad + \operatorname{Res}\nu^{-1}(i-1),
\end{aligned} \tag{4.38}$$

where the expression $\alpha(i-1)/|\alpha(i-1)|$ *is interpreted as* 1 *if* $\alpha(i-1) = 0$.

Proof. If $C(N)$ were a compact manifold, the first equality above would follow immediately from the standard argument which gives the analytic continuation of the zeta function. In our situation if suffices to modify the argument in a manner similar to the arguments of Theorem 4.1. The second equality now follows easily from (4.4) and Lemma 4.2.

EXAMPLE 4.1. Let N^m be the interval of length β with Dirichlet or Neumann boundary conditions. Then $C(N^m)$ is an angular sector with angle β.

The zeta function of N^m for $\Delta = -d^2/d\theta^2$ on functions is just given by

$$\zeta_\beta(s) = \left(\frac{\pi}{\beta}\right)^{-2s}\zeta(2s), \tag{4.39}$$

where $\zeta(s)$ is the ordinary riemann zeta function. Using

$$\begin{aligned}
&\zeta(-1) = -1/12, \\
&\zeta(2s) = \frac{1}{2s-1} + \cdots = \frac{1/2}{s-1/2} + \cdots, \\
&B_1 = 1/6,
\end{aligned} \tag{4.40}$$

(see [54]), Theorem 4.4 yields the value

$$-\frac{1}{2}\left[\frac{\pi}{\beta}\left(-\frac{1}{12}\right) + \frac{\beta}{\pi}\left(\frac{1}{6}\right)\left(\frac{1}{2}\right)\right] = \frac{\pi^2 - \beta^2}{24\pi\beta}. \tag{4.41}$$

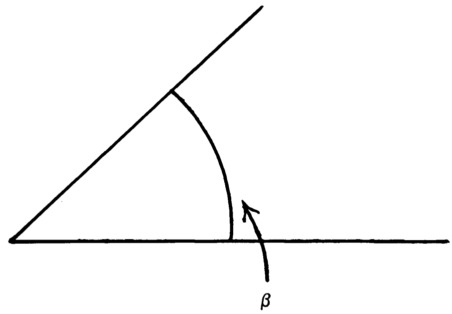

FIG. 4.1

This formula is due to Kac [33] (see also [3], [19]). Ray also gave a simple unpublished derivation based on the Kanterovitsch Lebedev transform. If we let $N^1 = S^1_\beta$, the circle of radius β, a similar analysis gives

$$\frac{(2\pi)^2 - \beta^2}{24\pi\beta}, \tag{4.42}$$

Note that the appearance of $1/\beta$ in (4.41), (4.42) implies that neither these expressions nor their derivatives under change of metric are locally computable geometric invariants. Every such invariant which is not constant is clearly proportional to β. Of course, in this very special case, (4.41) and (4.42) are rational functions of locally computable invariants.

If we take $N^m = S^m_1$ or more generally a lens space, the above results recover known values of the Hurwitz zeta function.

We point out that everything we have done up to this point extends immediately to the case of the Laplacian with coefficients in a vector bundle with connection over $X \setminus p$, provided the connection is locally flat in a neighborhood of p. This remark also applies to the results of §§5 and 6.

5. The Euler characteristic

Recall that in the case of a smooth compact manifold Y^{2k} without boundary the Chern-Gauss-Bonnet formula can be proved by the heat equation method, [27], [38], [43]. For this, one starts with the identity,

$$\chi(Y) = \sum (-1)^i \operatorname{tr}(E_i(t)). \tag{5.1}$$

Since (5.1) holds for all t, we can replace the integrated trace by the integrated asymptotic expansion to obtain

$$\int_{Y^{2k}} \sum (-1)^i a^i_j(x) = 0, \qquad j < 2k, \tag{5.2}$$

(5.3) $$\int_{Y^{2k}} \sum (-1)^i a_{2k}^i(x) = \chi(Y^{2k}).$$

The results of [1], [27], [43] show that in fact one has the pointwise identities

(5.4) $$\sum (-1)^i a_j^i(x) = 0,$$

(5.5) $$\sum (-1)^i a_{2k}^i(x) = P_\chi(\Omega),$$

where $P_\chi(\Omega)$ denotes the Chern-Gauss-Bonnet form, and Ω is the curvature form. In view of Theorem 4.4, if $\partial X^{2k} = \varnothing$, we can apply the same argument to X^{2k}, a manifold with conical singularities, to calculate the L^2-Euler characteristic $\chi_{(2)}(X^{2k})$. When we take the alternating sum, the nonlocal spectral invariants in (4.38) cancel, and we obtain the formula.

Theorem 5.1.

(5.6) $$\chi_{(2)}(X) = \int_M P_\chi(\Omega) + \int_{N^{2k-1}} \sum_{i=1}^{m} (-1)^i \operatorname{Res} \sum_j \nu_j^{-1}(i-1)\frac{d\phi_i}{\sqrt{\mu_j}} \wedge * \frac{d\phi}{\sqrt{\mu_j}}$$
$$+ \frac{1}{2}\chi_{(2)}(C_{0,1}(N^{2k-1})) + \frac{1}{2}\chi_{(2)}(C_{0,1}(N^{2k-1}), N^{2k-1}).$$

Proof. One checks that $\int_{C_{0,1}(N)} P_\chi(\Omega) = 0$. Inspection of (4.38) shows that it suffices to vertify that

(5.7) $$\sum_{i=0}^{k-1} (-1)^i b_i = \chi_{(2)}(C_{0,1}(N^{2k-1})),$$

(5.8) $$-\sum_{i=k}^{2k-1} (-1)^i b_i = \chi_{(2)}(C_{0,1}(N^{2k-1}), (\tfrac{1}{2}, 1) \times N^{2k-1}).$$

This follows from the Poincaré Lemma of [9, §2]:

(5.9) $$H_{(2)}^i(C_{0,1}(N^m)) = \begin{cases} H^i(N^m), & i \leqslant m/2, \\ 0, & i > m/2, \end{cases}$$

and the corresponding relative Poincaré Lemma:

(5.10) $$H_{(2)}^i(C_{0,1}(N^m), (\tfrac{1}{2}, 1) \times N^m) = \begin{cases} 0, & i < m + 1 - [m/2], \\ H^{i-1}(N^m), & i \geqslant m + 1 - [m/2], \end{cases}$$

which is proved in similar fashion.

Although (5.6), as it stands, is a formula for $\chi_{(2)}(X^{2k})$, it can easily be recast as a formula for $\frac{1}{2}(\chi(M) + \chi(M, N))$; for the case $\partial N = \varnothing$, we have $\chi(M^{2k}) = \chi(M^{2k}, N^{2k-1})$. In fact, by [9], the usual exact cohomology sequences hold for the $H^*_{(2)}$ theory; they imply

$$\chi_{(2)}(X) = \chi_{(2)}(C_{0,1}(N)) + \chi_{(2)}(M, (\tfrac{1}{2}, 1) \times N), \tag{5.11}$$

$$\chi_{(2)}(X) = \chi_{(2)}(C_{0,1}(N), (\tfrac{1}{2}, 1) \times N) + \chi_{(2)}(M). \tag{5.12}$$

Since $\chi_{(2)}(M, (\frac{1}{2}, 1) \times N) = \chi(M, N)$ and $\chi_{(2)}(M) = \chi(M)$, by averaging (5.11), (5.12) and substituting in (5.6) we obtain

$$\begin{aligned} &\frac{1}{2}(\chi(M) + \chi(M, N)) \\ &= \int_M P_\chi(\Omega) + \int_N \sum_i \sum_j \operatorname{Res}(-1)^i \nu_j^{-1}(i-1)\frac{d\phi_j}{\sqrt{\mu_j}} \wedge \tilde{*}\frac{d\phi_j}{\sqrt{\mu_j}}. \end{aligned} \tag{5.13}$$

For the case $\partial N = \varnothing$, the essential content of (5.13) is to express the boundary integral $\int_N SP_\chi(\theta, \Omega)$ in the usual Chern-Gauss-Bonnet formula as a spectral invariant. Let Ω_{ij} denote the curvature form of M^{2k} relative to some orthonormal basis $\{e_i\}$, chosen so that e_{m+1} is the inward normal to the boundary. Then if $w_{i,j}$ is the connection form, $w_{i,m+1}$ is the second fundamental form of the boundary. The standard formula for the boundary term expresses $SP_\chi(\theta, \Omega)$ in terms of Ω_{ij}, $w_{i,m+1}$, see [28, p. 338]. In our case $\partial M = (1, N)$ and one easily calculates that $w_{i,m+1} = w_i$, where $\{w_i\}$ is dual to $\{e_i\}$. Thus we can use the Gauss equation

$$\Omega_{ij} = \tilde{\Omega}_{ij} + w_i \wedge w_j$$

to rewrite this formula in terms of the intrinsic curvature forms $\tilde{\Omega}_{ij}$ of N^m. If we employ the identity

$$\begin{aligned} &\sum_{i=0}^{(m-2j-1)/2} \frac{(-1)^i}{1 \cdot 3 \cdots (m-2j-2i)2^i i!} \\ &= \frac{1}{(m-2j)2^{(m-2j-1)/2}((m-2j-1)/2)!}, \end{aligned} \tag{5.14}$$

we find that

$$SP_\chi(\theta, \Omega) = \sum_{j=0}^{(m-1)/2} \frac{1}{(m-2j)((m-2j-1)/2)!\pi^{1/2}} R_j, \tag{5.15}$$

where

$$\begin{aligned} R_j = \frac{(-1)^j}{(4\pi)^{m/2} j!} \sum_\sigma (-1)^{|\sigma|} \Omega_{\sigma(1),\sigma(2)} \wedge \cdots \wedge \Omega_{\sigma(2j-1),\sigma(2j)} \\ \wedge w_{\sigma(2j+1)} \wedge \cdots \wedge w_{\sigma(m)}. \end{aligned} \tag{5.16}$$

Thus R_0 is a multiple of the volume form, R_1 is a multiple of the scalar curvature, and in general the R_j are multiples of the so-called Lipschitz Killing curvatures. Note that R_j is multiplied by l^{m-2j} when the metric tensor on N is multiplied by l^2; i.e., when lengths are multiplied by l. On the other hand,

$$(5.17)\qquad \nu^{-1} = \mu^{-1/2}\left(1 - \frac{\alpha^2(i-1)}{\mu}\right) + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2!}\,\frac{\alpha^4(i-1)}{\mu^2} + \cdots .$$

Since the equality (5.15) holds for all metrics, it follows that terms of the same homogeneity must be equal. Thus

$$(5.18)\qquad \sum_{i=0}^{m-1} (-1)^i \alpha^{m-2j-1}(i)\,\mathrm{Res}\,\Gamma\left(\frac{m-2j}{2}\right) {}_i\mu^{-(m-2j)/2} = \frac{(-1)^j}{m-2j} R_j .$$

Let ${}_i a_j$ be the coefficient of $t^{-m/2+j}$ in the expansion for i-forms. If we make the substitution

$$(5.19)\qquad \mathrm{Res}\,\Gamma\left(\frac{m-2j}{2}\right) {}_i\mu^{-(m-2j)/2} = \left({}_i a_j - {}_{i-1}a_j + \cdots \pm {}_0 a_j\right)$$

in (5.18), then we can show that the resulting equation holds locally. In fact, Gilkey's theorem [28] implies that it suffices to check this for metrics of the form $M^{2j} \times \mathbf{R}^{m-2j}$. This is carried out in §8 (in the piecewise flat context) by means of the identity (8.28). Thus (5.14) and (8.28) are in a sense equivalent identities. We also point out that the individual terms under the summations in (5.18) contain higher covariant derivatives of curvature. Thus (5.18) exhibits a cancellation phenomenon.

We can obtain supplements to (5.18) by substituting our explicit formulas for the pointwise coefficients and Theorem 4.1(i) into (5.4), (5.5), and multiplying the metric by l^2 as above. This yields

$$(5.20)\qquad \sum (-1)^i \alpha^{2(k-j)}(i) \operatorname*{Res}_{s=1/2+k} e^{\mu_{i-1}^{-s}} = 0, \qquad k = 0, \cdots, \frac{1}{2}(m-1), j < k.$$

For the even dimensional analogs of (5.18), (5.20), we simply apply these formulas to the odd dimensional manifold $N^{2k} \times S^1$. The fact that the R_j are spectral invariants was observed by Donnelly [18]. Relations (5.18), (5.20) should also be compared to similar formulas of Günther and Schimming [32].

The discussion given up to this point has an immediate generalization to the case in which the boundary of N^m is not empty. A boundary integral appears on the right-hand side of (5.6), (5.13) and the spectral invariant $\Sigma(-1)^i \,\mathrm{Res}\,\nu^{-1}(i-1)$ can be split into a contribution coming from $\mathrm{int}(N)$ and one attached to ∂N. The latter corresponds to the contribution due to the nonsmoothness of the boundary of M at $(1, N)$. Rather than given further details we will illustrate this discussion with the following down to earth example.

Example 5.1. Let X^2 be the interior of a polygon with angles $\gamma_1, \cdots, \gamma_n$. Fix Dirichlet boundary conditions. Then at each vertex $b_0 = 0$, $b_1 = 1$. Thus the topological term equals $\frac{1}{2}$. Also $P_\chi(\Omega) \equiv 0$ since X is flat, and the boundary integrand is easily seen to vanish as well. As in Example 4.1 the zeta function on coexact zero forms is $(\pi/\gamma)^{-2s}\zeta(2s)$ where $\zeta(s)$ is the Riemann zeta function. At each vertex we get a contribution

$$-\operatorname{Res} \nu^{-1-2s}(0) = \tfrac{\gamma}{2\pi}. \tag{5.21}$$

Combining this with the topological term $\frac{1}{2}$ above and summing over the vertices give

$$\chi_{(2)}(X^2) = (0 - 0 + 1) = 1, \tag{5.22}$$

$$1 = \sum \frac{1}{2\pi}(\pi - \gamma_j), \tag{5.23}$$

which of course is a well-known elementary formula.

6. The η-invariant and signature

If $m + 1 = 4l$, N^{4l-1} is oriented, and $\partial N^{4l-1} = \varnothing$, then we can consider the index problem for the signature operator on $X^{4l} = C_{0,1}(N^{4l-1}) \cup M^{4l}$. Since according to [8], the middle dimensional L^2-cohomology of X^{4l} is canonically identified with

$$\operatorname{im}\big(H^{2l}(M^{4l}, N^{4l-1})\big) \subset H^{2l}(M^{4l}), \tag{6.1}$$

the L^2-signature of X^{4l} is easily seen to coincide with the Novikov signature of M^{4l} as a manifold with boundary.

When we apply the heat equation method to the calculation of the signature, a contribution to the constant term naturally enters, which is analogous to the term in (4.38). This term, as it stands, is equal to the η-invariant of Atiyah-Patodi-Singer, plus a sum of residues of the η function at points in the half plane $\operatorname{Re} s > -\frac{1}{2}$. By examining the effect of multiplying the metric on N^{4l-1} by a constant factor, one sees that the integrated residues are all zero. This gives the Atiyah-Patodi-Singer formula, together with the result that $\eta(s)$ is holomorphic in the half plane $\operatorname{Re} s > -\frac{1}{2}$. In fact, it is known by Gilkey's theorem [28] that this latter result is true locally on N^{4l-1}.

Our main purpose here is not only to give a new proof of the η-invariant formula, but also to exhibit this formula as the natural signature formula for spaces with conical singularities. In §9, we extend our argument to the case of pseudomanifolds with piecewise smooth metrics of constant positive curvature.

Since $e^{-\Delta_{2l}t}$ is trace class and $*: \Lambda^{2l}(X^{4l}) \to \Lambda^{2l}(X^{4l})$ is a bounded operator, it follows that $*e^{-\Delta_{2l}t}$ is trace class. Applying the standard formalism, we see that the L^2-signature of X^{4l} is given by

$$\operatorname{sig}(X^{4l}) = a_{2l}, \tag{6.2}$$

where a_{2l} is the constant term in the asymptotic expansion of $\operatorname{tr}(*e^{-\Delta_{2l}t})$.

By an argument analogous to those of §§1–3, the calculation of a_{2l} can be split into 2 terms. By the Abramov-Gilkey-Patodi Theorem, the first of these is the integral of the L-form $P_L(\Omega)$ over M^{4l}. The second term, which we write as $a_{2l}(C_{0,1}(N^{4l-1}))$, includes any possible contribution from the integral of $P_L(\Omega)$ over $C_{0,1}(N^{4l-1})$, together with a contribution to the constant term at p.

We begin the calculation of $a_{2l}(C_{0,1}(N^{4l-1}))$ by recalling that a metric of the form

$$dr^2 + f^2(r)g(x) \tag{6.3}$$

on $(a, b) \times N^{4l-1}$ is conformally equivalent to a product. To see this, multiply (6.3) by $f^{-2}(r)$ and set $s = \int_1^r f^{-1}(u)\,du$. Since the Pontrjagin forms are conformally invariant, it follows that

$$P_L(\Omega) \,|\, C_{0,1}(N^{4l-1}) \equiv 0. \tag{6.4}$$

More generally, choose $f > 0$ such that $f(r) = r$ near $r = 1$, and $f(r) = cr$ near $r = \frac{1}{2}$. It then follows that the integrated contribution to the constant term, coming from the singular point p, is invariant under multiplying the metric on N^{4l-1} by c^2.

Let $(*_{2l}\zeta_{2l}(s)(r, x))$ denote the trace of the pointwise zeta function on $2l$-forms of $C_{0,1}(N^{4l-1})$ composed with $*_{2l}$. An argument like that of §2 shows that the contribution to the constant term due to the singularity is equal to

$$\int_{N^{4l-1}} \frac{1}{2} \frac{d}{ds}\Big(s\big(\Gamma(s)*_{2l}\zeta_{2l}(s)\big)\Big)\Big|_{s=0}, \tag{6.4}$$

half the constant term in the Laurent expansion at $s = 0$. We can calculate this by a slight modification of the computation of §4. Note that

$$\alpha(2l-1) = \tfrac{1}{2}[1 + 2(2l-1) - (4l-1)] = 0. \tag{6.5}$$

Thus $\nu_j(2l-1) = \mu_j^{1/2}$. Moreover, the coexact eigenforms on N^{4l-1} in dimension $(2l-1)$ can be chosen to satisfy $d\phi_j = (\pm\nu_j) * \phi_j$. Since

$$\phi_j^{2l-1} \wedge \tilde{*}^2\, \phi_j^{2l-1} = d\phi_j^{2l-1} \wedge \tilde{*}^2\, d\phi_j^{2l-1} = 0,$$

forms of types 1 and 4 will not contribute to the zeta function (nor will there be contributions from the harmonic forms in dimensions $2l$, $2l-1$ on N^{4l-1}). Using $*_{2l-1}^2 = 1$, one sees that the contribution for type 3 equals the contribution for type 2 and that these are each given by

$$\Gamma(s)\sum_j \int_0^\infty \lambda^{-1-2s} d\left[J_{\nu_j(2l-1)}(\lambda r)\phi_j(x_1)\right] \wedge d\left[J_{\nu_j(2l-1)}(\lambda r_2)\phi_j(x_2)\right] d\lambda. \tag{6.6}$$

At $r_1 = r_2 = 1$, the sum of the contributions for types 2 and 3 becomes

$$4\Gamma(s)\sum_j \int_0^\infty \lambda^{-2s} J'_{\sqrt{\mu_j}}(\lambda) J_{\sqrt{\mu_j}}(\lambda)\, d\lambda\, dr \wedge \phi_j \wedge d\phi_j. \tag{6.7}$$

Using $J'_{\sqrt{\mu_j}} = \frac{1}{2}(J_{\sqrt{\mu_j}-1} - J_{\sqrt{\mu_j}+1})$ and applying the Weber-Schaftheitlin formula [53, p. 401] gives for $\sqrt{\mu_j} - 1$,

$$\frac{1}{2^{2s-1}} \frac{\Gamma(2s)}{\Gamma(s+1)} \sum_j \frac{\Gamma\left(\sqrt{\mu_j} - s\right)}{\Gamma\left(\sqrt{\mu_j} + s\right)} \phi_j \wedge d\phi_j. \tag{6.8}$$

For $\sqrt{\mu_j} + 1$ we get

$$\frac{1}{2^{2s-1}} \frac{\Gamma(2s)}{\Gamma(s+1)} \sum_j \frac{\Gamma\left(\sqrt{\mu_j} - s + 1\right)}{\Gamma\left(\sqrt{\mu_j} + s\right)} \phi_j \wedge d\phi_j, \tag{6.9}$$

where from now on we drop dr. Subtracting (6.9) from (6.8) and using Legendre's duplication formula for the gamma function yields

$$\Gamma(s) *_{2l}\zeta(s) = \frac{2}{\sqrt{\pi}} \sum_j \frac{\Gamma\left(\sqrt{\mu_j} - s\right)}{\Gamma\left(\sqrt{\mu_j} + s + 1\right)} \Gamma\left(s + \frac{1}{2}\right) \phi_j \wedge d\phi_j. \tag{6.10}$$

As in §4 we can use the asymptotic expansion of the gamma function to evaluate the constant term in the Laurent expansion at $s = 0$. Neglecting terms which are $O(s^2)$, and multiplying by $\frac{1}{2}$ as in (6.4), we obtain

$$\begin{aligned} \int_{N^{4l-1}} \Bigg\{ & \frac{d}{ds}\left(s \sum_j \mu_j^{-(s+1/2)} \phi_j \wedge d\phi_j\right)\Bigg|_{s=0} \\ & + \sum_{l=0} \operatorname*{Res}_{s=0}\left[(-1)^{l-1}\frac{B_l}{l}\mu_j^{l-(s+1/2)}\phi_j \wedge d\phi_j\right]\Bigg\} \\ = & \frac{d}{ds}(s\eta(s))\Bigg|_{s=0} + \sum_l (-1)^{l-1}\frac{B_l}{l}\operatorname*{Res}_{s=l}\eta(s). \end{aligned} \tag{6.11}$$

If we use the above result that $\eta(s)$ is holomorphic in the plane $\operatorname{Re} s < -\frac{1}{2}$, then the expression in (6.11) simply reduces to $\eta(0)$, the η-invariant of N^{4l-1}. The integrated form of this result is also a direct consequence of the above fact that the total contribution in (6.11) is invariant under scaling of the metric on N^{4l-1}. In fact, if we multiply the metric on N^{4l-1} by c, we replace $\eta(s)$ by $c^{-2s}\eta(s)$. Now for $l > 0$,

$$\operatorname*{Res}_{s=l} c^{-2s}\eta(s) = c^{-2l}\operatorname*{Res}_{s=l}\eta(s), \tag{6.12}$$

and since near zero $c^{-2s} = (1 - 2s\log c + O(s^2))$,

$$\frac{d}{ds}(sc^{-2s}\eta(2s))\Big|_{s=0} = \frac{d}{ds}(s\eta(s))\Big|_{s=0} - 2\log c \operatorname*{Res}_{s=0}\eta(s). \tag{6.13}$$

Since it is known that $B_l \neq 0$ for all l, it follows that the expression is (6.11) can be independent of c, only if the integrated residues vanish for $s = 0, 1, \cdots$. Thus we obtain the η-invariant formula of Atiyah-Patodi-Singer,

$$\operatorname{sig}(M^{4l}) = \int_{M^{4l}} P_L(\Omega) + \eta(N^{4l-1}), \tag{6.14}$$

or equivalently,

$$\operatorname{sig}(X^{4l}) = \int_{X^{4l}} P_L(\Omega) + \eta(N^{4l-1}), \tag{6.15}$$

where $\eta(N^{4l-1}) = \eta(0)$.

It is natural from our present viewpoint to consider as well, the η-invariant for $(4l - 1)$-manifolds with conical singularities. We can interpret this as a definition of the η-invariant for *manifolds with boundary* where in effect we impose global boundary conditions through the device of attaching a cone to the boundary. In case $H^{2l-1}(N^{4l-2}, R) \neq 0$, we must choose $*$-invariant ideal boundary conditions as in (1.11). Then the η-invariant can be defined as the analytic continuation to $s = \frac{1}{2}$ of

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\operatorname{tr}(*dE_{2l-1}(t))\,dt, \tag{6.16}$$

$s > (4l - 1)/2$. Although we will not pursue the analog of (6.15) in detail here (see however §9), we note that the η-invariant is still finite for $X^{4l-1} = C_{0,1}(N^{4l-1}) \cup M^{4l-1}$. As usual we can reduce to considering the fundamental solution $\mathfrak{E}_{2l-1}(t)$ for $C(N^{4l-2})$. By considering the various types of forms, (1–4), it is easily checked that in fact, $\operatorname{tr}(*d\mathfrak{E}_{2l-1}(t)) \equiv 0$. The finiteness of the η-invariant for X^{4l-1} immediately follows.

7. Pseudomanifolds

7.1 Introduction. The methods of the previous sections can be exploited in the study of the spectral geometry simplicial complexes with piecewise smooth metrics of constant curvature. Although the general case is similar, for simplicity we will restrict attention to pseudomanifolds. Recall that an *n-dimensional pseudomanifold* X^n is a finite simplicial complex such that every point p is contained in a closed n-simplex and every $(n-1)$-simplex is the face either of one or two n-simplices (if the number is one, p is a point of ∂X^n).

We will assume that $X^n \setminus \Sigma^{n-2}$ is a flat riemannian manifold such that any simplex σ^n is isometric to the interior of some linear n-simplex. More generally, we consider metrics *piecewise smoothly riemannian equivalent* to g in the sense of [9], i.e., piecewise smoothly *quasi-isometric* to g. In particular, the links of simplices inherit metrics of this type.

Here by the *link*, $L(\sigma^k)$ of a simplex $\sigma^k \subset X$, we mean the following. Let $p \in \sigma^k$ and consider the union of line segments in X, emanating from p, of length $\leqslant \epsilon$, normal to σ^k. If ϵ is sufficiently small, it is easy to see that this set is isometric to some truncated cone $C_{0,\epsilon}(L(\sigma^k))$ where $L(\sigma^k)$ is by definition of the links of σ^k. We call $C(L(\sigma^k))$, the *normal cone* to σ^k. If the metric on X is piecewise flat, then $L(\sigma^k)$ is a union of simplices of curvature $\equiv 1$ with totally geodesic faces. Such a metric is of course quasi-isometric to a piecewise flat metric. Then p has a neighborhood which is isometric to $\mathcal{U}^k \times (C_{0,\epsilon}L(\sigma^k))$, where $\mathcal{U}^k \subset \sigma^k$, and the metric is the product metric. The appearance of $C_{0,\epsilon}(L(\sigma^k))$ makes our previous analysis relevant.

We emphasize that the results of subsections 7.2, 7.3, depend only on the quasi-isometry type of the metric.

7.2 Review of Hodge theory. In [9] a number of basic results on the L^2-cohomology and Hodge theory were given in the case where X is admissible in the sense of that paper. Essentially this means that ideal boundary conditions do not enter. Equivalently in our context, $\bar{d} = \bar{\delta}^*$, where the bar denotes closure, and the star denotes adjoint. Here we will use Gaffney's terminology and say that X has *negligible boundary*. Unfortunately even in this case the Laplacian on forms of compact support may fail to be essentially selfadjoint; see [7].

(i) If X^n has negligible boundary, the selfadjoint extensions Δ_D, Δ_N of the Laplacian Δ_0 on forms of compact support coincide. The operators Δ_D, Δ_N correspond to the generalized Dirichlet and Neumann problems respectively.

(ii) the L^2-cohomology spaces $H^i_{(2)}(X) \overset{\text{def}}{=} H^i_{(2)}(X \setminus \Sigma^{n-2})$ are finite dimensional. Thus $\bar{d}$, $\bar{\delta}$ have closed range and the strong Hodge Theorem that $\mathcal{H}^i \simeq H^i_{(2)}(X)$ holds. Here $\mathcal{H}^i = \{h \in L^2 \mid dh = \delta h = 0\}$.

(iii) In fact, $H^i_{(2)}$ is naturally isomorphic to the dual of the middle intersection homology group $\mathrm{IH}_i(X)$ introduced by Goresky and MacPherson [29].

7.3 The Kunneth formula. We also mention a consequence of the above which was implicit in [9], namely the Kunneth formula; see also [55]. Let Y be arbitrary. Define the *reduced L^2-cohomology* by

$$\bar{H}^i_{(2)}(Y) = \ker \bar{d}_i / \overline{\mathrm{range}\, d_{i-1}}. \tag{7.1}$$

By Gaffney [25],

$$(\bar{d}_{i-1})^* = \bar{\delta}_{i,0}, \tag{7.2}$$

where $\bar{\delta}_{i,0}$ is the closure of the operator δ_i restricted to forms of compact support. As in [7] we have the self-adjoint Laplacian, the generalized Neumann problem,

$$\bar{\Delta}_N = \bar{\delta}_{i-1,0}\bar{d}_i + \bar{d}_{i-1}\bar{\delta}_{i,0}, \tag{7.3}$$

and it follows immediately that

$$i_{\mathcal{H}}: \ker \Delta_N \to \bar{H}^i_{(2)}(Y) \tag{7.4}$$

is always an isomorphism. Thus if

$$Y = Y_1 \times Y_2, \tag{7.5}$$

we can consider the operator

$$\sum_{i+j=k} {}_1\Delta_{N,i} \otimes {}_2I_j \oplus {}_1I_i \otimes \Delta_{N,j} = A_k \tag{7.6}$$

on $L^2 \cap \Lambda^i(Y)$, where ${}_l\Delta_{N,i}$ is the Neumann Laplacian on i-forms of Y_l. The operator A_k in (7.6) is known to be essentially selfadjoint by Hilbert space theory; see [46, p. 300]. However, it is trivial to check that the domain of A_k is contained in that of $\Delta_{N,k}$ on Y. Since $\Delta_{N,k}$ is always essentially selfadjoint, we have

$$\bar{A}_k = \bar{\Delta}_{N,k}. \tag{7.7}$$

The relation (7.7) together with (7.4) implies

Theorem 7.1 (*Kunneth formula*).

$$\bar{H}^k_{(2)}(Y) \cong \sum_{i+j=k} \bar{H}^i_{(2)}(Y_1) \otimes \bar{H}^j_{(2)}(Y_2). \tag{7.8}$$

For the case of closed pseudomanifolds with negligible boundary, by §7.2(ii) we can replace $\bar{H}_{(2)}$ by $H_{(2)}$. By (i) we have $\Delta_N = \Delta_D = \Delta$. The discussion can also be modified in a straightforward fashion to include the case of ideal boundary conditions.

7.4 Inductive arguments invariant under quasi-isometry. In §§1–6 we discussed spaces $X^{m+1} = \overline{C_{0,1}(N^m)} \cup M^{m+1}$ with conical singularities by detailing the following observations.

(i) *Local analysis in dimension* $(m+1)$. In dimension $m+1$, local analysis away from the singular point p is as in the nonsingular case. The singularity manifests itself in the nonuniformity of this local behavior as the singularity is approached.

(ii) *Semilocalization in dimension* $(m+1)$. Aspects of global analysis on X^{m+1} can be reduced to (a) local analysis in dimension $(m+1)$ and (b) global analysis on the cone $C(N^m)$.

(iii) *Functional calculus on cones* $C(N^m)$. Global analysis on the cone $C(N^m)$ can be reduced to global analysis on the cross section by means of a functional calculus based on the Hankel inversion formula and the functional calculus for the Laplacian on the cross section.

At this point we have obtained enough information about X^{m+1} so that we can replace the cross section N^m with the space X^{m+1}, and repeat the whole process. In this way, the results on cones can be generalized to pseudomanifolds. In this subsection, we will give the initial portion of this discussion—the part which is invariant under (piecewise smooth) quasi-isometry. This includes the fact that $e^{-\Delta t}$ is trace class, and the estimate $\mathrm{tr}(e^{-\Delta t}) \leqslant kt^{-n/2}$, as $t \to 0$. The complete asymptotic expansion is treated in the next subsection.

We now introduce some terminology and give a flow chart for the inductive argument.

The subscripts $m+1, l$; $m+1, sl$; m, g; will refer to the following regions respectively.

I. *Regions.*

1. $m+1, l$; (local in dimension $m+1$) refers to

$$\mathfrak{U}^k \times C_{0,\epsilon}(L(\sigma^k)) \subset C(L(q)), \quad \text{where } \mathfrak{U}^k \subset \sigma^k, q \in \mathfrak{U}^k, k > 0.$$

2. $m+1, sl$; (semilocal in dimension $m+1$) refers to

$$C_{0,\epsilon}(L(p)) \subset C(L(p)) \quad \text{where } p = \sigma^0.$$

3. m, g; (global in dimension m) refers to X^m.

We will be concerned with the following properties on these regions.

II. *Properties.*

1. $\mathrm{tr}_{m+1,l}$, $\mathrm{tr}_{m+1,sl}$, $\mathrm{tr}_{m,g}$ are the estimates (a)–(c) below, on the integrated pointwise trace.

(7.9)

(a) For all fixed $\mathfrak{U}_0^k$ with $\rho(\mathfrak{U}_0^k, \Sigma^{k-1}) > \delta$, as $t \to 0$,

$$\int_{\mathfrak{U}_0 \times C_{0,\epsilon}(L(\sigma^k))} \operatorname{tr} \mathcal{E}(t) \leqslant K_\epsilon t^{-(m+1)/2}.$$

(b) $$\int_{C_{0,\epsilon}(L(p))} \operatorname{tr} \mathcal{E}(t) \leqslant K_\epsilon t^{-(m+1)/2},$$

(c) $$\int_{X^m} \operatorname{tr} E(t) \leqslant K_{X^m} t^{-m/2}.$$

2. Let $P_1(d_1, \delta_1)$, $P_2(d_2, \delta_2)$ denote polynomials in d_1, δ_1, and d_2, δ_2 respectively. The *decay* conditions $D_{m,l}$, $D_{m+1,sl}$ refer to the statements $(\alpha), (\beta)$ below where $n > 0$ is arbitrary. The norm is the L^2-norm with respect to both variables.

(α) For almost all[4] u_1, u_2 such that

$$\rho(u_j, \Sigma^{k-1}) > \delta, \qquad |u_1 - u_2| > \delta/2,$$

we have

(7.10) $$\int_{u_1 \times C_{0,\epsilon_1}(L(\sigma^k))} \int_{u_2 \times C_{0,\epsilon_2}(L(\sigma^k))} \|P_1 P_2 \mathcal{E}(t)\| \leqslant K_{\delta,N} t^N.$$

(β) If $\epsilon > \delta$, $\rho(C_{0,\epsilon_1}(L(p)), u_2 \times C_{0,\epsilon_2}(L(\sigma^k))) > \delta/2$, then

$$\int_{C_{0,\epsilon_1}(L(p))} \int_{u_2 \times C_{0,\epsilon_2}(L(\sigma^k))} \|P_1 P_2 \mathcal{E}(t)\| \leqslant K_{\delta,N} t^N.$$

There is a corresponding estimate with the roles of u_1, u_2 reversed.

The inductive interconnections between the various properties of II will be exhibited with the help of the following principles which are established in all dimensions simultaneously without inductive arguments. Of these, only Principles 1 and 6 will require proof.

III. *Principles.*

1. (q.i.i.) Whether or not $e^{-\Delta t}$ is trace class for $t > 0$ and satisfies the inequality $\operatorname{tr}(e^{-\Delta t}) < Kt^{-n/2}$, as $t \to 0$, depends only on the piecewise smooth quasi-isometry, i.e., riemannian equivalence, class of X.
2. (f.c.c.) functional calculus on cones.
3. (f.c.p.) functional calculus on products $R \times Y$.
4. ($\mathfrak{D}$) denotes any argumennt based on Duhamel's principle.
5. (symm) denotes the fact that the heat kernel $E(x, y, t)$ is symmetric on x and y.

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[4] It is not difficult to see that "almost all" can be replaced by "all."

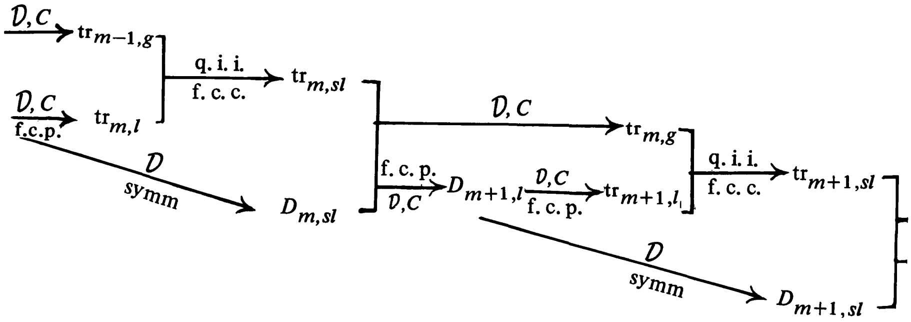

FIG. 7.1

6. ($\mathcal{C}$) denotes the existence of a cut-off function g on the piecewise linear cone $C(L(\sigma))$ such that multiplication by g preserves the domain of all powers of Δ on $C(L(\sigma))$.

Given the above terminology, the inductive argument can be explained by means of Figure 7.1 which contains five distinct types of entries, say $\mathrm{tr}_{m+1,sl}$, $D_{m+1,sl}$, $\mathrm{tr}_{m,g}$, $\mathrm{tr}_{m+1,l}$, $D_{m+1,sl}$. We begin by establishing Principles 1 and 6 above, and then proceed to the entries in Figure 7.1.

Lemma 7.1 (q.i.i.). *Let Y_j^n be riemannian manifolds with metrics g_j, $j = 1, 2$, and let $f\colon Y_1 \to Y_2$ be a quasi-isometry (for example piecewise smooth) such that $f^*\bar{d} = \bar{d}f^*$. Let ${}_j\Delta_i$ denote the Laplacian on i-forms of Y_j^n for generalized Neumann conditions ${}_j\bar{\Delta}_i = {}_j\bar{\delta}_0\bar{d} + \bar{d}_j\bar{\delta}_0$. Then $e^{-{}_1\Delta_i t}$ is trace class for all i and satisfies $\mathrm{tr}(e^{-{}_2\Delta_i t}) < K_1 t^{-n/2}$, if and only if $e^{-{}_2\Delta_i t}$ is trace class for all i and satisfies $\mathrm{tr}(e^{-{}_2\Delta_i t}) < K_2 t^{-n/2}$.*

Proof. As in subsection 7.3, $\dim\ker {}_1\Delta_1 = \dim\ker {}_2\Delta_i$. Let $\bar{d}^{-1}$ denote the map which vanishes on $\overline{\ker\bar{\delta}_0}$, and assigns to each form $\bar{d}\alpha \in \mathrm{range}\,\bar{d}$, the unique form $P_{ce}\alpha$ in $\overline{\delta_0 \wedge_0^{i+1}}$, $\bar{d}$ of which is $\bar{d}\alpha$. The spectrum of ${}_j\Delta_i$ is discrete if and only if the bilinear forms defined on ranges $\bar{d}_{i-1}$, $\bar{d}_i$ by

$$(7.11) \qquad g_j\left(\bar{d}_{i-1}^{-1}\gamma, {}_j\bar{d}_{i-1}^{-1}\gamma\right), \qquad g_j\left({}_j\bar{d}_i^{-1}\gamma, {}_j\bar{d}_i^{-1}\gamma\right)$$

have discrete spectrum with respect to g_1, g_2. If $\{{}_j\mu_{i-1}\}$, $\{{}_j\mu_i\}$ are the nonzero spectra of these bilinear forms, the nonzero spectrum of ${}_j\bar{\Delta}_i$ is $\{{}_j\mu_{i-1}^{-1}\} \cup \{{}_j\mu_i^{-1}\}$. But since the operators

$$(7.12) \qquad {}_2P_{ce}|\,\overline{{}_1\delta_0 \wedge_0^{i+1}}, \qquad {}_1P_{ce}|\,\overline{{}_2\delta_0 \wedge_0^{i+1}}$$

are bounded with respect to both g_1 and g_2, it follows that the forms in (7.13) are mutually bounded. Since the same is true of g_1, g_2, the lemma easily follows.

($\mathcal{C}$) The possibility of localizing follows in particular from the existence of a cut-off function on $\mathbf{R}^k \times C(L(\sigma^k))$, multiplication by which preserves the domain of all powers of $\bar{\Delta}$. To construct such a function, clearly it suffices to construct a function with the analogous property with respect to the Laplacian on $C(L(\sigma^k))$. Since $L(\sigma^k)$ is itself a space with conical singularities, it will *not* immediately suffice simply to employ a function of the radial variable, when considering i-forms with $i > 0$. However, a direct way to construct a suitable cut off g, for the piecewise linear cone $C(L(\sigma^k))$, is the following.[5] Fix $\epsilon > 0$ small. Let $\{\tau_j^i\}$ denote the i-simplices of $L(\sigma^k)$. Define g on $\bigcup_j C(\tau_j^0)$ to be $\phi(r)$, where $\phi\,|\,(0, \epsilon/2) \equiv 1$, and $\phi\,|\,(\epsilon, \infty) \equiv 0$. Extend g to a small product neighborhood T^0 of $\bigcup_j C_{\epsilon/3,\infty}(\tau_j^0)$ in $C(L(\sigma^k))$ by composing g with orthogonal projection onto $\bigcup_j C_{\epsilon/3,\infty}(\tau_j^0)$. Next extend $g\,|\,T^0 \cap \bigcup_j C(\tau_j^1)$ to all of $\bigcup_j C(\tau_j^1)$ in the obvious fashion depicted in Figure 7.2; the level surfaces of g have been indicated. Extend g to a product neighborhood of $\bigcup_j C(\tau_j^1)$ in $C(L(\sigma^k))$ by composing with orthogonal projection onto the various $C(\tau_j^1)$. Note that this extends $g\,|\,T^0$. By proceeding inductively, we construct a cut-off function g which is compatible with the local product structures on the T^i. Then multiplication by g will clearly preserve dom Δ^j for all j.

We now consider the five distinct types of entries in Figure 7.1.

($\mathrm{tr}_{m+1,sl}$). If we assume that for piecewise flat m-dimensional pseudomanifolds we have $\mathrm{tr}(e^{-\Delta t}) \leqslant Kt^{-m/2}$, then by Lemma 7.1 (q.i.i.), this estimate also holds for m-dimensional pseudomanifolds with piecewise smooth metrics of constant curvature $\equiv 1$. The arguments of §§2–4 now apply. In particular, let p be a vertex of the piecewise flat pseudomanifold X^{m+1}, and $L(p)$ the link of p. By $\mathrm{tr}_{m,g}$, one sees that for $t > 0$ we have convergent series representations for $\mathfrak{S}_i(t)$ on $C_{0,1}(L(p))$ in terms of the modified Bessel function I_ν as in (3.42) and (4.1). The convergence is uniform pointwise if we stay away from lower dimensional skeleta. More important, from (3.43) it follows that the convergence is uniform in the Sobolev spaces corresponding to dom $\tilde{\Delta}^i$ for all i. For fixed t, these representations imply that

$$(7.13)\qquad \int_0^1 \int_{L(p)} \mathrm{tr}\,\mathfrak{S}(r,t) r^u\,dr = \frac{1}{2}\int_t^\infty \int_{L(p)} \mathrm{tr}\,\mathfrak{S}(1,s) s^{-1}\,ds,$$

where, in particular, it follows from the series representation that the integrals converge; the integrals in (7.13) are regarded as L^2 valued. In order to obtain the estimate for small t most directly from $\mathrm{tr}_{m+1,l}$, we fix t small and define

[5] Analogous constructions will play a role in §9.

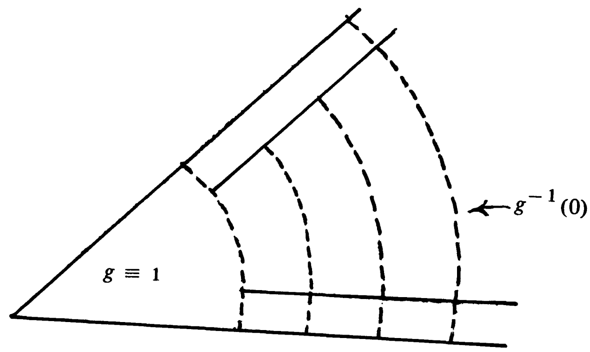

FIG. 7.2

$a \in [\frac{1}{4}, \frac{1}{2}]$ and N by $t = a^{2N}$. We can write

(7.14)

$$\int_0^1 \int_{L(p)} \operatorname{tr} \mathfrak{E}(r, t) r^m \, dr = \int_0^{t^{1/2}} \int_{L(p)} \operatorname{tr} \mathfrak{E}(r, t) r^m \, dr + \int_{t^{1/2}}^1 \int_{L(p)} \operatorname{tr} \mathfrak{E}(r, t) r^m \, dr.$$

The first integral is given by

$$\int_0^{t^{1/2}} \int_{L(p)} \operatorname{tr} \mathfrak{E}(rt^{-1/2}, 1) t^{-(m+1)/2} r^m \, dr = \int_0^1 \int_{L(p)} \operatorname{tr} \mathfrak{E}(u, 1) u^m \, du, \tag{7.15}$$

and thus is independent of t. The second integral can be written as

$$\begin{aligned} \sum_{i=0}^{N-1} \int_{a^{i+1}}^{a^i} \int_{L(p)} \operatorname{tr} \mathfrak{E}(r, t) r^m \, dr &= \sum_{i=0}^{N-1} \int_a^1 \int_{L(p)} \operatorname{tr} \mathfrak{E}(ua^i, t) a^{(m+1)i} u^m \, du \\ &= \sum_{i=0}^{N-1} \int_a^1 \int_{L(p)} \operatorname{tr} \mathfrak{E}(u, t/a^{2i}) u^m \, du \\ &\leqslant K \sum_{i=0}^{N-1} \left(t/a^{2i}\right)^{-(m+1)/2} \\ &\leqslant K_1 t^{-(m+1)/2}, \end{aligned} \tag{7.16}$$

where the next to the last step follows from $\operatorname{tr}_{m+1,l}$. This completes the verification of $\operatorname{tr}_{m+1,sl}$.

$(D_{m+1,l})$. Let $E_l^k(t)$ denote the heat kernel of l-forms of R^k, and $\mathfrak{E}_{i-l}^{m-k+1}(t)$ denote the heat kernel of $(i-l)$-forms of $C(L(\sigma^k))$. Then by $D_{m,sl}$, $\operatorname{tr}_{m,l}$ and the analogous properties for E^k, it follows that

$$E_i^{m+1,x} = \sum_{j=0}^{i} E_j^i \otimes \mathfrak{E}_{i-j}^{m-k+1} \tag{7.17}$$

has the decay property in (7.10)(α). Thus it suffices to show that the heat kernel $\mathcal{E}_i^{m+1}$ on $C(L(p))$ can be localized to $E_i^{m+1,x}$ on $\{u_1 \times C_{0,\epsilon}(L(\sigma^k))\} \times \{u_2 \times C_{0,\epsilon}(L(\sigma^k))\}$. If we set

$$\mathcal{P}_i = (h \times g)E_i^{m+1,x}, \tag{7.18}$$

where h is a cut-off function on $\mathbf{R}^k$, and g is the cut-off function described in $(\mathcal{C})$, then we get

$$\mathcal{E}_i^{m+1}(t) = \mathcal{P}_i(t) + \int_0^t e^{-\Delta(t-s)}Q_i(s)\,ds, \tag{7.19}$$

where

$$Q_i(z, x_2, s) = (\Delta_z + \partial/\partial s)\mathcal{P}_i(z, x_2, s), \tag{7.20}$$

and $e^{-\Delta(t-s)}$ is the heat kernel on $C(L(p))$. Observe that

$$P(d_{x_1}, \delta_{x_2})e^{-\Delta(t-s)} = e^{-\Delta(t-s)}P(\delta_z, d_z), \tag{7.21}$$

and that $e^{-\Delta(t-s)}$ does not increase L^2-norm. From the Stokes' theorem for collars proved in [9, §2], it follows that we can apply Duhamel's principle and integrate over $\{u_1 \times C_{0,\epsilon}(L(\sigma^k))\} \times \{u_2 \times C_{0,\epsilon}(L(\sigma^k))\}$ for almost all boundaries in the shaded region in Figure 7.3. In this way we obtain the desired localization.

$$u_1 \times C_{0,\epsilon}(L(\sigma^k)) \quad u_2 \times C_{0,\epsilon}(L(\sigma^k))$$

$(D_{m+1,sl})$. We can argue as in the proof of $(D_{m+1,l})$ above, and then use (symm) to reverse the roles of the variables.

$(\mathrm{tr}_{m+1,l})$. If we assume $\mathrm{tr}_{m,sl}$, then it is clear that $E_i^{m+1,x}$ satisfies the required estimate (7.9)(a). But E_i^{m+1} can be localized to $E_i^{m+1,x}$ by the argument used in the proof of $(D_{m+1,l})$.

$(\mathrm{tr}_{m,g})$. In view of what has been established above, the argument can proceed as in §2.

7.5 The asymptotic expansion. We now derive the complete asymptotic expansion on a piecewise flat pseudomanifold X^n. Intuitively, it is easy to see what the general form of this expansion must be. Note that the geometry of X^n is locally constant on the open simplices σ^k of X^n, and is completely determined by the links $L(\sigma^k)$. Thus any *locally computable* invariant of X^n must be of the form

$$\sum_{\sigma_\alpha^k} \phi_k^n(L(\sigma_\alpha^k))A(\sigma_\alpha^k), \tag{7.22}$$

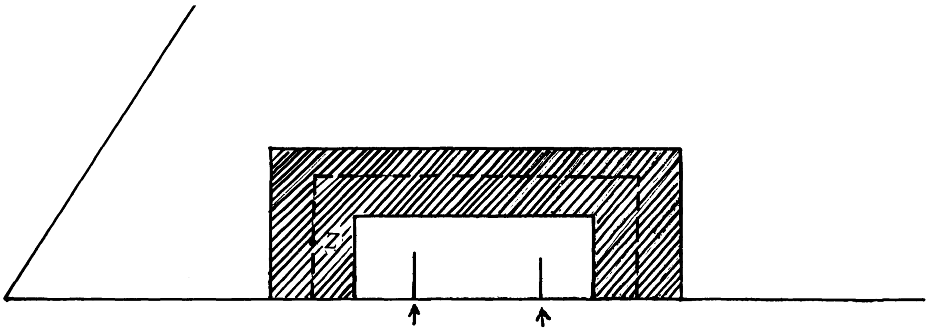

FIG. 7.3

where $A(\sigma_\alpha^k)$ denotes the k-dimensional area of σ_α^k, and $\phi(L(\sigma_\alpha^k))$ is some (quite possibly *global*) invariant of $L(\sigma_\alpha^k)$. Now multiply the metric tensor on X^n by c^2, i.e., multiply distances by c. Then the $L(\sigma_\alpha^{n-j})$ remain invariant. However, for fixed j, the areas $A(\sigma_\alpha^{n-j})$ are multiplied by c^{n-j}. Since multiplying the metric by c^2 has the effect of replacing t by t/c^2 in the trace of the heat kernel, the coefficient $a_{j/2}$ of a term of the form $t^{-n/2+j/2}$ in the asymptotic expansion would also exhibit this behavior. Thus the expansion on i-forms should involve only terms of the form ${}_ia_{j/2}t^{-n/2+j/2}$, $j = 0, 1, \cdots, n$. Moreover, for fixed j we should have

$$(7.23)\qquad {}_ia_{j/2} = \sum_\alpha {}_i\Psi^n_{j/2}\big(L\big(\sigma_\alpha^{n-j}l\big)\big)A\big(\sigma_\alpha^{n-j}\big),$$

for some invariant ${}_i\Psi^n_{j/2}(L(\sigma_\alpha^{n-j}))$. Finally, keeping in mind the form of the local parametrix in (7.17), it is tempting to assume that ${}_i\Psi^n_{j/2}(L(\sigma_\alpha^{n-j}))$ has the following description. Let ${}_l\psi(L(\sigma_\alpha^{n-j}))$ denote the constant term in the asymptotic expansion of the trace of the heat kernel $\mathcal{E}_l^j(t)$ for l-forms on $C_{0,\epsilon}(L(\sigma_\alpha^{n-j}))$; of course we have yet to show that this expansion exists. For the case $C_{0,1}(N^m)$, ${}_l\psi(N^m)$ is computed as a spectral invariant of N^m in Theorem 4.3. Let $\vec{\Psi}_{nj/2}(L(\sigma_\alpha^{n-j}))$, $\vec{\psi}(L(\sigma_\alpha^{n-j}))$ denote the column vectors whose entries are ${}_i\Psi^n_{j/2}(L(\sigma_\alpha^{n-j}))$, ${}_l\psi_{j/2}(L(\sigma_\alpha^{n-j}))$.

Define the $(n+1) \times n$ matrix M_n by

$$(7.24)\qquad M_n = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 \\ 1 & 1 & 0 & \cdot & \cdot \\ 0 & 1 & 1 & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & 1 \\ 0 & 0 & 0 & \cdot & 1 \end{bmatrix}.$$

Let I_{n+1} be the identity matrix, and put

$$\mathfrak{M}_{n,k} = \begin{cases} M_n \cdot M_{n-1} \cdots M_{k-1}, & 1 \leq k \leq n-1, \\ I_{n+1}, & k = n \end{cases} \tag{7.25}$$

(the jth column of the $M_{n,k}$ consists of $(j-1)$ zeros, followed by $\binom{n-k}{i}$ ones, $i = 0 \cdots n-k$, followed by $k - j + 1$ zeros). Then from the local product structure, we might guess that

$$\vec{\Psi}^n_{j/2}\left(L\left(\sigma_\alpha^{n-j}\right)\right) = \frac{1}{(4\pi)^{(n-j)/2}} M_{n,j}\vec{\psi}\left(L\left(\sigma_\alpha^{n-j}\right)\right). \tag{7.26}$$

In fact, (7.26) does give a correct description of the trace locally on regions of the form $\mathfrak{U}_\alpha^{n-j} \times C_{0,\epsilon}(L(\sigma_\alpha^{n-j}))$. However, it is *not* correct globally on X^n, because X^n cannot be covered by a family of such regions whose interiors do not overlap. One way of dealing with this point is to replace the normal cones $C_{0,\epsilon}(L(\sigma_\alpha^{n-j}))$ by "dual sets" D_α^j with *piecewise flat* boundary (see below for the description of D_α^j). Then X^n can be covered by nonoverlapping regions $\mathfrak{U}_\alpha^{n-j} \times D_\alpha^j$. In this way we obtain a formula which looks like (7.26). However, $\psi(L(\sigma_\alpha^{n-j}))$ gets replaced by $\vec{\phi}(D^j)$, the constant term in the asymptotic expansion for the heat kernel $\vec{\mathcal{E}}^j(t)$ of $C(L(\sigma^{n-j}))$ restricted to $D_\alpha^j \subset (L(\sigma_\alpha^{n-j}))$ (rather than to $C_{0,\epsilon}(L(\sigma_\alpha^{n-j}))$). Thus this simpler formula is not expressed directly in terms of spectral invariants of links.

When one calculates the $\vec{\phi}(D_\alpha^j)$ in terms of such invariants, a much more complicated formula results. However, for the particular combinations of coefficients which occur in applications to the Euler characteristic and signature in §§8 and 9, the distinction between $C_{0,\epsilon}(L(\sigma_\alpha^{n-j}))$ and D_α^j turns to be unimportant.

We now give the details of the above discussion. Let L^k be a complex with a metric which is piecewise of constant curvature $\equiv 1$, and let $\{\tau_\alpha^j\}$ denote the simplices of L^k. After possible subdivision we can assume that each τ_α^j is isometric to a subset of a hemisphere. Let $1 - 2\epsilon^2$ be small and let the *dual set* D^{k+1} be the set

$$\left\{q \in C(L^k) \mid \overline{q, p}^2 - \overline{q, \tau_\alpha^0}^2 \leq 1 - 2\epsilon^2 \text{ for all } \alpha\right\}. \tag{7.27}$$

Then D^{k+1} has a piecewise flat boundary and its codim 1 faces D_α^k can be regarded as subsets of the normal cone to the ray from p to τ_α^0. If L^k is homeomorphic to a sphere, D^{k+1} is a cell which is "dual" to p.

Now let X^n be piecewise flat, and for each σ^0 choose a small D^n as above. We proceed inductively to choose dual sets in the normal cones to the k-simplices σ^k, $k = 1, \cdots, n$, subject to the following condition. Note that for each k-simplex σ^k and each face $\sigma^{k'} \subset \sigma^k$, there is a unique $(n-k)$-dim face of $D^{n-k'}$ normal to σ^k. Regard all these as subsets of a fixed normal cone to σ^k. At each stage choose the dual set D^{n-k} to σ^k so small that it is contained in all such $(n-k)$-dim faces of the various $D^{n-k'}$. Also, set

$$\mathfrak{U}^k = \sigma^k \setminus \bigcup_{\sigma^{k'} \subset \sigma^k} \mathfrak{U}^k \times D^{k'}. \tag{7.28}$$

Then the sets $\mathfrak{U}^{k_1}_{\alpha_1} \times D^{k_1}_{\alpha_1}$, and $\mathfrak{U}^{k_2}_{\alpha_2} \times D^{k_2}_{\alpha_2}$ corresponding to $\sigma^{k_1}_{\alpha_1}$, $\sigma^{k_2}_{\alpha_2}$ have disjoint interiors unless $\sigma^{k_1}_{\alpha_1} = \sigma^{k_1}_{\alpha_2}$. Moreover,

$$X^n = \bigcup_{\sigma^k} \mathfrak{U}^k \times D^{n-k}. \tag{7.29}$$

We now let D^n be the dual set to σ^0, and consider the trace of the heat kernel $\vec{\mathfrak{E}}^n(t)$ of $C(L(\sigma^0))$ on D^n. We divide D^n into subsets $\mathcal{C}^n_\alpha$ consisting of all segments joining σ^0 to points of the codim 1 face D^{n-1}_α. Assume for convenience that the distance from D^{n-1}_α to σ_0 is 1, and set $x_\alpha = \tau^0_\alpha \in L(\sigma^0)$. Let $\pi : D^{n-1}_\alpha \to L(\sigma^0)$ denote radial projection, and let $\rho(x, x_\alpha)$ be the distance in $L(\sigma^0)$ from x to x_α. Note that if dx is the volume element on $L(\sigma^0)$, and dA is the volume element on D^{n-1}, then

$$\begin{aligned} dA(\pi^{-1}(x)) &= (\sec \rho(x, x_\alpha))^{n-1} \sec \rho(x, x_\alpha) d\pi^*(dx) \\ &= (\sec \rho(x, x_\alpha))^n d\pi^*(dx). \end{aligned} \tag{7.30}$$

Thus if we let the density for $\operatorname{tr} \vec{\mathfrak{E}}^n(t)$ be the column vector $f(r, x, t)$, we have

$$\begin{aligned} \int_{\mathcal{C}^n_\alpha} \operatorname{tr} \vec{\mathfrak{E}}^n(t) &= \int_{\pi(D^{n-1}_\alpha)} \int_0^{\sec \rho(x, x_\alpha)} f(r, x, t) r^{n-1}\, dr\, dx \\ &= \int_{\pi(D^{n-1}_\alpha)} \int_0^{\sec \rho(x, x_\alpha)} f\left(\sec \rho(x, x_\alpha), x, \frac{\sec^2 \rho(x, x_\alpha) t}{r^2} \right) \\ &\qquad \times r^{-1}\, dr (\sec \rho(x, x_\alpha))^n\, dx \\ &= \int_{D^{n-1}_\alpha} \frac{1}{2} \int_t^\infty f(\sec \rho(x, x_\alpha), x, u) u^{-1}\, du\, dA \\ &= \frac{1}{2} \int_t^\infty \left(\int_{D^{n-1}_\alpha} f(\sec \rho(x, x_\alpha), x, u) dA \right) u^{-1}\, du. \end{aligned} \tag{7.31}$$

Now suppose that for u small,

$$\int_{D^{n-1}_\alpha} f(\sec \rho(x, x_\alpha), x, u)\, dA \sim \sum_{j=0}^{n-1} a_{j/2} u^{-(n-1)/2 + j/2 - 1/2}. \tag{7.32}$$

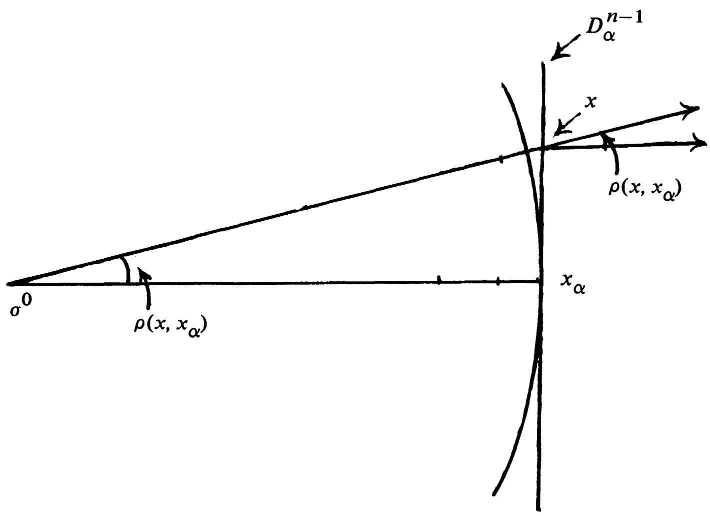

FIG. 7.4

Then as in §2 the last integral in (7.31) is asymptotic to

$$\frac{1}{2}\int_1^\infty \int_{D_\alpha^{n-1}} f(\sec\rho(x, x_\alpha), x, u)\, dA\, u^{-1}\, du$$

$$(7.33)\qquad + \frac{1}{2}\int_0^1 \left(\int_{D_\alpha^{n-1}} f(\sec\rho(x, x_\alpha), x, u)\, dA - \sum_{j=0}^{n-1} a_{j/2} u^{-(n-1)/2+j/2-1/2} \right) u^{-1}\, du$$

$$+ \frac{1}{2}\int_t^1 \sum_{j=1}^{n-1} a_{j/2} u^{-(n-1)/2+j/2-3/2}\, du.$$

Now for u small, as in (7.17), $\vec{\mathfrak{E}}^n(u)\,|\,D_\alpha^{n-1}$ can be replaced by $\vec{E}^{n,x}(u)$, where in this case $k = 1$. We can assume by induction that $\operatorname{tr} \vec{\mathfrak{E}}^{n-1}(u)\,|\,D_\alpha^{n-1}$ has an asymptotic expansion of the form

$$(7.34)\qquad \sum_{j=0}^{n-1} b_{j/2} u^{-(n-1)/2+j/2}.$$

Then (7.32) will hold with

$$(7.35)\qquad a_{j/2} = \frac{1}{(4\pi)^{1/2}} M_n b_{j/2}.$$

We can further assume by induction that

$$(7.36)\qquad b_{j/2}=\frac{1}{(4\pi)^{(n-1-j)/2}}\sum_{\beta}\mathfrak{M}_{n-1,j}\vec{\phi}\left(D_\beta^j\right)A\left(\theta_\beta^{n-1-j}\right),$$

where θ_β^{n-1-j} varies over the $(n-1-j)$-simplices of D_α^{n-1}. From the lower dimensional case of (7.32) and the existence of decompositions such as (7.29) it follows easily that $\vec{\phi}(D_\beta^j)$ is independent of the choices involved for D_β^j, i.e., of possible subdivision of the link and distance from the vertex. Note that the $(n-j)$-simplices θ^{n-j} of D^n which contain σ^0 are just the joins to σ^0 of the various θ^{n-1-j}. Also the dual set to θ^{n-j} in D^n can be taken as the corresponding dual set for $\theta^{n-1-j}\subset D_\alpha^{n-1}$. Finally

$$(7.37)\qquad A(\theta^{n-j})=\left(\int_0^1 u^{n-j-1}\right)A(\theta^{n-j-1})=\frac{A(\theta^{n-j-1})}{n-j}.$$

By combining (7.33)–(7.37), we find that

$$(7.38)\qquad \int_{D^n}\operatorname{tr}\vec{\mathcal{E}}(t)\sim\sum_{j=0}^{n-1}\frac{t^{-n/2+j/2}}{(4\pi)^{(n-j)/2}}\sum_\alpha\mathfrak{M}_{n,j}\vec{\phi}(D_\alpha^j)A(\theta_\alpha^{n-j})+\vec{\phi}(D^n).$$

Finally, if we use the decomposition (7.29) and the parametrices (7.17), we easily get

Theorem 7.2. *On the piecewise flat pseudomanifold X^n we have*

$$(7.39)\qquad \operatorname{tr}\vec{E}^n(t)\sim\sum_{j=0}^{n}\frac{t^{-n/2+j/2}}{(4\pi)^{(n-j)/2}}\sum_\alpha\mathfrak{M}_{n,j}\vec{\phi}(D_\alpha^j)A(\sigma_\alpha^{n-j}).$$

We now consider the more complicated expansion for $\vec{\mathcal{E}}^n(t)$ on $C_{0,1}(L^{n-1})$. This will enable us to calculate the constant term in the asymptotic expansion for D^n in terms of links (some details will be omitted). We also obtain the analytic continuation of the functions ν^{-2s} on the links. It will be convenient to work with slightly more general dual sets to which the preceding discussion immediately applies. We choose these as follows. Given $x\in L^{n-1}$, set

$$(7.40)\qquad \mathcal{K}(x,\epsilon)=\left\{q\in\overline{C_{0,1}(L^{n-1})}\mid\overline{q,p}^{\,2}-\overline{q,x}^{\,2}<1-2\epsilon^2\right\},$$

where the bar denotes distance. Choose points $x_\beta^k\in\bigcup\tau_\alpha^k$ and $\epsilon_\beta^k>0$, and set

$$(7.41)\qquad \mathcal{V}^k=\bigcup_{\substack{j\le k\\ \beta}}\mathcal{K}\left(x_\beta^j,\epsilon_\beta^j\right)\setminus\bigcup_{\substack{j<k\\ \gamma}}\mathcal{K}\left(x_\gamma^j,\epsilon_\gamma^j\right).$$

It is easy to see that x_β^k, ϵ_β^k can be chosen so that

(i) $\bigcup_{j\le k}\mathcal{V}^j$ is a neighborhood of $\bigcup_{j\le k,\alpha}\tau_\alpha^j$ in $C_{0,1}(L^{n-1})$,

(ii) for fixed $\alpha, j\le k$, we have $\mathcal{K}(x_\alpha^k,\epsilon_\alpha^k)\cap\tau_\beta^j=\varnothing$ unless $j=k$, $\alpha=\beta$.

(iii) $\mathcal{K}(x_\alpha^k, \epsilon_\alpha^k) \cap \tau_\beta^j \neq \varnothing$ for $j > k$ implies τ_α^k is a face of τ_β^j. Put

$$W_\alpha^k = \mathcal{V}^k \cap C_{0,1}(\tau_\alpha^{k-1}).$$

For $w \in W_\alpha^k$, set $\overline{w, p} = s(w)$. Then the components Z_α^k of $\mathcal{V}^k$ are of the form

$$\{(w, z) \mid w \in W^k, z \in C_{0,(1-s^2(w))^{1/2}}(L(\tau^{k-1}))\}, \tag{7.42}$$

where the metric is induced from the product metric on $W_\alpha^k \times C_{0,1}(L(\tau_\alpha^{k-1}))$. If we set

$$D^n = C_{0,1}(L^{n-1}) \setminus \bigcup_{k=1}^{n-1} \mathcal{V}^k, \tag{7.43}$$

then D^n is a dual set of essentially the type we have previously considered, and

$$\int_{C_{0,1}(L^{n-1})} \operatorname{tr} \vec{\mathcal{E}}^n(t) = \int_{D^n} \operatorname{tr} \vec{\mathcal{E}}^n(t) + \sum_{k,\alpha} \int_{Z_\alpha^k} \operatorname{tr} \vec{\mathcal{E}}^n(t). \tag{7.44}$$

We now calculate the asymptotic expansion of the integrals in (7.44) corresponding to the Z_α^k. To simplify notation we drop the subscript α. Set

$$A^k(\underline{s}) = \operatorname{Area}(\{w \in W^k \mid s(w) = \bar{s}\}), \tag{7.45}$$

$$l_k = \min_{w \in W^k} s(w); \tag{7.46}$$

we will just write l for l_k. On Z_α^k we have the parametrix

$$\operatorname{tr} \vec{E}^{n,x}(t) = \frac{1}{(4\pi t)^{k/2}} \mathfrak{M}_{n,n-k} \operatorname{tr} \vec{\mathcal{E}}^{n-k}(t). \tag{7.47}$$

Set

$$\int_{C_{0,c}(L(\tau_\alpha^{k-1}))} \operatorname{tr} \vec{\mathcal{E}}^{n-k}(t) = F^{n-k}(c, t), \tag{7.48}$$

where $F^{n-k}(c, t)$ is a column vector. If we put $F^{n-k}(1, t) = F^{n-k}(t)$, then

$$F^{n-k}(c, t) = F^{n-k}(t/c^2). \tag{7.49}$$

Then using (7.42), (7.49), we have

$$\int_{Z^k} \operatorname{tr} \vec{E}^{n,x}(t) = \frac{1}{(4\pi t)^{k/2}} \int_l^1 \mathfrak{M}_{n,n-k} A^k(s) F^{n-k}(t(1-s^2)^{-1})\, ds. \tag{7.50}$$

Set

$$\frac{t}{1-s^2} = y, \quad s = \left(1 - \frac{t}{y}\right)^{1/2}, \quad ds = \frac{1}{2}\left(1 - \frac{t}{y}\right)^{-1/2} \frac{t}{y^2} ds. \tag{7.51}$$

Then (7.50) becomes

$$\frac{k}{2t(4\pi t)}\int_{t/(1-l^2)}^{\infty}\mathfrak{M}_{n,n-k}A^k\left(\left(1-\frac{t}{y}\right)^{1/2}\right)\cdot\frac{1}{2}\left(1-\frac{t}{y}\right)^{-1/2}\left(\frac{t}{y}\right)^2F^{n-k}(y)\,dy. \tag{7.52}$$

Assume now that

$$F^{n-k}(y)\sim\sum_{j=0}^{\infty}a_jy^{\lambda_j}, \tag{7.53}$$

where $\lambda_j<\lambda_{j+1}$; this assumption will have to be generalized later. Also let

$$A^k\left(\left(1-\frac{t}{y}\right)^{1/2}\right)\frac{1}{2}\left(1-\frac{t}{y}\right)^{-1/2}\sim\sum_{\gamma=0}^{\infty}b_\gamma\left(\frac{t}{\gamma}\right)^{\gamma} \tag{7.54}$$

be the Taylor expansion, and set

$$\eta_{N'}\left(\frac{t}{y}\right)=A^k\left(\left(1-\frac{t}{y}\right)^{1/2}\right)\frac{1}{2}\left(1-\frac{t}{y}\right)^{-1/2}-\sum_{\gamma=0}^{N'}b_\gamma\left(\frac{t}{\gamma}\right)^{\gamma}. \tag{7.55}$$

Then the integral in (7.52) can be written as

$$\begin{aligned}&\int_1^{\infty}\mathfrak{M}_{n,n-k}A^k\left(\left(1-\frac{t}{y}\right)^{1/2}\right)\frac{1}{2}\left(1-\frac{t}{y}\right)^{-1/2}\left(\frac{t}{y}\right)^2F^{n-k}(y)\,dy\\&+\int_{t/(1-l^2)}^{1}\mathfrak{M}_{n,n-k}A^k\left(\left(1-\frac{t}{y}\right)^{1/2}\right)\frac{1}{2}\left(1-\frac{t}{y}\right)^{-1/2}\left(\frac{t}{y}\right)\rho_N(y)\,dy\\&+\int_{t/(1-l^2)}^{1}\mathfrak{M}_{n,n-k}A^k\left(\left(1-\frac{t}{y}\right)^{1/2}\right)\frac{1}{2}\left(1-\frac{t}{y}\right)^{-1/2}\left(\frac{t}{y}\right)^2\sum_{j\leqslant N}a_jy^{\lambda_j}\,dy.\end{aligned} \tag{7.56}$$

For the first term in (7.56) we can use (7.54) (7.55) to get

$$\sum_{\gamma\leqslant N'}b_\gamma t^{\gamma+2}\int_1^{\infty}\mathfrak{M}_{n,n-k}y^{-(\gamma+2)}F^{n-k}(y)\,dy+O(t^{N'+3}). \tag{7.57}$$

If we choose $N'<\lambda_N-3$, for the second term in (7.56) we have

$$\sum_{\gamma=0}^{N'}b_\gamma t^{\gamma+2}\int_{t(1-l^2)}^{1}\mathfrak{M}_{n,n-k}y^{-(\gamma+2)}\rho_N(y)\,dy+O(t^{N'+3}). \tag{7.58}$$

The third term in (7.56) can be rewritten as

$$\sum_{\gamma\leq N''} b_\gamma t^{\gamma+2}\int_{t/(1-l^2)}^1 \mathfrak{M}_{n,n-k} y^{-(\gamma+2)} \sum_{j\leq N} a_j y^{\lambda_j}\,dy$$
$$+\int_{t/(1-l^2)}^1 \mathfrak{M}_{n,n-k}\eta_{N''}\left(\frac{t}{y}\right)\left(\frac{t}{y}\right)^2 \sum_{j\leq N} a_j y^{\lambda_j}\,dy, \tag{7.59}$$

where we choose $N'' > \lambda_N$. The first term in (7.59) equals

$$\sum_{\gamma\leq N''} b_\gamma t^{\gamma+2} \sum_{\substack{\lambda_j-\gamma-1\neq 0\\ \lambda_j\leq N}}\left[\frac{\mathfrak{M}_{n,n-k}a_j}{\lambda_j-\gamma-1}-\frac{\mathfrak{M}_{n,n-k}a_j}{\lambda_j-\gamma-1}\left(\frac{t}{1-l^2}\right)^{\lambda_j-\gamma-1}\right]$$
$$-\sum_{\gamma\leq N''} b_\gamma t^{\gamma+2}\sum_{\substack{\lambda_j-\gamma-1=0\\ \lambda_j\leq N}} \mathfrak{M}_{n,n-k}a_j \log\left(\frac{t}{1-l^2}\right). \tag{7.60}$$

For the second term in (7.59) we set

$$\frac{t}{y}=u,\quad -\frac{t}{u^2}du=dy, \tag{7.61}$$

to get

$$\int_0^{1-l^2} \mathfrak{M}_{n,n-k}\eta_{N''}(u)\sum_{j\leq N} a_j t^{\lambda_j+1}u^{-\lambda_j}\,du + O(t^{N''+3}). \tag{7.62}$$

The expressions in (7.58), (7.59), (7.60), (7.62) provide the required asymptotic expansion of (7.50); however, this was derived under the assumption that $F^{n-k}(y)$ has an expansion of the form (7.53). This holds if $k = n-2$, and in particular if $n = 3$. For the latter case however positive powers of t arise in (7.60). Thus if $n = 4$, $k = 2$, then the second line in (7.60) can give rise to logarithmic terms in the expansion for F^4. So for the case $n-k = 4$, (7.53) will no longer hold. The general assumption which replaces (7.53) is

$$F^{n-k}(y)\sim \sum_{0\leq i\leq i}\sum_j a_{ji}y^{\lambda_j}\log^i y. \tag{7.63}$$

Then corresponding to (7.60) we have

$$\int_{t/(1-l^2)}^1 \sum_{\gamma\leq N'} b_\gamma\left(\frac{t}{y}\right)^{\gamma+2}\mathfrak{M}_{n,n-k}\sum_{i\leq i}\sum_{j\leq N''} a_{ji}y^{\lambda_j}\log^i y\,dy. \tag{7.64}$$

If $-(\gamma+2)+\lambda_j\neq -1$, integration by parts gives a sum of terms of the form

$$t^{\lambda_j+1}\log^\beta t,\qquad \beta = 0,\cdots,i. \tag{7.65}$$

If $-(\gamma + 2) + \lambda_j = -1$, we also get a term

$$\log^{i+1} t. \tag{7.66}$$

Thus the expansion in dimension $n \geqslant 4$ can involve logarithmic terms at most of the form $t^a \log^{n-3} t$.

By using

$$\int_{C_0(L^{n-1})} \operatorname{tr} \mathcal{E}^n(t)\, dx = \frac{1}{2} \int_t^\infty \int_{L^{n-1}} \operatorname{tr} \mathcal{E}^n(1, x, u) u^{-1}\, dx\, du, \tag{7.67}$$

we see that an asymptotic expansion of essentially the same form holds for

$$\int_{L^{n-1}} \operatorname{tr} \mathcal{E}^n(1, x, u)\, dx. \tag{7.68}$$

Now consider the function $\psi(\nu(i), s)$ on L^{n-1}, where we mean the trace integrated over L^{n-1}. The estimates of subsection 7.4, together with standard arguments, show that this function is analytic in the half plane $\operatorname{Re} s > \frac{1}{2}(n - 1)$. Using the arguments of §4 and the asymptotic expansion just established we can now get

Theorem 7.3. *The functions $\psi(\nu(i), s)$ have analytic continuations to meromorphic functions in* $\mathbf{C}$. *The possible poles are points $j/2$, where $j = n, (n - 1), (n - 2), \cdots$; they are simple for $n \leqslant 3$ and have order at most $n - 3$ for $n > 3$.*

Proof. The arguments of §4 and the asymptotic expansion just established show that the expression show that the expression in (4.4) has such an analytic continuation, where the possible logarithmic terms give rise to the higher order poles. Let $\operatorname{Re} s > a$ be a maximal half plane in which $\psi(\nu, s)$ is meromorphic. The terms involving $\psi(\nu(i - 1), s + 1)$ are then meromorphic in $\operatorname{Re} s > a - 1$. The finitely many harmonic terms are meromorphic in $\mathbf{C}$. The remaining terms can be written as

$$\mathfrak{M}_{n,n-1} \vec{\psi}(\nu, s), \tag{7.69}$$

where $\vec{\psi}(\nu, s)$ is the column vector whose ith entry is $\nu(i, s)$, $i = 0, \cdots, n - 2$. Since the total expression in (4.4) is meromorphic in $\mathbf{C}$; the expression in (7.69) is meromorphic in $\operatorname{Re} s > a - 1$. Multiplying (7.69) by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & \cdot \\ 3 & -2 & 1 & \cdot \\ \vdots & \vdots & \vdots & 0 \\ \pm n & \mp(n-1) & \pm(n-2) & 1 \end{pmatrix} \tag{7.70}$$

gives back $\vec{\psi}(\nu, s)$. Thus $\vec{\psi}(\nu, s)$ is meromorphic in $\operatorname{Re} s > a - 1$, which implies $a = -\infty$.

By using (4.25), an argument like that of Theorem 4.4 now shows that the functions ν^{-2s} are also meromorphic in $\mathbf{C}$ with poles at $j/2$, where $j = n-1, n-2, \cdots$. Again the poles are of order at most $n-3$. Using the techniques of §4, the terms in (7.58)–(7.62) can be expressed as spectral invariants of the links $L(\tau_\alpha^k)$. Applying (7.42) we can now calculate the asymptotic expansion of

$$\int_{D^n} \operatorname{tr} \vec{\mathcal{E}}^n(t), \tag{7.71}$$

and in particular, the constant term in that expansion, in terms of spectral invariants of links.

By letting the number of faces increase to infinity we can choose a sequence of dual sets D_ϵ^n, such that $D_\epsilon^n \to C_{0,1}(L^{n-1})$. The corresponding term in (7.42) then gives a contribution

$$\sum_{j=1}^{n} \frac{t^{-n/2+j/2}}{(4\pi)^{(n-j)/2}} \sum_\alpha \mathfrak{M}_{n,j} \vec{\phi}\left(D_\alpha^j\right) \cdot A\left(C_{0,1}\left(\tau_\alpha^{n-j-1}\right)\right) \tag{7.72}$$

as in (7.38). The volume of a region $\bigcup_\alpha Z_{\alpha,\epsilon}^k$ goes to zero as the region shrinks to the $(k-1)$-skeleton of L^{n-1}. The contribution to the asymptotic expansion corresponding to $\bigcup_\alpha Z_{\alpha,\epsilon}^k$ for fixed k will blow up in general. However, these blow ups must cancel when we sum over k, since the expression in (7.72) remains finite. Using our previous computations we can extract a "finite part" for each fixed k, which can be thought of as the correction term to (7.72) associated to the $(k-1)$-skeleton of L^{n-1}. Although we will omit further details, we make the following observation which will be needed in §8.

Remark 7.1.. The correction terms associated to the $(k-1)$-skeleton of L^{n-1} can effect only terms of order $\geqslant t^{-n/2+k/2+1-\epsilon}$ in the expansion. To see this, note that as in (7.50) we have

$$\int_{\bigcup Z_{\alpha,\epsilon}^k} \operatorname{tr} \vec{E}^{n,x}(t) = \frac{1}{(4\pi t)^{k/2}} \int_l^1 \mathfrak{M}_{n,n-k} A^k(s) F^{n-k}\left(\frac{t}{1-s^2}\right) ds. \tag{7.73}$$

Replace $F^{n-k}(t/(1-s^2))$ in (7.73) by its asymptotic expansion up to terms which are $O((t/(1-s^2))^{1-\epsilon})$ plus an error which is $o((t/(1-s^2))^{1-\epsilon})$. Since the resulting integrals with respect to s converge, and $\operatorname{Vol}(\bigcup Z_{\alpha,\epsilon}^k) \to 0$, our claim easily follows.

8. The Chern-Gauss-Bonnet formula for pseudomanifolds

In this section we give a concrete application of the results of §7 by showing how they lead to an explicit formula for the Euler characteristic of a piecewise flat pseudomanifold. The formula is entirely in terms of *interior* dihedral angles of simplices. Naturally the derivation which is based on the heat equation method leads directly to a formula for the L^2-Euler characteristic $\chi_{(2)}(X^n)$. But from this we can easily derive the corresponding formula for the ordinary Euler characteristic $\chi(X^n)$.

Remark 1. Although for convenience we have only considered pseudo-manifolds, it is possible to modify our whole discussion so that it holds for arbitrary simplicial complexes with piecewise flat metric. In particular, the formula to be derived in this section holds in that generality; compare also (8.24)–(8.26).

Remark 2. There is an almost obvious formula,

$$\chi(X^n) = \sum_{\sigma^0} \sum_{\sigma^k \subset \sigma^0} (-1)^k \operatorname{ext}_\alpha(\sigma^k), \tag{8.1}$$

where $\operatorname{ext}_\alpha(\sigma^k)$ is the *exterior* angle of σ^k at its vertex σ^0_α; see [4] for further discussion. However, insofar as we are aware, our formula (8.18), (8.19) in terms of *interior* angles is new. As one might expect, in fact, it is possible to obtain our formula from (8.1) by "elementary" arguments. But it was only much later that we realized how to do this. We refer to [14] for details.

Remark 3. The underlying philosophy of this section and the next is that in order to generalize curvature invariants of smooth spaces to piecewise smooth spaces, it suffices to express these invariants as spectral invariants of links. More generally it suffices to find some analytic or geometric interpretation of the particular curvature invariant under consideration, which makes sense for piecewise smooth spaces.

Remark 4. Spectral invariants of links may themselves be local on the links, as is the case for the Lipschitz-Killing curvatures; see (8.27) and [13], [14]. However, for the Pontrjagin classes of §9 the relevant η-invariants are not local on the links. For essentially this reason, the problem of generalizing curvature invariants to piecewise smooth spaces is more subtle than one might initially suppose.

We now derive the formula for $\chi_{(2)}(X^n)$ where X^n is piecewise flat. By a standard argument [38], together with the results of §7, we have

$$\chi_{(2)}(X^n) = \sum_{i=0}^{n} (-1)^i {}_i a_{n/2} = \sum_{\beta} \sum_{i=0}^{n} (-1)^i {}_i\phi(D^n_\beta). \tag{8.2}$$

Now fix $D^n_\beta = D^n$ which we choose so as to satisfy the conditions preceding (7.42). On any region $Z^k_\alpha \subset C(L(\sigma^0))\setminus D^n$, we can use the parametrix $\vec{E}^{n,x}(t)$ of (7.50). Then (7.50) and the relation

$$\sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} = 0, \tag{8.3}$$

imply that all coefficients in the asymptotic expansion of

$$\sum(-1)^i \operatorname{tr} E_i^{n,x}(t) \tag{8.4}$$

vanish identically. Hence the right-hand side of (8.2) can be replaced by

$$\sum_k \sum_{i=0}^{n} (-1)^i {}_i\psi(L(\sigma_k^0)), \tag{8.5}$$

where ${}_i\psi(L(\sigma_k^0))$ is as in (7.26). By taking the alternating sum of the expression in (4.4) for $L(\sigma^0)$, we see in the same way

$$\chi_{(2)}(C_{0,1}(L(\sigma^0))) - \tfrac{1}{2}\chi_{(2)}(L((\sigma^0))) + \sum_{i=0}^{n-2} (-1)^{i-1} 4s\psi(\nu(i), s+1) \tag{8.6}$$

is a holomorphic function; compare §5. If we introduce the notation

$$\chi_{(2)}^{\perp}(\sigma^k) = \chi_{(2)}(C_{0,1}(L(\sigma^k))) - \tfrac{1}{2}\chi_{(2)}(L(\sigma^k)), \tag{8.7}$$

in the same way we see that

$$\sum_{i=0}^{n} (-1)^i {}_i\psi(\sigma^0) = \chi_{(2)}^{\perp}(\sigma^0) + 4 \operatorname*{Res}_{s=0} \sum_{i=0}^{n-2} (-1)^i \psi(\nu(i), s+1) \tag{8.8}$$

is the value of this function at zero.

We will need an expression analogous to (8.8) in other dimensions. For this purpose, let $\sigma^{l'}$ be a face of σ^l, and introduce the notation

$$[\sigma^{l'}, \sigma^l], \tag{8.9}$$

for the normalized dihedral angle of $\sigma^{l'}$ at σ^l. Thus $[\sigma^{l'}, \sigma^l]$ is the volume on the $(l - l' - 1)$-sphere, of the unit vectors at a point of $\sigma^{l'}$ pointing into σ^l, subject to the normalization that the volume of $S^{l-l'-1}$ is 1.

Let $\mathfrak{M}_{n,k}$ be as in (7.25), and let K_{n+1} be the $n \times (n+2)$ matrix in (7.70) with two columns of zeros added at the right. Then it is easily checked that for $j \leqslant k-1$

$$K_{n-2j} \cdots K_n \cdot \mathfrak{M}_{n,n-2k} = \mathfrak{M}_{n-2(j+1),n-2k}, \tag{8.10}$$

where

$$\mathfrak{M}_{n-2k,n-2k} = I_{n-2k+1}. \tag{8.11}$$

Now consider the coefficient of $t^{-n/2+j}$ for

$$\int_{C_{0,1}(L(\sigma^0))} \operatorname{tr} \vec{\mathfrak{E}}(t). \tag{8.12}$$

Multiply (8.12) on the left by

$$K_{n-2(j-1)} \cdots K_{n-2} \cdot K_n, \tag{8.13}$$

take the alternating sum of the entries of the resulting column vector, and consider the coefficients of $t^{-n/2+j}$. If we use the parametrix $E^{n,x}(t)$, it is clear from (8.3) that the terms in (7.44) corresponding to Z_α^β, for $\beta > 2j$ do not contribute. However, in view of Remark 7.1 at the end of §7, the terms corresponding to Z_l^β, $\beta \leqslant 2j$, make an arbitrarily small contribution, provided the volumes of these regions are chosen arbitrarily small. Thus by passing to the limit $D_\epsilon^n \to C_{0,1}(L(\sigma^0))$, we find as in (7.39) that the expression just calculated is just equal to

$$\frac{1}{(n-2j)} \sum_{\sigma^0 \subset \sigma^{2j}} [\sigma^0, \sigma^{2j}] \frac{V(S_1^{2j-1})}{(4\pi)^j} \sum_i (-1)^i{}_i\phi(D^{n-2j}), \tag{8.14}$$

where $V(S_1^{2j-1}) = 2\pi^j/\Gamma(j)$.

Let $\langle\ \rangle$ denote the operation which replaces a vector with $n-1$ components by a vector with $n+1$ components defined by

$$\left\langle \begin{matrix} v_1 \\ \vdots \\ v_{n-1} \end{matrix} \right\rangle = \begin{pmatrix} 0 \\ v_1 \\ \vdots \\ v_{n-1} \\ 0 \end{pmatrix}.$$

Then

$$K_{n-2(j-1)} \cdots K_n \langle V \rangle = \langle K_{n-2j} \cdots K_n \cdot \mathfrak{M}_{n,n-2} V \rangle. \tag{8.15}$$

If we drop the harmonic terms in (4.4), which does not effect the operation Res, the remaining expression can be written as

$$\mathfrak{M}_{n,n-2}\vec{\psi}(\nu, s) + 4s\langle \vec{\psi}(\nu, s+1)\rangle. \tag{8.16}$$

Multiply this expression on the left by $K_{n-2(j-1)} \cdots K_n/(n-2j)$. Then take the alternating sum of the elements in each column and take $\text{Res}_{s=j}$. The resulting expression is also equal to (8.14). Thus multiplying through by $(n-2j)(4\pi)^j/V(S_1^{2j-1})$ we get

$$\begin{aligned} &\sum_{\sigma^0 \subset \sigma^{2j}} [\sigma^0, \sigma^{2j}] \sum_i (-1)^i{}_i\phi(D^{n-2j}) \\ &= \frac{4^j\Gamma(j)}{2} \operatorname*{Res}_{s=j} \sum (-1)^i K_{n-2(j-1)} \cdots K_n \\ &\quad \cdot \{ M_{n,n-2}\vec{\psi}(\nu, s) + 4s\vec{\psi}(\nu, s+1)\}, \end{aligned} \tag{8.17}$$

where the symbol $\Sigma(-1)$ on the right-hand side indicates taking the alternating sum of the elements of a column vector. Now add (8.17) for $j = 1, \cdots, [n/2]$. Since $4^j\Gamma(j) \cdot 4 \cdot j = 4^{j+1} \cdot \Gamma(j+1)$, in view of (8.15) all terms on the right-hand side cancel, save for

$$(8.18)\qquad 4\operatorname{Res}_{s=1}\sum(-1)^i\vec{\psi}(\nu, s).$$

This is just the second term appearing on the right-hand side of (8.8). Thus by (8.8),

$$(8.19)\qquad \sum_{i=0}^{n}(-1)^i{}_i\psi(\sigma^0) = \chi_{(2)}^{\perp}(\sigma^0) - \sum_{\sigma^0\subset\sigma^{2j}}[\sigma^0, \sigma^{2j}]\sum_i(-1)^i{}_i\psi(D^{n-2j}).$$

The argument can now be iterated. Introduce the notation

$$(8.20)\qquad \Delta(\sigma^{2j}) = \chi_{(2)}^{\perp}(\sigma^{2j}) - \sum_{\sigma^{2j}\subset\sigma^{2k}}[\sigma^{2j}, \sigma^{2k}],$$

where $2k$ is the largest even integer $< n$, and $2j < 2k$. Thus $\Delta(\sigma^{2j})$ is a "defect" which will vanish if X^n is flat at σ^{2j}. Then we get

Theorem 8.1 (*Chern-Gauss-Bonnet*). *Let $2k$ be as above. Then*

$$\chi_{(2)}(X^n) = \sum_{\sigma^0}\chi_{(2)}^{\perp}(\sigma^0)$$

$$(8.21)\qquad + \sum_{\sigma^0\subset\cdots\subset\sigma^{2l_j}}(-1)^j[\sigma^0, \sigma^{2l_1}][\sigma^{2l_1}, \sigma^{2l_2}]\cdots[\sigma^{2l_{j-1}}, \sigma^{2l_j}]\chi_{(2)}^{\perp}(\sigma^{2l_j}),$$

$$2l_j \leqslant 2k$$

$$(8.22)\qquad = \sum_{\sigma^0\subset\cdots\subset\sigma^{2l_j}}(-1)^j[\sigma^0, \sigma^{2l_1}]\cdots[\sigma^{2l_{j-1}}, \sigma^{2l_j}]\Delta(\sigma^{2l_j}), \qquad 2l_j < 2k.$$

Note that if X^n is a piecewise linear *manifold*, we can replace $\chi_{(2)}$ by χ and just write $\chi^{\perp}$ for $\chi_{(2)}^{\perp}$. If n is odd, then the invariants $\chi^{\perp}(\sigma^{2k})$ are all zero at interior points. If n is even, then $\chi^{\perp}(\sigma^{2k}) = 1$ at interior points. In either case, $\chi^{\perp}(\sigma^{2k})$ is equal to $\frac{1}{2}$ if σ^{2k} is contained in the boundary, unless $2k = n$, ($\chi^{\perp}(\sigma^n) = 1$); see (8.26).

Example 8.1. Consider the case of a 3-dimensional cube. At each vertex we get a contribution of $\frac{1}{2}$ from the vertex itself, and a contribution of $\frac{1}{4}\times\frac{1}{2}$ for each of the 3 faces of dimension 2 which contain the vertex. Thus

$$(8.23)\qquad \hat{\chi}(\text{3-cube}) = \chi(\text{3-cube}) = \sum_{\sigma^0}\frac{1}{2} - 3\left(\frac{1}{4}\times\frac{1}{2}\right) = 8\times\frac{1}{8} = 1.$$

If X is a k-simplex σ^k, we obtain from (8.21),

$$(8.24)\qquad 1 = \sum_{\sigma^0\subset\cdots\subset\sigma^{2l_j}\subset\sigma^k}(-1)^j[\sigma^0, \sigma^{2l_1}]\cdots[\sigma^{2l_{j-1}}, \sigma^{2l_j}]\chi^{\perp}(\sigma^{2l_j}),$$

where $\chi^{\perp}(\sigma^{2l_j})$ is equal to $\frac{1}{2}$ unless $\sigma^{2l_j} = \sigma^k$, in which case $\chi^{\perp} = 1$. If X^n is a piecewise flat simplicial complex, by applying (8.24) to each σ^k we easily get

$$\chi(X^n) = \sum_{\sigma^0 \subset \cdots \subset \sigma^{2l_j}} (-1)^j [\sigma^0, \sigma^{2l_1}] \cdots [\sigma^{2l_{j-1}}, \sigma^{2l_j}] \times \sum_{\sigma^{2l_j} \subset \sigma^k} (-1)^k \chi^{\perp}(\sigma^{2l_j}). \tag{8.25}$$

But clearly,

$$\begin{aligned} \sum_{\sigma^{2l_j} \subset \sigma^k} (-1)^k \chi^{\perp}(\sigma^{2l_j}) &= \frac{1}{2} + \frac{1}{2}\{\chi(C_{0,1}(L(\sigma^{2l_j})) - (L(\sigma^{2j}))))\} \\ &= 1 - \frac{1}{2}\chi(L(\sigma^{2j})). \end{aligned} \tag{8.26}$$

This yields for the ordinary Euler characteristic $\chi(X^n)$, the formula which corresponds to the expression (8.21) for $\chi_{(2)}(X^n)$.

We now consider the Lipschitz-Killing curvatures. Recall that R_1 is just the scalar curvature up to normalization (the curvatures which are denoted here by $R^{j/2}$ are denoted by R^j in [14]). From the formula (5.15) it follows that R_j is multiplied by

$$\frac{(n+p-2j)!}{(4\pi)^{p/2}(n-2j)} \operatorname{Vol}(Y^p),$$

if we take the metric product of X^n with a flat space Y^p. Since the R_j should be locally computable, it is then reasonable to set

$$\begin{aligned} R_j = \frac{(m-2j)!}{(4\pi)^{(m-2j)/2}} \sum_{\sigma^{m-2j}} \sum_{\sigma^{m-2j} \subset \sigma^{2l_1} \cdots \subset \sigma^{2l_i}} & (-1)^i [\sigma^{m-2j}, \sigma^{2l_1}] \cdots [\sigma^{2l_{i-1}}, \sigma^{2l_i}] \\ & \times \Delta(\sigma^{2l_i}) A(\sigma^{m-2j}), \qquad 2l_i < 2j, \end{aligned} \tag{8.27}$$

in the piecewise flat case. For the case of scalar curvature, this coincides with the proposal of Regge [47]. Moreover, it plays the expected role in the piecewise flat version of Chern's kinematic formula; see [14]. Finally, note that definition (8.27) has the correct scaling property when the metric is multiplied by a constant.

We can now see that the relations (5.18), (5.20) continue to hold in the piecewise flat case; as in §5 we will only treat the case n odd, since the case n even, then follows easily by taking products with a circle.

Theorem 8.2. *Let X^n be a piecewise flat pseudomanifold. Then relations* (5.18), (5.20) *hold.*

Proof. If we use the formulas

$$(8.28)\quad \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} (n-i+l)^{2n} = (-1)^j (2n)!, \qquad l = 0, \pm 1, \cdots,$$

$$(8.29)\quad \sum_{i=1}^{2n} (-1)^i \binom{2n}{i} (n+i+l)^{2(n-k)} = 0,$$
$$l = 0, \pm 1, \cdots ; k = 1, \cdots, n,$$

the claim follows immediately from Theorem 7.2 and the proof of Theorem 8.1.

In view of Theorem 8.2 it is natural to conjecture that if we consider a suitable sequence of piecewise flat approximations X_ϵ^n converging to a *smooth* riemannian space X^n, then the expression (8.27) should converge to the corresponding R_j for X^n. Indeed, those physicists who have discussed Regge calculus seem to have accepted this statement for R_1 without proof. The assertion is correct, but not obvious; see [13], [14]. In this connection, note that for piecewise flat manifolds the form of the asymptotic expansion (of Theorem 7.2) is quite different from that in the smooth case; it contains extra negative half powers of t, and no positive powers. Thus the corresponding approximation property for arbitrary combinations of coefficients in the asymptotic expansion cannot hold.

9. The η-invariant and combinatorial formulas for Pontrjagin classes

In this section we discuss the generalization to pseudomanifolds of the results of §6 concerning the η-invariant and signature. As a consequence we obtain a canonical (if not readily computable) local formula for the L-classes of piecewise linear manifolds, and more generally a *local definition* of these classes for pseudomanifolds.

As we indicated in §6, nontrivial ideal boundary conditions play a significant role in the discussion of the signature, and it is also important to consider coefficients in a vector bundle. We can then make a connection with Morgan's unpublished work on the geometric realization of K-homology; in conversations with Morgan (in 1977) it became apparent that our $*$-invariant boundary conditions were the metric version of his choice of selfannihilating subspaces for the intersection pairing on links.

In the present work, the L-classes, which are ordinary homology classes, are defined in terms of η-variants of links. We take the point of view (natural in our context) that the statements that the chains representing these classes are cycles and that corresponding homology classes are independent of piecewise linear structure, are formal consequences of the fact that the derivative of the η-invariant under change of metric is locally computable at the vertices of the link.

Of course, there are various discussions of the existence of local formulas for Pontrjagin classes of piecewise linear manifolds in the literature (see e.g., [22], [35], [45]) as well as an explicit formula for $p_1(x)$, due to Gabrielov, Gelfand and Losik [23], [24], [37], [51]. The relation between these papers and the present work has yet to be clarified. Also Goresky and MacPherson [29] have independently defined L-classes (equivalent to those discussed here) for pseudomanifolds admitting a stratification by strata of even codimension. However, their definiton (which generalizes Thom's for the case of piecewise linear manifolds; see [39]) is not local, and in particular they do not show that the signature is multiplicative under coverings; perhaps the *existence* of a local formula in their context can also be proved by extending the arguments of [35].

A natural consequence of our methods is a generalization of the η-invariant formula of Atiyah-Patodi-Singer [2] to piecewise flat pseudomanifolds with boundary. We can also define piecewise linear invariant analogs $\hat{\rho}_E(Y)$, $\hat{\sigma}_1(\tilde{Y})$ of the invariants $\rho_E(Y)$, $\sigma_1(\tilde{Y})$ defined in [2] for the smooth case. If Y is smooth, it is clear that $\hat{\sigma}_1(\tilde{Y}) = \sigma_1(\tilde{Y})$. We conjecture that if Y is smooth, then $\hat{\rho}_E(Y) = \rho_E(Y)$ which would imply the piecewise linear invariance of $\rho_E(Y)$. The latter follows from [2], provided there exists a finite cover for which the pullback of E^k is globally flat. It seems to be unknown in general.

Finally, we discuss briefly the extra step (not carried out here) which when combined with the methods of [14] would show that in the limit under subdivision (in the sense described in §8), our formulas for (the duals of) the L-classes go over into the Pontrjagin forms in the smooth case. This is a special case of the corresponding conjecture for η-invariants with coefficients in a flat bundle E^k. The latter would imply that $\rho_E(Y) = \hat{\rho}_E(Y)$ in general.

We now begin with the case of a closed oriented piecewise flat pseudomanifold X^{4l} with negligible boundary and trivial coefficients. As in §6 we can consider $\operatorname{tr}(*e^{-\Delta_{2l}t})$. This operator is trace class, as follows from the results of §7 and the fact that $*$ is a bounded operator. By the standard argument,

$$\operatorname{sig}(X^{4l}) = \operatorname{tr}\left(*e^{-\Delta_{2l}t}\right) = \lim_{t\to 0} \operatorname{tr}\left(*e^{-\Delta_{2l}t}\right). \tag{9.1}$$

We can localize the trace in a preliminary fashion by using dual cells as in Theorem 7.2. We observe

Lemma 9.1. *The only nontrivial term in the asymptotic expansion of* $\operatorname{tr}(*e^{-\Delta_{2l}t})$ *is the constant term.*

Proof. On sets $\mathcal{U}_\alpha^i \times D_\alpha^{n-i}$ as in (7.28), a local parametrix for $*e^{-\Delta_{2l}t}$ vanishes identically. This follows because such sets admit local orientation reversing isometries; the $\mathcal{U}_\alpha$ are flat.

Note that the possibility of localizing the asymptotic expansion of $\operatorname{tr}(*e^{-\Delta_{2l}t})$ already yields

Theorem 9.2. $\operatorname{sig}(X^{4l})$ *is multiplicative under coverings.*

Since $\operatorname{tr}(*e^{-\Delta_{2l}t}) = O(t^N)$ for all N on $\mathcal{U}_\alpha \times D_\alpha$, we can pass from a dual cell D^n to $C_{0,\epsilon}(L(\sigma^0))$ with changing the asymptotic expansion. In this way as in (6.10) we find that on $L(\sigma^0)$ the function

$$\Gamma(s)*_{2l}\zeta(s) = \int_{L(\sigma^0)} \frac{2}{\sqrt{\pi}} \sum_j \frac{\Gamma\left(\sqrt{\mu_j} - s\right)}{\Gamma\left(\sqrt{\mu_j} + s + 1\right)} \Gamma\left(s + \frac{1}{2}\right) \phi_j \wedge d\phi_j \tag{9.2}$$

converges for $\operatorname{Re} s > (4l - 1)/2$, and continues analytically to a holomorphic function in $\mathbf{C}$, whose value at zero is equal to the constant term in the asymptotic expansion for $\operatorname{tr}(*e^{-\Delta_{2l}t})$ on D^n. By arguing as at the end of §7 we find that the function

$$\eta(s) = \int_{L(\sigma^0)} \frac{2}{\sqrt{\pi}} \sum_j \mu_j^{-(s+1/2)} \phi_j \wedge d\phi_j \tag{9.3}$$

for $s > (4l - 1)/2$ also continues to a holomorphic function in $\mathbf{C}$, and we obtain the following explicit local formula for the signature.

Theorem 9.3. *Let X^{4l} be a closed oriented admissible riemannian pseudomanifold with piecewise flat metric. Then*

$$\operatorname{sig}(X^{4l}) = \sum_{\sigma^0} \eta\left(L(\sigma^0)\right). \tag{9.4}$$

Theorem 9.4. *Formula* (9.4) *gives rise to a canonical local combinatorial formula for triangulated pseudomanifolds and in particular for piecewise linear manifolds by choosing the metric for which all edge lengths are equal to* 1.

Remark 9.1. The above formula is combinatorial in the sense that it is invariant under (orientation preserving) combinatorial symmetries. Unfortunately, the invariants $\eta(0)$ for $L(\sigma^0)$ are not readily computable and may well be irrational numbers in many cases (approximate computer calculations seem possible in principle).

Remark 9.2. Actually, we can obtain a 1-parameter family of "canonical" formulas by considering metrics of arbitrary constant curvature on X, rather than restricting attention to piecewise flat metrics; compare the proof of Theorem 9.5.

The formula of Theorem 9.3 can be promoted to a local definition for L-classes as follows. Let X^n be a closed oriented pseudomanifold with piecewise flat metric. For each k, define an $n-4k$ chain $c_{n-4k}(X^n)$ by

(9.5) $$c_{n-4k}(X^n) = \sum_{\sigma^{n-4k}} \eta\big(L(\sigma^{n-4k})\big)\sigma^{n-4k},$$

where the orientations $L(\sigma^{n-4k})$ of σ^{n-4k} are compatible with the orientation of X^{4n}.

Theorem 9.5. *The chains c_{n-4k} are cycles whose homology class L_{n-4k} depends only on the piecewise linear structure.*

Proof. At this point we will only give the formal argument, assuming the result that the derivative of the η-invariant under change of metric on $L(\sigma^{n-4k})$ is supported at the vertices of σ^{n-4k}. We will then derive the explicit formula for the derivative in detail.

In order to show that c_{n-4k} is a cycle we compute its boundary. For a fixed $(n-4k-1)$-simplex σ^{n-4k-1} we get the contribution

(9.6) $$\sum_{\sigma^{n-4k}\supset^{n-4k}} \eta\big(L(\sigma^{n-4k})\big).$$

We claim that the sum in (9.5) is equal to

(9.7) $$\operatorname{sig}\big(L(\sigma^{n-4k-1})\big).$$

Since we are still assuming negligible boundary, we have

(9.8) $$\operatorname{sig}\big(L(\sigma^{n-4k-1})\big) = 0;$$

this will be generalized below.

The equality of (9.6) and (9.7) would just be the content of Theorem 9.3 if the metric on $L(\sigma^{n-4k-1})$ were flat. After subdividing the link sufficiently finely, we can assume that for each $4k$-simplex of $L(\sigma^{n-4k-1})$ there exists a unique simplex τ_v^{4k} (up to isometry) in the space of constant curvature v, $0 \leqslant v \leqslant 1$, with the same edge lengths as $\tau^{4k} = \tau_1^{4k}$. Now each 1-simplex of $L(\sigma^{n-4k-1})$ contains a pair of 0-simplices in its boundary. The derivative of the sum of the η-invariants of the 0-simplices for the metrics g_v on $L(\sigma^{n-4k-1})$ corresponding to τ_v contributes once to each 1-simplex of $L(\sigma^{n-4k-1})$ for each point in its boundary, but with opposite orientations. Thus the sum of the η-invariants remains constant throughout the deformation, and for $v=1$ we are reduced to Theorem 9.3; the derivative formula applies since $\eta(0)$ is independent of the scaling factor v.

We now observe that the homology class

(9.9) $$[c_{n-4k}] = L_{n-4k}$$

depends only on the piecewise linear structure. By definition any two piecewise flat metrics on X^n giving rise to the same piecewise linear structure have subdivisions which are piecewise linearly isomorphic. Since the η-invariant of the standard $(4k-1)$-sphere vanishes, our formula is invariant under combinatorial subdivision, (keeping the underlying metric fixed). Clearly any two piecewise flat metrics g_0, g_1 on X^n with the same underlying combinatorial triangulation can be connected by a 1-parameter family of piecewise flat metrics g_v. Write

$$(9.10)\quad c_{n-4k,1}-c_{n-4k,0}=\int_0^1 \dot{c}_{n-4k,v}\,dv=\sum_{\sigma^{n-4k}}\int_0^1 \dot{\eta}\left(L(\sigma^{n-4k})\right)dv\,\sigma^{n-4k},$$

where the dot denotes differentiation with respect to v. Each term $\dot{\eta}(L(\sigma^{n-4k}))$ is a sum of contributions at each of the vertices τ^0 of $L(\sigma^{n-4k})$, which depend only on the metric and variation induced on $L(\tau^0)$, the link of τ^0 in $L(\sigma^{n-4k})$. The vertices of $L(\sigma^{n-4k})$ correspond to the σ^{n-4k+1} containing σ^{n-4k}, and $L(\sigma^{n-4k+1})$ in X^n is isometric to $L(\tau^0)$ in $L(\sigma^{n-4k})$. Let $e_v(L(\sigma^{n-4k+1}))$ be the contribution to $\dot{\eta}(L(\sigma^{n-4k}))$ for some fixed $\sigma^{n-4k}\subset\sigma^{n-4k+1}$. Then it is easy to see that the orientations are such that in fact

$$(9.11)\qquad c_{n-4k,1}-c_{n-4k,0}=\partial\sum_{\sigma^{n-4k+1}}\int_0^1 e_v(\sigma^{n-4k+1})\,dv\cdot\sigma^{n-4k+1}.$$

Before proceeding to derive the explicit formula for e_v in (9.11), we wish to generalize the above discussion to allow for coefficients in a nontrivial riemannian vector bundle. In order to avoid the necessity of further generalizing our analytic arguments, we restrict attention to connections which are *partially flat* in the sense which we now describe. Given a vector bundle F^m over X^n, we begin by choosing a decomposition of X^n as described after (7.27). Choose an orthonormal basis for the fibre over each σ^0, and extend it to D^n by using radial projection. Next extend the induced trivial connection on $F^m|(\cup D^n)\cap(\cup\sigma^1)$ to a smooth connection over $\cup\sigma^1$. Then extend to the $\mathfrak{U}^1\times D^{n-1}$ by using radial projection normal to the σ^1 onto the $\mathfrak{U}^1$. Since the various projections are compatible, by proceeding inductively in this fashion we construct a "partially flat connection" θ on F^m over X^n. If X^n is embedded in some $\mathbf{R}^N$, it is easy to extend θ to a smooth connection on a smooth regular neighborhood of X^n by a similar construction. Thus it follows that the cochain which assigns to each σ^{2i}, the integral of the corresponding characteristic form over it, represents the ith Chern class of E^m.

Now consider the Laplacian on forms with coefficients in F^m relative to the connection θ. By the construction of θ, it follows that on each $U_\alpha^i\times D_\alpha^{n-i}$ we have a parametrix for the heat kernel $E^x=E_\theta^i\times\mathfrak{S}^{n-i}$. Thus by using the standard results on the heat equation in the smooth case (see [27]) we get

Theorem 9.6. *If X^{4l}, F^m are as above, then the signature* $\mathrm{sig}_{F_\theta}(X^{4l})$ *is defined and is given by*

$$\mathrm{sig}_{F_\theta}(X^{4l}) = \mathrm{ch}(F)[L(X^{4l})]. \tag{9.12}$$

Also note that we can extend the above discussion to the case of $*$-invariant boundary conditions as in §6. Consider first a closed pseudomanifold X^3 of dimension 3. The links of vertices in X^3 are pseudomanifolds of dimension 2. Since each $L(\sigma^0)$ has signature 0, we can locally choose $*$-invariant boundary conditions. In the case of general X^n one can continue the above construction by induction (starting with the links of codim-3 simplices), provided that the L^2-signature of the links of the σ^{n-4k-1} are zero at each stage. In this way we arrive at a general class of pseudomanifolds for which the signature with coefficients in F^m is defined and is given by Theorem 9.6.

The class of pseudomanifolds so defined was considered independently by Morgan [40] who showed that the bordism theory determined by using these spaces as cycles is a geometric representation for K-homology away from the prime 2. However, in this work, the pairing between a cycle and a vector bundle was not given explicitly. The result of Theorem 9.6 shows that this pairing arises naturally. Moreover, suppose that $\partial Y^{4l+1} = X^{4l}$ where Y^{4l+1}, X^{4l} are pseudomanifolds of the above type. Choose a piecewise flat metric on Y^{4l+1} which is a product near X^{4l}. If $c_1(Y^{4l+1})$ is defined as above, but using only those 1-simplices which are interior to Y^{4l+1}, then as in the proof of Theorem 9.5, we see that $\partial c_1(Y^{4l+1}) = c_0(X^{4l})$. This immediately implies

Theorem 9.7 (*Cobordism invariance*). *If $\partial Y^{4l+1} = X^{4l}$, and F extends over Y^{4l+1}, then* $\mathrm{sig}_{F_\theta}(X^{4l}) = 0$.

The previous discussion of η-invariants and the proofs which follow generalize immediately to the case of coefficients in a flat orthogonal or unitary bundle F^k. Correspondingly, there is a generalization of the notion of partially flat connection in which the connection is only assumed to be *locally* flat over the normal cones; the corresponding bundles are not defined over the whole space in general.

We now derive the formula for the derivative of the η-invariant under change of metric for spaces of piecewise constant curvature, the form of which was required in Theorem 9.5. For simplicity we will treat only the case of a pseudomanifold L^{4k-1} with negligible boundary. The general case of $*$-invariant ideal boundary conditions is similar but technically slightly more complicated.

Consider a pseudomanifold X^n with a fixed triangulation. We can specify a one-parameter family of piecewise flat spaces X^n_v with metric g_v and a fixed underlying combinatorial structure by picking smooth positive functions–$\{l_\xi(V)\}$ for the edge lengths of the 1-simplices $\{\sigma_\alpha^1\}$ (provided the $l_\alpha(v)$ satisfy all relevant triangle inequalities; see [14]). If $I: X_0^n \to X_v^n$ is induced by the identity map on X^n, then $I^*(g_v)$ is a one-parameter family of piecewise flat metrics on X_0^n. However this parameterization is not convenient, because it does not respect the local product structures $\mathfrak{U}_\alpha^i \times C(L(\sigma^i))$. Our first goal is the observation that there does exist a smooth family of maps $f_v: X_0 \to X_v$ such that the family $f_v^*(g_v)$ preserves the local product structures and radial coordinates in $C(L(\sigma_\alpha^i))$. By considering the points at small distance from vertex of X^n (and rescaling) we obtain a corresponding reparameterization for variations L_v^{n-1} of piecewise constant curvature $\equiv 1$ metrics.

Lemma 9.8. *There exists $f_v: X_0^n \to X_v^n$ such that f is continuous in x and v, and the restriction of $f_v(x)$ to each open i-simplex $\sigma_{\alpha,0}^i$ is smooth in x, and v. $f_v: \sigma_{\alpha,0}^i \to \sigma_{\alpha,v}^i$ is a diffeomorphism. Moreover, there exists $\epsilon_i > 0$ and maps*

$$g_{\alpha,v}^i : \sigma_{\alpha,0}^i \to \sigma_{\alpha,v}^i,$$
$$h_{\alpha,v}^i : C_{0,\epsilon_i}\big(L(\sigma_{\alpha,v}^i)\big) \to C_{0,\epsilon_i}\big(L(\dot\sigma_{\alpha,v}^i)\big)$$

such that if $x \in \sigma_{\alpha,0}^i$, then on some neighborhood

$$W_{\alpha,0}^i \times C_{0,\bar\epsilon}\big(L(\sigma_{\alpha,0}^i)\big), \quad f_v = g_{\alpha,v}^i \times h_{\alpha,v}^i. \tag{9.13}$$

Finally, $h_{\alpha,v}^i$ preserves the radial coordinate.

Proof. Pick a sufficiently small sequence $0 < \epsilon_{n-1} < \cdots < \epsilon_0$ and contradictable open sets $\mathfrak{U}_{\alpha,v}^i \subset \sigma_{\alpha,v}^i$ such that the sets

$$\mathfrak{V}_{\alpha,v}^i = \mathfrak{U}_{\alpha,v}^i \times \overline{C_{0,\epsilon_i}\big(L(\sigma_{\alpha,v}^i)\big)} \tag{9.14}$$

cover X_v^n, and for $i \leq k$,

$$\mathfrak{V}_{\alpha,v}^i \cap \mathfrak{V}_{\beta,v}^i = \varnothing, \tag{9.15}$$

unless $\sigma_{\alpha,v}^i$ is a face of $\sigma_{\beta,v}^k$. Define f_v on $\{\sigma_{\alpha,0}^0\}$ by

$$f_v\big(\sigma_{\alpha,0}^0\big) = \sigma_{\alpha,v}^0. \tag{9.16}$$

Assume by induction that f_v has been extended to the closed j-skeleton such that the conditions of the lemma are satisfied on the sets $\mathfrak{V}_{\alpha,0}^i \cap \sigma_{\beta,0}^i$, $(\sigma_\alpha^i \subset \sigma_\beta^j)$ with $\epsilon = \epsilon_i$ for $x \in \mathfrak{U}_{\alpha,0}^i$. We claim that f_v can be extended to the $(j+1)$-skeleton so that the induction hypothesis continues to hold. This is proved by descending induction starting with $\{\mathfrak{V}_{\alpha,0}^j \cap \sigma_{\beta,0}^{j+1}\}$ and working downward to $\{\mathfrak{V}_{\gamma,0}^0 \cap \sigma_{\beta,0}^{j+1}\}$. For $\{\mathfrak{V}_{\alpha,0}^j \cap \sigma_{\beta,0}^{j+1}\}$, the definition is forced by the condition

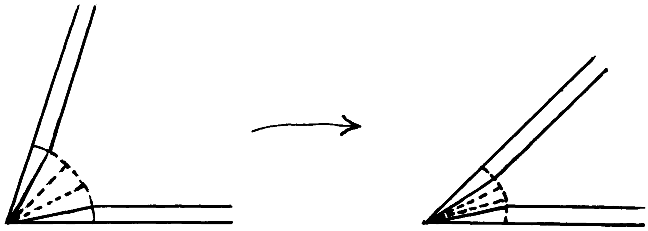

FIG. 9.1

that $h^j_{\alpha,v}$ preserves radial coordinates. Suppose f_v has been extended to $\{\mathcal{V}^i_{\alpha,0} \cap \sigma^{j+1}_{\beta,0}\}$ such that the induction hypothesis holds. Then it is easy to see that it also holds on

$$\left\{\mathcal{V}^l_{\alpha,0} \cap \sigma^{j+1}_{\beta,0}\right\} \cap \left\{\mathcal{V}^i_{\gamma,0} \cap \sigma^{j+1}_{\beta,0}\right\}, \quad \text{for } l < i.$$

This is a direct consequence of the following obvious fact. Let $\mathbf{R}^{i_1} \subset \mathbf{R}^{i_2} \subset \mathbf{R}^{i_3}$ and let $p_3 \notin \mathbf{R}^{i_2}$. Let $p_{i_2} \in \mathbf{R}^{i_2}$ be the point closed to p_3. Assume $p_2 \notin \mathbf{R}^{i_1}$ and let $p_1 \in \mathbf{R}^{i_1}$ be closest to p_2. Then p_1 is the point in $\mathbf{R}^{i_1}$ closest to p_3, and $\overline{p_3, p_1}^2 = \overline{p_3, p_2}^2 + \overline{p_2, p_1}^2$. Thus if $\pi^{i-1}_{\alpha,0} : \mathcal{V}^{i-1}_{\alpha,0} \to \mathcal{U}^{i-1}_{\alpha,0}$ is the projection, the extension of f_v to

$$\left(\pi^{i-1}_{\alpha,0}\right)^{-1}\left[\left\{\mathcal{V}^{i-1}_{\alpha,0} \cap \sigma^{j+1}_{\beta,0}\right\} \cap \left\{\mathcal{V}^i_{\gamma,0} \cap \sigma^{j+1}_{\beta,0}\right\}\right]$$

is determined by the condition that f_v preserves fibres and radial coordinates. Clearly we can pick an extension to all of $\{\mathcal{V}^{i-1}_{\alpha,0}\}$ so that this continues to hold. After completing the second induction, we pick any continuous extension which is smooth in x, v and such that $f_v(x)$ is a diffeomorphism on that part of $\{\mathcal{U}^{j+1}_{\alpha,0}\}$ on which it is yet to be defined. This completes the first induction.

From now on we will assume that variations are parametrized as in Lemma 9.8, and simply write g_v for $f^*_v(g_v)$. Let $*_v$ denote the $*$ operator corresponding to g_v and in general let a dot denote differentiation with respect to v. A basic consequence of the above construction, which is clear by inspection (and induction), is

Corollary 9.9. *For a variation X^n_v or L^{n-1}_v with g_v as above, the operator $\dot{*}$ is uniformly bounded in pointwise norm. In particular $\dot{*} : \Lambda^i \cap L^2 \to \Lambda^{n-i} \cap L^2$ is a bounded operator.*

The proof of Theorem 9.5 now follows easily from the following three propositions. Objects on L^{4k-1}_v will be denoted by a tilda.

Proposition 9.10.

$$\operatorname{tr}\left(\tilde{*} d \frac{e^{-\sqrt{\tilde{\Delta}}\,t}}{\sqrt{\tilde{\Delta}}}\right)^{\bullet} = (-t)\operatorname{tr}\left(\dot{\tilde{*}} d e^{-\sqrt{\tilde{\Delta}}\,t}\right). \tag{9.17}$$

Proposition 9.11. *As $t \to 0$, for all N*

$$\operatorname{tr}\left(\tilde{*}d\frac{e^{-\sqrt{\tilde{\Delta}}t}}{\sqrt{\tilde{\Delta}}}\right) \sim b_0 + O(t^N). \tag{9.18}$$

Proposition 9.12. *As $t \to 0$,*

$$\operatorname{tr}\left(\tilde{*}e^{-\sqrt{\tilde{\Delta}}t}\right) \sim c_{-1}t^{-1} + O(t^{-1/2}), \tag{9.19}$$

and c_{-1} is locally computable at the vertices of L^{4k-1}.

Completion of the proof of Theorem 9.5.

For $s > 0$, by Proposition 9.11 we have

$$\begin{aligned}\eta(s) &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\operatorname{tr}\left(\tilde{*}d\frac{e^{-\sqrt{\tilde{\Delta}}t}}{\sqrt{\tilde{\Delta}}}\right)dt\\ &= \frac{1}{\Gamma(s)}\left\{\int_1^\infty t^{s-1}\operatorname{tr}\left(\tilde{*}d\frac{e^{-\sqrt{\tilde{\Delta}}t}}{\sqrt{\tilde{\Delta}}}\right)dt\right.\\ &\qquad\left.+\int_0^1 t^{s-1}\left[\operatorname{tr}\left(\tilde{*}d\frac{e^{-\sqrt{\tilde{\Delta}}t}}{\sqrt{\tilde{\Delta}}}\right)-b_0\right]dt+\frac{b_0}{s}\right\}.\end{aligned} \tag{9.20}$$

Thus $\eta(0)$, the analytic continuation of $\eta(s)$ to $s = 0$ is given by

$$\eta(0) = b_0. \tag{9.21}$$

Then by Propositions 9.10, 9.11,

$$\begin{aligned}\eta_1(0) - \eta_0(0) &= b_{0,1} - b_{0,0}\\ &= \lim_{t\to 0}\left\{\operatorname{tr}\left(\tilde{*}_1 d\frac{e^{-\sqrt{\tilde{\Delta}_1}t}}{\sqrt{\tilde{\Delta}_1}}\right) - \operatorname{tr}\left(\tilde{*}_0 d\frac{e^{-\sqrt{\tilde{\Delta}_0}t}}{\sqrt{\tilde{\Delta}_0}}\right)\right\}\\ &= \lim_{t\to 0}\int_0^1 \operatorname{tr}\left(\dot{*}d\frac{e^{-\sqrt{\tilde{\Delta}_v}t}}{\sqrt{\tilde{\Delta}_v}}\right)dv = \int_0^1 c_{-1,v}\,dv.\end{aligned} \tag{9.22}$$

It remains to establish Propositions 9.10–9.12. Propositions 9.11 and 9.12 are proved by "the method of descent." At the core of this method is the metric identity $R \times C(X) = C(S(X))$ where $S(X)$, the *suspension* of X, is the space $(0, \pi) \times X$ with metric $d\theta^2 + \sin^2\theta g$. The proof of Proposition 9.10

follows from two lemmas which are quite general. Before proving these, we give as motivation a formal argument based on "first order perturbation theory".

Let ϕ be an eigenform of $*\,d$ with $\|\phi\| = 1$. If

$$\xi\phi = *d\phi, \tag{9.23}$$

then

$$\dot{\xi}\phi + \xi\dot{\phi} = \dot{*}d\phi + *d\dot{\phi}. \tag{9.24}$$

Taking inner products with ϕ and using

$$\langle *d\dot{\phi}, \phi\rangle = \langle \dot{\phi}, \delta * \phi\rangle = \xi\langle \dot{\phi}, \phi\rangle, \tag{9.25}$$

we get

$$\dot{\xi} = \langle \dot{*}d\phi, \phi\rangle. \tag{9.26}$$

Also, if $\tilde{\Delta}\phi = \mu\phi$, $\mu = \xi^2$, then

$$\dot{\mu} = 2\xi\dot{\xi}. \tag{9.27}$$

Thus

$$\frac{e^{-\sqrt{\mu}\,t}}{\sqrt{\mu}}\xi = \int_t^\infty e^{-\sqrt{\mu}\,s}\xi\, ds, \tag{9.28}$$

$$\begin{aligned}\left(\frac{e^{-\sqrt{\mu}\,t}}{\sqrt{\mu}}\xi\right)^{\cdot} &= \int_t^\infty - s\frac{1}{2\sqrt{\mu}}2\xi\dot{\xi}e^{-\sqrt{\mu}\,s}\xi\, ds + \int_t^\infty e^{-\sqrt{\mu}\,s}\dot{\xi}\, ds \\ &= \int_t^\infty - s\sqrt{\mu}\, e^{-\sqrt{\mu}\,s}\dot{\xi}\, ds + \int_t^\infty e^{-\sqrt{\mu}\,s}\dot{\xi}\, ds.\end{aligned} \tag{9.29}$$

Integrating by parts and using (9.26), we obtain

$$-te^{-\sqrt{\mu}\,t}\dot{\xi} = -te^{-\sqrt{\mu}\,t}\langle \dot{*}d\phi, \phi\rangle, \tag{9.30}$$

which implies

$$\operatorname{tr}\left(*d\frac{e^{-\sqrt{\Delta}\,t}}{\sqrt{\Delta}}\right)^{\cdot} = (-t)\operatorname{tr}\left(\dot{*}de^{-\sqrt{\Delta}\,t}\right). \tag{9.31}$$

In order to make the above argument rigorous, we begin with a formula for the variation of the Green's operator on i-forms of L^{4k-1}. The essential components of the Green's operator which change with v are the projection operators on the spaces of closed and exact forms. We will write these as $\mathcal{P}$ and P respectively, with the dependence on v understood. For operators corresponding to $v = 0$, we will add the subscript 0. We have

$$\mathcal{P} = P + H, \tag{9.32}$$

where H denotes orthogonal projection on the space of coclosed forms. Write

$$\langle \phi, \psi \rangle_v = \langle *_0^{-1} *_v \phi, \psi \rangle = \langle A\phi, \psi \rangle, \tag{9.33}$$

where A $(= A_v)$ is a smooth family of bounded selfadjoint operators on $\Lambda^i \cap L^2$. Let

$$\dot{A}_0 = \alpha, \tag{9.34}$$

where α is a bounded operator by Corollary 9.9.

Lemma 9.13. *Let g_v be a 1-parameter family of metrics on a riemannian manifold such that the operators d_0^{-1}, δ_0^{-1} are bounded on i-forms and such that $*_0^{-1} *_v$ is a smooth family of bounded operators. Then the Green's operators G_v form a smooth family of bounded operators, and at $v = 0$,*

$$\dot{G} = G\alpha(I - P) - P\alpha G + \delta^{-1} d^{-1} - d^{-1}\alpha\delta^{-1} \tag{9.35}$$
$$- H\alpha d^{-1}\delta^{-1} - d^{-1}\delta^{-1}\alpha H,$$

$$\dot{H} = H\alpha - H\alpha\mathcal{P} - P\alpha H. \tag{9.36}$$

Proof. It is easy to see that

$$\mathcal{P} = (\mathcal{P}_0 A \mathcal{P}_0)^{-1} A, \qquad P = (P_0 A P_0)^{-1} A, \tag{9.37}$$

where the notation means that $(\mathcal{P}_0 A \mathcal{P}_0)^{-1}$, $(P_0 A P_0)^{-1}$ vanish on the coexact (respectively coclosed) forms for g_0, and

$$(\mathcal{P}_0 A \mathcal{P}_0)^{-1}(\mathcal{P}_0 A \mathcal{P}_0) = \mathcal{P}_0, \qquad (P_0 A P_0)^{-1}(P_0 A P_0) = P_0. \tag{9.38}$$

In fact, for the metric g_v the coexact and coclosed spaces are obtained by applying A^{-1} to the corresponding spaces for g_0. Thus the operators in (9.37) clearly vanish on these spaces, Also, if ϕ is say closed, then

$$(\mathcal{P}_0 A \mathcal{P}_0)^{-1} A = (\mathcal{P}_0 A \mathcal{P}_0)^{-1} A \mathcal{P}_0 \phi$$
$$= (\mathcal{P}_0 A \mathcal{P}_0)^{-1} \mathcal{P}_0 A \mathcal{P}_0 \phi = \mathcal{P}_0 \phi = \phi, \tag{9.39}$$

where the second equality follows since $(\mathcal{P}_0 A \mathcal{P}_0)^{-1} \,|\, \ker \mathcal{P}_0 \equiv 0$.

We have

$$\left((\mathcal{P}_0 A \mathcal{P}_0)^{-1}\right)^{\cdot} = -\mathcal{P}_0 \alpha \mathcal{P}_0, \qquad \left((P_0 A P_0)^{-1}\right)^{\cdot} = -P_0 \alpha P_0. \tag{9.40}$$

Thus by (9.37),

$$\dot{\mathcal{P}} = \mathcal{P}_0 \alpha - \mathcal{P}_0 \alpha \mathcal{P}_0, \qquad \dot{P} = P_0 \alpha - P_0 \alpha P_0. \tag{9.41}$$

Since $\dot{H} = \dot{\mathcal{P}} - \dot{P}$, (9.36) follows easily from (9.41).

As usual let d^{-1} denote the operator which assigns the unique coexact form β to an exact form ρ such that $d\beta = \rho$. Then

$$d^{-1} = (I - \mathcal{P})d_0^{-1}P, \tag{9.42}$$

$$\begin{aligned}(d^{-1})^{\cdot} &= -(\mathcal{P}_0\alpha - \mathcal{P}_0\alpha\mathcal{P}_0)d_0^{-1} + d_0^{-1}(P_0\alpha - P_0\alpha P_0),\\ &= d_0^{-1}\alpha - \mathcal{P}_0\alpha d_0^{-1} - d_0^{-1}\alpha P_0,\end{aligned} \tag{9.43}$$

where we have used $\mathcal{P}_0 d_0^{-1} = 0$, $d_0^{-1}P_0 = d_0^{-1}$. To derive the corresponding formula for $(\delta_0^{-1})^{\cdot}$, note that

$$\delta^{-1} = (d^{-1})*_v = A^{-1}(d^{-1})*_0 A, \tag{9.44}$$

where $*_v$, $*_0$ denote adjoints with respect to g_v, g_0. Thus

$$\begin{aligned}(\delta^{-1})^{\cdot} &= -\alpha\delta_0^{-1} + \left((d^{-1})^{\cdot}\right)^{*0} + \delta_0^{-1}\alpha\\ &= -\alpha\delta_0^{-1} + \left(\alpha\delta_0^{-1} - \delta_0^{-1}\alpha\mathcal{P}_0 - P_0\alpha\delta_0^{-1}\right) + \delta_0^{-1}\alpha\\ &= \delta_0^{-1}\alpha - \mathcal{P}_0^{-1}\alpha\mathcal{P}_0 - P_0\alpha\delta_0^{-1}.\end{aligned} \tag{9.45}$$

By combining (9.43), (9.45), we can derive the formula for the variation of the Green's operator. We have

$$G = \delta^{-1}d^{-1} + d^{-1}\delta^{-1}, \tag{9.46}$$

$$\begin{aligned}G &= (\delta^{-1})^{\cdot}d_0 + \delta_0^{-1}(d^{-1})^{\cdot} + (d_0^{-1})^{\cdot}\delta_0^{-1} - d_0^{-1}(\delta_0^{-1})^{\cdot}\\ &= \left[\delta_0^{-1}\alpha - \delta_0^{-1}\alpha\mathcal{P}_0 - P_0\alpha\delta_0^{-1}\right]d_0^{-1} + \delta_0^{-1}\left[d_0^{-1}\alpha - \mathcal{P}_0\alpha d_0^{-1} - d_0^{-1}\alpha P_0\right]\\ &\quad + \left[d_0^{-1}\alpha - \mathcal{P}_0\alpha d_0^{-1} - d_0^{-1}\alpha P_0\right]\delta_0^{-1} + d_0^{-1}\left[\delta_0^{-1}\alpha - \delta_0^{-1}\alpha\mathcal{P}_0 - P_0\alpha\delta_0^{-1}\right].\end{aligned} \tag{9.47}$$

Simplifying (9.47) gives (9.35).

Lemma 9.14. *Let C_v be a smooth 1-parameter family of compact operators with real spectrum, and let B_v be a smooth family of bounded operators such that $C_vB_v = B_vC_v$. Let $\{\lambda_{i,v}\}$ be the spectrum of C_v, and let f be a smooth function on a set containing the spectra of all the C_v such that the series*

$$\sum f(\lambda_{i,v}), \qquad \sum f'(\lambda_{i,v}) \tag{9.48}$$

are uniformly absolutely convergent with respect to v. Then $B_vf(C_v)$ is a smooth family of trace class operators, and

$$\operatorname{tr}(Bf(C)) = \operatorname{tr}(\dot{B}f(C)) + \operatorname{tr}(Bf'(C)\dot{C}). \tag{9.49}$$

Proof. This follows from the corresponding formula in the finite dimensional case by a standard approximation argument.

Proof of Proposition 9.10. Write

$$* de^{-\sqrt{\Delta}t} = * \delta^{-1}(\Delta + H)^{1/2}e^{-\sqrt{\Delta+H}t} \tag{9.50}$$

(where we omit the tilda's), and apply Lemma 9.13 with $B_v = * \delta^{-1}$, $C_v = G + H$. Using Lemma 9.12 and (9.45) we get

$$\operatorname{tr}\left(\dot{*}\, \delta^{-1}(\Delta + H)e^{-\sqrt{\Delta+H}t}\right) + \operatorname{tr}\left[*\,(\delta^{-1}\alpha - \delta^{-1}\alpha\mathcal{P} - P\alpha\delta^{-1})(\Delta + H)e^{-\sqrt{\Delta+H}t}\right]$$

$$(9.51) \qquad + \operatorname{tr}\left[*\,\delta^{-1}\left\{-(\Delta + H)^2 e^{-\sqrt{\Delta+H}t} + \frac{t}{2}(\Delta + H)^{5/2}e^{-\sqrt{\Delta+H}t}\right\}\right.$$

$$\times\{G\alpha(I - P) - P\alpha G + \delta^{-1}\alpha d^{-1} - d^{-1}\alpha\delta^{-1}$$

$$\left.-H\alpha d^{-1}\delta^{-1} - d^{-1}\delta^{-1}\alpha H + H\alpha - H\alpha\mathcal{P} - P\alpha H\}\right].$$

The first term (in 9.51) is equal to

$$(9.52) \qquad \operatorname{tr}\left(\dot{*}e^{-\sqrt{\Delta}t}\right).$$

Since

$$(9.53) \qquad \delta^{-1}H = d^{-1}H = 0,$$

$\operatorname{tr}(D_1D_2) = \operatorname{tr}(D_2D_1)$ if D_1 is bounded and D_2 is trace class, and $\alpha = \dot{*} = -\dot{*}*$, the second term equals

$$(9.54) \quad \left(-\dot{*}*(*\delta^{-1}) - \dot{*}*\mathcal{P}*^{-1} - \dot{*}\delta^{-1}*P*\right)e^{-\sqrt{\Delta+H}t}\Big) = -2\operatorname{tr}\left(\dot{*}de^{-\sqrt{\Delta}t}\right),$$

where we have used (9.53) and

$$(9.55) \qquad \mathcal{P}*\delta^{-1} = P * \delta^{-1} = 0.$$

Similar manipulations show that all terms involving H in the third piece of (9.51) drop out, and this piece becomes

$$(9.56) \qquad 2\operatorname{tr}\left(\dot{*}de^{-\sqrt{\Delta}t}\right) - t\operatorname{tr}\left(\dot{*}d\sqrt{\Delta}\, e^{-\sqrt{\Delta}t}\right).$$

The proposition now follows by arguing as in (9.29).

Proof of Proposition 9.11. Let L^{4k-1} have piecewise constant curvature $\equiv 1$. Consider the cone $C(L^{4k-1})$, and let s denote the radial variable. The Green's operator $G(s_1, x_1, s_2, x_2)$ on $(2k - 1)$-forms of $C(L^{4k-1})$ has a coexact type 1 piece given by

$$(9.57) \qquad \sum \frac{s_1^{a^+} s_2^{a^-}}{2\nu}\phi \otimes \phi,$$

together with log terms corresponding to $\mu = 0$; see [8]. Since

$$(9.58) \qquad (2k - 1) = \tfrac{1}{2}[1 + 2(2k - 1) - (4k - 1)] = 0,$$

applying d_1, d_2 to (9.57) and evaluating at $(s_1, x_1, 1, x_2)$ give

$$\sum \frac{s_1^{\sqrt{\mu}}}{2}\left(\frac{d\phi}{\sqrt{\mu}} + s_1^{-1}ds_1 \wedge \phi\right) \otimes \left(d\phi + \sqrt{\mu}\, ds_2 \wedge \phi\right), \tag{9.59}$$

(where $d\phi$, $d\phi/\sqrt{\mu}$ are absent if $\mu = 0$). This is closely related to the kernel $*de^{-\sqrt{\tilde{\Delta}}t}$ on L^{4k-1} in which we are interested, via the substitution $s_1 = e^{-t}$.

Next observe that the expression in (9.59) can be obtained by applying $d_1 d_2$ to the full Green's operator G, and not just the type 1 piece. To see this, first note that G also contains an exact type 4 piece which is killed by applying $d_1 d_2$. Moreover, one can check that in general the remaining piece of G is given by

$$\begin{aligned} &\sum \frac{1}{4\nu(\nu+1)a^-}\left[s_1^{a^+ +2}d\phi + a^- s_1^{a^+ +1}ds_1 \wedge \phi\right] \\ &\qquad\qquad \otimes\left[s_2^{a^-}d\phi + a^- s_2^{a^- -1}ds_2 \wedge \phi\right] \\ &+\sum \frac{1}{4\nu(\nu-1)a^+}\left[s_1^{a^+}d\phi + a^+ s_1^{a^+ -1}ds_1 \wedge \phi\right] \\ &\qquad\qquad \otimes\left[s_2^{a^- +2}d\phi + a^+ s_2^{a^- +1}ds_2 \wedge \phi\right]. \end{aligned} \tag{9.60}$$

Since each term contains an exact form on either the right or the left, the expression in (9.60) is also killed by applying $d_1 d_2$.

Let $T_{s_1}^*$ map the cotangent space (s_1, x) to the cotangent space at $(1, x)$, and be given by parallel translation along the radial segment from (s_1, x) to $(1, x)$ followed by multiplication by s_1^{2k}. Then letting tr denote the pointwise trace at $(1, x)$, we have

$$\operatorname{tr}\left(*_1{}_1 T_{s_1}^* d_1 d_2(G)\right) = \sum s_1^{\sqrt{\mu}} ds_1 \wedge d\phi \wedge \phi. \tag{9.61}$$

For the remainder of the calculation, it will suffice to assume that $(1, x)$ lies in a small neighborhood of a vertex τ^0 of L^{4k-1}, where L^{4k-1} is identified with $(1, L^{4k-1}) \subset C(L^{4k-1})$. In any case, this can be achieved by picking an appropriate subdivision of L^{4k-1}. Let $E(t)$ be the heat kernel on $(2k-1)$-forms of $C(L^{4k-1})$. Then from the relation

$$G = \int_0^\infty E(t)\, dt, \tag{9.62}$$

and an argument like that of Theorem 4.1 if there exist small eigenvalues of $\tilde{\Delta}$, it follows easily that as $s_1 \to 1$,

$$\int_{D^{4k-1}} \operatorname{tr}\left(*_1 T_{s_1}^* d_1 d_2 G\right) \sim \int_{D^{4k-1}} \int_0^1 \operatorname{tr}\left(*T_{s_1}^* d_1 d_2 E(t)\right) dt + \text{const} + o(1), \tag{9.63}$$

where D^{4k-1} is the "spherical dual cell" (with totally geodesic faces) dual to τ^0. If $\mathfrak{S}(t)$ denotes the heat kernel of the normal cone to the ray of $C(L^{4k-1})$ through $(1, \tau^0)$, then we have a local parametrix $E^x(t)$ for the heat kernel of $C(L^{4k-1})$, of the form

$$(9.64)\qquad \frac{e^{-(u_1-u_2)^2/4t}}{(4\pi t)^{1/2}}\mathfrak{S}_{2k-1}(t) + \frac{e^{-(u_1-u_2)^2/4t}}{(4\pi t)^{1/2}} du_1 \otimes du_2 \mathfrak{S}_{2k-2}(t).$$

Clearly, the second term will be killed by applying $\operatorname{tr} *_1 {}_1T_s^*$, since $du \wedge du = 0$. Similarly, after applying $d_1 d_2$ to the first term, we need only consider

$$(9.65)\qquad \frac{e^{-(u_1-u_2)^2/4t}}{(4\pi t)^{1/2}}\left[\frac{(u_2-u_1)}{2t} du_1 \tilde{d}_2 \mathfrak{S}_{2k-1}(t) + \frac{(u_1-u_2)}{2t} du_2 \tilde{d}_1 \mathfrak{S}_{2k-1}(t)\right].$$

After applying $\operatorname{tr} *_1 {}_1T_s^*$ we see that the two terms in (9.65) cancel due to the opposing signs of the factors $(u_2 - u_1)$, $(u_1 - u_2)$. Thus

$$(9.66)\qquad \lim_{s_1 \to 1} \int_{D^{4k-1}} \sum s_1^{\sqrt{\mu}} d\phi \wedge \phi \sim \text{const} + o(1),$$

and so, setting $s_1 = e^{-t}$, as $t \to 0$

$$(9.67)\qquad \operatorname{tr}\left(*de^{-\sqrt{\Delta}\, t}\right) \sim \text{const}.$$

Finally

$$(9.68)\qquad \operatorname{tr}\left(*d\frac{e^{-\sqrt{\Delta}\, t}}{\sqrt{\Delta}}\right) = \int_t^\infty \operatorname{tr}\left(*de^{-\sqrt{\Delta}\, w}\right) dw \sim b_0 + o(1).$$

Note that since, as we observed earlier in this section, $\eta(s)$ is holomorphic (and vanishes at the poles of $\Gamma(s)$), by taking the Mellin transform of $\operatorname{tr}(*de^{-\sqrt{\Delta}\, t}/\sqrt{\Delta})$ and arguing as in §4 if necessary, it follows that in fact

$$(9.69)\qquad \operatorname{tr}\left(*d\frac{e^{-\sqrt{\Delta}\, t}}{\sqrt{\Delta}}\right) \sim b_0 + O(t^N),$$

for all N, which completes the proof.

Proof of Proposition 9.12. Consider a variation L_v^{4k-1}, and let s be the radial coordinate of the associated variation $C(L_v^{4k-1})$ which preserves ds. Moreover,

$$(9.70)\qquad *_1 \dot{*}_1 i_{ds_1} ds_1 \wedge {}_1T_{s_1}^* d_1 d_2 G = \frac{s^{\sqrt{\mu}}}{2\sqrt{\mu}} *_1 \dot{*} d\phi \otimes \left(d\phi + \sqrt{\mu}\, ds_2 \wedge \phi\right),$$

where i_{ds_1} denotes interior product with ds_1.

Thus

$$\mathrm{tr}\left(*i_{ds_1}ds_1 \wedge {}_1T^*_{s_1}d_1d_2G\right) = \sum \frac{s^{\sqrt{\mu}}}{2} ds \wedge \tilde{*}\,\mathring{*}d\phi \wedge \phi. \tag{9.71}$$

Let $(1, x)$, where $x = x_1 = x_2$, be a point which lies in a neighborhood $\mathcal{U}^i_{\alpha,0} \times C(L(\sigma^i_{\alpha,0}))$. The variation on $C(L^{4k-1})$ preserves the product structure and the radial coordinate in each factor. At the point $(1, x)$ we introduce rectangular coordinates (u, w) in $\mathcal{U}^i_{\alpha,0}$ with $(1, x)$ at the origin, u in the radial direction of $C(L(\sigma^i_{\alpha,w}))$, and $w = (w_2, \cdots, w_i)$. Then at this point,

$$\mathring{*}\left(\gamma_j + du \wedge \psi_{j-1}\right) = (-1)^j du \wedge \mathring{*}\gamma_j + \mathring{*}\psi_{j-1}. \tag{9.72}$$

Moreover, if the coordinates of $(1, x)$ are $(0, 0, r_2)$, then the coordinates of (s_1, x) are $(1 - s_1, 0, s_1 r_2)$.

Near $(0, 0, r_1)$ the heat kernel of $C(L_0^{4k-1})$ has a parametrix of the form

$$\sum_j \frac{e^{-(u_1-u_2)^2/4t}}{(4\pi t)^{1/2}} E_j(t)\mathfrak{S}_{2k-j-1}(t)$$

$$+ \sum_j du_1 \otimes du_2 \frac{e^{-(u_1-u_2)^2/4t}}{(4\pi t)^{1/2}} E_j(t)\mathfrak{S}_{2k-j-2}(t), \tag{9.73}$$

where $E_j(t)$ is the Euclidean heat kernel in the w direction on j-forms, and $\mathfrak{S}(t)$ is the heat kernel of $C(L(\sigma^i_{\alpha,0}))$.

Let d'_j, d''_j, d'''_j denote d in the u, w and $C(L(\sigma^i_{\alpha,0}))$ directions respectively. Since $**$ preserves the splitting and $ds = du$ at $(0, 0, r_2)$, after applying $*_1 *_1 i_{ds_1} ds_1 \wedge {}_1T^*_{s_1} d_1 d_2$ the only possibilities for a nonzero contribution correspond to

$$d''_1 d'_2, \qquad d'''_1 d'_2, \tag{9.74}$$

acting on the first term in (9.73) with

$$\begin{aligned} 2j + 1 = i - 1, \quad 2(2k - 1 - j) = 4k - i, \\ 2j = i - 1, \quad 2(2k - 1 - j) + 1 = 4k - i, \end{aligned} \tag{9.75}$$

respectively. Suppose $i \geqslant 2$. Then the first possibility gives zero since

$$d''_1 \frac{e^{-\Sigma(w_j^1 - w_j^2)^2/4t}}{(4\pi t)^{i/2-1}} = \sum_l \frac{-\left(w_l^1 - w_l^2\right)}{4t} dw_l^1 \frac{e^{-\Sigma(w_j^1 - w_j^2)/4t}}{(4\pi t)^{i/2-1}}, \tag{9.76}$$

which vanishes when we set $w^1 = w^2 = 0$. The second possibility also gives zero for $i \geqslant 2$. To see this let $*$ denote the $*$ operator on j-forms of the w-factor, $2j = i - 1$. Let $f_1, \cdots, f_{\bar{j}}$ be an orthonormal basis for the j-forms of the w-factor, and let ω be the volume form. The contribution of this factor to the trace is

$$\begin{aligned}(9.77)\qquad \operatorname{tr}(\dot{*}\underline{*})\omega &= \sum_l \langle \underline{*}\dot{*}f_l, f_l\rangle\omega \\ &= \sum_l \langle \dot{*}f_l, \underline{*}f_l\rangle\omega = \sum_l \langle \dot{*}\underline{**}f_l, \underline{*}f_l\rangle\omega \\ &= -\sum_l \langle \underline{*}\dot{*}\underline{*}f_l, \underline{*}f_l\rangle\omega = -\sum_l \langle \underline{*}\dot{*}g_l, g_l\rangle\,\omega,\end{aligned}$$

where g_l is the orthonormal basis $\underline{*}f_l$, and we have used $\dot{*}\underline{*} = -\underline{*}\dot{*}$, which follows from $\underline{*}^2 = I$.

If we now argue as in (9.62), (9.63), we see that for $i \geq 2$, $\operatorname{tr}(\dot{*}de^{-\sqrt{\tilde{\Delta}}t})$ stays finite as $t \to 0$, and, in particular, *any contribution to the coefficient of t^{-1} must be computable near the vertices of L^{4k-1}. This actually suffices to establish Theorem* 9.5.

Now consider the case $i = 1$. Take a local coordinate system as above with the origin at $\tau^0 \subset L^{4k-1}$ where L^{4k-1} is identified with $(1, L^{4k-1}) \subset C(L^{4k-1})$. Set $(1 - u_2) = z$. As in (9.75) we are reduced to consideration of the expression

$$(9.78)\qquad 2(1-z)^{2k}\int_0^\infty \frac{2z}{4t}\frac{e^{-z^2/4t}}{(4\pi t)^{1/2}}dz\int_{C_{0,\epsilon}(L(\tau^0))}\operatorname{tr}\left(\dot{*}_1 d_1 E_{2k-1}(t)\right)dt,$$

where the factor z in front of the expression corresponds to the factor 2 in the denominator of (9.73). If we recall (see [6]) the identity

$$(9.79)\qquad e^{-\sqrt{\Delta_{2k-1}}z} = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{z}{t^{3/2}}e^{-z^2/4t}e^{-\Delta_{2k-1}t}\,dt,$$

where Δ_{2k-1} is the Laplacian on $C(L(\tau^0))$, we see that we have reduced our problem from the curvature 1 case to the curvature 0 case (up to the factor $(1 - z)^{2k}$ which will play no role).

Let $F(r, t)$ represent the pointwise trace of the kernel of $\dot{*}de^{-\sqrt{\Delta_{2k-1}}t}$. Then it is easy to check that $F(r, t)$ satisfies the scaling property

$$(9.80)\qquad F(r,t) = \frac{1}{r^{4k}}F\left(1, \frac{t}{r}\right).$$

Thus

$$\begin{aligned}(9.81)\qquad \int_{C_{(0,\epsilon)}(L(\tau^0))}\operatorname{tr}\left(\dot{*}de^{-\sqrt{\Delta}t}\right) &= \int_0^\epsilon\int_{L(\tau^0)}F(r,t)r^{4k-2}\,dr \\ &= \int_0^\epsilon\int_{L(\tau^0)}F\left(1,\frac{t}{r}\right)dr.\end{aligned}$$

Setting

$$\frac{t}{r} = s, \qquad -\frac{t}{r^2}dr = ds, \tag{9.82}$$

this becomes

$$\frac{1}{t}\int_{\epsilon t}^{\infty}\int_{L(\tau^0)} F(1, s)\, ds. \tag{9.83}$$

Since

$$\lim_{s\to 0}\int_{L(\tau^0)} F(1, s)\, ds = 0 \tag{9.84}$$

(the case $i \geqslant 2$), we see from (9.78), (9.83) that Proposition 9.11 holds with

$$c_{-1} = \sum_{\tau^0 \in L^{4k-1}} \int_0^{\infty}\int_{L(\tau^0)} F(1, s)\, ds. \tag{9.85}$$

This completes the proof of Proposition 9.11, and hence the proof of Theorem 9.4.

In fact, the integral in (9.85) can be calculated as a spectral invariant of $L(\tau^0)$. By arguing as in §§4 and 6 we see that the integral in (9.85) can be identified with the analytic continuation to $s = 1$ of

$$\Gamma\left(\frac{s}{2}\right)\int_{L(\tau^0)} \operatorname{tr}\left(\ast d\Delta_{2k-2}^{-s/2}\right). \tag{9.86}$$

If we apply the Weber-Schaftheitlin formula, it is clear that it is only necessary to consider forms of types 1 and 3. For forms of type 1 we get

$$\begin{aligned} &\Gamma\left(\frac{s}{2}\right)\int_0^{\infty}\lambda^{1-s}\left((-1)^{2k}J_\nu(\lambda)\, dr \wedge \mathring{\ast} d\phi + \lambda J_\nu'(\lambda)\mathring{\ast}\phi\right) \\ &\qquad \wedge \left((-1)^{2k-1}J_\nu(\lambda)\, dr \wedge \mathring{\ast}\phi\right) d\lambda \\ &= \Gamma\left(\frac{s}{2}\right)\int_0^{\infty}\left(\lambda^{2-s}J_\nu'(\lambda)J_\nu(\lambda)\, d\lambda\right)\mathring{\ast}\phi \wedge \ast\phi. \end{aligned} \tag{9.87}$$

As in (6.7)–(6.10), this gives

$$\frac{\Gamma(s/2)}{2^s}\,\frac{\Gamma(s-2)}{\Gamma(s/2)\Gamma(s/2-1)}\,\frac{\Gamma(\nu - s/2 + 1)}{\Gamma(\nu + s/2)}\mathring{\ast}\phi \wedge \tilde{\ast}\phi\Bigg|_{s=1}. \tag{9.88}$$

A simple calculation shows that the contribution from type 3 forms is also given by (9.88). Thus we get a total contribution of

$$\frac{2}{\sqrt{\pi}}\Gamma(s-1)\frac{\Gamma(\nu - s/2 + 1)}{\Gamma(\nu + s/2)}\mathring{\ast}\phi \wedge \tilde{\ast}\phi\Bigg|_{s=1}. \tag{9.89}$$

Now the fact that on $L(\tau^0)$, $\mathrm{tr}(*de^{-\sqrt{\Delta}\,t}) \sim \mathrm{const} + o(1)$ implies that

$\Gamma(\nu - s/2 + 1)/\Gamma(\nu + s/2)$ is holomorphic for $\mathrm{Re}\, s \geqslant 1$ and vanishes at $s = 1$. Thus we get

Theorem 9.15.

$$c_{-1} = \sum_{\tau^0} \frac{2}{\sqrt{\pi}} \left(\mathrm{tr}(*\Delta_{ce}^{-s})\right)'\Big|_{s=0}, \tag{9.90}$$

where Δ_{ce} is the Laplacian on coexact $(2k-1)$-forms of $L(\tau^0)$.

Note that the expression in (9.90) is *not* locally computable on $L(\tau^0)$, due to the fact that we project on coexact forms. Also, performing further differentiations with respect to v does not lead to locally computable expressions. This is an obstacle to making our combinatorial formulas more explicitly computable.

As mentioned in §6 we can also consider the η-invariant for manifolds with isolated conical singularities, or equivalently for manifolds with boundary. A straightforward modification of the arguments leading to (9.90) shows that (9.90) remains valid in this case. This implies a generalization of (6.14) to the case in which N^{4l-1} itself has conical singularities, *the cross-sections of which are spheres*. On the other hand the device of attaching a cone to the boundary can also be used to define the η-invariant for pseudomanifolds with boundary.

We now consider the geometric index formula (6.15) for piecewise flat pseudomanifolds X^{4k} with boundary. First consider the case where the metric is a product near the boundary. Form a closed manifold Y_l^{4k} by adding one new vertex p and an n-simplex containing p for each $(n-1)$-simplex of ∂X^n. Take the lengths of all new 1-simplices to the same constant l. Now apply Theorem 9.5 to Y_l^n and let $l \to \infty$. In the limit the metric approaches a product $R \times g'$ near the new interior vertices which were formerly boundary vertices. So these vertices do not contribute. The link $L_l(p)$ with metric rescaled to have curvature $1/l^2$ approaches "geometrically" ∂X^{4k}. Moreover, by modifying our previous analysis it is not difficult to show that $\eta(\partial X^{4k})$ exists and that

$$\lim_{l\to\infty} \eta(L_l(p)) = \eta(\partial X^{4k}). \tag{9.91}$$

Thus we get

Theorem 9.16. *Let X^{4k} be a piecewise flat pseudomanifold with boundary such that Y_l^{4k} is of the class described after* (9.12). *Let the metric be a product at ∂X^{4k}. Then*

$$\mathrm{sig}(X^{4k}) = \sum_{\sigma^0} \eta(L(\sigma^0)) + \eta(\partial X^{4k}),$$

The case in which the metric is not a product at the boundary can now be handled using (9.90). In particular, the analog of the second fundamental form term in the smooth case is locally computable at the vertices of ∂X^{4k}.

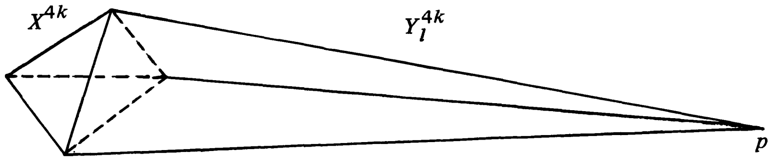

FIG. 9.2

Now suppose L^{4k-2} is the unit sphere. It is easy to check (by a symmetry argument) that the expression in (9.90) vanishes for all variations. We conjecture in fact that the first $(2k-1)$ derivatives with respect to v of the η-invariant vanish for S_1^{4k-1}. This together with an appropriate expression for the $2k$th derivative (compare [14]) would imply via the methods of [14] that in the limit under subdivision of our formula for the top L-class goes over into the dual of corresponding characteristic form in the smooth case. More generally we have

Conjecture 9.1. *Let $X_\epsilon^n \to X^n$ be a sequence of uniformly fat piecewise flat approximations to the smooth compact riemannian manifold X (as in* [13], [14]*). Then for all $\omega^{n-4k} \in \Lambda^{n-4k}(X)$,*

$$\lim_{\epsilon\to 0} \omega^{n-4k}(C_{n-4k,\epsilon}) = \int_X \omega^{n-4k} \wedge L_k(\Omega). \tag{9.92}$$

Conjecture 9.2. *Let $X_\epsilon^{4l-1} \to X^{4l-1}$ as above. Then*

$$\lim_{\epsilon\to 0} \eta_{E^k}(X^{4l-1}) = \eta_{E^k}(X^{4l-1}). \tag{9.93}$$

The fact that the derivative under change of metric of the η-invariant $\eta_{E^k}(X)$ with coefficients in E^k is locally computable at the vertices implies that we have piecewise linear invariants

$$\hat\rho_{E^k}(X) = \eta_{E^k}(X) - k\eta(X) \tag{9.94}$$

corresponding to the smooth invariants $\rho_{E^k}(X)$ of [2]. Conjecture 9.2 implies

Conjecture 9.3. *For X^{4l-1} compact smooth, $\hat\rho_{E^k}(X^{4l-1}) = \rho_{E^k}(X^{4l-1})$. In particular $\rho_{E^k}(X^{4l-1})$ is a piecewise linear invariant. One might further conjecture that $\rho_{E^k}(X^{4l-1})$ is a topological invariant.*

Finally we mention that there does not as yet exist a satisfactory interpretation of the η-invariant for pseudomanifolds in terms of generalized Chern-Simons invariants. In any case one would like to have the nonimmersion theorems for pseudomanifolds which such an interpretation would imply; see [2].

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