On the Characteristic Numbers of Complete Manifolds of Bounded Curvature and Finite Volume

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0. Introduction

Let M^n be a non-compact complete Riemannian manifold, whose sectional curvature K, and volume $\mathrm{Vol}(M)$ satisfy

$$|K| \leqq 1, \tag{0.1}$$

$$\mathrm{Vol}(M) < \infty. \tag{0.2}$$

Sometimes we will assume that M^n is diffeomorphic to the interior of a compact manifold $\bar{M}^n$ with boundary N^{n-1}.

Example 0.1. The simplest examples of manifolds of the above type are two dimensional. A neighborhood of infinity looks like several copies of $(A, \infty) \times S^1$, with metric

$$dr^2 + f^2(r)\,\tilde{g}, \tag{0.3}$$

where $\tilde{g}$ is the usual metric on S^1, $f > 0$, and

$$\frac{|f''|}{f} = |K| \leqq 1, \qquad \int_A^\infty f < \infty. \tag{0.4}$$

By a standard argument $|f'|/|f|$ is also bounded in this situation.

Let P denote an invariant polynomial of degree k, $(n = 2k)$ and $P(\Omega)$ the corresponding characteristic form in the curvature Ω of M. Here, we will assume that $P(\Omega)$ is either the Euler form $P_\chi(\Omega)$ or some Pontrjagin form, and for the most part we will restrict attention to the Pontrjagin form $P_L(\Omega)$, corresponding to the L-polynomial of the Hirzebruch Signature Theorem. Since $|K| \leqq 1$, $\mathrm{Vol}(M) < \infty$, the integral

$$\int_M P(\Omega) = P(M, g) \tag{0.5}$$

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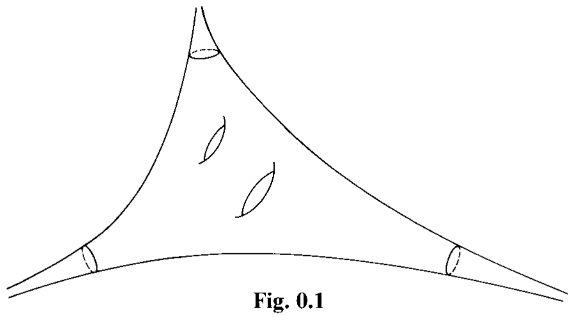

Fig. 0.1

defining the *geometric characteristic number* $P(M,g)$ is absolutely convergent (for the case of Pontrjagin forms, we assume M^{4l} is oriented). The numbers $P_\chi(M,g)$ and $P_L(M,g)$ will simply be denoted by $\chi(M,g)$ and $\sigma(M,g)$ respectively. We ask:

A) What values can $P(M,g)$ assume?
B) To what extent is $P(M,g)$ independent of the particular metric satisfying (0.1), (0.2)?
C) What is the purely topological significance of $P(M,g)$?

These questions were first considered for $\chi(M,g)$ in [CV], [Hu] and [Har], in the 2-dimensional and locally symmetric cases (see also [Ros]). They were also considered for $\chi(M,g)$ in [G_3] under the assumption that for some *profinite* covering space $\tilde{M}$ of M, the pulled back metric has *bounded geometry* (the definitions are given below). Our main concerns in this paper are to provide the details of a basic technical result (Theorem 2.1) which were not given in [G_3], to extend the discussion to the case of $\sigma(M,g)$ and to the case of normal coverings which are not necessarily profinite.

Observe that since $P(M,g)$ is a locally computable invariant, and thus behaves multiplicatively under coverings, it is natural to consider the effect of placing geometric hypotheses on various coverings $\tilde{M}$ of M. A covering $\tilde{M}$ is said to be *profinite* if there exists a decreasing sequence of subgroups of finite index, $\Gamma_j \subset \pi_1(M)$, such that $\cap \Gamma_j = \pi_1(\tilde{M})$.

Before describing answers to questions A)–C), we recall the situation for M^n closed. In this case, $P(M,g)$ is independent of g and equal to the *topological characteristic number*, $P(M)$, corresponding to P under the Chern-Weil homomorphism. Thus $P(M,g)$ is an integer if P comes from an integral class. Moreover, by the Gauss-Bonnet-Theorem and Hirzebruch-Signature-Theorem,

$$\chi(M,g) = \chi(M) \tag{0.6}$$

$$\sigma(M,g) = \sigma(M); \tag{0.7}$$

(since P_L corresponds to a rational class under the Chern-Weil homomorphism, (0.7) entails an integrality statement).

Since $\chi(M,g)$, $\sigma(M,g)$ are multiplicative under coverings, by (0.6), (0.7), the same holds for $\chi(M)$, $\sigma(M)$. Of course, since there is a local combinatorial formula for the Euler characteristic, $\chi(X)$ actually behaves multiplicatively for any space X. But at present, there is no *elementary* proof of the existence a local formula for

$\sigma(M)$, if M^n is closed. Moreover, for manifolds with boundary, $\sigma(M)$ does *not*, behave multiplicatively. Thus, there is no local formula in that case.

The generalizations of (0.6), (0.7) to manifolds with boundary are

$$\chi(M,g) + \mathrm{II}_\chi(N,g) = \chi(M), \tag{0.8}$$

$$\sigma(M,g) + \eta(N,g) + \mathrm{II}_\sigma(N,g) = \sigma(M). \tag{0.9}$$

Here, $\mathrm{II}_\chi(N,g)$, $\mathrm{II}_\sigma(N,g)$ are certain *locally computable* expressions involving the second fundamental form of N, and $\eta(N,g)$ is the η-invariant of Atiyah-Patodi-Singer, a *global* (spectral) invariant of N; see [APS_1] and Sect. 4. Formula (0.9) easily implies a crucial property of the η-invariant. If g_t is a 1-parameter family of metrics, the derivative, $\dot\eta = \frac{d}{dt}\eta(N,g_t)$, is in fact given by a locally computable formula involving $g, \dot g$. Similarly, the η-invariant $\eta_{E^k}(N,g)$ can be defined with coefficients in a flat orthogonal bundle E^k and

$$\dot\eta_{E^k} = k\dot\eta. \tag{0.10}$$

Thus

$$\frac{1}{k}\eta_{E^k}(N,g) - \eta(N,g) = \rho_{E^k}(N) \tag{0.11}$$

is independent of g. This invariant was introduced in [APS_2]. We will study its significance in our context.

We can now give some answers to questions A)–C).

A) The values of $P(M,g)$, for P an integral class, are discussed in [CGY]. The number $\chi(M,g)$ is always an *integer* but the geometric Pontrjagin numbers $P(M,g)$ can be irrational; see Example 1.8. The relation between the rationality of $P(M,g)$ and the geometry of M is studied in [CG_3].

B) Essentially, the standard argument for closed manifolds shows that $P(M,g_t)$ is independent of t, provided the family of metrics g_t satisfies (0.1), (0.2) and a growth property at infinity. But even the Euclidean spaces R^n, ($n \geqq 3$), admit metrics g_0, g_1 satisfying (0.1) and (0.2), which can not be connected by such a deformation, and for which $\chi(R^n,g_0) \neq \chi(R^n,g_1)$, $\sigma(R^n,g_0) \neq \sigma(R^n,g_1)$ in appropriate dimensions; see Sect. 1. However $\chi(M,g)$, $\sigma(M,g)$ *are independent of* g *for metrics satisfying* (0.1), (0.2) and the following

Additional Hypothesis. For some neighborhood of infinity $U \subset M$, some profinite or normal covering space $\tilde U$ has injectivity radius at least (say) 1 for the pulled back metric,

$$i(\tilde U) \geqq 1. \tag{0.12}$$

Since also $|K| \leqq 1$ on U we say that U has *bounded geometry*, $\mathrm{geo}_{\tilde\infty}(M) \leqq 1$. If $U = M$ we write $\mathrm{geo}(\tilde M) \leqq 1$.[3] In this paper the notation $\mathrm{geo}_{\tilde\infty}(M) \leqq 1$, $\mathrm{geo}(\tilde M) \leqq 1$ will indicate that in addition $\tilde U$, $\tilde M$ are assumed to be profinite or

[3] To simplify the exposition, most statements and all proofs will be given only for the case $\mathrm{geo}(\tilde M) \leqq 1$

normal. When the distinction between the two cases is important it will be mentioned explicitly.

Even for metrics with $\mathrm{geo}_{\infty}(M) \leqq 1$, $P(M,g)$ may *not* be independent of g for arbitrary Pontrjagin numbers; see Example 1.9. Nevertheless, in certain *special* situations, one can show that *$P(M\,g)$ is independent of g and even prove analogous results for Pontrjagin classes*, by bringing in ideas related to the Novikov Conjecture; see [CG_4].

C) If $\mathrm{geo}_{\infty}(M) \leqq 1$, then $\chi(M,g)$ is a homotopy invariant and $\sigma(M,g)$ is a *proper homotopy invariant* of M; see Theorems 3.1, 5.1, 6.1, 6.2. The topological significance of these invariants is most easily explained if one adds further assumptions. If M has *finite topological type*, i.e. M is diffeomorphic to the interior of a compact manifold with boundary $\bar{M}$, then (0.6) holds, $\chi(M,g) = \chi(M)$. Now suppose in addition, that the (not necessarily normal) covering space $\tilde{M}$, is profinite with $\mathrm{ind}(\Gamma_j) = d_j$. Then for the corresponding covering spaces $p_j\colon \tilde{M}_j \to M$, we have

$$\sigma(M,g) = \lim_{j\to\infty} \frac{1}{d_j}\,\sigma(\tilde{M}_j); \tag{0.13}$$

see Theorem 5.1. The existence of the limit on the right hand side of (0.13) is not obvious a priori. Although the limit can be shown to exist under more general circumstances (see Theorem 7.3) whether it exists for arbitrary compact manifolds with boundary seems difficult to decide.

If one continues to assume that M is profinite but drops the assumption of finite topological type, there are still expressions for $\chi(M,g)$, $\sigma(M,g)$ which generalize (0.13).

Finally, suppose $\tilde{M}$ is normal but not necessarily profinite. We begin by observing that the L^2-Index-Theorem for normal coverings of compact manifolds (see [A], [S]) can be extended to our situation. Thus we have

$$\begin{aligned} \chi(M,g) &= \chi_{(2)}(M), \\ \sigma(M,g) &= \sigma_{(2)}(M) \end{aligned} \tag{0.14}$$

where $\chi_{(2)}(M)$, $\sigma_{(2)}(M)$ are the L^2-Euler characteristic and signature; see Sect. 6. If M is compact, Dodziuk has shown that $\chi_{(2)}(M)$, $\sigma_{(2)}(M)$, as well as the corresponding L^2-Betti numbers $b^i_{(2)}(M)$, are homotopy invariants of M; see [D_1]. Here and in [CG_1] we show that these numbers are *homotopy invariants* in our context.[4]

Parts of the general picture presented so far can be easily grasped on the basis of the following (simply stated but difficult to establish) generalization of the situation described in Example 0.1.

Assertion 0.1. If M^n is complete with $|K| \leqq 1$, and $\mathrm{Vol}(M^n) < \infty$ then M^n admits an exhaustion by compact manifolds with smooth boundary, M^n_k, such that $\mathrm{Vol}(\partial M^n_k) \to 0$ and for which the second fundamental forms $\mathrm{II}(\partial M^n_k)$ are uniformly bounded.

[4] As above, $\sigma_{(2)}(M)$ is only a proper homotopy invariant

If we grant the above assertion, it follows immediately from (0.8) that

$$\chi(M^n, g) = \chi(M_k^n) \in \mathbb{Z}, \tag{0.15}$$

for k sufficiently large. The point is

$$\lim_{k\to\infty} \mathrm{II}_\chi(\partial M_k^n, g) = 0. \tag{0.16}$$

However, according to the discussion of **B)** above, different metrics satisfying (0.1), (0.2) can give rise to *topologically distinct exhaustions*, if we omit the assumption $\mathrm{geo}_\infty(M) \leqq 1$.

Assume now that $\mathrm{geo}_\infty(M) \leqq 1$, and also that M^n has finite topological type. We can then explain (0.13) (which implies that $\sigma(M^n, g)$ is independent of g); the analogous result for $\chi(M^n, g)$ follows similarly. Recall that the signature of an (oriented) manifold with boundary X^{4l} is defined as the signature of the cup product pairing restricted to the group.

$$j(H^{2l}(X^{4l}, \partial X^{4l})) \subset H^{2l}(X^{4l}), \tag{0.17}$$

where j is the natural inclusion. In general, if we set

$$\mathbf{b}^i(A) = \dim\{j(H^i(A, \partial A) \subset H^i(A)\} \tag{0.18}$$

(where $H^i(A, \partial A)$ denotes cohomology with compact supports) then $A_1 \subset A_2$ is easily seen to imply

$$\mathbf{b}^i(A_1) \leqq \mathbf{b}^i(A_2). \tag{0.19}$$

It follows that if M^{4l} has finite topological type and M_k^{4l} is *any* exhaustion, for all sufficiently large k,

$$\mathbf{b}^i(p_j^{-1}(M_k^{4l})) = \mathbf{b}^i(\tilde{M}_j). \tag{0.20}$$

Similarly,

$$\sigma(p_j^{-1}(M_k^{4l})) = \sigma(\tilde{M}_j). \tag{0.21}$$

Thus, if we use the exhaustion supplied by Assertion 0.1, together with (0.9), it suffices to establish that for all $\varepsilon > 0$, there exists $k_0, N(k)$ such that for $k > k_0$, $j > N(k)$,

$$\left| \frac{1}{d_j} \eta(p_j^{-1}(\partial M_k^{4l})) \right| < \varepsilon. \tag{0.22}$$

This is a direct consequence of the following basic estimate for the η-invariant; see Sect. 4.

Theorem 0.1. There exists a constant[5] $c(4l-1)$ such that if N^{4l-1} is compact and satisfies $\mathrm{geo}(N) \leqq 1$, then

$$|\eta(N^{4l-1})| \leqq c(4l-1)\ \mathrm{Vol}(N^{4l-1}). \tag{0.23}$$

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[5] Throughout the paper we make the following convention. We indicate the dependence of constants appearing in estimates on parameters by writing e.g. $c(n)$ for any constant depending only on n. Thus if any parameter does not appear, it means that the constant can be estimated independent of this parameter

As we mentioned, the simple picture provided by Assertion 0.1 is actually technically difficult to establish. The proof depends on a generalization of the arguments of [G_1] and will not be attempted here. But for our present purposes, a much less delicate result will suffice. This is the analog of Assertion 0.1 for the covering space $\tilde{M}$; see Theorem 2.1.

The rationality or irrationality of $P(M,g)$ is related to the properties of a generalized torus action ($\mathcal{F}$-structure) which can be shown to exist outside of a compact subset of M; see Sect. 1 for examples, and [CG_2] and [CGY] for details.

The remainder of this paper will consist of seven sections as follows.

1. Examples
2. Approximation Theorems
3. The Euler Characteristic and Stable Acyclicity of the Boundary
4. An Estimate for the η-Invariant
5. The η-Invariant and Signature
6. L^2 Theory for Normal Coverings
7. L^2 Theory for Profinite Normal Coverings

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1. Examples

An indication of richness of the class of complete Riemannian manifolds with $|K| \leqq 1$, $\mathrm{Vol}(M) < \infty$, is provided by examples such as the following, which are constructed "by hand". These examples also give some feeling for the geometry of manifolds in this class. Of course, certain classical examples such as locally symmetric spaces have been studied in enormous detail.

Example 1.1 (R^2, g). By forming the surface of revolution generated by a suitable curve as in Fig. 1.1, we obtain a metric g with $|K| \leqq 1$, $\mathrm{Vol}(R^2) < \infty$ on R^2.

For this metric, clearly $\chi(R^2, g) = 1$.

Example 1.2 (R^{2m}, g^m). If g is as in Example 1.1, the metric $g_0 = g \times \ldots g$ (m factors) on R^{2m} satisfies

$$\chi(R^{2m}, g_0) = \chi(R^{2m}) = 1. \tag{1.1}$$

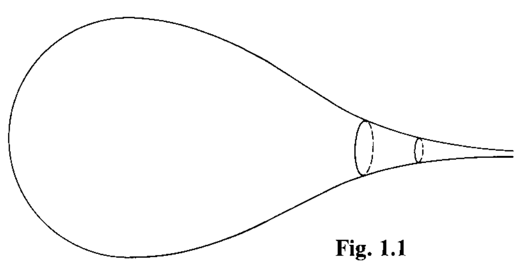

Fig. 1.1

We now recall a construction from [G_3] for producing metrics with $|K| \leqq 1$, $\mathrm{Vol}(M^3) < \infty$ on a large class of 3-manifolds called *graph manifolds*. We then show how this can be applied in a simple fashion to $\mathbb{R}^3$. Finally we generalize to $\mathbb{R}^n$, yielding in particular, a metric g_1 with $\chi(\mathbb{R}^4, g_1) = 0$.

Example 1.3 (graph manifolds). Let Σ_i^2, $i = 1, 2, \ldots$ be a sequence of compact 2-manifolds whose boundaries are unions of circles, S_{ij}^1, $j = 1, \ldots, j(i)$. Topologically, the graph manifolds in question are obtained as follows. Take an infinite sequence of circles S_i^1 and form $\Sigma_i^2 \times S_i^1$. Then form a non-compact manifold without boundary, M^3, by identifying the boundary components $S_{ij}^1 \times S_i^1$, $S_{i'j'}^1 \times S_{i'}^1$ in pairs, preserving the product structure but interchanging the roles of the factors.[6]

The metric on M^3 is obtained by gluing together metrics on the pieces $\Sigma_i^2 \times S_i^1$ by isometries of their boundaries. The metric on $\Sigma_i^2 \times S_i^1$ is a product metric where the S^1 factor has length ε_i. Given any sequence $(\delta_i) = \delta(i, l), \ldots, \delta(i, j(i))$ we can find a metric $g_{(\delta_i)}$ on Σ_i^2 with

$$\mathrm{Vol}(\Sigma_i^2, g_{(\delta_i)}) < c(\Sigma_i^2), \tag{1.3}$$

$$\mathrm{Length}(S_{ij}^1) = \delta(i, j), \tag{1.4}$$

such that $g_{(\delta_i)}$ splits isometrically as a product near the boundary $\cup S_{ij}^1$. This is done by a slight modification of the construction of Example 0.1. By taking $\delta(i, j) = \varepsilon_{j'}$ where i, j, i', j' are as above and choosing ε_i such that

$$\Sigma\, \varepsilon_i \times c(\Sigma_i^2) < \varepsilon < \infty, \tag{1.5}$$

we get the required metric.

Remark 1.1. If all Σ_i^2 have non-positive Euler characteristic, the above metric can be chosen to have non-positive curvature.

Example 1.4 $(\mathbb{R}^3, h)$. Write $\mathbb{R}^3$ as an increasing union of solid tori, $D_i^2 \times S^1$ (each contractible in the next) as in Fig. 1.2.

If we set $\Sigma_1^2 \times S^1 = D_1^2 \times S^1$, it suffices to decompose the region between each pair $D_i^2 \times S^1$, $D_{i+1}^2 \times S^1$ into two pieces $\Sigma_{2i}^2 \times S^1 \cup \Sigma_{2i+1}^2 \times S^1$. To do this, view $D_{i+1}^2 \times S^1$ as a solid cylinder C about the x-axis, with ends identified. Let A denote the axis of C and S a circle which links A. Identify $D_i^2 \times S_i^1$ with a small tubular neighborhood $T_\varepsilon(S)$ of S, and put $\Sigma_{2i+1}^2 \times S^1 = D_{2i+1}^2 \times S^1 = T_\varepsilon(A)$.

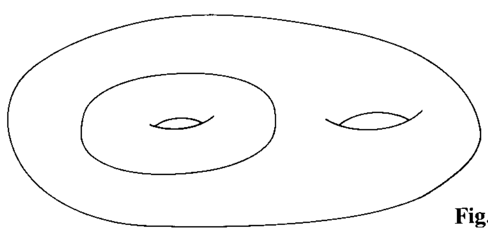

Fig. 1.2

[6] More generally one can use pieces which admit locally free circle actions; see Example 1.7. Still more generally one can consider polarized f-structures; see [CG_2]

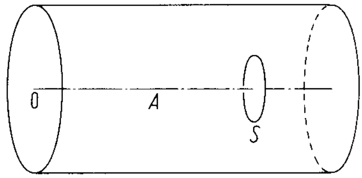

Fig. 1.3

Then $C\backslash T_\varepsilon(A)\backslash T_\varepsilon(S)$ split as a product $\Sigma^2_{2i} \times S^1$. Here the S^1 factors are circles linking A. Each such circle intersects the positive quadrant of the $x - z$ plane in a unique point. It follows that Σ^2_{2i} is a rectangle with a disc deleted and ends identified (i.e., a band F^2 with a disc deleted).[7]

Example 1.5 ($R^{2m+3k}, g^m \times h^k$). As a consequence of the local product structure, it follows that for k positive and even, any characteristic from vanishes identically. In particular for n even, $n \geqq 6$, we obtain metrics $\underline{g}$ on R^n with

$$\chi(R^n, \underline{g}) = 0 \neq \chi(R^n). \tag{1.6}$$

Example 1.6 (R^n, g_1). The construction of Example 1.4 generalizes to give a family of metrics on R^n, $n \geqq 3$. Thus for n even, $n \geqq 4$, we get metrics satisfying

$$\chi(R^n, g_1) = 0. \tag{1.7}$$

To obtain g_1, write R^n as a union of solid tori $D^2 \times T^{n-2}$ where $T^{n-2} = S^1 \times \ldots \times S^1$ ($j - 2$ factors). Replace A of Example 1.4 by T^{n-2}. Let T^{n-3} be contractibly imbedded in T^{n-2} and replace C by the product of T^{n-3} and a circle linking T^{n-2}. The rest of the construction proceeds as before yielding

$$\Sigma^2_{2i+1} \times T^{n-2} = D^2 \times T^{n-2}, \qquad \Sigma^2_{2i} \times T^{n-2} = F^2 \times T^{n-2}.$$

Example 1.7[8] (2-plane bundles, $\sigma(\mathbb{R}^4, g_f) = \frac{2}{3}$). Let $E^2 \to X^{n-2}$ be a 2-plane bundle with connection θ and let $S^1 \to N^{n-1} \xrightarrow{\pi} X^{n-2}$ be the associated circle bundle. We will construct a metric g_f with $|K| \leqq 1$, $\mathrm{Vol}(M^n) < \infty$ on the total space M^n of E^2, for suitable $f\colon \mathbb{R}^+ \to \mathbb{R}^+$.

Let X have metric h and let H denote the horizontal subbundle for θ. Let z be the infinitesimal generator of the generator of the S^1 action and assume $\theta(z) = 1$. We consider the metric

$$\rho_f = \pi^* h + f^2 \theta^2 \tag{1.7}$$

on N and the metric

$$g_f = dr^2 + \rho_{f(r)} \tag{1.8}$$

on M^n.

If u is a vector field on X, let $\underline{u}$ denote its horizontal lift. Define the skew symmetric transformation $S\colon H \to H$ by

$$-\Omega(u, v) = \langle S(\underline{u}), \underline{v}\rangle = \langle [\underline{u}, \underline{v}], z\rangle. \tag{1.9}$$

[7] The fact that the region between two solid tori decomposes as above is used in [So] for example, to show that the connected sum of 3-dimensional graph manifolds is again a graph manifold. More generally, as observed in [G_3], this argument works in odd dimensions.

[8] See [CGY] for a simplification and generalization of Examples 1.7 and 1.8

Then the Euler form of E^2 is

$$P_\chi(\Omega) = \frac{1}{2\pi}\,\Omega. \tag{1.10}$$

Let $\tilde{R}_f, \mathscr{R}$ denote the curvature tensors of N, and X respectively. Let $\bar{\nabla}$ be the pullback of the Riemannian connection on X. Then by a straightforward calculation [based, for example, on formula (2.66) of Sect. 2]

$$\begin{aligned}\tilde{R}_f(u,v)\,w = \underline{\mathscr{R}(u,v)\,w} + \frac{f^2}{4}\,\{&\langle S(\underline{u}),\underline{w}\rangle\, S(\underline{v}) - \langle S(\underline{v}),\underline{w}\rangle\, S(\underline{u})\\ &+ \langle S(\underline{u}),\underline{v}\rangle\, S(\underline{w}) - \langle S(\underline{v}),\underline{u}\rangle\, S(\underline{w})\}\\ &+ \langle \bar{\nabla}_{\underline{u}} S(\underline{v}),\underline{w}\rangle\, z_f - \langle \bar{\nabla}_{\underline{v}} S(\underline{u}),\underline{w}\rangle\, z_f,\end{aligned} \tag{1.11}$$

$$\tilde{R}_f(\underline{u},z_f)\,z_f = -\frac{f^2}{4}\,S^2(\underline{u}), \tag{1.12}$$

where $z_f = z/f$. In particular, as $f \to 0$, $\mathrm{Vol}(N) \to 0$ but $|K_f|$ remains bounded. This is essentially the Berger example; see e.g. [CE].

A similar calculation shows that the curvature R of g_f satisfies

$$R(u,v)\,w = \tilde{R}_f(u,v)\,w, \tag{1.13}$$

$$R(u,z_f)\,v = \tilde{R}_f(u,z_f)\,v + f'\langle S(u),v\rangle\, T, \tag{1.14}$$

$$R(u,z_f)\,z_f = \tilde{R}_f(u,z_f)\,z_f, \tag{1.15}$$

$$R(u,T)\,T = 0, \tag{1.16}$$

$$R(z_f,T)\,T = -\frac{f''}{f}\,z_f \tag{1.17}$$

where we have put $T = \partial/\partial r$. In particular, we have $|K| < 1$, $\mathrm{Vol}([A,\infty) \times N) < \infty$, for suitable f; e.g. $f = e^{-r}$ or $f = r^{-c}$, $c > 1$.

Formulas (1.13)–(1.17) together with (0.9) can be used to calculate $\sigma(Y^4, g)$ if Y^4 is isometric to $([1,\infty) \times N, g)$ near infinity. Recall that in dim 4,

$$P_L(R) = \frac{1}{3}\,p_1(R) = \frac{-1}{24\pi^2}\,\mathrm{tr}(R \wedge R). \tag{1.18}$$

Assume now for simplicity that $\bar{\nabla} S = 0$ where S is as in (1.8) and put

$$\Omega = -s \cdot \omega \tag{1.19}$$

where ω is the volume form on X^2. If $\mathscr{K}$ denotes the curvature of X^2 a routine computation gives

$$P_L(R) = \frac{1}{6\pi^2}\,\{f^3 f' s^3 - \mathscr{K} f f' s + f'' f' s\}\, dr \wedge \omega \wedge \theta. \tag{1.20}$$

By choosing $f = r$ near $r = 0$, we obtain a metric g_f on M^4 for which

$$\sigma(M^4, g_f) = \tfrac{1}{3}\chi(E^2). \tag{1.21}$$

By (0.9), we get

$$\lim_{f\to 0} \eta(N^3, \rho_f) = \sigma(M^4) - \sigma(M^4, g) = -\operatorname{sign}\chi(E^2) + \tfrac{1}{3}\chi(E^2), \tag{1.22}$$

where

$$\operatorname{sign}\chi(E^2) = \begin{cases} \chi(E)^2/|\chi(E^2)|, & \chi(E^2) \neq 0 \\ 0, & \chi(E^2) = 0. \end{cases} \tag{1.23}$$

Thus if Y^4 is as above

$$\sigma(Y^4, g_f) = \sigma(Y^4) + \operatorname{sign}\chi(E^2) - \tfrac{1}{3}\chi(E^2). \tag{1.24}$$

In particular by considering the Hopf fibration $S^1 \to S^3 \to S^2$, we get

$$\sigma(R^4, g_f) = \tfrac{2}{3}. \tag{1.25}$$

Example 1.8 $\left(\sigma(R^4, g_\gamma) = \frac{1}{3}\left(\gamma + \frac{1}{\gamma}\right), 0 < \gamma < \infty\right)$. We now construct a family of metrics g_γ on $\mathbb{R}^4, 0 < \gamma < \infty$, with bounded curvature and finite volume, such that $\sigma(M, g) = \frac{1}{3}\left(\gamma + \frac{1}{\gamma}\right)$ ($\gamma = 1$ corresponds to the metric in Example 1.7).

First consider a local orthonormal frame field $e_1, \ldots, e_m$ on a Riemannian manifold N^m. Recall that if

$$w_i = e_i^* \tag{1.26}$$

$$w_{ij}(x) = \langle \nabla_x e_j, e_i \rangle \tag{1.27}$$

$$\Omega_{ij}(x, y) = \langle R(x, y)\, e_j, e_i \rangle, \tag{1.28}$$

then we have the structural equations

$$d w_i = -\sum_j w_{ij} w_j \tag{1.29}$$

$$d w_{ij} = -\sum_k w_{ik} w_{kj} + \Omega_{ij}. \tag{1.30}$$

(Here and below the wedge product symbol is omitted.) Consider a 1-parameter family of metrics h_t for which $e_1, \ldots, e_{m-1}, \frac{1}{t} e_m$ is orthonormal. Since the connection forms are the unique forms satisfying (1.29) (for h_t) which are antisymmetric in i, j, we easily obtain

$$w_{ij}^t = w_{ij} + (1 - t^2)\, b_{ij} w_m, \qquad 1 \leqq i \leqq m-1 \tag{1.31}$$

$$w_{im}^t = \frac{1}{t}\left(w_{im} + \sum_1^n (1 - t^2)\, b_{ij} w_j\right). \tag{1.32}$$

Here b_{ij} are the unique functions for which

$$d w_m = \sum_{i,j=1}^m b_{ij} w_i w_j \tag{1.33}$$

and

$$b_{ij} = -b_{ji}. \tag{1.34}$$

If however, $e_m = f \cdot v$ for some smooth function f and (non-vanishing) Killing field v, then using (1.27) and

$$\langle \nabla_x v, y\rangle + \langle \nabla_y v, x\rangle = 0, \tag{1.35}$$

it follows that

$$w_{im} = -\sum_{i=1}^{m} b_{ij} w_j. \tag{1.36}$$

Thus, (1.32) is replaced by

$$w^t_{i,m} = t w_{i,m} \tag{1.37}$$

in this case. By (1.30) (for h_t)

$$\Omega^t_{ij} = \Omega_{ij} + (1+t^2)\left[d(b_{ij}) w_m + \sum_{k=1}^{m-1} b_{jk} w_{ki} w_m - b_{ik} w_{kj} w_m - w_{im} w_{mj}\right], \tag{1.38}$$

$$\Omega^t_{im} = t\Omega_{im} - t(1-t^2)\sum b_{ij} w_{jm} w_m. \tag{1.39}$$

Note that (1.37)–(1.39) exhibit the basic fact that as $t \to 0$, h_t converges to a metric which is the warpped product of a *smooth* metric on some U^{m-1} and an interval (whose length approaches zero).

Now suppose N^{4l-1} is compact, oriented and that v (non-vanishing) is globally defined. Then (1.37)–(1.39) can be used to calculate the derivative with respect to t of the secondary invariant corresponding to P_L, and hence of the η-invariant, η_t (see [APS$_1$], [APS$_2$] and [CS]). In particular, in dim 3, we have

$$\lim_{t\to 0} \eta_t = \eta_1 - \frac{1}{12\pi^2}\int_0^1 \int_{N^3} \operatorname{tr}(\Omega^t \wedge (w^t)')\,dt, \tag{1.40}$$

where $(w^t)'$ denotes the derivative of the connection form with respect to t. For the case in which h_1 is the standard metric on S^3, we have $\eta_1 = 0$, $\Omega_{ij} = w_i w_j$,

$$(w_{12})' = -2t b_{12} w_3, \tag{1.41}$$

$$(w_{13})' = w_{13} = -b_{12} w_2 - b_{13} w_3, \tag{1.42}$$

$$(w_{23})' = w_{23} = -b_{21} w_1 - b_{23} w_3, \tag{1.43}$$

$$\Omega^t_{12} = w_1 w_2 + (1-t^2)\,[d(b_{12} w_3) - w_{13} w_{32}] \tag{1.44}$$

$$\Omega^t_{13} = t w_1 w_2 - t(1-t^2)\, b_{12} w_{23} w_3 \tag{1.45}$$

$$\Omega^t_{23} = t w_2 w_3 - t(1-t^2)\, b_{21} w_{13} w_3. \tag{1.46}$$

Using

$$d w_3 = 2 b_{12} w_1 w_2 \tag{1.47}$$

in (1.44), and (1.43), (1.44) in (1.45), (1.46) one checks that

$$-\frac{1}{12\pi^2}\operatorname{tr}(\Omega^t \wedge w') = \frac{1}{12\pi^2}[\Omega^t_{12} w'_{12} + \Omega^t_{13} w'_{13} + \Omega^t_{23} w'_{23}]$$
$$= -\frac{2}{3\pi^2} t(1-t^2)\, b_{12}^3 w_1 w_2 w_3. \tag{1.48}$$

Hence by (1.40),

$$\lim_{t\to 0} \eta_t = -\frac{1}{3\pi^2}\int_{S^3} b_{12}^3. \tag{1.49}$$

Now let $(r_1, \theta_1, r_2, \theta_2)$ be polar coordinates on $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$, and introduce coordinates (s, θ_1, θ_2) on the unit sphere S^3 by putting

$$\tan s = r_1/r_2, \qquad 0 \leqq s \leqq \tfrac{\pi}{2}. \tag{1.50}$$

Then the standard metric on S^3 is

$$h_1 = ds^2 + \cos^2 s\, d\theta_1^2 + \sin^2 s\, d\theta_2^2. \tag{1.51}$$

Up to isometry any Killing field v on S^3 is given by

$$v = \alpha \frac{\partial}{\partial\theta_1} + \beta \frac{\partial}{\partial\theta_2}, \tag{1.52}$$

where α, β are constants. Take

$$e_3 = \frac{1}{(\alpha^2\cos^2 s + \beta^2 \sin^2 s)^{1/2}}\left(\beta \tan s \frac{\partial}{\partial\theta_1} - \alpha \cot s \frac{\partial}{\partial\theta_2}\right). \tag{1.53}$$

Then

$$b_{12} = \tfrac{1}{2} dw_3(e_1, e_2) = \tfrac{1}{2}\langle [e_1, e_2], e_3\rangle = \frac{\alpha\beta}{(\alpha^2\cos^2 s + \beta^2\sin^2 s)}. \tag{1.54}$$

Since

$$\operatorname{area}(S^1 \times S^1) = 4\pi^2 \sin s \cos s, \tag{1.55}$$

we get

$$\int_{S^3} b_{12}^3 = 4\pi^2\alpha^4\beta^4 \int_0^{\frac{\pi}{2}} \frac{\sin s \cos s\, ds}{(\alpha^2\cos^2 s + \beta^2\sin^2 s)^3}$$
$$= \frac{2\pi^2\alpha^3\beta^3}{(\beta^2-\alpha^2)}\int_{\beta^2}^{\alpha^2}\frac{du}{u^3} = \pi^2\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right). \tag{1.56}$$

Thus if $\alpha/\beta = \gamma$, (1.49) gives

$$\lim_{t\to 0}\eta_t = -\tfrac{1}{3}\left(\gamma + \frac{1}{\gamma}\right). \tag{1.57}$$

Finally, for fixed γ, consider a metric on R^4 which, near infinity, looks like

$$g_\gamma = dr^2 + h_{e^{-r}}. \tag{1.58}$$

Then (0.9) and (1.57) give

$$\sigma(R^4, g_\gamma) = \tfrac{1}{3}\left(\gamma + \frac{1}{\gamma}\right). \tag{1.59}$$

Example 1.9 [Non-invariance of $P(M^{4l}, g)$]. Let N^{4l-1} be compact and oriented. Suppose (M^{4l}, g) is isometric to $([0,\infty) \times N^{4l-1}, g_0)$ near infinity, with $\mathrm{Vol}([0,\infty) \times N, g_0) < \infty$, $\mathrm{geo}([0,\infty) \times \tilde{N}) \leqq 1$. After changing g_0 on a compact set, an operation which preserves $P(M, g)$, we can assume that on $[0,1] \times N$,

$$g_0 = dt^2 + \bar{g}, \tag{1.60}$$

for some metric on $\bar{g}$ on N.

Construct a second metric g, as follows. Let $\phi\colon N \to N$ be an orientation preserving diffeomorphism (which might not be isotopic to the identity). Let

$$g_1 = \begin{cases} g_0 & [0, \frac{1}{3}] \times N \\ h & [\frac{1}{3}, \frac{2}{3}] \times N \\ (1 \times \phi)^*(g_0) & [\frac{2}{3}, \infty) \times N \end{cases} \tag{1.61}$$

where h is any smooth interpolation between g_0 and $(1 \times \phi)^*(g_0)$. Since for any P we have $P([0,1] \times N, g_0) = 0$, it follows that

$$P([0,\infty) \times N, g_1) - P([0,\infty) \times N, g_0) = P([0,1] \times N, g_1). \tag{1.62}$$

Form a closed manifold X^{4k} by identifying $0 \times N$ with $1 \times \phi(N)$. Clearly $g_1 | [0,1] \times N$ pushes down to a metric on X^{4k}. Thus if P corresponds to an integral class, the difference in (1.62) is an integer. But if $P \neq P_L$, in general

$$P([0,1] \times N, g_1) = P(X^{4k}, g_1) \neq 0. \tag{1.63}$$

Thus, $P(M, g)$ may depend on g even if $\mathrm{geo}_{\tilde{\infty}}(M) \leqq 1$.

For further examples and related constructions the reader is referred to [G_3].

2. Approximation Theorems

We begin this section with a theorem which asserts that any subset of a manifold of bounded geometry is contained in a (top dimensional) submanifold the geometry of whose boundary is controlled. Although this suffices for the application to $\chi(M^n, g)$, a second result (Theorem 2.5) concerning the regularization of metrics of bounded geometry is required in order to treat $\sigma(M^n, g)$.

Theorem 2.1 (Neighborhoods of bounded geometry). Let V^n be a complete Riemannian manifold and $W_0 \subset V$, an arbitrary subset. Assume that on the 1-tubular neighborhood $T_1(W_0)$, we have $\mathrm{geo}[T_1(W_0)] \leqq 1$. Then given $\varepsilon > 0$, there exists a submanifold with boundary $W_\varepsilon^n \subset V^n$ such that

1) $W_0 \subset W_\varepsilon^n$.
2) W_ε^n is contained in the ε-tubular neighborhood $T_\varepsilon(W_0)$ of W_0.

3) There are constants $c(n)$, $c(n,i,\varepsilon)$ such that

$$\mathrm{Vol}(\partial W_\varepsilon^n) \leqq c(n)\,\mathrm{Vol}[T_\varepsilon(\partial W_0)], \tag{2.1}$$

$$\|\nabla^i\, \mathrm{II}(\partial W_\varepsilon^n)\| \leqq c(n,i,\varepsilon). \tag{2.2}$$

If $V = \mathbb{R}^n$, then W_ε^n can be obtained as follows. Divide $\mathbb{R}^n$ into closed cubes of side ε, with disjoint interiors. Let $\tilde{W}_\varepsilon$ consist of those cubes whose intersection with W_0 is non-empty, and smooth the corners of $\tilde{W}_\varepsilon$ appropriately to obtain W_ε. Although this approach can be made to work in general (by using the triangulation lemma of [CMS], Sect. 7 or related unpublished results of Calabi) it is more efficient to proceed by combining the following three lemmas.

Let Ric denote the Ricci curvature of M^n.

Lemma 2.2 (Covering lemma). Let M^n be complete with $\mathrm{Ric} \geqq -1$. Fix $\rho_0 > 0$ and $\lambda > 1$. Then for all $0 < \rho \leqq \rho_0$, there is a covering of M^n by sets $U_1, \ldots, U_N$, such that

1) Each U_i is a union of (possibly infinitely many) disjoint metric balls of radius ρ, and the distance between the centers of each pair of balls is at least $2\lambda\rho$.
2) $N \leqq c(n, \rho_0, \lambda)$.

To state the next lemma we need a definition. Let $X^k \subset M^n$ be a submanifold with empty boundary, and let $N_x(X^k)$ denote the normal space to X^k at x. A metric ball $B_\rho(p)$ is called *ε-transversal* to X^k if the following holds. If $x \in X^k$ and γ is a minimal geodesic from x to p whose length $L[\gamma]$ satisfies,

$$\rho - \varepsilon \leqq L[\gamma] \leqq \rho + \varepsilon,$$

then the angle between $\gamma'(0)$ and $N_x(X^k)$ is at least ε.

Lemma 2.3 (Transversality lemma). Let $\mathrm{geo}[T_{2\rho}(X^k)] \leqq 1$ and $\|\mathrm{II}(X^k)\| \leqq \Lambda$. Then for all $\delta > 0$, $B_\delta(q)$ contains a point p such that $B_\rho(p)$ is ε-transverse to X^k with

$$\varepsilon = \frac{c(n,\Lambda)\,\delta^n e^{-n\rho}}{\mathrm{Vol}(W^k)}. \tag{2.3}$$

Lemma 2.4 (Smoothing corners). Let $V_1^n, V_2^n \subset M^n$ be compact manifolds with smooth boundary. Assume $\mathrm{geo}(V_j) \leqq 1$, $\|\mathrm{II}(\partial V_j)\| \leqq \Lambda$, and for all $x \in \partial V_1^n \cap \partial V_2^n$

$$\measuredangle\,(N_x(\partial V_1^n), N_x(\partial V_2^n)) \geqq \varepsilon. \tag{2.4}$$

Then for all $\varepsilon' > 0$, there exists a smooth manifold X^n such that

$$X^n \subset V_1^n \cup V_2^n \subset T_{\varepsilon'}(X^n), \tag{2.5}$$

$$\mathrm{Vol}(\partial X^n) \leqq c(n,\varepsilon,\Lambda)\,[\mathrm{Vol}(V_1) + \mathrm{Vol}(V_2)]^2 \tag{2.6}$$

$$\|\mathrm{II}(\partial X^n)\| \leqq c(n,\varepsilon,\varepsilon',\Lambda). \tag{2.7}$$

Proof of Theorem 2.1. Take $\rho = \varepsilon/4$, $\lambda = 10$ and let $U_1, \ldots, U_N$ be as in Lemma 2.2 with

$$N \leqq c(n, \varepsilon/4, 10). \tag{2.8}$$

Let $S_i = \bigcup_j B_{\varepsilon/4}(\tilde{p}_{i,j})$ consist of those balls of U_i whose intersection with W_0 is non-empty, and let $\tilde{V}_i = \bigcup_j B_{3\varepsilon/4}(\tilde{p}_{i,j})$. Let $\tilde{V}_1 = V_1$. By Lemma 2.3, we can find points $p_{2,j}$ with $\mathrm{dist}(\tilde{p}_{2,j}, p_{2,j}) \leqq \varepsilon/4$, and such that $V_2 = \bigcup_j B_{3\varepsilon/4}(p_{2,j})$ is ε-transverse to V_1 with ε as in Lemma 2.3. Since in particular

$$\| \mathrm{II}(\partial V_j) \| \leqq c(n)\, \varepsilon^{-1}, \tag{2.9}$$

applying Lemma 2.4 with $\varepsilon' = \varepsilon/4N$ gives

$$S_1 \cup S_2 \subset W_2 \subset T_\varepsilon(W_0). \tag{2.10}$$

Now replace $\tilde{V}_1, \tilde{V}_2$ by $W_2, \tilde{V}_3$ and repeat the construction to get W_3. By proceeding in this fashion, after N steps we obtain required manifold $W_\varepsilon^n = W_N^n \supset W_0$.

Proof of Lemma 2.2. Take a maximal set of balls of radius ρ such that 1) is satisfied; i.e. the centers of each pair of balls are at distance $\geqq 2\lambda\rho$. Let U_1 be the union of these balls. Then for all $p \in V$,

$$\mathrm{dist}(p, U_1) < 2\lambda\rho - \rho. \tag{2.11}$$

Now choose a set U_2 of balls of radius ρ such that 1) is satisfied, the centers of all balls lie in the complement U_1' of U_1, and the set is maximal with respect to these properties. Then if $p \in U_1'$

$$\mathrm{dist}(p, U_2) < 2\lambda\rho - \rho. \tag{2.13}$$

By repeating the process with U_1 replaced by $U_1 \cup U_2$ we construct U_3 such that if $p \in (U_1 \cup U_2)'$ then

$$\mathrm{dist}(p, U_3) < 2\lambda\rho - \rho. \tag{2.14}$$

We can proceed in this fashion to construct *non-empty* sets $U_1, \ldots, U_{N+1}$ provided $M^n \neq U_1 \cup \ldots \cup U_N$. But if $p \in (U_1 \cup \ldots \cup U_N)'$, by induction, for $j \leqq N$,

$$\mathrm{dist}(p, U_j) < 2\lambda\rho - \rho. \tag{2.15}$$

Let $B_\rho(q_j)$ be a ball from U_j such that $\mathrm{dist}(p, q_j) \leqq 2\lambda\rho$. Note that the condition $q_j \in (U_1 \cup \ldots \cup U_{j-1})'$ for all j, implies that the balls $B_{\rho/2}(q_j)$ are all disjoint. Then since $\mathrm{dist}(p, q_j) \leqq 2\lambda\rho$ implies

$$B_{2\lambda\rho+\rho/2}(p) \supset \bigcup_1^N B_{\rho/2}(q_j), \tag{2.16}$$

we have

$$\mathrm{Vol}(B_{2\lambda\rho+\rho/2}(p)) \geqq \sum_{j=1}^N \mathrm{Vol}(B_{\rho/2}(q_j)). \tag{2.17}$$

By the relative volume estimate (see [G$_2$] or [CGT])

$$c((2\lambda + \tfrac{1}{2})\,\rho_0, n)\ \mathrm{Vol}(B_{\rho/2}(q_j)) \geqq \mathrm{Vol}(B_{2\lambda\rho+\rho/2}(p)), \tag{2.18}$$

from which it follows that

$$N \leqq c((2\lambda + \tfrac{1}{2})\,\rho_0, n) = c(\rho_0, \lambda, N). \tag{2.19}$$

Proof of Lemma 2.3. Let $V_{\rho,\varepsilon}(X^k)$ denote the set of points m, such that $B_\rho(m)$ is not ε-transversal to X^k. We must show that for ε as above,

$$B_\delta(q) \not\subset V_{\rho,\varepsilon}(X^k). \tag{2.20}$$

For this it suffices to have

$$\mathrm{Vol}(V_{\rho,\varepsilon}(X^k)) < \mathrm{Vol}(B_\delta(q)). \tag{2.21}$$

Let $m \in V_{\rho,\varepsilon}(X^k)$. By definition, there is a minimal geodesic γ from some $x \in X^k$ to m that $\rho - \varepsilon \leqq L[\gamma] \leqq \rho + \varepsilon$, and $\measuredangle [N, \gamma'(0)] \leqq \varepsilon$, for some $N \in N_x(X^k)$, $\| N \| = 1$. A standard Rauch Comparison argument shows

$$\mathrm{dist}(\exp_x \rho N, m) \leqq 2 e^\rho \varepsilon. \tag{2.22}$$

In other words, if

$$X(\rho) = \{\exp_x \rho N \mid N \in N_x(X^k)\}, \tag{2.23}$$

then

$$V_{\rho,\varepsilon}(X^k) \subset T_{2e^\rho \varepsilon}(X(\rho)). \tag{2.24}$$

Suppose we can show

$$\mathrm{Vol}(T_{2e^\rho\varepsilon}(X(\rho)) \leqq \underset{\sim}{c}(n, \Lambda)\, \mathrm{Vol}(X^k)\, e^{n\rho} \varepsilon. \tag{2.25}$$

Then since we can assume $\delta < 1$, we have

$$c(n)\, \delta^n \leqq \mathrm{Vol}(B_\delta(q)), \tag{2.26}$$

and we obtain (2.21) with

$$\varepsilon = \frac{c(n)\, \delta^n}{\underset{\sim}{c}(n, \Lambda)\, \mathrm{Vol}(W^k)\, e^{n\rho}} = \frac{c(n, \Lambda)\, \delta^n e^{-n\rho}}{\mathrm{Vol}(W^k)} \tag{2.27}$$

as claimed.

To prove (2.25), we observe that if $y_1, \ldots, y_N$ is an $e^\rho \varepsilon$ dense in $X(\rho)$, then

$$T_{2e^\rho\varepsilon}(X(\rho)) \subset \bigcup_1^N B_{3e^\rho\varepsilon}(y_i), \tag{2.28}$$

which implies

$$\mathrm{Vol}(T_{2e^\rho\varepsilon}(X(\rho))) \leqq c(n)\, (e^\rho \varepsilon)^n N. \tag{2.29}$$

Choose $y_1, \ldots, y_N$ as follows. Let $x_1, \ldots, x_{N_1}$ be a maximal set of points of X^k which are at mutual distance at least ε. Then $x_1, \ldots, x_{N_1}$ is ε-dense and

$$N_1 = \frac{\mathrm{Vol}(X^k)}{\underset{\sim}{c}(n, \Lambda)\, \varepsilon^k}. \tag{2.30}$$

For each x_i, let $z_{i,1}, \ldots, z_{i,N_2}$ be an ε-dense set of points in the unit sphere of the fibre $N_{x_i}(X^k)$, with

$$N_2 = \frac{c(n)}{\varepsilon^{n-k-1}}. \tag{2.31}$$

If we take

$$\{y_1, \dots, y_N\} = \{\exp_{x_i} \rho z_{i,j}\}, \tag{2.32}$$

by a standard Rauch Comparison, $y_1, \dots, y_N$ is $e^\rho \varepsilon$-dense in $X(\rho)$. Since

$$N = N_1 \cdot N_2 = \frac{c(n)}{\underline{c}(n,\Lambda)} \frac{\mathrm{Vol}(X^k)}{\varepsilon^{(n-1)}}, \tag{2.33}$$

(2.25) follows from (2.29). This suffices to complete the proof.

Proof of Lemma 2.4. Let $\partial V_1 \cap \partial V_2 = Y^{n-2}$.

Claim:

$$\mathrm{Vol}(Y^{n-2}) \leqq c(n,\varepsilon^{-1},\Lambda)\,\mathrm{Vol}(\partial V_1) \cdot \mathrm{Vol}(\partial V_2), \tag{2.34}$$

$$\| \mathrm{II}(Y^{n-2}) \| \leqq c(n,\varepsilon^{-1},\Lambda). \tag{2.35}$$

Granting this for the moment, a standard Rauch Comparison shows that if $\hat{W}_\varepsilon \subset V_1 \cup V_2$ denotes the set of points at distance $\geqq \varepsilon$ from $(V_1 \cup V_2)$ (ε small) then

$$\mathrm{Vol}(\partial \hat{W}_\varepsilon) \leqq c(n,\Lambda)\,[\mathrm{Vol}(\partial V_1) + \mathrm{Vol}(\partial V_2)]^2. \tag{2.36}$$

Moreover, ∂W_ε is C^1 and $\mathrm{II}(\partial \hat{W}_\varepsilon)$ is well defined except along the set $\exp_Y \varepsilon N_j$, where N_j is the inward normal to ∂V_j, $j = 1,2$. Off this set, a standard Rauch Comparison argument shows

$$\| \mathrm{II}(\partial \hat{W}_\varepsilon) \| \leqq c(n,\varepsilon^{-1},\Lambda). \tag{2.37}$$

Now, by an elementary argument (the details of which will be omitted) we can approximate $\partial \hat{W}_\varepsilon$ by a manifold ∂W such that the conditions of Lemma 2.4 are satisfied for $i = 0$. The case $i > 0$ can then be handled using the argument of Theorem 2.5 below.

Proof of (2.35). Let T_j denote unit normal fields to Y^{n-2}, which are tangent to ∂V_j. If ∇ is the ambient connection and y is tangent to Y, $\|y\| = 1$, we have

$$(\nabla_y y)^\perp = \alpha_1 T_1 + \beta_1 N_1 = \alpha_2 T_2 + \beta_2 N_2, \tag{2.38}$$

where $|\beta_j| \leqq \Lambda$. Taking the inner product of (2.38) with N_2 yields

$$\alpha_1 = \frac{-\beta_1 \langle N_1, N_2 \rangle + \beta_2}{\langle T_1, N_2 \rangle}. \tag{2.39}$$

Thus

$$\alpha_1^2 + \beta_1^2 = \frac{\beta_1^2 - 2\beta_1 \beta_2 \langle N_1, N_2 \rangle + \beta_2^2}{\langle T_1, N_2 \rangle^2} \tag{2.40}$$

which gives (2.35).

Proof of (2.34). Since $\mathrm{geo}(M) \leqq 1$, $\| \mathrm{II}(\partial V_j) \| \leqq \Lambda$, a standard argument shows that there exists $r = r(n,\varepsilon^{-1},\Lambda)$ with the following property. Let Z_j^{n-1} be a connected open submanifold of ∂V_j which is a relatively closed subset of some $B_{2r}(p) \subset M$, $j = 1,2$. Then if $Z_1 \cap Z_2 \cap B_r(p)$ is non-empty, there exists a diffeomorphism $\psi: B_{2r}(p) \to B_{2r}(p)$ carrying Z_1, Z_2 onto a pair of transversally intersecting hyperplanes (in normal coordinates). Moreover, ψ can be chosen so that

$$\|\psi\|_{C^1} \leqq c(n, \varepsilon^{-1}, \Lambda). \tag{2.41}$$

Let $Z_{2,1} Z_{2,2} \ldots$ denote the components of $\partial V_2 \cap B_{2r}(p)$ whose intersection with $Z_1 \cap B_r(p)$ is non-empty. It follows in particular that $Z_1 \cap Z_{2,i} \cap B_r(p)$ is connected. Thus by (2.41),

$$\mathrm{Vol}(\partial V_2) \geqq \mathrm{Vol}(\cup Z_{2,i}) \geqq c(n, \varepsilon^{-1}, \Lambda) \sum_i \mathrm{Vol}(Z_1 \cap Z_{2,i} \cap B_r(p)). \tag{2.42}$$

Since $\mathrm{geo}(M) \leqq 1$, $\|(\partial V_j)\| \leqq \Lambda$, we have $\mathrm{geo}(\partial V_j) \leqq c(n, \Lambda)$ and there exists a covering of ∂V_1 by $c(n, \varepsilon^{-1}, \Lambda)\,\mathrm{Vol}(\partial V_1)$ balls of radius r (in the metric of ∂V_1). Then for some ball, say $\tilde{B}_r(p) \subset \partial V_1$,

$$\mathrm{Vol}(Y^{n-2} \cap B_r(p)) \geqq \mathrm{Vol}(Y^{n-2} \cap \tilde{B}_r(p)) \geqq \frac{\mathrm{Vol}(Y^{n-2})}{c(n, \varepsilon^{-1}, \Lambda)\,\mathrm{Vol}(\partial V_1)}. \tag{2.43}$$

If we take Z_1 to be the component of $\partial V_1 \cap B_{2r}(p)$ containing $\tilde{B}_r(p)$, combining (2.42), (2.43) gives (2.34).

We now give a result concerning the regularization of metrics of bounded geometry. Let g_1, g_2 be Riemannian metrics on M^n. Put

$$B = g_2 - g_1, \tag{2.44}$$

$$D(x, y) = {}_2\nabla_x y - {}_1\nabla_x y \tag{2.45}$$

where ${}_j\nabla$ is the Riemannian connection of g_j. Note that

$$g_2(D(x, y), z) = \tfrac{1}{2}\{{}_1\nabla_x B(y, z) + {}_1\nabla_y B(x, z) - {}_1\nabla_z B(x, y)\} \tag{2.46}$$

$${}_2\nabla_x B(Y, z) = g_1(D(x, y), z) + g_1(y, D(x, z)). \tag{2.47}$$

The proof of the following theorem is closely related to (and easily yields) the theorem concerning finiteness up to *diffeomorphism* discussed in [C_1], [G_1], [GLP], [P].

Theorem 2.5. Let g be a metric on M^n with $\mathrm{geo}(M, g) \leqq 1$.[9] Then for all $\varepsilon > 0$, there exists a metric g_ε on M^n such that if $B_\varepsilon = g_\varepsilon - g$, $D_\varepsilon = {}_\varepsilon\nabla - \nabla$, we have

1) $\|B_\varepsilon\| \leqq \varepsilon$.
2) $\|D_\varepsilon\| \leqq c(n, \varepsilon^{-1})$.
3) For all $i \geqq 0$.

$$\|{}_\varepsilon\nabla^i R_\varepsilon\| \leqq c(n, i, \varepsilon^{-1}). \tag{2.48}$$

4) There is a constant $c(n) > 0$ such that

$$i(M^n, g_\varepsilon) \geqq c(n). \tag{2.49}$$

Proof: Step 1: Let the positive number ρ be determined as follows. If p, q_1, q_2 are points on the unit 2-sphere such that $\mathrm{dist}(p, q_1) = \frac{1}{4} - \rho$, $\mathrm{dist}(q_1, q_2) \leqq \rho$, then the

[9] See [BMR] for a deeper result concerning manifolds with $|K| \leqq |$

angle between the minimal geodesics γ_1, γ_2 from p to q_1, q_2 is at most say $\frac{\pi}{4}$. Apply the Covering Lemma 2.1 with $2\lambda\rho = \frac{1}{2}$ to construct $U_1, \ldots, U_N$, where $U_i = \bigcup_j B_\rho(p_{ij})$ and $N = c(n, \rho, \lambda) = c(n)$.

Step 2: Let ϕ: $[0, \frac{1}{2}] \to [0, 1]$ be a smooth non-increasing function with $|\phi'| \leqq 12$ such that $\phi \,|\, [0, \frac{1}{4}] \equiv 1$ and $\phi \,|\, [\frac{3}{8}, \frac{1}{2}] \equiv 0$. If

$$\rho_{ij}(z) = \operatorname{dist}(z, p_{ij}), \qquad f_i = \sum_j \rho_{ij},$$

then f: $z \to (f_1, \ldots, f_N)$ is an immersion of M^n into $\mathbb{R}^N$. Clearly, if g_1 is the induced metric,

$$\tfrac{1}{2} g \leqq g_1 \leqq c(n)\, g \tag{2.50}$$

where the first inequality follows from the choice of ρ and Rauch's Comparison Theorem. Let II denote the second fundamental form of f, and ${}_1\nabla$ the Riemannian connection of g_1.

If γ is a geodesic with tangent vector T, then

$$\left.\begin{matrix} \| \mathrm{II}(T, T) \| \\ \| {}_1\nabla_T T \| \end{matrix}\right\} \leqq \left(\sum_1^N (f_i''(t)^2) \right)^{1/2}. \tag{2.51}$$

Using $|K| \leqq 1$, each term on the right hand side of (2.51) can be estimated by a standard Rauch Comparison argument based on the second variation formula. Thus, for g, g_1 we have

$$\left.\begin{matrix} \| D \| \\ \| \mathrm{II} \| \end{matrix}\right\} \leqq c(n). \tag{2.52}$$

Step 3: Let Z^n denote the zero section of the normal bundle, $v = v(f(M^n))$. In view of the definition of f and the bound (2.52), there exists $r = r(n)$ such that the restriction of the exponential map of v to a ball $B_r(x)$, $x \in Z^n$ is a diffeomorphism. We can equip the r-tubular neighborhood $T_r(Z^n)$ with the flat metric pulled back from R^N via exp. Moreover, the tangent bundle of $T_r(Z^n)$ is the pull back $\exp^*(T(R^N))$, and thus is canonically trivial. In particular, let B be the subbundle whose fibres F_x are the tangent spaces to the fibres of v. The orthogonal projections P_x onto the F_x can be represented by a field of matrices $A(x)$, with $A = A^*$, $A^2 = I$. The estimate (2.52) easily implies a bound

$$\| A \| + \| dA \| \leqq c(n) \tag{2.53}$$

on the C^1-norm of $A(x)$, measured with respect to the flat structure on $T_r(Z^n)$. Similarly, the field $r \dfrac{\partial}{\partial r}$ along the fibres of v induces a section s of B, and

$$\| s \|_1 + \| ds \|_1 \leqq c(n). \tag{2.54}$$

Step 4: If we identify Z^n with M^n, then the induced metric on Z^n coincides with g_1. Moreover, Z^n is canonically identified with the transversal intersection of s and the zero section θ of B. The angle between s and θ along Z^n is $\frac{\pi}{4}$.

Let $\phi_\delta = \delta^{-n}\phi((x-y)/\delta)$ be an C^∞ approximate identity, where ϕ is supported on a ball of radius 1. For $\delta < r/2$ the convolution

$$\phi_\delta * A = A_\delta \tag{2.55}$$

is well defined at points of $T_{r/2}(Z^n)$. For δ small, the sum of the dimensions of the eigenspaces of A_δ corresponding to the eigenvalues $\geqq \frac{1}{2}$ equals $N-n$. A simple argument based on the finite dimensional Spectral Theorem (Cauchy Integral Formula) shows that the matrix $C_\delta(x)$ representing orthogonal projection onto the direct sum of these eigenspaces satisfies

$$\| C_\delta \|_{C^i} \leqq c(n,i) \tag{2.56}$$

for all i. Similarly,

$$\| C_\delta(\phi_\delta * s) \|_{C^i} \leqq c(n,i), \tag{2.57}$$

and for $\delta < \delta(n)$, $C_\delta(\phi_\delta * s)$ makes an angle $> \frac{\pi}{4} - \psi(\delta)$, where $\psi(\delta) \to 0$ as $\delta \to 0$. Thus, $C_\delta(\phi_\delta * s) \cap \theta$ is a submanifold Z^n. It follows from (2.56) and (2.57) that the curvature R_δ of the metric h_δ induced on Z^n satisfies

$$\| {}_\delta\nabla^i R_\delta \|_\delta \leqq c(n,i,\delta^{-1}). \tag{2.58}$$

Moreover, by using the normal projection of $T_r(Z^n)$ onto Z^n we see that for $\delta < \delta(n)$, Z^n_δ is diffeomorphic to Z^n and

$$\| h_\delta - g_1 \|_1 + \| h_\delta - g_1 \|_\delta \to 0 \tag{2.59}$$

as $\delta \to 0$. Finally,

$$\| {}_\delta\nabla - {}_1\nabla \|_1 \leqq c(n,\delta^{-1}). \tag{2.60}$$

Step 5: As in Step 3, identify Z^n_δ with M^n and put

$$g(x,y) = h_\delta(E_\delta x, y). \tag{2.61}$$

Extend E_δ to the orthogonal complement bundle $B^\perp$ of B, by parallel translation. If we choose $\delta = \delta(\varepsilon)$ sufficiently small and set

$$g_\varepsilon(x,y) = h_\delta((I-C_\delta)(\phi_\delta * E)(I-C_\delta)\,x, y), \tag{2.62}$$

then g_ε has the properties claimed in the statement of the theorem.

For the application in Sect. 5, we will also require the following rather standard result (the notation is as in (2.44), (2.45) and we put $\| \;\|_{1+2} = \| \;\|_1 + \| \;\|_2$).

Lemma 2.6. Let g_1, g_2 be Riemannian metrics on M^n. Let h be the metric

$$h = dt^2 + t g_1 + (1-t) g_2 \tag{2.63}$$

on $[0,1] \times M^n$. Then there is a constant c such that the ambient curvature $\bar{R}_t$ and second fundamental form II_t of (t, M) satisfy

$$\mathrm{II}_t(x,y) = D(x,y)\frac{\partial}{\partial t} \tag{2.64}$$

$$\| \bar{R}_t \|_t = \| R_1 \|_1 + \| R_2 \|_2 + c(\| B \|^2_{1+2} + \| D \|_{1+2} + \| D \|^2_{1+2}). \tag{2.65}$$

Proof: Let $\nabla, \tilde{\nabla}$ be connections on the tangent bundle of a manifold Y, with ∇ torsion free. If

$$\nabla_u v - \tilde{\nabla}_u v = \mathscr{D}(u, v) \tag{2.66}$$

denotes the difference tensor, then by a standard calculation, the difference of the curvature tensors is given by

$$\begin{aligned} R(u,v)w = \tilde{R}(u,v)w &+ \mathscr{D}(u, \mathscr{D}(v,w)) - \mathscr{D}(v, \mathscr{D}(u,w)) \\ &- \mathscr{D}(\mathscr{D}(u,v), w) + \mathscr{D}(\mathscr{D}(v,u), w) \\ &+ \tilde{\nabla}_u \mathscr{D}(v,w) - \tilde{\nabla}_v \mathscr{D}(u,w). \end{aligned} \tag{2.67}$$

Let ∇ be the Riemannian connection on $Y = (0,1) \times M^n$ with metric h. Let $\tilde{\nabla}$ be the Whitney sum of the connection for which $\frac{\partial}{\partial t}$ is parallel and the Riemannian connection ${}_t\nabla$ of $t g_1 + (1-t) g_2$ on the subbundle tangent to the factors (t, M^n). A computation based on (2.67) shows that

$$\| \bar{R}_t \|_t \leqq \| R_t \|_t + c\{ \| B \|_{1+2}^2 + \| D \|_{1+2} \}, \tag{2.68}$$

where R_t is the curvature of g_t.

In order to calculate $\| R_t \|_t$ we observe that if $M^n \times M^n$ is equipped with the product metric $t\pi_1^* g_1 + (1-t)\pi_2^* g_2$, then the metric g_t is the metric induced on the diagonal. A simple computation with the Gauss equation gives

$$\| R_t \|_t \leqq \| R_1 \|_1 + \| R_2 \|_2 + \| D \|_{1+2}^2 \tag{2.69}$$

which, together with (2.68) yields (2.65).

3. The Euler Characteristic and the Stable Acyclicity of the Boundary

In this section we state our main result on the geometric Euler characteristic $\chi(M^n, g)$ for coverings which are either profinite or normal. The proof and explicit interpretation will only be given for the case in which $\tilde{M}$ is profinite. In Sect. 6, we give a second (L^2) interpretation assuming M^n is normal. However, the proper homotopy invariance of this interpretation will only be proved if M^n is *both* profinite and normal (see Sect. 7). The proof for general normal coverings will be given in [CG$_1$].

Let $A_1 \subset A_2$ and set

$$\mathbf{b}^i(A_1, A_2) = \dim\{ j(H^i(A_2)) \subset H^i(A_1) \} \tag{3.1}$$

where j: $A_1 \to A_2$ is the inclusion map (and real coefficients are understood); compare (0.18). If $A_1 \subset A_2 \subset A_3 \subset A_4$, one easily checks that

$$\mathbf{b}^i(A_1) \leqq \mathbf{b}^i(A_2) \leqq \mathbf{b}^i(A_2, A_4) \leqq \mathbf{b}^i(A_3, A_4). \tag{3.2}$$

Moreover, let $A \subset Y$, be a finite complex and let f: $Y \to Z$, g: $Z \to Y$ be simplicial and determine a homotopy equivalence. Then

$$\mathbf{b}^i(A, Y) \leqq \mathbf{b}^i(f(A), Z) \leqq \mathbf{b}^i(g \circ f(A), Y). \tag{3.3}$$

Now let $p\colon \tilde{Y}^n \to Y^n$ be profinite. Put

$$\sup \tilde{\chi}(Y^n) = \overline{\lim_{A \to \infty}}\ \overline{\lim_{j \to \infty}} \sum_{i=1}^{n} (-1)^i \frac{1}{d_j} \mathbf{b}^i(p_j^{-1}(A), \tilde{Y}_j^n) \leqq \infty, \tag{3.4}$$

and define $\inf \tilde{\chi}(Y^n)$ similarly.[10] These are not, a priori, homotopy invariants in general. But by (3.2) and a diagonal argument, there are subsequences, $S = \{\tilde{Y}^n_{j(l)}\}$, such that

$$\infty \geqq \tilde{\mathfrak{b}}^i(Y^n, S) \stackrel{\text{def}}{=} \lim_{A \to \infty}\ \overline{\lim_{l \to \infty}} \frac{1}{d_{j(l)}} \mathbf{b}^i(p_{j(l)}^{-1}(A), \tilde{Y}^n_{j(l)}), \tag{3.5}$$

$$= \lim_{A \to \infty}\ \underline{\lim_{l \to \infty}} \frac{1}{d_{j(l)}} \mathbf{b}^i(p_{j(l)}^{-1}(A), \tilde{Y}^n_{j(l)}),$$

exists. Using (3.3), $\tilde{\mathfrak{b}}^i(Y^n, S)$ is a homotopy invariant. Thus, if

$$\tilde{\mathfrak{b}}^i(Y^n, S) < \infty, \qquad i = 0, \ldots n, \tag{3.6}$$

and $\sup \tilde{\chi}(Y^n) = \inf \tilde{\chi}(Y^n)$, this number is also a homotopy invariant. (In the proof of Theorem 3.1 below, (3.6), for $Y^n = M^n$, follows from (3.10) and the analog of (3.13) with $\overline{B_{jk} - A_{jk}}$ replaced by B_{jk}).

Theorem 3.1. Let M be complete, $\mathrm{Vol}(M^n) < \infty$. Suppose $\tilde{M}$ is either profinite or normal, and that $\mathrm{geo}(\tilde{M}) \leqq 1$.

1) Then $\chi(M^n, g)$ is a proper homotopy invariant of M.
2) In case $\tilde{M}$ is profinite,

$$\chi(M, g) = \sup \tilde{\chi}(M) = \inf \tilde{\chi}(M) \tag{3.7}$$

3) If, in addition, M has finite topological type,

$$\chi(M, g) = \chi(M). \tag{3.8}$$

Remark 3.1. As we indicated in the introduction Theorem 3.1 has a simple generalization to the case $\mathrm{geo}_\infty(\tilde{M}) \leqq 1$; the details will be omitted.

Remark 3.2. Note that for $\tilde{M}$ profinite (but not normal) we can view Theorem 3.1 as providing asymptotic information about the sequence of finite coverings M_j, in terms of $\chi(M, g)$. However, in contrast to the situation in which $\tilde{M}$ is normal, we do not obtain information about $\tilde{M}$ itself; compare Sect. 6.

Proof of Theorem 3.1. Let $\tilde{M}$ be profinite and let $\cup M_k = M$ be an exhaustion of M by compact submanifolds with boundary. Let $M_k - R$ denote the set of points of M_k at distance R from the boundary. For j sufficiently large, we can apply the approximation theorem (Theorem 2.1), to $p_j^{-1}(M_k) - 1$, $p_j^{-1}(M_k)$, with $\varepsilon = \frac{1}{2}$, to obtain submanifolds $A_{jk} \subset p_j^{-1}(M_k) \subset B_{jk}$. It follows from (0.8), (2.1) and (2.2) that

[10] The notation $A \to \infty$ refers to the partial ordering on finite subcomplexes induced by inclusion

for all $\varepsilon > 0$, k there exists k_0, $N(k)$ such that for $k > k_0, j > N(k)$,

$$\left|\chi(M^n, g) - \frac{1}{d_j}\chi(B_{jk})\right|$$
$$\leqq \left|\chi(M^n, g) - \frac{1}{d_j}\int_{B_{jk}} P_\chi(\Omega)\right| + \left|\frac{1}{d_j}\int_{B_{jk}} P_\chi(\Omega) - \frac{1}{d_j}\chi(B_{jk})\right| < \varepsilon. \quad (3.9)$$

By (3.2),

$$\mathbf{b}^i(A_{jk}) \leqq \mathbf{b}^i(p_j^{-1}(M_k)) \leqq \mathbf{b}^i(p_j^{-1}(M_k), \tilde{M}_j) \leqq b^i(B_{jk}). \quad (3.10)$$

However, the exact sequence of the pair $(B_{jk}, \overline{B_{jk} - A_{jk}})$ together with excision shows

$$|\mathbf{b}^i(A_{jk}) - b^i(B_{jk})| \leqq b^{i-1}(\overline{B_{jk} - A_{jk}}) + b^i(\overline{B_{jk} - A_{jk}}). \quad (3.11)$$

The manifold with boundary $\overline{B_{jk} - A_{jk}}$ has bounded geometry, for $j > N(k)$. Moreover, for k sufficiently large,

$$\mathrm{Vol}(\overline{B_{jk} - A_{jk}}) \leqq d_j \varepsilon. \quad (3.12)$$

By a standard argument, it follows that

$$\sum_i b^i(\overline{B_{jk} - A_{jk}}) \leqq c(n)\,\mathrm{Vol}(\overline{B_{jk} - A_{jk}}), \quad (3.13)$$

which together with (3.10)–(3.12), allows us to replace $\chi(B_{jk})$ in (3.9) by $\boldsymbol{\chi}(p_j^{-1}(M_k), M_j)$. This suffices to prove 1) and 2) in case M is profinite.

3) In case M has finite topological type, we note that for k sufficiently large, ∂M_k will be contained in the image of a tubular neighborhood of $\partial \bar{M}$. Then

$$\mathbf{b}^i(p_j^{-1}(M_k), \tilde{M}_k) = b^i(\tilde{M}_j) \quad (3.14)$$

and we can replace $\boldsymbol{\chi}(p_j^{-1}(M_k), \tilde{M}_j)$ by $\chi(\tilde{M}_j)$. Since

$$\frac{1}{d_j}\chi(\tilde{M}_j) = \chi(M_j), \quad (3.15)$$

(3.8) follows.

By a similar argument, we have the following result on the stable Betti numbers of the boundary for the case in which M has finite topological type, $\partial M = N$ and $\tilde{N}$ is profinite.

Theorem 3.2. Let N be compact and suppose $\tilde{N} \to N$ is profinite. If $[0, \infty) \times N$ admits a complete metric with $\mathrm{Vol}([0, \infty) \times N) < \infty$ and $\mathrm{geo}([0, \infty) \times \tilde{N}) < 1$, then for all i,

$$\lim_{j \to \infty} \frac{1}{d_j} b^i(\tilde{N}_j) = 0. \quad (3.16)$$

Proof: Take $M_k = [0, R] \times N$ in (3.10). Note that

$$\mathbf{b}^i(p_j^{-1}(M_k)) = 0, \quad (3.17)$$

$$\mathbf{b}^i(p_j^{-1}(M_k), p_j^{-1}(M_{k+1})) = b^i(N). \quad (3.18)$$

But as in the proof of Theorem 3.1,

$$\lim_{j\to\infty} \frac{1}{d_j} \mathbf{b}^i(M_k)) = \lim_{j\to\infty} \frac{1}{d_j} \mathbf{b}^i(p_j^{-1}(M_k), p_j^{-1}(M_{k+1})), \tag{3.19}$$

which together with (3.17), (3.18) completes the proof.

See Theorem 7.2 and [CG_1] for an L^2 version of Theorem 3.2 in case N is normal but not necessarily profinite.

As noted in [G_3] the argument given in the proof of Theorem 3.1 also applies to certain situations in which M^n has infinite volume. For example, suppose that M^n is *stable at infinity* in the sense that there exists an exhaustion by compact submanifolds with boundary M_k, such that for some fixed $\rho > 0$,

$$\mathrm{Vol}(M_k)/\mathrm{Vol}(T_\rho(M_k)) \to 1. \tag{3.20}$$

In this case M_k is called a *stable sequence*. Note that if M has subexponential growth

$$\varliminf_{r\to\infty} \log \mathrm{Vol}(B_r(p))/r = 0, \tag{3.21}$$

then M is stable at infinity.

Theorem 3.3. Let $\mathrm{geo}(V) \leqq 1$ and let V_k^n be a stable sequence for V^n. Then

$$\lim_{k\to\infty} \frac{1}{\mathrm{Vol}(V_k)} |\chi(V_k, V) - \chi(V_k, g)| = 0. \tag{3.22}$$

Proof: The argument is completely analogous to the proof of Theorem 3.1.

In the same way we obtain.

Corollary 3.4. Let $\tilde{M}$ be profinite and satisfy $\mathrm{geo}(\tilde{M}) \leqq 1$. If M_k is a stable sequence for M,

$$\lim_{k\to\infty} \frac{1}{\mathrm{Vol}(M_k)} |\chi(M_k, g) - \sup \tilde{\chi}(M_k)|$$
$$= \lim_{k\to\infty} \frac{1}{\mathrm{Vol}(M_k)} |\chi(M_k, g) - \inf \tilde{\chi}(M_k)| = 0. \tag{3.23}$$

Corollary 3.5. Let M^n be as in Corollary 3.4 and assume that for some constant $k > 0$, one of the following pointwise relations holds (where ω denotes the volume form)

$$P_\chi(\Omega) > k\omega, \tag{3.24}$$
$$P_\chi(\Omega) < -k\omega. \tag{3.25}$$

1) If M has finite topological type then $\mathrm{Vol}(M^n) < \infty$.
2) Suppose $n = 4$ and (3.24) holds. If one only assumes that M^4 admits a *CW* decomposition with finitely many 2-cells, it follows that M^4 has finite volume.

Example 3.1. Recall that if M^4 *has pinched negative curvature*, $-1 \leqq K \leqq c < 0$, then

$$P(\Omega) > k(c)\,\omega. \tag{3.26}$$

Also, if for example, M^{2m} *has constant negative curvature*, then (3.24) holds for n even and (3.25) holds for n odd. Thus, Corollary 3.25 applies in these cases. Conversely, Jørgensen has constructed an example of a 3-dimensional manifold having constant negative curvature, finite topological type, linear growth and infinite volume; see [J]. (In fact his manifold is an infinite cyclic covering of a compact manifold of constant negative curvature.) One suspects that such examples exist in all odd dimensions, but not in even dimensions (see Example 5.1 for the continuation of this discussion).

4. An Estimate for the η-Invariant

Recall that the η-invariant can be defined as follows. Let N^{4l-1} be an oriented Riemannian manifold and consider the self adjoint operator $*\,d$, which sends the space of coexact $(2l-1)$-forms to itself. On this subspace $*\,d$ is a certain square root of the Laplacian, $\triangle$. Thus, if $\{\lambda_j\}$ are the eigenvalues of $\triangle$, the eigenvalues μ_j of $*\,d$ satisfy $\mu_j^2 = \lambda_j$.

One defines the zeta functions $\zeta_\pm(s)$ by

$$\zeta_\pm(s) = \sum_{\pm\mu_j > 0} |\mu_j|^{-2s} \tag{4.1}$$

where the individual series converge for $\operatorname{Re} s > (4l-1)/2$. If we put

$$\eta(s) = \zeta_+(s) - \zeta_-(s), \tag{4.2}$$

then $\eta(s)$ extends by analytic continuation to a meromorphic function in the whole complex plane; see [APS_1]. A more refined analysis shows that $\eta(s)$ is holomorphic for $\operatorname{Re} s > -\frac{1}{2}$; see [Gil]. The value $\eta(0)$ is by definition the η-invariant $\eta(N^{4l-1})$ and the Atiyah-Singer-Patodi formula, (0.9), for the signature of a manifold with boundary holds.

Let $\Gamma(s)$ be the gamma function, defined by

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt, \tag{4.3}$$

for $s > 0$. If $e^{-\Delta t} = E(x, y, t)$ denotes the heat kernel on $(2l-1)$-forms, we have the function

$$\operatorname{tr}(*\,d e^{-\Delta t}) = \Sigma\, \mu_j e^{-\lambda_j t}, \tag{4.4}$$

which is smooth for $t > 0$, and exponentially decreasing as $t \to \infty$. The analysis mentioned above shows that

$$\operatorname{tr}(*\,d e^{-\Delta t}) \sim c_0 + c_1 t + \ldots + \tag{4.5}$$

as $t \to 0$. Hence, by a simple change of variables, we obtain the integral formula

$$\eta(s) = \frac{1}{\Gamma(s+\frac{1}{2})} \int_0^\infty t^{s-1/2} \operatorname{tr}(*de^{-\Delta t})\,dt, \tag{4.6}$$

valid for $s > -\frac{1}{2}$. The assertion of Theorem 0.1,

$$|\eta(N^{4l-1})| \leqq c(4l-1)\operatorname{Vol}(N^{4l-1}) \tag{4.7}$$

if $\operatorname{geo}(N) \leqq 1$, is based on (4.6).

Proof of Theorem 0.1. Set $s = 0$, in (4.6) and break the integral into $\int_0^1 + \int_1^\infty$. We estimate these pieces separately.

Estimate for $\int_0^1$: By combining (0.9) with the regularization theorem (Theorem 2.5) and Lemma 2.6, it suffices to assume that $\|\nabla^i R\| \leqq c(i, 4l-1)$, $i = 0, \ldots$. The argument is now very close to that of [C_2], [CGT], so we will be brief.

Let P be a parametrix for $e^{-\Delta t}$, which is compactly supported in space and time. Set

$$\left(\Delta_x + \frac{\partial}{\partial t}\right) P(x,y,t) = Q(x,y,t). \tag{4.8}$$

If we choose P sufficiently accurately, we will have

$$\|\Delta_x^i \Delta_y^j * dQ(x,y,t)\| \leqq c(4l-1) \tag{4.9}$$

for $i, j \leqq l$ and all $t > 0$. Here $\|*dQ\|$ and $c(4l-1)$ depend on $c(i, 4l-1)$ $i = 0, \ldots, 2l$. As in [C_2], Sect. 5, we can now employ Duhamel's principle

$$\begin{aligned}&\Delta_x^i \Delta_y^j(*dE(x,y,t) - *dP(x,y,t))\\ &\quad= \int_0^t e^{-\Delta(t-s)} \Delta_x^i \Delta_y^j * dQ(s)\,ds,\end{aligned} \tag{4.10}$$

and the elliptic estimate

$$\|f\| \leqq c(4l-1) \sum_{i=0}^{l} \|\Delta^i f\|, \tag{4.11}$$

to bound the pointwise norm of $(*dE - *dP)$, for $0 < t \leqq 1$ (the constant (4.11) is controlled by $\|R\|$; see [CGT]). If we write

$$\operatorname{tr}(*dE(t)) = \operatorname{tr}(*dE - *dP) + \operatorname{tr}(*dP) \tag{4.12}$$

(where the fact $\operatorname{tr}(*dP) \sim 0$ as $t \to 0$ plays no essential role) the estimate for $\int_0^1$ follows easily.

Estimate for $\int_1^\infty$: We have

$$\left|\int_1^\infty t^{-1/2}\operatorname{tr}(*de^{-\Delta t})\,dt\right| \leqq \sum_i \int_1^\infty t^{-1/2}\lambda_i^{1/2}e^{-\lambda_i t})\,dt$$
$$= \sum_i e^{-\lambda_i}\int_1^\infty e^{-\lambda_i(t-1)}\lambda_i^{1/2}t^{-1/2}\,dt$$
$$= \sum_i e^{-\lambda_i}\int_0^\infty e^{-u}(u+\lambda_i)^{-1/2}\,du$$
$$\leqq \sum_i e^{-\lambda_i}\int_0^\infty e^{-u}u^{-1/2}\,du$$
$$= \sqrt{\pi}\operatorname{tr}(P_{ce}e^{-\Delta}), \tag{4.13}$$

where P_{ce} denotes orthogonal projection on coexact forms. But by the same techniques as were employed above,

$$\operatorname{tr}(e^{-\Delta}) \leqq c(4l-1)\operatorname{Vol}(N^{4l-1}), \tag{4.14}$$

which suffices to complete the proof.

Note that the proof of Theorem 0.1 immediately generalizes to give the bound

$$|\eta_{E^k}(N)| \leqq c(4l-1)\,k. \tag{4.15}$$

5. The η-Invariant and Signature

Let M^{4l} be complete and $\tilde{M}^{4l}$ profinite. If $M_k^{4l} \subset M^{4l}$ is a compact submanifold with boundary set

$$\begin{aligned} \sup\tilde{\sigma}(M_k) &= \limsup_j \frac{1}{d_j}\sigma(p_j^{-1}(M_k)) \\ \sup\tilde{\sigma}(M) &= \limsup_{M_k}\sup\tilde{\sigma}(M_k). \end{aligned} \tag{5.1}$$

Similarly, we have $\inf\tilde{\sigma}(M_k)$, $\inf\sigma(M)$. Recall that $\sigma(M_k)$ can be defined as the signature of the cup product pairing on

$$j(H^{2l}(M_k^{4l}, M_k^{4l})) \subset H^{2l}(M_k^{4l}).$$

Thus we could also write $\boldsymbol{\tilde{\sigma}}(M_k)$, etc.

As in Sect. 3, $\sup\tilde{\sigma}(M)$, $\inf\tilde{\sigma}(M)$ are proper homotopy invariants. Moreover, we have the following generalization of (0.13) of the introduction (and analog of Theorem 3.1).

Theorem 5.1. Let M^{4l} be complete, Vol $M^n) < \infty$. Suppose $\tilde{M}$ is either profinite or normal and that $\operatorname{geo}(\tilde{M}) \leqq 1$.

1) Then $\sigma(M,g)$ is a proper homotopy invariant of M.
2) In case $\tilde{M}$ is profinite, for any exhaustion by compact subsets, $\cup M_k = M$,

$$\sigma(M,g) = \sup \tilde{\sigma}(M) = \inf \tilde{\sigma}(M). \tag{5.3}$$

3) If, in addition, M has finite topological type,

$$\sigma(M,g) = \lim_{j\to\infty} \frac{1}{d_j}\sigma(\tilde{M}_j). \tag{5.4}$$

Proof: 1) See Sect. 6 and 7 and [CG_1].

2) If $\tilde{M}$ is profinite, in view of (0.9) and Theorem 0.1, the proof follows by an argument analogous to that of Theorem 3.1.

3) The case in which M has finite topological type follows similarly.

Remark 5.1. There is also an easy generalization of Theorem 5.1 to the case $\mathrm{geo}_{\tilde{\infty}}(M) \leqq 1$.

Now recall the invariants $\rho_{E^k}(N^{4l-1})$ defined in (0.11). Note that these include as a special case invariants of finite coverings $\tilde{N}_j \to N$ of order $d_j < \infty$. To every such covering there corresponds a bundle $E^{d_j}(\tilde{N}_j)$ whose holonomy representation is the representation of $\pi_1(N)$ induced from the trivial representation of $\pi_1(\tilde{N}_j) \subset \pi_1(N)$. Analysis on $\tilde{N}^j$ is canonically identified with analysis on N_j with coefficients in $E^{d_j}(\tilde{N}_j)$.

We now observe that for certain degenerating sequences of metrics g_i which might exist on N, we have

$$\lim_{i\to\infty} \eta_{E^k}(N,g_i) = \rho_{E^k}(N); \tag{5.5}$$

compare Example 1.7.

Theorem 5.2. Let N^{4l-1} admit a sequence of metrics g_i such that $\mathrm{geo}(\tilde{N},g_i) \leqq 1$, for some covering space $\tilde{N}$, and such that

$$\mathrm{Vol}(N,g_i) \to 0. \tag{5.6}$$

Then for all E^k we have

$$\lim_{i\to 0} \frac{1}{k}\eta_{E^k}(N,g_i) = \rho_{E^k}(N). \tag{5.7}$$

Proof: If $\tilde{N}$ is profinite the claim follows immediately by applying (4.14) to a sequence of finite covering spaces $\tilde{N}_{j(i)}$, with say $\mathrm{geo}(\tilde{N}_{j(i)}) \leqq 2$.

Note that if there exists a manifold with boundary X^{4l} with $\partial X^{4l} = N^{4l-1}$, such that E^k extends to X^{4l} as a flat bundle and $\pi_1(\tilde{N})$ injects into $\pi_1(X)$, then ρ_{E^k} is a homotopy invariant of (X,N).

We also have the following analogs of the results of Sect. 3 for manifolds of infinite volume which are stable at infinity.

Theorem 5.3. Let $\operatorname{geo}(V) \leqq 1$ and let V_k^n be a stable sequence of V^n. Then

$$\lim_{k\to\infty} \frac{1}{\operatorname{Vol}(V_k)} |\sigma(V_k, g) - \sigma(V_k)| = 0. \tag{5.8}$$

Corollary 5.4. Let $\tilde{M}$ be profinite and satisfy $\operatorname{geo}(\tilde{M}) \leqq 1$. If M_k is a stable sequence for M,

$$\lim_{k\to\infty} \frac{1}{\operatorname{Vol}(M_k)} |\sigma(M_k, g) - \sup \tilde{\sigma}(M_k)|$$
$$= \lim_{k\to\infty} \frac{1}{\operatorname{Vol}(M_k)} |\sigma(M_k, g) - \inf \tilde{\sigma}(M_k)| = 0. \tag{5.9}$$

Corollary 5.5. Let M^{4l} be as in Corollary 5.3 and assume that for some constant $k > 0$, one of the following pointwise relations holds (where ω denotes the volume form)

$$P_L(\Omega) > k\,\omega, \tag{5.10}$$

$$P_L(\Omega) < -k. \tag{5.11}$$

Then if in addition M^{4l} admits a CW decomposition with finitely many cells in dimension $2l$, it follows that $\operatorname{Vol}(M) < \infty$.

Remark 5.2. If G is a semisimple group without $SO(k,1)$ or $SU(n,1)$ factors, then the following stronger assertion was pointed out to the authors by D. Kazhdan.

If G/Γ has subexponential growth, then $\operatorname{Vol}(G/\Gamma) < \infty$ for an *arbitrary* discrete subgroup $\Gamma \subset G$. This follows immediately from Kazhdan's T-property for G (see [K_1]).

Example 5.1. Corollary 5.5 is similar to Corollary 3.5 in dimension 4, but unlike the latter, it does not yield non-trivial information if M^{4l} has constant negative curvature. However, for spaces covered by the complex ball (the dual of complex projective space) it is stronger than Corollary 3.5.

6. L^2-Theory for Normal Coverings

In this section we assume only that $\tilde{M}$ is normal.[11] We begin by observing that the L^2-Index Theorem for coverings of compact manifolds (see [A], [S]) has an easy generalization to our situation. The proper homotopy invariance of $\chi(M,g)$, $\sigma(M,g)$ is then a consequence of the proper homotopy invariance of the corresponding L^2-Betti numbers, $\tilde{b}^i_{(2)}(M)$. The latter is proved in Sect. 7, under the additional hypothesis that $\tilde{M}$ profinite. The general case will be treated in [CG_1].

Before discussing the L^2-Index Theorem, we will briefly recall the relevant concept von Neumann dimension; for further details see [A], [Co], [Gui], [Nai]. Let Γ be a discrete countable group and let $L^2(\Gamma)$ be regarded as a Γ-module via the left regular representation $\{L_g\}$. If W is a Hilbert space with trivial Γ-action,

[11] We continue to assume $\operatorname{geo}(\tilde{M}) \leqq 1$, but drop the assumption that $\tilde{M}$ is profinite

$L^2(\Gamma) \otimes_\Gamma W$ splits naturally as a direct sum of copies of W, one for each element of Γ. Relative to this decomposition a bounded operator A can be written as a matrix (A_{g_1, g_2}) where A_{g_1, g_2}: $W_{g_2} \to W_{g_1}$. If A commutes with all $\{L_g\}$, it follows immediately that

$$A_{g_1, g_2} = A_{g_2^{-1} g_1} = A_{g_2^{-1} g_1, e} \tag{6.1}$$

for some A_g: $W \to W$. The operators R_g of the right regular representation satisfy

$$(R_g)_g = \begin{cases} 0 & g \neq g \\ I & g = g \end{cases}. \tag{6.2}$$

It follows that any A as above can be written as

$$A = \sum R_g \otimes A_g. \tag{6.3}$$

If A_e is trace class, define

$$\operatorname{tr}_\Gamma(A) = \operatorname{tr}(A_e). \tag{6.4}$$

Let $\mathcal{N}$ consist of all T commuting with $\{L_g\}$ such that $(T^* T)_e$ is trace class. Let $\mathcal{M} = \mathcal{N}^2$ be the set of all A of the form

$$A = \sum_{i=1}^{N} T_i S_i \tag{6.5}$$

where $T_i, S_i \in \mathcal{N}$. Then it is not difficult to see that if $A \in \mathcal{M}$ and B is a bounded operator commuting with $\{L_g\}$

$$\operatorname{tr}(AB) = \sum_g \operatorname{tr}(A_g B_{g^{-1}}) = \sum_g \operatorname{tr}(B_{g^{-1}} A_g) = \operatorname{tr}(BA). \tag{6.6}$$

In particular, if U is unitary and commutes with $\{L_g\}$,

$$\operatorname{tr}(UAU^{-1}) = \operatorname{tr}(A). \tag{6.7}$$

Let H be a Γ-module, i.e. Γ acts on the Hilbert space H by unitary operators. Let f_j: $H \to L^2(\Gamma) \otimes W_j$, $j = 1, 2$ be isometric Γ-imbeddings and put

$$L^2(\Gamma) \otimes W_j = f_j(H) \oplus f_j(H)^\perp. \tag{6.8}$$

Then since

$$\begin{array}{ccccccc} f_1(H) & \oplus & f_1(H)^\perp & \oplus & f_2(H)^\perp & \oplus & f_2(H) \\ \downarrow{\scriptstyle f_2 f_1^{-1}} & & \searrow & & \swarrow & & \downarrow{\scriptstyle f_1 f_2^{-1}} \\ f_2(H) & \oplus & f_2(H)^\perp & \oplus & f_1(H)^\perp & \oplus & f_1(H). \end{array} \tag{6.9}$$

It follows that a Γ-isomorphism $f_2 f_1^{-1}: f_1(H) \to f_2(H)$ extends stably to Γ-automorphism of

$$L^2(\Gamma) \otimes W_1 \oplus L^2(\Gamma) \otimes W_2 \simeq L^2(\Gamma) \otimes (W_1 \oplus W_2). \tag{6.10}$$

In view of (6.7), this immediately implies that if $\pi_{f_j}(H)$ denotes orthogonal projection on $f_j(H)$,

$$\dim_\Gamma H \stackrel{\text{def}}{=} \operatorname{tr}_\Gamma(\pi_{f_j(H)}) \tag{6.11}$$

is well defined independent of f_j.

If $H \cong H_1 \oplus H_2$ splits as an orthogonal direct sum,

$$\dim_\Gamma H = \dim_\Gamma H_1 \oplus \dim_\Gamma H_2. \tag{6.12}$$

Moreover, if $|\operatorname{tr} A_i| \leqq \text{const}$ and $A_i \to A$ weakly,

$$\operatorname{tr}(A_i) \to A. \tag{6.13}$$

Finally, if $A\colon H_1 \to H_2$ is a bounded injective map of Γ-modules with dense range,

$$A(A^* A)^{-1/2}\colon H_1 \to H_2 \tag{6.14}$$

is a Γ-isometry.

Now suppose that $\tilde{M} \to M$ is a normal covering, $M = \tilde{M}/\Gamma$, $\operatorname{Vol}(M) < \infty$, $\operatorname{geo}(\tilde{M}) \leqq 1$. Any choice of fundamental domain F determines an isomorphism

$$L^2(\tilde{M}) \simeq L^2(\Gamma) \otimes_\Gamma L^2(F). \tag{6.15}$$

Define the reduced L^2-cohomology of $\tilde{M}$ by

$$\bar{H}^i_{(2)}(\tilde{M}) = \ker \bar{d}_i / \overline{\operatorname{im} d_{i-1}} \tag{6.16}$$

where $\alpha \in \operatorname{dom} d$, if $\alpha, d\alpha \in L^2$. Since $\tilde{M}$ is complete, we have the Γ-isomorphism

$$\bar{H}^i_{(2)}(\tilde{M}) \simeq \tilde{\mathscr{H}}^i \subset L^2(\tilde{M}) \tag{6.17}$$

where $\tilde{\mathscr{H}}^i$ is the space of (necessarily closed and coclosed) L^2-harmonic forms. It follows from the above discussion that the L^2-Betti numbers,

$$\tilde{b}^i_{(2)}(M) \stackrel{\text{def}}{=} \dim_\Gamma \tilde{\mathscr{H}}^i, \tag{6.18}$$

depend only on the quasi-isometry class of g, as does the isomorphism class of the Γ-module $\tilde{\mathscr{H}}^i$ itself. Note that if $\Gamma_1 \subset \Gamma$, and $\operatorname{ind} \Gamma_1 = d < \infty$, then

$$\tilde{b}^i_{(2)}(\tilde{M}/\Gamma_1) = d \cdot \tilde{b}^i_{(2)}(\tilde{M}/\Gamma). \tag{6.19}$$

Thus, the $\tilde{b}^i_{(2)}$ behave multiplicatively under coverings, even though they are *not* locally computable.

If $\prod$ denotes orthogonal projection on $\tilde{\mathscr{H}}^i$, then

$$\prod(\omega) = \int_{\tilde{M}} \tilde{h}^i(x,y)\, \omega(y)\, dy \tag{6.20}$$

where $\tilde{h}^i(x,y)$ is a symmetric C^∞ kernel which for each fixed x, satisfies

$$\tilde{h}^i(x,y) \in L^2(V). \tag{6.21}$$

The pointwise norm of $\tilde{h}^i(x,y)$ can be bounded in terms of the constant in the elliptic estimate near the points x, y (see [CGT]). Hence, if we assume

$$\operatorname{geo}(\tilde{M}) \leqq 1, \tag{6.22}$$

it follows that

$$\sup \| \tilde{h}^i(x,y) \| \leqq c(n). \tag{6.23}$$

Since $\tilde{h}^i(x,y)$ is invariant under isometries, the pointwise trace, $\operatorname{tr}(\tilde{h}^i(x,x))$, is invariant under Γ and thus can be regarded as a function on M. It is not difficult to verify that

$$\tilde{b}^i_{(2)}(M) = \int_M \operatorname{tr}(\tilde{h}^i(x,x))\, dx. \tag{6.24}$$

Example 6.1. Let M/Γ be a compact manifold admitting a sequence of metrics g_i with $\operatorname{geo}(\tilde{M},\tilde{g}_i) \leqq 1$, $\operatorname{Vol}(M,g_i) \to 0$. Since (6.24) is independent of metric, (6.23) implies $\dim \mathscr{H}^i = 0$ for all i.

Set

$$\begin{aligned} \tilde{\chi}_{(2)}(M) &= \sum (-1)^i\, \tilde{b}^i_{(2)}(M) \\ \tilde{\sigma}_{(2)}(M^{4l}) &= \int_M \operatorname{tr}(* \tilde{h}^{2l}(x,x))\, dx. \end{aligned} \tag{6.25}$$

Then we have:

Theorem 6.1 (L^2-Index Theorem).[12]

$$\sigma(M,g) = \tilde{\chi}_{(2)}(M) \tag{6.26}$$

$$\sigma(M,g) = \tilde{\sigma}_{(2)}(M). \tag{6.27}$$

In case M is compact, Dodziuk [D] showed that the $\tilde{b}^i_{(2)}(M)$ are *homotopy invariants*, thus answering a question of Atiyah, [A]. Dodziuk's argument extends in a straightforward manner to manifolds with boundary and either relative or absolute boundary conditions. Then the homotopy invariants $\tilde{\mathbf{b}}^i_{(2)}(M)$, $\tilde{\mathbf{b}}^i_{(2)}(M_k, M_l)$ are defined in the obvious way.

Theorem 6.2. If M_k is any exhaustion of M by compact submanifolds with boundary, we have

$$\lim_{k\to\infty} \mathbf{b}^i_{(2)}(M_k) = \lim_{k\to\infty} \lim_{l\to\infty} \tilde{\mathbf{b}}^i_{(2)}(M_k, M_l) = \tilde{b}^i_{(2)}(M). \tag{6.28}$$

Theorems 6.1 and 6.2 formally imply Theorem 0.1 and 1) of Theorem 3.1. These assert the homotopy invariance (respectively proper homotopy invariance) of $\chi(M,g)$, $\sigma(M,g)$ in our situation. We now prove Theorem 6.1. The proof of Theorem 6.2 will be given in Sect. 7 under extra the assumption that $\tilde{M}$ is profinite. The general case will be dealt with in [CG$_1$].

Proof of Theorem 6.1. 1) $\chi(M,g) = \tilde{\chi}_{(2)}(M)$. Let $\tilde{E}^i(t)$ denote the heat kernel of $\tilde{M}$ on i-forms. The constructions and estimates of [CLY], [CGT] show that the

[12] Clearly, this theorem generalizes to other operators which are invariantly attached to the geometry e.g. the Dirac operator (compare also [CM])

pointwise trace $\mathrm{tr}(\tilde{E}^i(t))$ is uniformly bounded for $t > t_0 > 0$. Since the covering is normal, $\mathrm{tr}(\tilde{E}^i(t))$ is the pull back of a function on M, which we also denote by $\mathrm{tr}(\tilde{E}^i(t))$. Since $\mathrm{geo}(\tilde{M}) \leqq 1$, the estimates of [CLY], [CGT] show that for t small,

$$|\mathrm{tr}\,\tilde{E}^i(t) - t^{-n/2}(a_0^i + a_1^i t \dots a_n^i t^n)| \leqq c(n)\, t^{n+1} \tag{6.29}$$

pointwise. Hence as $t \to 0$, we have the uniform pointwise estimate

$$\left| \sum_{i=0}^{n} (-1)^i \,\mathrm{tr}\,\tilde{E}^i(t) - P_\chi(\Omega) \right| \leqq c(n)\, t. \tag{6.30}$$

Next, we claim that

$$\frac{d}{dt}\Big[\sum (-1)^i \int_M \mathrm{tr}\,\tilde{E}^i(t)\Big] = 0, \tag{6.31}$$

or equivalently,

$$\sum (-1)^i \int_M \mathrm{tr}(\triangle\, \tilde{E}^i(t)) = 0. \tag{6.32}$$

To see this formally, let ϕ be a coexact eigenform, $\|\phi\| = 1$, with eigenvalue $\lambda > 0$. Then by Stokes' Theorem,

$$\begin{aligned} \int_M e^{-\lambda t} \lambda \phi \wedge * \phi &= \int_M e^{-\lambda t}\, \delta d \phi \wedge * \phi \\ &= \pm \int_M e^{-\lambda t}\, d(* d\phi \wedge \phi) + \int_M e^{-\lambda t}\, d\phi \wedge * d\phi \\ &= \int e^{-\lambda t}\, \lambda \frac{d\phi}{\sqrt{\lambda}} \wedge * \frac{d\phi}{\sqrt{\lambda}}. \end{aligned} \tag{6.33}$$

In actuality, by the Spectral Theorem, we have the pointwise relation

$$d(\wedge *_y *_x d_x \tilde{E}^i(t)) = \mathrm{tr}(\delta d\, \tilde{E}^i(t)) - \mathrm{tr}(d \delta\, \tilde{E}^{i+1}(t)). \tag{6.34}$$

If M is complete and the forms in (6.34) as well as the form

$$\wedge *_y *_x d_x \tilde{E}^i(t) \tag{6.35}$$

are integrable over M, by [G] we obtain

$$\int_M \mathrm{tr}(\delta d\, \tilde{E}^i(t)) = \int_M \mathrm{tr}(d \delta\, \tilde{E}^i(t)) \tag{6.36}$$

which gives (6.26). In our case the forms in question are pointwise bounded and M has finite volume so integrability is obvious, q.e.d.

Suppose we attempt to show by a similar argument that

$$\int_M \mathrm{tr}(P_{ce} \tilde{E}^i(t)) = \int_M \mathrm{tr}(P_e \tilde{E}^{i+1}(t)), \tag{6.37}$$

where P_{ce}, P_e denote orthogonal projection on coexact and exact forms respectively. The form in (6.35) is now replaced by

$$\wedge *_y *_x d_x \tilde{G}_x \tilde{E}^i(t), \tag{6.38}$$

where $\tilde{G}$ denotes the Greens operator. Due to the possible unboundedness of G we can not control the pointwise norm of this form without further information. However, for any complete manifold V we have the following result.

Lemma 6.3. We have pointwise converge of kernels

$$\lim_{t\to\infty} E(x,y,t) \to h(x,y). \tag{6.39}$$

The convergence is uniform in the C^∞ topology on compact subsets K of $V \times V$.

If we grant Lemma 6.3, the proof of Theorem 6.1 is easily completed. If $\cup M_k = M$ is an exhaustion and $\pi_1(\tilde{M}) \subset \pi_1(M)$ is normal, then for each k, $\pi_1(M)/\pi_1(\tilde{M}) = \Gamma$ operates uniformly on $p^{-1}(M_k)$. Thus the convergence

$$\tilde{E}^i(t) \to \tilde{h}^i \tag{6.40}$$

is uniform on $p^{-1}(M_k)$. The assumption $\mathrm{geo}(\tilde{M}) \leqq 1$ implies pointwise bounds, independent of $t \geqq 1$, on

$$\mathrm{tr}(\tilde{E}^i(x,x,t)), \tag{6.41}$$

and on

$$\mathrm{tr}(\tilde{h}^i(x,x)). \tag{6.42}$$

Since M has finite volume, it follows that

$$\lim_{t\to\infty} \int_M \mathrm{tr}\, \tilde{E}^i(t) = \int_M \mathrm{tr}\, \tilde{h}^i, \tag{6.43}$$

which suffices to complete the proof.

Proof of Lemma 6.3. By Duhamel's principle we can write

$$E(x,y,t) = P(x,y,t) + \int_0^t e^{-\triangle(t-s)}\, Q(s)\, ds, \tag{6.44}$$

where

$$Q(z,y,s) = \left(\triangle_z + \frac{\partial}{\partial s}\right) P(z,y,s). \tag{6.45}$$

As y varies in a compact subset K the functions $Q(z,y,s)$ (viewed, for each s, as functions of z) vary in a compact subset of $L^2(V)$. Thus, using the existence of A such that $P(t)\,|\,[A,\infty) \equiv 0$, and the Spectral Theorem, we have pointwise convergence as $t \to \infty$,

$$E(x,y,t) \to \int_0^\infty \int_V h^i(x,z)\, Q(z,y,s)\, dz\, ds. \tag{6.46}$$

The convergence is uniformly C^k as y varies over K and x varies over any set on which the local C^k geometry is uniformly bounded. The right hand side of (6.46) can be rewritten as

$$\lim_{\varepsilon\to 0} \int_{\varepsilon}^{\infty} \int_{V} h^i(x,z)\left(\triangle_z + \frac{\partial}{\partial s}\right) P(z,y,s)$$

$$= \lim_{\varepsilon\to 0} \int_{\varepsilon}^{\infty} \int_{V} h^i(x,z)\frac{\partial}{\partial s} P(z,y,s) = \lim_{\varepsilon\to 0} \int_{V}^{\infty} h^i(x,z)\, P(z,y,\varepsilon)$$

$$= h^i(x,y). \tag{6.47}$$

We now observe that Theorem 6.2 is a direct consequence of Assertion 0.1. Essentially the same argument with Assertion 0.1 replaced by Theorem 2.1 is used in Sect. 7 to prove Theorem 6.2 if in addition M is profinite. In [CG_1], it is shown that this argument also applied to small balls of suitable radius. It can then be globalized to obtain Theorem 6.2 without the benefit of Assertion 0.1.

Let $A\colon H_1 \to H_2$ be a bounded map of Γ-modules. It is convenient to define

$$\operatorname{im} A = \overline{\operatorname{range} A}. \tag{6.48}$$

With this proviso, concepts such as a cochain complex C^* of Γ-modules, the cohomology $\bar{H}^i(C^*)$ and the cohomology sequence associated to an exact sequence of chain complexes have their usual meaning. Even though $\bar{H}^i(C^*)$ are *reduced* cohomology groups, one can see using (6.12), (6.13) and *the cohomology sequence*

$$\to \bar{H}^i(C_3) \to \bar{H}^i(C_2) \to \bar{H}^i(C_1) \to \bar{H}^i(C_3) \to \ldots \tag{6.49}$$

is exact. It follows without difficulty that there is an exact sequence of Γ-modules

$$\ldots \to \bar{H}^i_{(2)}(M, M_k) \to \bar{H}^i_{(2)}(M) \to \bar{H}^i_{(2)}(\overline{M\backslash M_k}) \to \ldots \tag{6.50}$$

The analog for manifolds with controlled boundary geometry of the pointwise bound (6.23), implies

$$\lim_{k\to\infty} \tilde{b}^i_{(2)}(M\backslash M_k) = 0. \tag{6.51}$$

Together with (6.50), this gives (6.28) of Theorem 6.2.

We close this section by noting a proportionality principle which holds for invariants obtained by integrating Γ-invariant functions on $\tilde{M}$.

Proposition 6.4 (Proportionality Principle). Let Γ_1, Γ_2 be discrete groups of isometries of (V, g) such that $\mathrm{Vol}(V/\Gamma_j) < \infty$. If $\tilde{f}$ is a bounded continuous function which is invariant under Γ_1, Γ_2, then

$$\frac{\int_{V/\Gamma_1} \tilde{f}}{\int_{V/\Gamma_2} \tilde{f}} = \mathrm{Vol}(V/\Gamma_1)/\mathrm{Vol}(V/\Gamma_2). \tag{6.52}$$

Moreover, the quantity in (6.52) is the same for all Γ_1, Γ_2-invariant metrics which are quasi-isometric to g.

Proof: Let G be the group of isometries generated by $\Gamma_1 \cup \Gamma_2$. Note that $\tilde{f}$ is invariant under G. If G is discrete (6.52) is clear. Otherwise we restrict attention to $\hat{V}$ the open dense subset of V which is the union of orbits of principal type. Let $Y = \hat{V}/G$ carry the induced Riemannian metric. Note that $\tilde{f}$ is constant on the orbits $K\backslash G$

$$\int_{V/\Gamma_j} f \tilde{d} y = \int_Y \tilde{f}(K\backslash G) \operatorname{Vol}(K\backslash G/\Gamma_j)\, d y. \tag{6.53}$$

Note that $K\backslash G$ is homogeneous and all principal orbits are equivalent as G spaces. Thus

$$\operatorname{Vol}(K\backslash G/\Gamma_1)/\operatorname{Vol}(K\backslash G/\Gamma_2) \tag{6.54}$$

is independent of the particular left invariant metric and of the particular orbit. It follows that for all $\tilde{f}$, the value of the left hand side of (6.52) equals the quantity in (6.54). Taking $\tilde{f} \equiv 1$ we get the right hand side of (6.52).

Corollary 6.5. Suppose $\operatorname{geo}(\tilde{M}) \leqq 1$, and $M_j = \tilde{M}/\Gamma_j$, $\operatorname{Vol}(M_j) < \infty$. If $H^i_{(2)}(\tilde{M}) \neq 0$ for some i, then $\operatorname{Vol}(M_1)/\operatorname{Vol}(M_2)$ is a homotopy invariant.

Observe that the conclusion of Corollary 6.5 fails for $\tilde{M}^n = \mathbb{R}^n$.

7. L^2-Theory for Profinite Normal Coverings

Let the covering $\tilde{M} \to M$ be profinite finite and normal. Thus, there exist finite coverings $\tilde{M}_j \to M$ and

$$\tilde{M} \to \ldots \tilde{M}_j \to \ldots \tilde{M}_1 \to M. \tag{7.1}$$

In this situation it is somewhat easier to discuss analysis on $\tilde{M}$.

Proof of Theorem 6.2 (for $\tilde{M}$ profinite). Let M_k be an exhaustion. Pick k so large that

$$\operatorname{Vol}(M\backslash(M_k - 1)) < \varepsilon. \tag{7.2}$$

For $j > N(k)$, there exists $A_{jk} \subset p_j^{-1}(M_k)$ with

$$\operatorname{Vol}(\partial A_{jk}) \leqq c(n) \cdot d_j \cdot \varepsilon, \tag{7.3}$$

$$\| \mathrm{II}(\partial A_{jk}) \| \leqq c(n). \tag{7.4}$$

As in (6.51), this gives

$$0 \leqq \tilde{b}^i_{(2)}(\tilde{M}_j) - \mathfrak{b}^i_{(2)}(A_{jk}) \leqq c(n) \cdot d_j \cdot \varepsilon. \tag{7.5}$$

If we use

$$\tilde{b}^i_{(2)}(M) = \frac{1}{d_j} \tilde{b}^i_{(2)}(\tilde{M}_j), \qquad \tilde{b}^i_{(2)}(M_k) = \frac{1}{d_j} \tilde{b}^i_{(2)}(p_j^{-1}(M_k)), \tag{7.6}$$

$$\frac{1}{d_j} \mathfrak{b}^i_{(2)}(A_{jk}) \leqq \frac{1}{d_j} \mathfrak{b}^i_{(2)}(p_j^{-1}(M_k)) \leqq \frac{1}{d_j} \tilde{b}^i_{(2)}(\tilde{M}_j) \tag{7.7}$$

for $A_{jk} \subset p_j^{-1}(M_k)$, Theorem 6.2 follows in this case.

We also have the following analog of Theorem 3.2 (by Remark 7.1 below it is a generalization Theorem 3.2 in the profinite case).

Theorem 7.1. Let N be compact, and $\tilde{N} \to \tilde{N}$ normal. If $[0, \infty) \times N$ admits a complete metric with $\mathrm{Vol}([0, \infty) \times N) < \infty$, and $\mathrm{geo}([0, \infty) \times \tilde{N}) \leqq 1$, then for all i,

$$\tilde{b}^i_{(2)}(N) = 0. \tag{7.8}$$

For the general case of Theorem 7.1, see [CG$_1$]. The profinite case is easily obtained by combining the idea of proof of Theorem 3.2 with that of the profinite version of Theorem 6.2.

In view of Theorem 7.1, the following result is a generalization of (5.4) of Theorem 5.1.

Theorem 7.2. Let M^{4l} be compact oriented, $\partial M^{4l} = N^{4l-1}$. Suppose that for some profinite normal covering $\tilde{M}^{4l}$, the induced covering $\tilde{N}^{4l-1}$ satisfies

$$\tilde{b}^{2l-1}_{(2)}(\tilde{N}^{4l-1}) = \tilde{b}^{2l}_{(2)}(\tilde{N}^{4l-1}) = 0. \tag{7.9}$$

Then the following limits exist.

$$\lim_{j \to \infty} \frac{1}{d_j} \sigma(M^{4l}) = \sigma(M^{4l}, g) + \lim_{j \to \infty} \frac{1}{d_j} \eta(\tilde{N}_j). \tag{7.10}$$

Proof: Observe that since $\tilde{N}$ is the limit of the $\{\tilde{N}_j\}$ we have the following: Let ρ_j: $\tilde{N} \to \tilde{N}_j$. For all R there exists $j(R)$ such that if $B_R(p) \subset \tilde{N}_j$), then ρ_j is a homeomorphism from each component of $\rho_j^{-1}(B_k(p))$ to $B_k(p)$ whenever $j \leqq j(R)$.

Now consider the integral for $\eta(\tilde{N}_j)$ given in (4.6). Since $\tilde{N}$ is a normal covering of N, we also have the periodic function $\mathrm{tr}(* d e^{-\tilde{\Delta} t})$ on $\tilde{N}$ corresponding to the integrand of (4.6). The estimates of [CGT], together with the fact mentioned above imply that if we identify $B_R(p_j)$ with some component of $B_R(\rho_j^{-1}(p_j))$ (j large) then we have uniform convergence

$$\mathrm{tr}(* d e^{-\tilde{\Delta}_j t}) \to \mathrm{tr}(* d e^{-\tilde{\Delta} t}), \tag{7.11}$$

$$\mathrm{tr}(e^{-\tilde{\Delta}_j t}) \to \mathrm{tr}(e^{-\tilde{\Delta} t}), \tag{7.12}$$

and any fixed interval $[0, A]$. Thus,

$$\lim_{j \to \infty} \frac{1}{d_j} \left[\frac{1}{\Gamma(1/2)} \int_0^A \int_{\tilde{N}_j} t^{-1/2}\, \mathrm{tr}(* d e^{-\tilde{\Delta}_j t})\, d\tilde{x}_j\, dt \right],$$

$$= \frac{1}{\Gamma(1/2)} \int_0^A \int_{\tilde{N}} t^{-1/2}\, \mathrm{tr}(* d e^{-\tilde{\Delta} t})\, d\tilde{x}\, dt \tag{7.13}$$

(this can also be seen by the arguments of [Don]).

In view of (7.9), we have uniform pointwise convergence

$$\lim_{A \to \infty} \mathrm{tr}(e^{-\tilde{\Delta} A}) = 0. \tag{7.14}$$

It now follows as in (4.13) (by use of the Spectral Theorem) that

$$\int_A^{\infty} \int_{\tilde{N}} t^{-1/2} \operatorname{tr}(*de^{-\tilde{\Delta}t}) \tag{7.15}$$

exists and thus that

$$\lim_{A\to\infty} \int_A^{\infty} \int_{\tilde{N}} t^{-1/2} \operatorname{tr}(*d^{-\tilde{\Delta}t}) = 0. \tag{7.16}$$

Using (7.12) and (7.14) we also get

$$\begin{aligned} &\lim_{A\to\infty} \overline{\lim} \frac{1}{d_j} \int_A^{\infty} t^{-1/2} \operatorname{tr}(*de^{-\tilde{\Delta}_j t}) \\ &\quad = \lim_{A\to\infty} \underline{\lim} \frac{1}{d_j} \int_A^{\infty} t^{-1/2} \operatorname{tr}(*de^{-\tilde{\Delta}_j t}) = 0. \end{aligned} \tag{7.17}$$

If we now put

$$\tilde{\eta}_{(2)}(N) \stackrel{\text{def}}{=} \frac{1}{\Gamma(1/2)} \int_0^{\infty} \int_N t^{-1/2} \operatorname{tr}(*de^{-\tilde{\Delta}t}), \tag{7.18}$$

it follows from (7.13), (7.16), (7.17) that

$$\lim_{j\to\infty} \frac{1}{d_j} \tilde{\eta}(N_j) \to \tilde{\eta}_{(2)}(N). \tag{7.19}$$

This suffices to complete the proof.

Remark 7.1. If N is compact and $\tilde{N}$ is profinite and normal, then Lemma 6.3 and (7.12) (for forms of arbitrary degree) imply

$$\overline{\lim} \frac{1}{d_j} b^i(\tilde{N}_j) \leqq \tilde{b}^i_{(2)}(N). \tag{7.20}$$

If $\tilde{b}^i_2(N) > 0$, it seems difficult to decide under what circumstances the inequality in (7.20) is an equality. A similar point occurs in connection with the hypothesis (7.9) of Theorem 7.2 (compare also [K_2]).

Example 7.1. If $\tilde{N}^{2k}$ is a symmetric space of rank 1 of the non-compact type, then $\tilde{b}^i_{(2)}(N^{2k}) = 0$ for $i \neq k$ but $\tilde{b}^k_{(2)}(N^{2k}) \neq 0$. However, by applying the L^2-Index Theorem for the Euler characteristic it follows that for all i,

$$\lim_{j\to\infty} \frac{1}{d_j} b^i(\tilde{N}_j) = \tilde{b}^i_{(2)}(N). \tag{7.21}$$

Remark 7.2. If $\tilde{N}$ is profinite but *not* necessarily normal, as in (7.12) one can see that for all finite t, we have uniform pointwise convergence

$$\lim_{j\to\infty} \operatorname{tr}(e^{-\tilde{\Delta}_j t}) \to \operatorname{tr}(e^{-\tilde{\Delta}t}). \tag{7.22}$$

However, in the absence of a group action on $\tilde{N}$ one can not use Lemma 6.3 to let $t \to \infty$ and obtain a definition of $\tilde{b}^i_{(2)}(N)$. Similarly, analogs of Theorem 6.1 and of (7.20) are lacking if $\tilde{N}$ is not assumed normal (compare [D$_2$]).

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