L^2-COHOMOLOGY AND INTERSECTION HOMOLOGY OF SINGULAR ALGEBRAIC VARIETIES

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§1. *Introduction*

In this largely expository paper we describe some generalizations to the singular case of the special cohomological properties of nonsingular complex projective algebraic varieties which are usually proven using Kähler geometry. One avenue of generalization is to suitably modify the properties so that they remain true of the ordinary cohomology of a singular variety (see 1.3 below). Here we take the point of view that the properties themselves need not be weakened provided that the cohomology theory to which they apply is suitably modified.

1.1. Let M be a compact complex algebraic manifold of dimension n embedded in complex projective space CP^N, and let Ω be the restriction to M of the Kähler form on CP^N. The cohomology of M with complex coefficients satisfies several remarkable theorems which we refer to collectively as the *Kähler package*:

1. Pure Hodge decomposition:

$$H^i(M) \simeq \bigoplus_{p+q=i} H^{p,q} \quad (\text{with } \overline{H^{p,q}} = H^{q,p}) . \tag{1.1}$$

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2. Hard Lefschetz: the map

$$H^{n-i}(M) \xrightarrow{\cup[\Omega]^i} H^{n+i}(M) \tag{1.2}$$

is an isomorphism.

3. Poincaré duality: the pairing

$$H^i(M) \times H^{2n-i}(M) \to \mathbb{C} \tag{1.3}$$

given by cup product and evaluate on the fundamental class is, nonsingular. (Equivalently the pairing

$$H_i(M) \times H_{2n-i}(M) \to \mathbb{C} \tag{1.4}$$

given by intersecting transverse cycles and counting points with their multiplicities, is nonsingular.)

4. The Lefschetz hyperplane theorem.
5. The Hodge signature theorem.

1.2. Aside from the Lefschetz hyperplane theorem, all the theorems of the Kähler package can be deduced using L^2 analytic methods. (Of course Poincaré duality and Hard Lefschetz [D4] can also be proved by other means.)

The first step is De Rham's theorem that the topological invariant $H^*(M)$ has an analytic expression as $H^*_{DR}(M)$, the cohomology of differential forms. Then the results can be viewed as formal consequences of a) the Hodge theorem that every cohomology class contains exactly one harmonic form, b) the fact that the action of the almost complex structure J on differential forms carries harmonic forms to harmonic forms; and c) the fact that the harmonic forms are just the forms which are closed and co-closed.

1.3. There have been several deep studies of how to extend various theorems of the Kähler package to the cohomology of a singular variety. The Zeeman spectral sequence [ZE], [MC], studies Poincaré duality; Ogus'

De Rham depth [O] studies the Lefschetz hyperplane theorem; and Deligne's mixed Hodge theory [D1], [D2], [D3] studies the Hodge (p,q) decomposition. All of these theories proceed by filtering the cohomology (roughly by how "tied to the singularities" a cocycle is). Then they express the "degree of failure" of the theorem as stated in the nonsingular case in terms of this filtration. The picture we present here does not oppose, but rather compliments, that of these theories, (see §7).

1.4. Let X be a complex n dimensional projective algebraic variety. The invariants of X which concern us in this paper are the middle intersection homology groups $IH_*(X)$. Roughly, $IH_*(X)$ is the homology of a certain subcomplex of the homology chain complex of X, defined by certain geometric conditions on how the chains enter the singularities of X. (These will be recalled in §2; see [GM1], [GM2], [GM3] for details.) For cycles of this special type the intersection pairing is well defined and leads to a perfect pairing (1.3). The groups $IH_*(X)$ are topological invariants but not homotopy invariants. There are natural maps

(1.5)
$$\begin{array}{ccc} & IH_i(X) & \\ \nearrow & & \searrow \\ H^{2n-i}(X) & \xrightarrow[\cap[X]]{} & H_i(X)\,. \end{array}$$

If X is nonsingular these are all isomorphisms, but in some cases $IH_*(X)$ is much bigger than either homology or cohomology.

The local calculation of $IH_*(X)$ has an interesting result. For any $x \in X$ let B be the intersection of X with a small open ball of radius ϵ about x and let S be the intersection of X with a sphere of radius $\epsilon/2$ about x. Then (using chains with compact support)

(1.6)
$$IH_i(B) = \begin{cases} IH_i(S) & \text{for} \quad i < n \\ 0 & \text{for} \quad i \geq n \end{cases} .$$

In fact, an appropriately stated version of this local calculation characterizes the groups $IH_*(X)$. (See §2.2.)

1.5. The main idea of this paper is the following program: the theorems of the Kähler package should hold without modification in the singular case, provided that intersection homology is used in place of ordinary homology.

We present several conjectures relating to this program and possible L^2 methods of proof. We also give a number of examples and consequences.

The current status of the program is summarized in §6. Briefly, all parts of the Kähler package but those relating directly to the Hodge (p, q) decomposition have been proved in general (by other than L^2 methods, for the most part). The (p, q) decomposition is known in many classes of examples.

1.6. The original motivation for the program was the existence of L^2 methods on certain singular varieties which turned out to be appropriate for the study of intersection homology theory [C1], [C2], [C3].

Let $X \subset \mathbf{CP}^N$ be a singular variety and let Σ be its singularity set. One studies the *smooth incomplete* Riemannian manifold $X - \Sigma$ with the metric Ω induced by the inclusion. The i^{th} L^2 cohomology group $H^i_{(2)}(X)$ of X is just the quotient space of smooth i forms θ on $X - \Sigma$ which are L^2, by the subspace $\{d\psi\}$, where ψ is a smooth $i-1$ form with ψ, $d\psi \in L^2$ (see §3).

If X has conical singularities (see §3 for definitions) then two somewhat surprising results hold [C3], §6.3:

1) $H^i_{(2)}(X)$ is isomorphic to the space of closed and co-closed harmonic forms.
2) The calculation of the local groups is dual to that of (§1.6) and therefore $H^*_{(2)}(X)$ is the cohomology theory dual to $IH_*(X)$.

These results are analogues of the Hodge theorem and the De Rham theorem so they lead one to expect L^2 proofs of the Kähler package theorem for intersection homology theory.

1.7. Although the possible existence of an appropriate L^2 theory on the incomplete Kähler manifold $X - \Sigma$ was the main source of motivation for the above conjectures, of course it is not the only possible method of attack. In fact, at present the most general results have been obtained by other methods. Moreover, as the first author realized after the publication of [C3], the assertion made there that the conjectures are formal consequences of Hodge Theory, is *not* correct in the context of *incomplete* metrics. The reason is that on incomplete manifolds, an L^2 harmonic form is not necessarily closed and co-closed. Thus although a) and b) of §1.3 above are automatic, c) is not. In particular, the assertion of [C3], that the Hodge Theorem proved there implies the "Kähler package" for algebraic varieties for which the induced metric has conical singularities, is still unsubstantiated, except in certain special cases in which it can be checked directly that J preserves the space $\mathcal{H}^i$ closed and co-closed L^2 harmonic forms; (see §6.4 and [C4]). A general proof of this assertion, (which seems extremely delicate), must make use of the assumption that the singularities of the metric are conical in a complex analytic (rather than just piecewise smooth) sense. Otherwise, the topological conclusions can definitely be false.

1.8. The above-mentioned difficulty disappears for the case of *complete* metrics on $X - \Sigma$, since on a complete manifold L^2 harmonic forms are automatically closed and co-closed; (see [DR], [AV] and §3). Thus another possible approach is to attempt to construct a complete metric on $X - \Sigma$ (or on $X - \Sigma'$ for suitable $\Sigma' \supset \Sigma$), for which the space $\mathcal{H}^i$ is dual to IH_i. But in return for the great advantage offered by completeness one pays a certain price in that $\mathcal{H}^i$ becomes more difficult to calculate in general.* In either program formidable difficulties arise at the outset because even in very simple cases, the riemannian geometry of singular algebraic varieties and their complements has been little studied and, it seems fair to say, is very poorly understood.

*See however, [M], [ZU1], and [ZU2].

1.9. In summary, although at present, the results on the "Kähler package" which can be obtained from L^2 methods are quite modest as compared to those which can be obtained by other techniques, we emphasize the L^2 theory here for several reasons. First, it was the original source of motivation for the conjectures and should be basic to an eventual complete understanding. Second, just as in the nonsingular case, these methods are based on the existence of a Kähler metric rather than the more special property of being an algebraic variety. Thus, when successful they work in more generality. Third, the Kähler metrics (complete and incomplete) on the nonsingular parts of algebraic varieties, provided a rich source of as yet unstudied problems in geometry and analysis.

1.10. *Historical note*

The ideas in this paper had three independent sources.

1) The intersection homology groups were defined in piecewise linear topology by Goresky and MacPherson to study intersection theory of cycles on a singular variety.
2) The study of L^2 cohomology on the nonsingular part of a variety with conical singularities was initiated by Cheeger; this was suggested by points which arose in the study of analytic torsion.
3) A procedure for extending to all of M, a variation of Hodge structure on $M-D$, (where M is a nonsingular complex algebraic variety, and D is a divisor with normal crossings), was discovered by Deligne, generalizing work of Zucker, and was used to study the mixed Hodge theory of sheaf cohomology.

All of these procedures led to the same seemingly strange local calculation of (§1.6). This coincidence was observed in 1976 by Sullivan (for 1) and 2)) and by Deligne (for 1) and 3)). Deligne then proposed a sheaf theoretic construction of intersection homology.

The natural hypothesis was that all three approaches give the same group—i.e., that a single invariant has definitions by topological, analytic, and algebraic means. This hypothesis led, in conversations between

MacPherson and Cheeger in 1977, to the conjecture that the theorems of the Kähler package hold for that invariant. Now the hypothesis has been proved ((§6.3) and [C3] for 1) and 2); [GM3] for 1) and 3)) and much of the Kähler package has been established as well.

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§2. *Intersection homology theory*

2.1. In this section we recall the definition and basic properties of the "middle" intersection homology groups $IH_k(X)$ ([GM1], [GM2]). Since there is no canonical piecewise-linear structure on an algebraic variety, it is technically convenient to use subanalytic chains in the definition of $IH_k(X)$. However, we remark that one could just as well choose a P.L. structure on X, use P.L. chains to define $IH_k(X)$; and the result is independent of the P.L. structure. This is the point of view which is adopted at the end of Chapter 3. The reader who is unfamiliar with the subanalytic category can think in terms of P.L. chains in this section as well.

Let X be an n-dimensional complex analytic variety contained in some nonsingular variety. We choose an analytic Whitney stratification ([MA], [T]). This consists of a filtration by (closed) analytic subvarieties

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$$

such that:

(a) $X_i - X_{i-1}$ is a (possibly empty) complex analytic i-dimensional manifold (the components of which are called the *strata* of complex dimension i).

(b) Whitney's condition B holds with respect to any pair of strata R and S, i.e., suppose $x_i \in S$ is a sequence of points converging to some point $x \in R$ and suppose $y_i \in R$ is a sequence converging to x. Suppose the secant lines $\overline{x_i, y_i}$ converge to some line ℓ and the tangent planes $T_{x_i}S$ converge to some limiting plane τ (with respect to any coordinate system on the ambient nonsingular variety). Then we demand that $\ell \subset \tau$.

The groups $IH_k(X)$ will be constructed using this stratification but the result is independent of the stratification.

Let $C_*(X)$ denote the chain complex of compact (real) subanalytic chains on X with complex coefficients. The homology of this complex coincides with the singular homology groups $H_*(X; \mathbb{C})$ by Hardt [H]. For any $\xi \in IC_i(X)$ we let $|\xi|$ denote the *support* of ξ; it will be a (real) i-dimensional subanalytic subset of X.

Define the subcomplex $IC_*(X)$ of *allowable* chains by the condition $\xi \in IC_i(X)$ if

$$\dim |\xi| \cap X_k \leq i - n + k - 1$$

and

$$\dim |\partial\xi| \cap X_k \leq i - n + k - 2 .$$

Definition. $IH_i(X)$ is defined to be the i^{th} homology group of this chain complex, $IC_*(X)$.

It is possible to define an intersection product $IH_i(X) \times IH_j(X) \to IH_{i+j-2n}(X)$ by following the original method of Lefschetz [L]. If $\xi \in IC_i(X)$ and $\eta \in IC_j(X)$ are transverse, then the intersection $|\xi| \cap |\eta|$ carries the structure of an $i+j-2n$ dimensional subanalytic chain such that

$$\partial(\xi \cap \eta) = (\partial\xi) \cap \eta + (-1)^i \xi \cap \partial\eta .$$

For any chain ξ, almost all chains η are transverse to ξ. Therefore transverse intersection induces a pairing on the intersection homology groups.

Theorem (Poincaré duality). *If X is compact, the intersection pairing in complementary dimensions is nonsingular, i.e., the composition*

$$IH_i(X) \times IH_{2n-i}(X) \to H_0(X) \to \mathbb{C}$$

induces an isomorphism

$$IH_i(X) \cong \mathrm{Hom}(IH_{2n-i}(X), \mathbb{C}) .$$

Relation with cohomology

The intersection homology groups lie "between" cohomology and homology, in the sense that for compact X we have the following:

THEOREM. *There are canonical homomorphisms*

$$H^i(X) \to IH_{2n-i}(X) \to H_{2n-i}(X)$$

which factor the Poincaré duality map $\cap [X] : H^i(X) \to H_{2n-i}(X)$. *If* X *is nonsingular, these are both isomorphisms. The intersection product extends to a module structure*

$$H^i(X) \times IH_j(X) \to IH_{j-i}(X)$$

which is compatible with cup and cap products.

These maps may be constructed for compact varieties X which are embedded in some nonsingular variety of dimension N as follows: choose a subanalytic neighborhood with boundary $(U, \partial U)$ of X in the ambient space, such that X is a deformation retraction of U, and ∂U is a topological manifold. By Lefschetz duality $H^i(X) \cong H_{2N-i}(U, \partial U)$ so any cohomology class may be represented by a (relative) subanalytic chain ξ on $(U, \partial U)$ which we may take to be transverse to X. The intersection $\xi \cap X$ then satisfies the allowability condition, so it determines a chain in $IC_{2n-i}(X)$, thus inducing $H^i(X) \to IH_{2n-i}(X)$.

Similarly if $\eta \in IC_j(X)$ then ξ may also be chosen transverse to η so $\xi \cap \eta \in IC_{j-i}(X)$, thus inducing the module structure.

Finally the map $IH_i(X) \to H_i(X)$ is induced by the inclusion of chain complexes $IC_*(X) \subset C_*(X)$.

Functorality

DEFINITION. Let $f: Y \to X$ be the inclusion of a subvariety Y. f is said to be a *normally nonsingular inclusion* (of relative complex dimension c) if f is proper and there is an analytic manifold M (of codimension c) in the ambient projective space, such that M is transverse to each stratum of X and $Y = M \cap X$. Such a subvariety inherits a stratification from that of X.

Such a map f determines a pushforward

$$f_*: IH_i(Y) \to IH_i(X)$$

since $f(\xi)$ is an allowable chain in $IC_i(X)$ whenever ξ is an allowable chain in $IC_i(Y)$. Dualizing we obtain a pullback:

$$f^*: IH_k(X) \cong \mathrm{Hom}(IH_{2n-k}(X), C) \to \mathrm{Hom}(IH_{2n-k}(Y), C) \cong IH_{k-2c}(Y).$$

DEFINITION. Let $f: Y \to X$ be a proper smooth map. This will be a topological fibre bundle whose fibres are complex manifolds. We shall assume they all have dimension d. Such a map is called a *normally nonsingular projection* of relative dimension $-d$. The stratification of X pulls back to a stratification of Y.

Such a projection f induces a pullback

$$f^*: IH_i(X) \to IH_{i+2d}(Y)$$

since the pre-image $f^{-1}(\xi)$ satisfies the allowability conditions on Y whenever $\xi \in IC_i(X)$. We obtain by duality a push forward map

$$f_*: IH_k(Y) \to IH_k(X).$$

DEFINITION. A normally nonsingular map f is any map which can be factored into a normally nonsingular inclusion followed by a normally nonsingular projection. The relative dimension of f is defined to be the sum of the relative dimensions of its factors.

For normally nonsingular maps the homology, cohomology, and intersection homology groups map *both* ways. ([FM], [GO]).

Branched coverings

If $f : Y \to X$ is a finite branched covering then f induces homomorphisms $f_* : IH_i(Y) \to IH_i(X)$ and $f^* : IH_i(X) \to IH_i(Y)$. Simply stratify X and Y compatibly with the map f and observe that for any allowable chain ξ in Y the image $f(\xi)$ is allowable in X. Similarly, for any allowable chain η in X, the pre-image $f^{-1}(\eta)$ is allowable in Y.

2.2. *The local calculation*

We now give an intuitive description of the local calculation (1.6). In this case, any allowable cycle $\xi \in IC_i(B)$ is the boundary of the allowable chain $c(\xi)$ (the *cone* on ξ) provided $i \geq n$. Thus $IH_i(B) = 0$ if $i \geq n$.

For $i < n$, any allowable chain $\xi \in IC_i(B)$ cannot contain the singular point p, so it may be deformed into a chain on S by "pushing along the cone lines." (This corresponds to the homotopy operator of (3.27).) Furthermore, any allowable chain $\eta \in IC_i(S)$ is allowable in B, so we conclude $IH_i(B) \cong IH_i(S)$ for $i < n$. This agrees with the analogous calculation for L^2 cohomology (3.23) since for nonsingular S we have $H_i(S) \cong IH_i(S)$.

Using this local calculation in the Mayer-Vietoris exact sequence, we obtain the following

COROLLARY. *Suppose* X *has a single isolated singularity* x. *Then*

$$IH_i(X) = \begin{cases} H_i(X) & \textit{if } i > n \\ \text{Image } (H_i(X-x) \to H_i(X)) & \textit{if } i = n \\ H_i(X-x) & \textit{if } i < n . \end{cases}$$

2.3. *Axiomatic characterization*

A *complex of sheaves* on X is a collection of sheaves $\{\underline{\underline{S}}^p\}$ of C-vector spaces, together with sheaf maps

$$\underline{\underline{S}}^p \xrightarrow{d} \underline{\underline{S}}^{p+1} \xrightarrow{d} \underline{\underline{S}}^{p+2}$$

such that $d \circ d = 0$. If each $\underline{\underline{S}}^p$ is *fine*, we shall use $H^p(X;\underline{\underline{S}}^\cdot)$ to denote the p^{th} cohomology group of the complex

$$\Gamma(X;\underline{\underline{S}}^0) \to \Gamma(X;\underline{\underline{S}}^1) \to \Gamma(X;\underline{\underline{S}}^2) \to \cdots$$

while the *local cohomology sheaf* $\underline{\underline{H}}^p(\underline{\underline{S}}^\cdot)$ denotes the sheaf $(\underline{\underline{\ker}}\, d^p/\underline{\underline{\mathrm{Im}}}\, d^{p-1})$.

Using the same method as that in [GM3], we obtain the following characterization of the intersection homology groups:

THEOREM. *Let $\underline{\underline{S}}^\cdot$ be a complex of fine sheaves on X such that:*

(1) $\underline{\underline{S}}^k = 0$ *for all* $k < 0$.

(2) *The local cohomology sheaves* $\underline{\underline{H}}^p(\underline{\underline{S}}^\cdot)$ *are locally constant on each stratum* $X_i - X_{i-1}$.

(3) *There is a sheaf map* $\underline{\underline{C}}_{X-X_{n-1}} \to \underline{\underline{S}}^\cdot|(X-X_{n-1})$ *which induces isomorphisms on the local cohomology sheaves, where* $\underline{\underline{C}}_{X-X_{n-1}}$ *denotes the constant sheaf on* $X - X_{n-1}$.

(4) *For each point* $x \in X_i - X_{i-1}$ *there is a neighborhood* $U \subset X - X_{i-1}$ *of* x, *such that*

$$H^k(U;\underline{\underline{S}}^\cdot|U) = 0 \quad \textit{for all } k \geq n-i$$

and $H^k(U;\underline{\underline{S}}^\cdot|U) \cong H^k(U-X_i;\underline{\underline{S}}^\cdot|(U-X_i))$ *for all* $k \leq n-i-1$ *where the isomorphism is induced by restriction.*

Then $H^k(X;\underline{\underline{S}}^\cdot) \cong IH_{2n-k}(X)$.

REMARKS (1) Condition (4) above is a functorial version of the local homology calculation (1.6) and §2.2.

(2) The above theorem remains valid when X is a pseudomanifold with a stratification by strata of even dimension (see §3.4).

§3. L^2-*Cohomology*

In this section we recall the basic definitions and facts concerning L^2-cohomology, and indicate the connection with IH_* in the conical case; (see [C3] for further details).

3.1. Let Y be any (in general incomplete) riemannian manifold with metric g. Assume $\partial Y = \phi$. Let Λ^i, Λ^i_0 denote the spaces of C^∞ i-forms and C^∞ i-forms of compact support, and let L^2 denote the space of square integrable i-forms with measurable coefficients. Let $d_i : \Lambda^i \to \Lambda^{i+1}$ be exterior differentiation. We can define an unbounded operator (in the Hilbert space sense), also to be denoted by d_i, by setting

$$\text{dom } d_i = \{\alpha \in \Lambda^i \cap L^2 | d\alpha \in \Lambda^{i+1} \cap L^2\} . \tag{3.1}$$

As usual,

$$\ker d_i = \{\alpha \in \Lambda^i | d\alpha = 0\} , \tag{3.2}$$

$$\text{range } d_i = \{\eta \in \Lambda^{i+1} | d_i\alpha = \eta , \text{ for some } \alpha \in \text{dom } d_i\} \tag{3.3}$$

The simplest definition of L^2-cohomology is

$$H^i_{(2)}(Y) = \ker d_i / \text{range } d_{i-1} . \tag{3.4}$$

Note that $H^i_{(2)}(Y)$ depends only on the quasi isometry class of g; (g' is quasi isometric to g if for some $k > 0$, $\frac{1}{k} g \le g' \le kg$).

Set $d_{i,0} = d_i | \Lambda^i_0$, $\delta_{i,0} = \delta_i | \Lambda^i_0$, where δ_i is the differential operator $(-1)^{n(i+1)+1} {}_* d_*$, and dom δ_i is defined as in (3.1). Let A^* denote the adjoint of an operator A. Then, since dom $\delta_{i+1,0} = \Lambda^{i+1}_0$ is dense, and by Stokes' Theorem,

$$< d\alpha, \beta > = < \alpha, \delta\beta > \tag{3.5}$$

whenever $\alpha \in \text{dom } d_i$, $\beta \in \text{dom } \delta_{i+1,0}$, it follows that d_i has a well-defined *weak* closure, $\delta^*_{i+1,0}$. There is also a *strong* closure $\overline{d}_i$, of $d_i \cdot \overline{d}_i \cdot \alpha = \eta$ means that $\alpha \in L^2$ and there exist $\alpha_j \in \text{dom } d_i$ such that $\alpha_j \to \alpha$, $d\alpha_j \to \eta$. Clearly $\overline{d}_i$, $\delta^*_{i+1,0}$ are closed operators (i.e. they have closed graphs) and dom $\overline{d}_i \subseteq$ dom $\delta^*_{i+1,0}$, $\delta^*_{i+1,0} | \text{dom } \overline{d}_i = \overline{d}_i$. In fact, one can show that in general, $\overline{d}_i = \delta^*_{i+1,0}$; (see [C3], [GA1]). Then one might also consider

(3.6) $$H^i_{(2),\#}(Y) = \ker \bar{d}_i / \text{range } \bar{d}_i$$

as a possible candidate for L^2-cohomology. However, as shown in [C3], the natural map $H^i_{(2)}(Y) \to H^i_{(2),\#}(Y)$ is always an isomorphism. Thus one can use either definition as convenience dictates. There are natural pseudo norms on $H^i_{(2)}(Y)$, $H^i_{(2),\#}(Y)$, given by

(3.7) $$\|U\| = \inf_{\alpha \in U} \|\alpha\| .$$

These are preserved by the isomorphism above. It follows immediately that the pseudo norm is a norm if and only if the range of $\bar{d}_i$ is a closed subspace; (i.e. if $\bar{d}_i \gamma_j \to \eta$ implies $\eta = \bar{d}_i \psi$ for some ψ). Since $\bar{d}_i$ is a closed operator, it is a standard consequence of the Open Mapping Theorem that range d_{i-1} is closed if $H^i_{(2)}(Y) = H^i_{(2),\#}(Y)$ is finite dimensional.

EXAMPLE 3.1. Let $Y = \mathbf{R}$, the real line. If f is a C^∞ function such that $f(x) = \frac{1}{x}$ for $|x| \geq 1$, then $f dx \in \ker \bar{d}_1$. Clearly, the most general function α such that $d\alpha = f(x)dx$ satisfies $\alpha = \log x + c$ for $x > 1$. But then, $\alpha \notin L^2$ so $f dx \notin$ range d_0. Since $H^1_{(2)} = H^1_{(2),\#}$, also $f dx \notin$ range $\bar{d}_0$. Let ϕ be a smooth function which is supported on $[-2,2]$, such that $\phi | [-1,1] \equiv 1$. Set $\phi_n(x) = \phi(x/n)$. Then easy estimates show $d(\phi_n \alpha) \to f dx$ in L^2. Thus range d_0 is not closed and $H^1_{(2)}(r)$ is infinite dimensional.

Let $\bar{V}$ denote the closure of a subspace V. Define $\mathcal{H}^i$ to be the space of i-forms h, such that $h \in L^2$, $dh = \delta h = 0$. Kodaira has observed that one always has the *Weak Hodge Theorem*

(3.8) $$L^2 = \overline{\text{range } \delta_{i+1,0}} \oplus \overline{\text{range } d_{i-1,0}} \oplus \mathcal{H}^i$$

where the sum is orthogonal and preserves $\Lambda^i \cap L^2$. This is a consequence of *local* elliptic regularity for the Laplacian $\Delta = d\delta + \delta d$.

Note that there is a natural map, $i_{\mathcal{H}}: \mathcal{H}^i \to H^i_{(2)}(Y)$. We say that the *Strong Hodge Theorem* holds for Y, if $i_{\mathcal{H}}$ is an isomorphism, or equivalently if $\text{range}\,\bar{d}_{i-1} = \overline{\text{range}\,d_{i-1,0}}$. Clearly, this property depends only on the quasi isometry class of the metric. The *surjectivity* of $i_{\mathcal{H}}$ is equivalent to $\text{range}\,\bar{d}_{i-1} \supseteq \overline{\text{range}\,d_{i-1,0}}$, and follows in particular if $\text{range}\,d_{i-1}$ is closed.

The *injectivity* of $i_{\mathcal{H}}$ follows if $\bar{d} = \delta^*$, or equivalently, (since $A^{**} = A$ for closed operators), if $\bar{d}^* = \bar{\delta}$. In this case, as usual,

$$<h, \bar{d}\beta> = <\bar{\delta}h, \beta> = 0\,. \tag{3.9}$$

As indicated above, in general, $\delta^*_{i+1,0} = \bar{\delta}^*_{i+1,0} = \bar{d}_i$. Thus,

$$\bar{d}_i = \bar{\delta}^*_{i+1} \Longleftrightarrow \bar{d}^*_i = \bar{\delta}_{i+1} \Longleftrightarrow \tag{3.10}$$

$$\bar{d}_i = \bar{d}_{i,0} \Longleftrightarrow \bar{\delta}_{i+1} = \bar{\delta}_{i+1,0}\,.$$

3.2. If Y is *complete* then by [GA2], (3.10) holds. We briefly indicate the argument under the assumption that there exists $y \in Y$, such that ρ_y, the distance function from y is smooth; in the general case one uses regularization to obtain a smooth approximation to ρ_y. Let ϕ_n be as in Example 3.1, and set $f_n = \phi_n \circ \rho_y$. Then one checks that if $\alpha \in \text{dom}\,\bar{d}_j$, then $(f_n\alpha) \to \alpha$, $d_{i,0}(f_n\alpha) \to d_i\alpha$, which implies $\bar{d}_{i,0} = \bar{d}_i$.

In certain incomplete cases of interest below, one can show $\bar{d}_i = \bar{\delta}^*_{i+1}$ without the availability of a cutoff function. However, in the complete case using f_n, one can prove the strong additional property that $h \in L^2$, $\Delta h = 0$ implies $h \in \mathcal{H}^i$; (i.e. $dh = \delta h = 0$); see [DR], [AV]. In the incomplete case, this property is quite delicate. It is not an invariant of the quasi isometry class of the metric and can fail to hold even if $\bar{d} = \bar{\delta}^*$; e.g. it fails for the double cover of the punctured plane with the pulled back (flat) metric. This phenomenon is responsible for the difficulties of the incomplete case which were described in the introduction.

In the complete case, if $h \in L^2$, $\Delta h = 0$, we have, by Stokes' Theorem,

$$<\delta dh, f_n^2 h> - <f_n dh, 2df_n \wedge h> = <f_n dh, f_n dh> \tag{3.11}$$

$$<d\delta h, f_n^2 h> \pm <f_n \delta h, 2*(df_n \wedge *h)> = <f_n \delta h, f_n \delta h> \tag{3.12}$$

Since $|<a,b>| \leq \frac{1}{2}<a,a> + \frac{1}{2}<b,b>$

$$\frac{1}{2}<f_n dh, f_n dh> + 2<df_n \wedge h, df_n \wedge h> \geq |<f_n dh, 2df_n \wedge h>| \tag{3.13}$$

$$\frac{1}{2}<f_n \delta h, f_n \delta h> + 2<df_n \wedge *h, df_n \wedge *h> \geq |<f_n \delta h, 2*(df_n \wedge *h)>| . \tag{3.14}$$

Adding (3.11), (3.12) and using (3.13), (3.14) and $\Delta h = 0$, we get

$$\begin{aligned} &\frac{1}{2}(<f_n dh, f_n dh> + <f_n \delta h, f_n \delta h>) \\ &\leq 2(<df_n \wedge h, df_n \wedge h> + <df_n \wedge *h, df_n \wedge *h>) . \end{aligned} \tag{3.15}$$

As $n \to \infty$, it is easy to see that the right-hand side of (3.15) $\to 0$. Since $f_n \to 1$, it follows that $dh = \delta h = 0$.

To account for the possibility that range $\overline{d}_{i-1}$ may not be closed, it is customary to define the reduced L^2-cohomology by setting

$$\overline{H}^i_{(2)}(Y) = \ker \overline{d}_i / \overline{\text{range } d_{i-1}} . \tag{3.16}$$

If $\overline{d} = \overline{\delta}^*$, then automatically, $\overline{H}^i_{(2)}(Y) \cong \mathcal{H}^i$. Suppose that one now takes as his objective to find *some* topological interpretation of the space $\mathcal{H}^i$ and then to derive properties of the resulting object from general properties of harmonic forms; e.g. if Y is *any* complete Kähler manifold then $\mathcal{H}^*$, (which might possibly be infinite dimensional for some i), satisfies the Kahler package. Then $\overline{H}^i_{(2)}(Y)$ can be viewed as a "bridge"; i.e. to interpret $\mathcal{H}^i$ it suffices to calculate $\overline{H}^i_{(2)}(Y)$. If in fact $\overline{H}^i_{(2)}(Y) = H^i_{(2)}(Y)$, one can calculate on open subsets and apply the usual exact cohomology sequences; see [C3]. But since $\overline{H}^*_{(2)}(Y)$ is not the cohomology of a complex of cochains, these sequences may not hold if $\overline{H}^i_{(2)}(Y) \neq H^i_{(2)}(Y)$; compare [APS].

3.3. We now describe the L^2-cohomology for the simplest singularity in the compact case, that of a metric cone. Let N^m be a riemannian manifold with metric $\tilde{g}$. The *metric cone* $C^*(N^m)$ is by definition the completion of the smooth incomplete riemannian manifold $C(N) = R^+ \times N$, with

$$g = dr \otimes dr + r^2\tilde{g} \, . \tag{3.17}$$

We denote by $C_{r_0,r}(N^m)$ the subset $(r_0, r)\times N \subset C(N^m)$: Suppose that X^{m+1} is a compact metric space such that for some finite set of points $\{p_j\}$, $X - \bigcup_{j=1}^{N} p_j$ is a smooth riemannian manifold. We say that X^{m+1} has *isolated metrically conical singularities* if there exist smooth compact riemannian manifold N_j^m, and neighborhoods U_j of p_j, such that $U_j - p_j$ is isometric to $C_{0,r_j}(N^m)$. We say that X has *isolated conical singularities* if the metric on $X - \cup p_j$ is quasi isometric to a metric of the above type. We define $H^*_{(2)}(X)$ by

$$H^i_{(2)}(X) = H^i_{(2)}(X - \bigcup_{j=1}^{N} p_j) \, . \tag{3.18}$$

In [C3], it is shown that $H^i_{(2)}(X)$ does not change if further points are removed from $X - \bigcup_{j=1}^{N} p_j$. Thus $H^i_{(2)}(X)$ is well defined.

The Poincaré Lemma in this situation takes the following form.

$$H^i_{(2)}(C_{0,1}(N^m)) = \begin{cases} H^i(N^m) & i \leq m/2 \\ 0 & i > m/2 \, . \end{cases} \tag{3.19}$$

This corresponds to the calculation in §2.2, for IH_* of a truncated cone. When combined with the standard exact sequences, (3.19) yields the tabulation of $H^i_{(2)}(X^{m+1})$ given in §2.2 for $m+1$ even. In particular, $H^i_{(2)}(X^{m+1})$ is finite dimensional which implies that $\overline{d}_{i-1}$ has closed range.

If $m+1 = 2k$, or in case $m+1 = 2k+1$, if $H^k(N^{2k}, R) = 0$, then $\bar{d} = \bar{\delta}^*$. Thus in these cases, the Strong Hodge Theorem holds. If $m+1 = 2k+1$ and $\dim H^k(N^{2k}, R) > 0$, then $d_k \neq \delta^*_{k+1}$. Moreover a calculation based on (3.19) shows that Poincaré duality also fails in this case. These interesting phenomena demonstrate that the global topology of the link plays a significant role in the theory. However, they do *not* occur for algebraic varieties, and thus will not be discussed further here; see however [C1], [C2], [C3], [C5]. (The point here is that algebraic varieties admit a stratification by strata of even codimension.)

The calculation in (3.19) can be made intuitively plausible as follows. Let π_2 be the natural projection of $C_{0,1}(N^m)$ onto N^m; ($C_{0,1}(N^m)$ is topologically $(0,1) \times N^m$). If ϕ is an i-form on N^m, then the pointwise norm of $\pi_2^*(\phi)$ at a point (r,x) is r^{-i} times the norm of ϕ at x. Since the cross sectional area of $C_{0,1}(N^m)$ varies as r^m, the condition for $\pi_2^*(\phi)$ to define an element of $H^i_{(2)}(C_{0,1}(N^m))$; (i.e. for $\pi_2^*(\phi)$ to be in L^2), is just $m - 2i > -1$, or equivalently $i \leq \frac{m}{2}$.

The proof of (3.19) uses the following homotopy operators. Let $\theta(r,x) = \phi(r,x) + dr \wedge \omega(r,x)$ and let $a \in (0,1)$. Set

$$K\theta = \begin{cases} \int_a^r \omega & i \leq \frac{m}{2} + 1 \\ \\ \int_0^r \omega & i > \frac{m}{2} + 1 \,. \end{cases} \tag{3.20}$$

Appropriate estimates show that the integrals converge and define bounded operators on L^2. A computation and some estimates show that

$$(\bar{d}K + K\bar{d})\theta = \begin{cases} \theta - \phi(a) & i < \frac{m+1}{2} \\ \\ \theta & i \geq \frac{m}{2} + 1 \,. \end{cases} \tag{3.21}$$

The cases $i = \frac{m+1}{2}$, $i = \frac{m}{2} + 1$ are a little more delicate, but the end result is (3.19).

3.4. Let X^n be a pseudomanifold—i.e. a simplicial complex such that each point is contained in a closed n-simplex, each n–1 simplex is a face of exactly two n-simplices, and the n-simplices can be compatibly oriented. We assume X^n is embedded as a subcomplex of some piecewise linear triangulation of $\mathbb{R}^n$. Let Σ^i denote the closed i-skeleton of X^n. The induced metric on $X^n - \Sigma^{n-2}$, gives $X^n - \Sigma^{n-2}$ the structure of a *smooth* flat riemannian manifold; (a neighborhood of a point on an (n–1)-simplex is isometric to an open subset of $\mathbb{R}^n$). If X^n is now given the structure of a metric space in such a way that the induced metric on $X^n - \Sigma^{n-2}$ is quasi-isometric to the flat metric then this metric space will be called a *riemannian pseudo-manifold with conical singularities*; (e.g. the metric on X^n induced from the embedding in $\mathbb{R}^n$ obviously has this property).

Define $H^i_{(2)}(X) = H^i_{(2)}(X-\Sigma)$. The results on isolated conical singularities: $\overline{d} = \overline{\delta}^*$ and the strong Hodge theorem extend to this case. If X also has a stratification by even dimensional strata, then the axioms of §2.2 are satisfied by the complex of sheaves of locally L^2-differential forms on $X-\Sigma$ which are in dom(d). We conclude that $H^k_{(2)}(X) \cong IH_{n-k}(X) \cong \mathrm{Hom}(IH_k(X), \mathbb{C})$. This isomorphism is also constructed directly by integration in [C3]. Here it is most convenient to use a certain subcomplex ($\mathcal{S}^*$ of [C3], p. 136) of the L^2 forms, which has the same cohomology, but for which integration over chains which satisfy the "middle intersection group allowability condition" is always defined.

Finally we mention the following three general facts. The usual cohomology groups $H^*(X^n)$ can be represented by forms whose norms are uniformly bounded on $X^n - \Sigma^{n-2}$; e.g. forms on an open neighborhood of $X^n \subset \mathbb{R}^N$. From this it follows that $H^*_{(2)}(X^n)$ is a module over $H^*(X^n)$.

For each pseudomanifold X there is an associated space $\hat{X}$, the *normalization* of X [GM2]. Since X and $\hat{X}$ have the same nonsingular parts, $H^*_{(2)}(\hat{X}) \cong H^*_{(2)}(X)$. (In fact, $\hat{X}$ is topologically just the metric completion of the nonsingular part $X-\Sigma$.)

Consider a finite covering of $X-\Sigma^{n-2}$. If one pulls back and completes the Riemannian metric on $X-\Sigma$, one obtains a space X' and a map $\pi: X' \to X$ which is a (topological) *branched covering* (see [F]). There is a natural map $\pi^*: H^i_{(2)}(X) \to H^i_{(2)}(X')$. By Poincaré duality, π^* is an injection. (Note that analytic properties on X do not carry over automatically to properties on X'. For example, the torus T^2 is a branched cover of the sphere S^2, so $C(T^2)$ is a branched cover of $C(S^2) = \mathbb{R}^3$. But $\dim H^1(T^2, \mathbb{R}) \neq 0$ so $\bar{d}_1 \neq \delta_2^*$ on $C(T^2)$.)

3.5. Let X be a compact analytic variety which admits an embedding in some compact Kähler manifold (e.g. complex projective space). Any choice $\rho: X \to M$ of such an embedding determines a metric Ω_ρ on the nonsingular part $X-\Sigma$ of X by restriction of the Kähler metric on M.

PROPOSITION. *The quasi-isometry class of* Ω_ρ *is independent of the embedding* ρ.

As in §3.4 we define $H^i_{(2)}(X)$ to be $H^i_{(2)}(X-\Sigma)$ with the metric Ω_ρ. By the proposition, it is independent of ρ.

We note that we do not know the general fact about normalization of §3.4 in the analytic context because we do not know that the metric on the nonsingular part of $\hat{X}$ is quasi-isometric to that of X. A similar remark applies to branched coverings.

Let $V \subset \mathbb{C}^N$ be an analytic subvariety. We say that V is a cone at $p \in V$ if for some union W of affine complex lines through p and some neighborhood U of p, we have $V \cap U = W \cap U$. For example any subvariety of $\mathbb{C}^N$ defined by homogeneous equations is conical at 0.

DEFINITION. An analytic variety X is *locally analytically conical* if each point $p \in X$ has a neighborhood U and an analytic embedding $\rho: U \to \mathbb{C}$ such that $\rho(U)$ is a cone at $\rho(p)$.

EXAMPLES 1. Not all cones at 0 are locally analytically conical: for example $X^2Z = Y^3$ fails the test at $p = (0, 0.1)$.

2. Single condition Schubert varieties (see §5.2) are locally analytically conical.

PROPOSITION. *Locally analytically conical varieties are conical.*

In other words, the nonsingular part of a locally analytically conical variety is quasi-isometric to an open dense subset of a polyhedron. The converse is false: all algebraic curves are conical in the quasi-isometry sense.

COROLLARY. *For a locally analytically conical variety* X, $H^i_{(2)}(X) \cong \mathrm{Hom}(IH_i(X), \mathbb{C})$.

As remarked in §1.7, this does not in itself imply the theorems of the "Kähler package" for locally analytically conical varieties. But it is known for other reasons: see the section on algebraically conical singularities of §6.2.

3.6. We close this section by noting the following construction of complete Kähler metrics. Let X be a projective algebraic variety. Then as is well known, there exists an algebraic map $\pi : X \to \mathbb{CP}^N$ which is a branched covering. Let $Z \subset \mathbb{CP}^N$ denote the branch locus, and let $\mathfrak{L}$ be the line bundle over $\mathbb{CP}^N$ corresponding to Z, equipped with a Hermition metric. Let Ω denote the Kähler form of the usual metric on $\mathbb{CP}^N$. Then, if σ is a holomorphic section of $\mathfrak{L}$ which vanishes on Z and ϕ is a smooth function such that $\phi|Z \equiv 1$, $\phi \equiv 0$ off and ϵ-tubular neighborhood of Z, then as in [CG], for small ϵ',

$$\epsilon' \sqrt{-1}\,(dd^c \log \log^2 \phi\|\sigma\|^2) + \Omega \tag{3.22}$$

defines the Kähler form of a complete Kähler metric on $\mathbb{CP}^N - Z$. We do not know if the L^2-cohomology of this metric is isomorphic to the usual cohomology of $\mathbb{CP}^N$. If so, it makes sense to ask if the L^2-cohomology of the pulled back metric on $X-\pi^{-1}(Z)$ is isomorphic to $IH^*(X)$. This would imply that the "Kähler package" holds for X.

§4. *Conjectures*

In this section X will denote a complex n-dimensional projective variety. Conjecture A states that the intersection homology groups $IH_*(X)$ satisfy the conditions of the "Kähler package." Conjectures B and C state that the De Rham and Hodge theorems hold for the L^2 differential forms on the nonsingular part of X.

Let Σ denote the singular set of X. We shall denote the L^2 cohomology of $X-\Sigma$ (with the metric induced from the embedding in projective space) by $H^*_{(2)}(X)$.

Conjecture A : The Kähler package

A.1. (*Pure Hodge* (p, q) *decomposition*). There is a natural direct sum decomposition

$$IH_k(X) \cong \bigoplus_{p+q=k} IH_{(p,q)}(X)$$

such that

$$IH_{(p,q)}(X) \cong \overline{IH_{(q,p)}(X)} .$$

This decomposition is compatible with f^* and f_* when f is a branched covering or is normally nonsingular. For example, if $f: Y \to X$ is normally nonsingular with relative dimension m then

$$f_*: IH_{(p,q)}(Y) \to IH_{(p,q)}(X)$$

and

$$f^*: IH_{(p,q)}(X) \to IH_{(p-m,q-m)}(Y) .$$

The map from cohomology $H^i(X) \to IH_{2n-i}(X)$ is a morphism of Hodge structures.

A.2. (*Hard Lefschetz*). Let H be a hyperplane in the ambient projective space, which is transverse to a Whitney stratification of X. Let $\Omega \in H^2(X)$ denote the cohomology class represented by $H \cap X$ (as in §2.1) and let $L : IH_i(X) \to IH_{i-2}(X)$ denote multiplication by this class. Then the map

$$L^k : IH_{n+k}(X) \to IH_{n-k}(X)$$

is an isomorphism for each k.

If we define the primitive intersection homology $P_{n+k}(X) = \ker(L^{k+1})$ then we have the Lefschetz decomposition $IH_m(X) = \bigoplus_k L^k(P_{m+2k}(X))$. This decomposition is compatible with the Hodge decomposition.

A.3. *Poincaré duality.* The intersection pairing

$$IH_i(X) \times IH_{2n-i}(X) \to \mathbb{C}$$

is nonsingular in complementary dimensions.

A.4. *Lefschetz Hyperplane Theorem*: If H is a hyperplane which is transverse to each stratum of X, then the homomorphism induced by inclusion

$$IH_k(X \cap H : \mathbb{Z}) \to IH_k(X; \mathbb{Z})$$

is an isomorphism for $k < n-1$ and a surjection for $k = n-1$.

A.5. *Hodge Signature Theorem.* For $\xi \in P_{(p,q)}$ a primitive intersection homology class of dimension $k = p+q$, we have

$$(\sqrt{-1})^{p-q}(-1)^{(n-k)(n-k-1)/2} L^k(\xi \cap \bar{\xi}) > 0 .$$

If $\sigma(X)$ denotes the signature of the intersection pairing (A.3) on $IH_n(X)$ then

$$\sigma(X) = \sum_{p+q \equiv 0 \pmod 2} (-1)^p \dim IH_{(p,q)}(X) .$$

Conjecture A would follow from the stronger Conjectures B and C below. (In fact, for this implication, it would suffice to substitute $\overline{H}^*_{(2)}$ for $H^*_{(2)}$ in what follows.)

Conjecture B. The L^2 cohomology group $H^k_{(2)}(X)$ is finite dimensional and is isomorphic to the subspace $\mathcal{H}^k$ of $\Lambda^k \cap L^2$ which consists of the square summable differential k-forms which are closed and co-closed. Furthermore, the operator J preserves this subspace $\mathcal{H}^k$.

Conjecture C. For almost any chain $\xi \in C_k(X)$ and almost any differential form $\theta \in \Lambda^k \cap L^2$, the integral $\int_\xi \theta$ is finite, and $\int_{\partial\eta} \theta = \int_\eta d\theta$ whenever both sides are defined. The induced homomorphism

$$H^k_{(2)}(X) \xrightarrow{\iota} \mathrm{Hom}(IH_k(X), \mathbb{C})$$

is an isomorphism.

§5. Examples

In this section we consider two classes of examples: varieties with isolated conical singularities and single condition Schubert varieties. These examples will be used as illustrations throughout the rest of the paper.

In each case we will give a stratification and a resolution. We also calculate the cohomology and the intersection homology of these examples and verify that the intersection homology has a Hodge (p,q) decomposition.

5.1. *Isolated algebraically conical singularities*

We will treat the case where X has a unique algebraically conical singular point x. The case of several isolated algebraically conical singular points is entirely similar.

DEFINITION. The isolated singularity $x \in X$ is said to be algebraically conical if the Hopf blowup $\pi : \tilde{X} \to X$ of X at x is nonsingular and the exceptional division $D = \pi^{-1}(x)$ is nonsingular.

EXAMPLE. A locally analytically conical varieties (§3.5) with one singular point is algebraically conical.

Let $L \to D$ be the normal complex line bundle of D and let $S \to D$ be its circle bundle, which is the boundary of a tubular neighborhood of D in $\tilde{X}$. Consider the Gysin sequence

$$H_i(D) \xrightarrow{\cap C_1 L} H_{i-2}(D) \xrightarrow{\beta} H_{i-1}(S) \xrightarrow{\alpha} H_{i-1}(D) \xrightarrow{\cap C_1 L} H_{i-3}(D)$$

where $C_1 L$ denotes the first chern class of L. Since $D = \pi^{-1}(x)$ is the exceptional divisor it must satisfy the following

Blowing down condition: $C_1 L$ can be represented by a Kähler form on D, so $\cap C_1 L$ satisfies hard Lefschetz.

Since X is a space with isolated singularities we have:

$$IH_i(X) = \begin{cases} H_i(X) & \text{for } i > n \\ H_i(X-x) & \text{for } i < n \\ \mathrm{Im}(H_i(X-x) \to H_i(X)) & \text{for } i = n\,. \end{cases}$$

The column and rows in the following diagrams are exact:

I

$$\begin{array}{ccccccccc} & & & & & H_{i-1}(S) & & & \\ & & & & \nearrow & & \searrow{\scriptstyle\alpha} & & \\ H_i(D) & \longrightarrow & H_i(\tilde{X}) & \longrightarrow & H_i(X) & & \longrightarrow & H_{i-1}(D) & \longrightarrow H_{i-1}(\tilde{X}) \end{array}$$

II

$$\begin{array}{ccccccccc} & & & & & H_{i-1}(S) & & & \\ & & & & {\scriptstyle\beta}\nearrow & & \searrow & & \\ H_i(\tilde{X}-D) & \longrightarrow & H_i(\tilde{X}) & \longrightarrow & H_{i-2}(D) & & \longrightarrow & H_{i-1}(\tilde{X}-D) & \longrightarrow H_{i-1}(\tilde{X}) \\ H^{2n-i}(X) & & & & & & & & \end{array}$$

III

$$\begin{array}{ccccc} & & H_i(D) & & \\ & & \downarrow & \searrow{\scriptstyle \cap C_1 L} & \\ H^{2n-i}(X) = H_i(\tilde{X}-D) & \longrightarrow & H_i(\tilde{X}) & \longrightarrow & H_i(\tilde{X}, \tilde{X}-D) = H_{i-2}(D) \\ & {\scriptstyle\gamma}\searrow & & & \\ & & H_i(\tilde{X}, D) = H_i(X)\,. & & \end{array}$$

If $i > n$ it follows from the blowing down condition and the Gysin sequence that $\alpha = 0$. Therefore

$$IH_i(X) = \operatorname{coker}(H_i(D) \to H_i(\tilde{X}))$$

which has a Hodge (p, q) decomposition induced from that on $\tilde{X}$.

Similarly if $i < n$ then $\beta = 0$ so

$$IH_i(X) = \ker(H_i(\tilde{X}) \to H_{i-2}(D))$$

which is also pure.

Finally, if $i = n$, the map $\cap C_1 L$ of diagram III is an isomorphism, so $IH_n(X) = \ker(H_n(\tilde{X}) \to H_{n-1}(D))$ which is pure.

5.2. *Single condition Schubert varieties*

Fix integers i, j, k, ℓ such that $j + k \leq \ell$, and

$$\begin{array}{ccccc} & & j & & \\ & \leq & & \leq & \\ i & & & & \ell. \\ & \leq & & \leq & \\ & & k & & \end{array}$$

Let $F^j \subset \mathbb{C}^\ell$ denote a fixed j-dimensional subspace and let $G_k(\mathbb{C}^\ell)$ denote the Grassmann variety of k planes in $\mathbb{C}^\ell$. Define

$$\mathcal{S} = \{V^k \in G_k(\mathbb{C}^\ell) \mid \dim(V^k \cap F^j) \geq i\}.$$

Such an $\mathcal{S}$ is called a single condition Schubert variety. It has complex dimension $i(j-i)+(k-i)(\ell-k)$. Define

$$\tilde{\mathcal{S}} = \left\{ \text{partial flags } (W^i \subset V^k \subset \mathbb{C}^\ell) \mid W^i \begin{array}{ccccc} & \subset & F^j & \subset & \\ & & & & \mathbb{C}^\ell \\ & \subset & V^k & \subset & \end{array} \right\}.$$

The map $\pi : \tilde{\mathcal{S}} \to \mathcal{S}$ (given by $\pi(W^i \subset V^k) = V^k$) is a resolution of singularities. $\mathcal{S}$ is stratified by the single-condition Schubert subvarieties

$$\mathcal{S}_p = \{V^k \in G_k(\mathbb{C}^\ell) \mid \dim(V^k \cap F^j) \geq p\}$$

for $i \leq p \leq \min(j,k)$. The codimension of $\mathcal{S}_p$ in $\mathcal{S}$ is $C = (p-i)(p+i+\ell-j-k)$. If $x \in \mathcal{S}_p - \mathcal{S}_{p+1}$, then $\dim(\pi^{-1}(x)) = i(p-i)$. Since $i(p-i) \leq \frac{1}{2}C$, $\tilde{\mathcal{S}}$ is a small resolution of $\mathcal{S}$ and consequently $IH_*(\mathcal{S}) \cong H_*(\tilde{\mathcal{S}})$ (see [GM3]). It follows that $IH_*(\mathcal{S})$ inherits a Hodge (p,q) decomposition from that of $\tilde{\mathcal{S}}$. (It is known that $H_{p,q}(\tilde{\mathcal{S}}) = 0$ unless $p=q$ so the same is true of $IH_{p,q}(\mathcal{S})$.)

We now give the Poincaré polynomials for these spaces. Define $P_n(t) = 1(1+t^2)(1+t^2+t^4)\cdots(1+t^2+\cdots+t^{2n-2})$. The Poincaré polynomial $Q_\ell^k(t)$ for $G_\ell(\mathbb{C}^k)$ is

$$Q_\ell^k(t) = \frac{P_\ell(t)}{P_k(t)\,P_{\ell-k}(t)}.$$

The Poincaré polynomial for $IH_*(\mathcal{S})$ is $Q_i^k(t)\,Q_{j-i}^{\ell-i}(t)$. The Poincaré polynomial for $H^*(\mathcal{S})$ is

$$\sum_{i\leq p\leq \min(j,k)} Q_p^k(t)\,Q_{j-p}^{\ell-k}(t)\,t^{2j(j-p)}.$$

(Each term in this sum is a contribution arising from a stratum.)

The map $H^*(\mathcal{S}) \to H^*(\tilde{\mathcal{S}}) \cong IH_*(\mathcal{S})$ is an injection. One may verify from the Poincaré polynomials that it is in general far from being a surjection.

§6. *Status of the conjectures*

In this section, we present the currently available evidence for the conjectures of Chapter 4.

In §6.1 we discuss the parts of the Kähler package that have been proved for all complex projective varieties. The methods of proof are topological or algebraic: they do not use L^2 analysis. The main gap at present is that there is no general proof of the existence of a pure Hodge (p,q) decomposition.

Even in the nonsingular case, establishing a conjugation invariant (p,q) decomposition requires analysis. Two possible approaches to the singular case are to reduce it to a related nonsingular space—e.g. a resolution of singularities, or to develop L^2 analysis directly on the singular variety. As for the first possibility there is a conjecture that the intersection homology of a variety X is always a direct summand of the homology of any resolution $\tilde{X}$ and that the inclusion of $IH_*(X)$ into $H_*(\tilde{X})$ respects the Hodge decomposition of $H(\tilde{X})$ (see [GM3]). This conjecture of course would imply the existence of a pure Hodge decomposition on X, but the conjecture itself appears to be very difficult in general. In §6.2 we give a number of special cases in which this conjecture can be established.

As for analysis on X itself, there are again two approaches as indicated in §1.8 and §3. One is to use the incomplete metric on $X-\Sigma$ induced by the inclusion $X \subset CP^N$, and the other is to fabricate a complete metric. In both cases the analysis is extremely delicate and it depends on as yet unexplored aspects of the metric structure of X near a singularity. To carry this out for general X may well be even more difficult than the procedure using $\tilde{X}$, but the resulting understanding of the differential geometry of the singularities of X would be extremely interesting in itself. The progress to date on this is sketched in §6.4.

6.1. *Results for general varieties*

Poincare Duality: As mentioned in §2 the Poincaré duality theorem ($IH_i(X) \times IH_{2n-i}(X) \to \mathbb{C}$ is nonsingular) is true for all projective varieties X.

Weak Lefschetz: The weak Lefschetz theorem (§4.A4) has been proven for all complex projective varieties. There is a sheaf-theoretic proof [GM3] following ideas of Artin [A]. There is also a proof using the Morse-theoretic techniques of [GM4].

Hard Lefschetz: We have been informed by O. Gabber that he has found a proof of the hard Lefschetz theorem (4.A.2) for the ℓ-adic analogue of $IH_*(X)$ when X is a variety defined over a field of characteristic p. This implies the same result in characteristic O.

Purity in characteristic p: The proof of Gabber also shows that the ℓ-adic analogues of $IH_*(X)$ are pure in the sense of Deligne (all eigenvalues of the Frobenius action have the same absolute value). According to the heuristic dictionary of Deligne [D1] §3, this is the characteristic p analogue of conjecture 4.A.1. But it is moral evidence only. It does not imply the existence of a (p, q) decomposition.

Mixed Hodge structures: Verdier has informed us of the existence of sheaf-theoretic techniques which may be used to put a mixed Hodge structure on $IH_*(X)$. It would remain to show that this mixed Hodge structure is pure.

6.2. *Special classes of varieties*

Curves and surfaces: The Hodge (p, q) decomposition conjecture (4.A.1) is true for varieties X with complex dimension 1 or 2. For curves this is true because $IH_*(X) = IH_*(\hat{X})$ where $\hat{X}$ is the normalization of X, which is always nonsingular. For surfaces the (p, q) decomposition of $IH_*(X)$ may be deduced from that of $H_*(\tilde{X})$ where $\tilde{X}$ is the minimal resolution of X. The proof is similar to that in §5.1 but uses Grauert's blowing down condition in place of the ampleness condition on the normal bundle of the exceptional divisor.

Schubert varieties: For the single condition Schubert varieties of §5.2, $IH_*(X)$ is isomorphic to the cohomology of the small resolution $\tilde{X}$ and therefore has a Hodge (p, q) decomposition.

Algebraically conical singularities:

DEFINITION. A monoidal transformation $\pi : \tilde{Y} \to Y$ with center $Z \subset Y$ is called a *clean blowup* if

1. Z is nonsingular.
2. $D \to Z$ is a topological fibration, where $D = \pi^{-1}(Z)$ is the exceptional divisor.
3. The inclusion $D \subset \tilde{Y}$ is normally nonsingular (§2).

A variety X is said to have *algebraically conical singularities* if it can be desingularized by a sequence of clean blowups. (The sequence of centers Z may be chosen so as to have increasing dimension.) For example, a locally analytically conical variety (§3.5) has algebraically conical singularities, but not vice-versa.

For varieties with algebraically conical singularities, the (p,q) decomposition of $IH_*(X)$ is induced from that of $IH_*(\tilde{X})$ where $\tilde{X}$ is the resolution produced by the sequence of clean blowups. This may be proven in a manner analogous to the case of isolated conical singularities (§5.1) using as ingredients the hard Lefschetz theorem of Gabber, and Deligne's criterion for the degeneration of a spectral sequence [D6] related to the fibrations $D \to Z$.

Rational homology manifolds: An example of a rational homology manifold is the quotient of any smooth compact manifold by the smooth action of any finite group.

Suppose X is an algebraic variety which is a rational homology manifold. Since (for any rational homology manifold) the cohomology coincides with the intersection homology [GM2], it suffices to find a Hodge (p,q) decomposition of the cohomology. However Deligne shows this exists by observing that Poincaré duality on X implies that the cohomology of X injects into the cohomology of any resolution of X, and therefore inherits a Hodge structure from the resolution.

6.3. *Results in special dimensions*

The first Betti number: For all projective varieties X, $\dim(IH_1(X))$ is even (as would be predicted by the existence of a conjugation invariant (p,q) decomposition). This follows from inductive application of the weak Lefschetz theorem and a direct verification for surfaces (see §6.2). It is always true that $IH_1(X) \cong H_1(\hat{X})$ where $\hat{X}$ is the normalization of X [GM2]. Therefore we obtain the following corollary, which was pointed out to us by Horrocks:

COROLLARY. *For any normal projective variety* X, *the first Betti number of* X *(for ordinary homology) is even.*

Varieties with small singular sets: Suppose the singular set of X has dimension $\leq p$. Then if we iterate the process of taking a generic hyperplane section $p+1$ times, we obtain a nonsingular variety. By repeated application of the Lefschetz hyperplane theorem, we have

THEOREM. $IH_k(X)$ *has a pure Hodge* (p,q) *decomposition for all* $k < n-p-1$ *and all* $k > n+p+1$.

REMARKS 1. If we also use the fact that $IH_*(X)$ has an appropriately natural mixed Hodge structure (see §6.1), this theorem can be extended to the cases $k = n-p-1$ and $k = n+p+1$.

2. The same idea shows that the hard Lefschetz map $L^k : IH_{n+k}(X) \to IH_{n-k}(X)$ is an isomorphism for all $k > p+1$.

6.4. L^2-*cohomology*

As explained in §3, for compact spaces with conical singularities $H^*_{(2)}(X) \simeq IH^*(X)$ and the Strong Hodge Theorem holds. However, if in addition the metric on the nonsingular part of X is Kähler this is still not enough to imply the "Kähler package" because the almost complex structure J may not preserve the space $\mathcal{H}^i$; (we still conjecture that J does preserve $\mathcal{H}^i$ if the singularities are conical in a suitable complex analytic sense, e.g. if X is an algebraic variety with metric induced from its embedding in CP^N; see §4). At present, there are two cases when J can be shown to preserve $\mathcal{H}^i$, see [C4] for details.

Isolated metrically conical singularities

Let $C(N^m)$ be a metric cone, where $m = 2k-1$ is odd. Then it can be shown that $h \in L^2$, $\Delta h = 0$, implies $h \in \mathcal{H}^i$, with the possible exception of the cases $i = \frac{m-1}{2}, \frac{m+1}{2}, \frac{m+3}{2}$. Thus if the metric on $C(N^m)$ is Kähler, $J(\mathcal{H}^i) = \mathcal{H}^i$ except possibly in these dimensions. Now assume further that the complex structure is invariant under the 1-parameter group of dilations of $C(N^m)$; e.g. suppose $C(N^m)$ is a complex affine cone. Then it can be checked directly that J preserves the space of forms θ such that $\theta, d\theta, \delta\theta, d\delta\theta, \delta d\theta \in L^2$. This suffices to show that for compact Kähler manifolds with isolated metrically conical singularities, such that J commutes with dilations, $J(\mathcal{H}^i) = \mathcal{H}^i$. More generally, the same follows if the metric and complex structure satisfy these conditions to sufficiently high order at the singular point.

Piecewise flat spaces

The arguments in the example above can be generalized to certain piecewise flat spaces by induction, and "the method of descent"; (compare [CT], example 4.5). Rather than giving a general definition of this class of spaces we will indicate how to construct some examples. Let Y be a compact Kähler manifold such that the metric g is flat and let Z be an arbitrary union of compact totally geodesic complex hypersurfaces. Let $\pi: X \to Y$ be a finite branched covering of Y, branched along Z. Then the completion of the metric $\pi^*(g)$ on $X - \pi^{-1}(Z)$ gives X the structure of a piecewise flat space with metrically conical singularities, and $J(\mathcal{H}^i) = \mathcal{H}^i$ on X. More generally, Y and Z might be quotients of piecewise flat spaces in this construction. For example, let Y be the space obtained by dividing $\overbrace{C \times \cdots \times C}^{n}$ by the group generated by the standard lattice, together with multiplication by -1 in each factor and permutations of the factors. Then Y is homeomorphic to CP^n.

Note that in both of the above cases, it is only the Kähler property that is relevant. Thus X need not be an algebraic variety.

Complete metrics (see [M], [ZU1], [ZU2])

In [ZU2],* Zucker considers $H^*_{(2)}(\Gamma\ X)$, the L^2 cohomology of of quotients of symmetric spaces by arithmetic groups, for which the natural metric is complete and has finite volume. In the Hermitian cases, the metric is Kähler. He shows that in certain cases $H^*_{(2)}(\Gamma \backslash X)$ is naturally isomorphic to $IH^*(\Gamma \backslash X^*)$, where $\Gamma \backslash X^*$ is the Baily Borel compactification of $\Gamma \backslash X$.

*Other strong evidence is provided by [ZU1].

§7. *Relations with mixed Hodge theory*

In [D1], [D2], [D3] Deligne defines a mixed Hodge structure on the cohomology of any algebraic variety X. This gives a filtration

$$w_0 \subset w_1 \subset \cdots \subset w_{2i} = H^i(X)$$

such that w_j/w_{j-1} has a Hodge (p, q) decomposition with $p+q = j$ ("w_j/w_{j-1} has pure weight j"). He shows:

(7.1) $$X \text{ compact} \Longrightarrow w_i = w_{i+1} = \cdots = w_{2i}$$

(7.2) $$X \text{ smooth} \Longrightarrow w_0 = w_1 = \cdots = w_{i-1} = 0 .$$

In §7.1 we give a (conjectural) relation between the mixed Hodge structure on the cohomology of X and the (conjectured) pure Hodge structure on $IH_*(X)$. In §7.2 we deduce both structures from the pure Hodge structure of a resolution of X, when X has isolated singularities. We find that an additional criterion is needed for the procedure to work with intersection homology. This criterion is sharpened in §7.3 and gives rise to new (conjectural) necessary conditions for blowing down.

7.1. *Conjecture.* $w_{i-1}(H^i(X)) = \ker(H^i(X) \to IH_{2n-i}(X))$ for compact X. Notice that the kernel always contains w_{i-1} if the Hodge (p, q) decomposition conjecture (4.A.1) is true (because the map is strictly compatible with the filtration [D2] 2.3.5). A consequence of this conjecture is that the (conjectural) pure Hodge structure on $IH_{2n-i}(X)$ determines the one from mixed Hodge theory on w_i/w_{i-1}. The reverse is not true. For single condition Schubert varieties (§5.2) the map $w_i/w_{i-1} \to IH_{2n-i}(X)$ is not surjective.

Conjecture 7.1 is true for the examples of §5 by direct calculation.

Deligne has suggested [D5] the existence of a technique whereby the Hodge structures on the other w_j/w_{j-1} could be similarly determined using the pure Hodge structures on other intersection homology groups.

This technique would apply to the hypercohomology of complexes of algebraic sheaves (as well as to the ordinary cohomology) thereby extending mixed Hodge theory to such hypercohomology groups.

7.2. In this section we describe Deligne's construction of the weight filtration on the cohomology of a space with an isolated singularity. This induces a mixed Hodge structure on intersection homology.

Let D be any compact subvariety of a nonsingular n-dimensional compact complex variety $\tilde{X}$. We first construct a mixed Hodge structure on the cohomology of $\tilde{X}/D$ (the space obtained by collapsing D to a point). In the case that $\tilde{X}/D$ admits the structure of an algebraic variety X (compatibly with the projection $\tilde{X} \to X$), this calculation gives the mixed Hodge structure on X. Consider the exact sequence of the pair (diagram II of 5.1):

$$\to H^{i-1}(X) \to H^{i-1}(\tilde{X}) \xrightarrow{\theta} H^{i-1}(D) \to H^i(X) \to H^i(\tilde{X}) \xrightarrow{\theta} H^i(D) \to .$$

Here, $w_i = H^i(X)$

$$w_{i-1} = \operatorname{coker}(H^{i-1}(X) \to H^{i-1}(D))$$

$$w_j = w_j(H^{i-1}(D)) \quad \text{for} \quad j < i-1 .$$

One can see directly from the exact sequence (and the fact that each homomorphism is strictly compatible with the filtration) that w_j/w_{j-1} has a pure Hodge (p,q) decomposition of weight j.

Since $H^i(X) \cong IH_{2n-i}(X)$ for $i > n$ and $\operatorname{Hom}(H^i(X), \mathbb{C}) \cong IH_i(X)$ for $i < n$, we obtain mixed Hodge structures on $IH_j(X)$ for all $j \neq n$. (However (7.1) is not satisfied even though X is compact.) It is easy to see from diagram III of §5.1 that $IH_n(X)$ has a pure Hodge structure of weight n. Note: this mixed Hodge structure on $IH_*(X)$ depends only on the algebraic structure of $\tilde{X} - D$, the nonsingular part of X.

This gives the following result:

PROPOSITION. *A necessary and sufficient condition that* $IH_*(X)$ *have a pure Hodge structure, is that the map*

$$\theta : H^i(\tilde{X}) \to H^i(D)$$

be a surjection for all $i \geq n$, *or equivalently*

$$H_j(D) \to H_j(X) \text{ is an injection for all } j \geq n .$$

Observe that in the example of §5.1, this condition is guaranteed by the blowing down condition. However even in this example, the cohomology has only a *mixed* Hodge structure. Thus, to prove that the intersection homology of a variety with an isolated singularity has a pure Hodge structure, one must verify the above condition on any resolution. We do not know how to do this in general, although the preceding construction of the *mixed* Hodge structure (on H^* and IH_*) requires no further condition. Thus the existence of a pure Hodge structure on $IH_*(X)$ appears to involve more subtle structure of the variety than does the existence of a mixed Hodge structure on cohomology.

7.3. We now turn the question around and ask what blowing down conditions are implied by these ideas.

Let D be an arbitrary (compact) subvariety of a nonsingular compact n-dimensional variety $\tilde{X}$ and suppose $X = \tilde{X}/D$ is algebraic.

Conjecture. $H_j(D) \to H_j(\tilde{X})$ is an injection for all $j \geq n$ and this holds for local reasons near D, i.e. if T is a tubular neighborhood of D in $\tilde{X}$, with boundary S, then $H_j(T) \to H_j(T, S)$ is an injection for all $j \geq n$. (This conjecture is a consequence of the "direct sum conjecture" in [GM3].)

REMARKS. The local condition is stronger than the global condition because of the factorization

$$\begin{array}{ccccc} H_j(D) \cong H_j(T) & \longrightarrow & H_j(\tilde{X}) & \longrightarrow & H_j(\tilde{X}, \tilde{X}-D) \\ & \searrow & & & \wr\!\Vert \\ & & & & H_j(T, S) \end{array}$$

This conjecture has two interesting consequences:

Consequence 1. For all $j \geq n$ the mixed Hodge structure on

$$H_j(D) = \ker(H_j(\tilde{X}) \to IH_j(X)) \text{ is actually pure.}$$

Consequence 2. The map given by pushing into $\tilde{X}$ and then restricting to D,

$$H_i(D) \cong H_i(T) \longrightarrow H_i(T, S) \xrightarrow{\cong} H^{2n-i}(T) \cong H^{2n-i}(D)$$

is an injection.

Consequence 2 is part of the blowing down condition from the example in §5.1 since the map $H_i(D) \to H^{2n-i}(D) \to H_{i-2}(D)$ coincides with $\cap C_1 L$.

If $\tilde{X}$ is a surface and D is a divisor with normal crossings, Grauert's necessary and sufficient blowing down criterion is that the intersection form be negative definite. Our necessary condition (2) is that the intersection form be nonsingular.

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