Homological ring epimorphisms and recollements from exact pairs, I.

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Abstract

Homological ring epimorphisms are often used in modern representation theory and algebraic K-theory. In this paper, we give some new characterizations of when a universal localization related to an 'exact' pair of ring homomorphisms is homological. These characterizations are flexible and applicable to many cases, thus give rise to a wide variety of new recollements (of derived module categories) which have become of interest in and attracted increasing attentions towards to understanding invariants in algebra and geometry. As a consequence, we show that if $\lambda: R \to S$ is an injective homological ring epimorphism between commutative rings R and S, then the derived module category of the endomorphism ring of the R-module $S \oplus S/R$ always admits a recollement of the derived module categories of R and the tensor product $S \otimes_R \mathrm{End}_R(S/R)$. In particular, this result is applicable to localizations of integral domains by multiplicative sets in commutative rings.

Contents

1 Introduction

The investigation of homological ring epimorphisms has a long history, and there is a large variety of literature. For example, in representation theory, homological ring epimorphisms were used to study perpendicular categories and sheaves, recollements and stratifications of derived module categories of rings (see [9], [5]), and to construct infinitely generated tilting modules (see [2]). In algebraic K-theory, Neeman and Ranicki used homological ring epimorphisms to establish a useful long exact sequence of algebraic K-groups (see [12]), which generalizes many earlier results in the literature. Also, in Banach algebra, homological ring epimorphisms were topologically modified to investigate the analytic functional calculus (see [17]), where they were called "localizations".

Let R be an associative ring with identity. Suppose that $\lambda: R \to S$ and $\mu: R \to T$ are two homomorphisms of rings. We may form the coproduct $S \sqcup_R T$ of S and T over R. Let $\rho: S \to S \sqcup_R T$ and $\phi: T \to S \sqcup_R T$ be the canonical ring homomorphisms given by the definition of coproducts of R-rings. Then one may define a homomorphism θ of the following rings:

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$$\theta : \begin{pmatrix} S & S\otimes_R T \\ 0 & T \end{pmatrix} \longrightarrow \begin{pmatrix} S\sqcup_R T & S\sqcup_R T \\ S\sqcup_R T & S\sqcup_R T \end{pmatrix},$$

$$\begin{pmatrix} s_1 & s_2\otimes t_2 \\ 0 & t_1 \end{pmatrix} \mapsto \begin{pmatrix} (s_1)\rho & (s_2)\rho(t_2)\phi \\ 0 & (t_1)\phi \end{pmatrix}$$

for $s_i \in S$ and $t_i \in T$ with $i = 1,2$.

For simplicity, we denote by B the former 2×2 triangular matrix ring, and by C the latter 2×2 full matrix ring $M_2(S\sqcup_R T)$ over $S\sqcup_R T$.

The ring homomorphism $\theta : B \to C$ is of particular interest in representation theory: It can be regarded as the universal localization of B at a homomorphism between finitely generated projective B-modules, and therefore it is a ring epimorphism with $\mathrm{Tor}_1^B(C,C) = 0$ (see [16]), and yields a fully faithful exact functor $\theta_* : C\text{-Mod} \to B\text{-Mod}$, called the restriction functor, between the category of all left C-modules and the one of all left B-modules.

Generally speaking, θ is not always homological. In [5], there is a sufficient condition for θ to be homological. Concisely, assume that $\lambda : R \to S$ is injective, and choose T to be the endomorphism ring of the R-module S/R and $\mu : R \to T$ to be the ring homomorphism defined by $r \mapsto (x \mapsto xr)$ for $r \in R$ and $x \in S/R$. If λ is additionally a ring epimorphism with $\mathrm{Tor}_1^R(S,S) = 0$, then B is isomorphic to the endomorphism ring of the R-module $S\oplus S/R$. Moreover, if ${}_RS$ has projective dimension at most 1, then θ is homological. In particular, the (unbounded) derived category $\mathscr{D}(B)$ of the ring B admits a recollement with $\mathscr{D}(C)$ on the left-hand side and $\mathscr{D}(R)$ on the right-hand side (see [5]). This was used in [5] to establish the so-called Happel's Theorem for infinitely generated tilting modules and to show that the Jordan-Hölder theorem fails for stratifications of derived module categories by derived module categories. Here, the condition on ${}_RS$ only ensures that the R-module $S\oplus S/R$ is a tilting R-module of projective dimension at most 1 (see [2, 9]), and consequently, the homomorphism λ itself is homological.

However, in general, for an arbitrary homological ring epimorphism $\lambda : R \to S$, the projective dimension of ${}_RS$ may be greater than 1 (see the examples in the last section). Thus, it is certainly of interest in stratifications of derived categories and in algebraic K-theory to find some other new and applicable criterions for θ to be homological under a more general setting. Namely, the following question arises naturally:

Question. *Given a pair (λ,μ) of ring homomorphisms with λ being homological, when is $\theta : B \to C$ homological, or equivalently, when is the derived functor $D(\theta_*) : \mathscr{D}(C) \to \mathscr{D}(B)$ fully faithful ?*

In the present paper, we shall provide some answers to this question. Here, we assume neither that λ is injective, nor that $S\oplus S/R$ is a tilting R-module of projective dimension at most 1. Furthermore, we allow some flexibilities for the choice of the ring homomorphism $\mu : R \to T$. Under these general settings, we shall provide some new and handy characterizations for the universal localization θ to be homological. Our characterizations will be given in terms of vanishing of homology groups of R-modules T and S, or in terms of another ring homomorphism between two rings which are related to both (λ,μ) and the coproduct of S and T over R. In particular, the vanishing condition on homology groups can be applied in many cases.

As a consequence of these characterizations, we can produce a large variety of new recollements which could be used to understand stratifications of derived module categories, or of derived categories of coherent sheaves over geometric manifolds as well as to calculate algebraic K-theory of rings (see [2], [6], [9], [12]). Moreover, we show in the present paper that if λ is an injective homological ring epimorphism between two commutative rings R and S and if μ is the ring homomorphism from R to the endomorphism ring of the R-module S/R, defined by the right multiplication, then θ is always homological, and therefore, the derived category of the ring B is a recollement of the derived module categories of R and the tensor product of S and $\mathrm{End}_R(S/R)$ over R.

To state our results more precisely, let us first introduce some definitions that will be employed throughout the paper.

Given a pair of ring homomorphisms $\lambda : R \to S$ and $\mu : R \to T$, there are R-R-bimodule structures on S and T, respectively, and natural homomorphisms of R-R-bimodules: $\lambda' = \lambda \otimes T : T \to S \otimes_R T$ defined by $t \mapsto 1 \otimes t$ for $t \in T$, and $\mu' = S \otimes \mu : S \to S \otimes_R T$ defined by $s \mapsto s \otimes 1$ for $s \in S$. The pair (λ, μ) of ring homomorphisms is called *semi-exact* if the map $\begin{pmatrix} \mu' \\ -\lambda' \end{pmatrix} : S \oplus T \to S \otimes_R T$ is surjective. The kernel of this map is denoted by K. Then one can check that K is indeed a ring and that there is a canonical ring homomorphism $\zeta : R \to K$, defined by $r \mapsto \big((r)\lambda, (r)\mu\big)$ for $r \in R$. The pair (λ, μ) is called *exact* if it is semi-exact and ζ is an isomorphism of rings. There is a recipe to get semi-exact pairs: Let I_i be an arbitrary ideal of R for $i = 1, 2$. Define $S := R/I_1$ and $T := R/I_2$. Let λ and μ be the canonical surjective ring homomorphisms. Then the pair (λ, μ) is always semi-exact, and it is exact if and only if $I_1 \cap I_2 = 0$.

Now we can state our main result in this paper as follows.

Theorem 1.1. *Let (λ, μ) be an exact pair of ring homomorphisms $\lambda : R \to S$ and $\mu : R \to T$. Suppose that $\lambda : R \to S$ is homological. Then the following assertions are equivalent:*

(1) *The ring homomorphism $\theta : B \to C$ is homological.*

(2) *The ring homomorphism $\phi : T \to S \sqcup_R T$ is homological.*

(3) $\mathrm{Tor}_i^R(T, S) = 0$ *for all $i \geq 1$.*

If one of the above assertions holds, then there exists a recollement of derived module categories:

$$\begin{tikzcd} \mathscr{D}(\mathrm{End}_T(T \otimes_R S)) \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

Note that the ring $\mathrm{End}_T(T \otimes_R S)$ in Theorem 1.1 is isomorphic to the coproduct $S \sqcup_R T$ (see Lemma 3.8(2)), which is Morita equivalent to the ring C.

Clearly, $\mathscr{D}(B)$ is always a recollement of $\mathscr{D}(T)$ and $\mathscr{D}(S)$, in which the category $\mathscr{D}(R)$ is not involved. However, Theorem 1.1 provides us a different recollement for $\mathscr{D}(B)$. A remarkable feature of this recollemnt is that it contains $\mathscr{D}(R)$ as its member, and thus provides a way to understand properties of the ring R through those of the rings B, S and T. This idea will be discussed in another paper.

As a consequence of Theorem 1.1, we obtain the following result which can be seen as a concrete realization of Theorem 1.1.

Corollary 1.2. (1) *Let R be a ring, and let I_1 and I_2 be ideals of R with $I_1 \cap I_2 = 0$. If the canonical surjective ring homomorphism $R \to R/I_1$ is homological (for instance, the ideal I_1 is idempotent and projective as a left module), then so is the canonical surjective ring homomorphism $R/I_2 \to R/(I_1 + I_2)$, and therefore there is a recollement of derived module categories:*

$$\begin{tikzcd} \mathscr{D}(R/(I_1 + I_2)) \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} ,$$

where $B := \begin{pmatrix} R/I_1 & R/(I_1 + I_2) \\ 0 & R/I_2 \end{pmatrix}$.

(2) *Suppose that $\lambda : R \to S$ is a homomorphism of rings and M is an S-S-bimodule. Let $\widetilde{\lambda} : R \ltimes M \to S \ltimes M$ be the ring homomorphism between trivial extensions induced from λ. Then λ is homological if and only if so is $\widetilde{\lambda}$. In particular, if λ is homological, then there is a recollement of derived module categories:*

$$\begin{tikzcd} \mathscr{D}(S \ltimes M) \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} ,$$

where $B := \begin{pmatrix} S & S \ltimes M \\ 0 & R \ltimes M \end{pmatrix}$.

Another realization of Theorem 1.1 occurs in universal localizations.

Given a ring homomorphism $\lambda : R \to S$, we may consider λ as a complex $Q^\bullet$ of left R-modules with R and S in degrees -1 and 0, respectively. Then there exists a distinguished triangle $R \xrightarrow{\lambda} S \xrightarrow{\pi} Q^\bullet \longrightarrow R[1]$ in the homotopy category $\mathscr{K}(R)$ of the category of all R-modules. This triangle induces a canonical ring homomorphism from R to the endomorphism ring of $Q^\bullet$ in $\mathscr{K}(R)$, and therefore yields a ring homomorphism $\overline{\mu}$ from R to the endomorphism ring of $Q^\bullet$ in $\mathscr{D}(R)$, which depends on λ (see Section 4.2). Let $S' := \mathrm{End}_{\mathscr{D}(R)}(Q^\bullet)$. Observe that if λ is injective, then $Q^\bullet$ can be identified in $\mathscr{D}(R)$ with the R-module S/R, and consequently, the map $\overline{\mu}$ coincides with the map from R to $\mathrm{End}_R(S/R)$ by the right multiplication.

Further, let $\Lambda := \mathrm{End}_{\mathscr{D}(R)}\big(S \oplus Q^\bullet\big)$, and let π^* be the induced map $\mathrm{Hom}_{\mathscr{D}(R)}(S \oplus Q^\bullet, \pi) : \mathrm{Hom}_{\mathscr{D}(R)}(S \oplus Q^\bullet, S) \to \mathrm{Hom}_{\mathscr{D}(R)}(S \oplus Q^\bullet, Q^\bullet)$, which is a homomorphism of finitely generated projective Λ-modules. Let $\lambda_{\pi^*} : \Lambda \to \Lambda_{\pi^*}$ stand for the universal localization of Λ at π^* in the sense of Cohn and Schofield (see [8, 16]).

If λ is a ring epimorphism such that $\mathrm{Tor}_1^R(S,S) = 0 = \mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)\big)$, then we shall prove in Section 4.2 that the pair $(\lambda, \overline{\mu})$ is exact. Hence, the following corollary follows from Theorem 1.1.

Corollary 1.3. *If $\lambda : R \to S$ is a homological ring epimorphism such that $\mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)\big) = 0$, then the following assertions are equivalent:*

(1) *The universal localization $\lambda_{\pi^*} : \Lambda \to \Lambda_{\pi^*}$ of Λ at π^* is homological.*

(2) *The ring homomorphism $\phi : S' \to S \sqcup_R S'$ is homological.*

(3) $\mathrm{Tor}_i^R(S', S) = 0$ *for any $i \geq 1$.*

In particular, if one of the above assertions holds, then there exists a recollement of derived module categories:

$$\begin{tikzcd} \mathscr{D}(\mathrm{End}_{S'}(S' \otimes_R S)) \arrow[r] & \mathscr{D}(\Lambda) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

As an application of Corollary 1.3, we obtain the following result which generalizes the first statement of [5, Corollary 6.6(1)] since we do not require that the ring epimorphism λ is injective. In this general case, the module $S \oplus\ S/R$ may not be a tilting R-module.

Corollary 1.4. *Let $\lambda : R \to S$ be a homological ring epimorphism such that $\mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)\big) = 0$. Then we have the following:*

(1) *If ${}_RS$ has projective dimension at most 1, then $\lambda_{\pi^*} : \Lambda \to \Lambda_{\pi^*}$ is homological.*

(2) *The ring Λ_{π^*} is zero if and only if there is an exact sequence $0 \to P_1 \to P_0 \to {}_RS \to 0$ of R-modules such that P_i is finitely generated and projective for $i = 0, 1$. In this case, the rings R and Λ are derived equivalent.*

As another application of Corollary 1.3, we have the following result, which extends greatly the second statement of [5, Corollary 6.6(1)] since we do not impose any restriction on the projective dimension of ${}_RS$.

Corollary 1.5. *Suppose that $\lambda : R \to S$ is an injective homological ring epimorphism between commutative rings R and S. Then there exists a recollement of derived module categories:*

$$\begin{tikzcd} \mathscr{D}(S \otimes_R S') \arrow[r] & \mathscr{D}(\mathrm{End}_R(S \oplus S/R)) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} ,$$

where $S' := \mathrm{End}_R(S/R)$ is a commutative ring.

We remark that if R is a commutative ring and Φ is a multiplicative subset of R (that is, $\emptyset \neq \Phi$ and $st \in \Phi$ whenever $s, t \in \Phi$), then the localization $R \to \Phi^{-1}R$ of R at Φ is always homological. If $f : R \to S$ is a homomorphism from the ring R to another commutative ring S, then the image of a multiplicative subset of R under f is again a multiplicative set in S. So, as a direct consequence of Corollary 1.5, we obtain the following result which may be of its own interest in commutative algebra.

Corollary 1.6. *Suppose that R is a commutative ring with Φ a multiplicative subset of R. Let S be the localization $\Phi^{-1}R$ of R at Φ, and let $\lambda : R \to S$ be the canonical ring homomorphism. If the map λ is injective (for example, if R is an integral domain), then there exists a recollement of derived module categories:*

$$\begin{tikzcd} \mathscr{D}(\Psi^{-1}S') \arrow[r] & \mathscr{D}(\mathrm{End}_R(S \oplus S/R)) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} ,$$

where $S' := \mathrm{End}_R(S/R)$, and Ψ is the image of Φ under the ring homomorphism $\overline{\mu} : R \to S'$ associated to λ.

The contents of this paper are outlined as follows. In Section 2, we fix notation and recall some definitions and basic facts which will be used throughout the paper. In particular, we shall recall the definitions of universal localizations, recollements and coproducts of rings, and prepare several lemmas for our proofs. In Section 3, we prove Theorem 1.1. In Section 4, we prove all corollaries mentioned in Section 1. Also, in this section, we show a few other consequences of our results. Finally, in Section 5, we give several examples to explain the necessity of some assumptions in our results.

2 Preliminaries

In this section, we shall recall some definitions, notation and basic results which are closely related to our proofs.

2.1 Notation

Let $\mathcal{C}$ be an additive category.

Throughout the paper, a full subcategory $\mathcal{B}$ of $\mathcal{C}$ is always assumed to be closed under isomorphisms, that is, if $X \in \mathcal{B}$ and $Y \in \mathcal{C}$ with $Y \simeq X$, then $Y \in \mathcal{B}$.

Given two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, we denote the composite of f and g by fg which is a morphism from X to Z. The induced morphisms $\mathrm{Hom}_{\mathcal{C}}(Z, f) : \mathrm{Hom}_{\mathcal{C}}(Z,X) \to \mathrm{Hom}_{\mathcal{C}}(Z,Y)$ and $\mathrm{Hom}_{\mathcal{C}}(f,Z) : \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$ are denoted by f^* and f_*, respectively.

We denote the composition of a functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ with a functor $G : \mathcal{D} \to \mathcal{E}$ between categories $\mathcal{D}$ and $\mathcal{E}$ by GF which is a functor from $\mathcal{C}$ to $\mathcal{E}$. The kernel and the image of the functor F are denoted by $\mathrm{Ker}(F)$ and $\mathrm{Im}(F)$, respectively.

Let $\mathcal{Y}$ be a full subcategory of $\mathcal{C}$. By $\mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(-,\mathcal{Y}))$ we denote the left orthogonal subcategory with respect to $\mathcal{Y}$, that is, the full subcategory of $\mathcal{C}$ consisting of the objects X such that $\mathrm{Hom}_{\mathcal{C}}(X,Y) = 0$ for all objects Y in $\mathcal{Y}$. Similarly, $\mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(\mathcal{Y},-))$ stands for the right orthogonal subcategory of $\mathcal{C}$ with respect to $\mathcal{Y}$.

Let $\mathscr{C}(\mathcal{C})$ be the category of all complexes over $\mathcal{C}$ with chain maps, and $\mathscr{K}(\mathcal{C})$ the homotopy category of $\mathscr{C}(\mathcal{C})$. When $\mathcal{C}$ is abelian, the derived category of $\mathcal{C}$ is denoted by $\mathscr{D}(\mathcal{C})$, which is the localization of $\mathscr{K}(\mathcal{C})$ at all quasi-isomorphisms. It is well known that both $\mathscr{K}(\mathcal{C})$ and $\mathscr{D}(\mathcal{C})$ are triangulated categories. For a triangulated category, its shift functor is denoted by $[1]$ universally.

If $\mathcal{T}$ is a triangulated category with small coproducts (that is, coproducts indexed over sets exist in $\mathcal{T}$), then, for each object U in $\mathcal{T}$, we denote by $\mathrm{Tria}(U)$ the smallest full triangulated subcategory of $\mathcal{T}$ containing U and being closed under small coproducts. We mention the following properties related to $\mathrm{Tria}(U)$:

Let $F : \mathcal{T} \to \mathcal{T}'$ be a triangle functor of triangulated categories, and let $\mathcal{Y}$ be a full subcategory of $\mathcal{T}'$. We define $F^{-1}\mathcal{Y} := \{X \in \mathcal{T} \mid F(X) \in \mathcal{Y}\}$. Then

(1) If $\mathcal{Y}$ is a triangulated subcategory, then $F^{-1}\mathcal{Y}$ is a full triangulated subcategory of $\mathcal{T}$.

(2) Suppose that $\mathcal{T}$ and $\mathcal{T}'$ admit small coproducts and that F commutes with coproducts. If $\mathcal{Y}$ is closed under small coproducts in $\mathcal{T}'$, then $F^{-1}\mathcal{Y}$ is closed under small coproducts in $\mathcal{T}$. In particular, for an object $U \in \mathcal{T}$, we have $F(\mathrm{Tria}(U)) \subseteq \mathrm{Tria}(F(U))$.

In this paper, all rings considered are assumed to be associative and with identity, and all ring homomorphisms preserve identity. Unless stated otherwise, all modules are referred to left modules.

Let R be a ring. We denote by R-Mod the category of all unitary left R-modules. By our convention of the composite of two morphisms, if $f: M\to N$ is a homomorphism of R-modules, then the image of $x\in M$ under f is denoted by $(x)f$ instead of $f(x)$. The endomorphism ring of the R-module M is denoted by $\mathrm{End}_R(M)$.

As usual, we shall simply write $\mathscr{C}(R)$, $\mathscr{K}(R)$ and $\mathscr{D}(R)$ for $\mathscr{C}(R\text{-Mod})$, $\mathscr{K}(R\text{-Mod})$ and $\mathscr{D}(R\text{-Mod})$, respectively, and identify R-Mod with the subcategory of $\mathscr{D}(R)$ consisting of all stalk complexes concentrated in degree zero.

Let $(X^\bullet, d_{X^\bullet})$ and $(Y^\bullet, d_{Y^\bullet})$ be two chain complexes over R-Mod. The mapping cone of a chain map $h^\bullet: X^\bullet\to Y^\bullet$ is usually denoted by $\mathrm{Cone}(h^\bullet)$. In particular, we have a triangle $X^\bullet \xrightarrow{h^\bullet} Y^\bullet \longrightarrow \mathrm{Cone}(h^\bullet)\longrightarrow X^\bullet[1]$ in $\mathscr{K}(R)$, called a *distinguished triangle*. For each $n\in\mathbb{Z}$, we denote by $H^n(-):\mathscr{D}(R)\to R\text{-Mod}$ the n-th cohomology functor. Certainly, this functor is naturally isomorphic to the Hom-functor $\mathrm{Hom}_{\mathscr{D}(R)}(R,-[n])$.

The Hom-complex $\mathrm{Hom}_R^\bullet(X^\bullet,Y^\bullet)$ of $X^\bullet$ and $Y^\bullet$ over R is defined to be the complex $\big(\mathrm{Hom}_R^n(X^\bullet,Y^\bullet), d^n_{X^\bullet,Y^\bullet}\big)_{n\in\mathbb{Z}}$ with

$$\mathrm{Hom}_R^n(X^\bullet,Y^\bullet) := \prod_{p\in\mathbb{Z}} \mathrm{Hom}_R(X^p, Y^{p+n})$$

and the differential $d^n_{X^\bullet,Y^\bullet}$ of degree n given by

$$(f^p)_{p\in\mathbb{Z}} \mapsto \big(f^p d^{p+n}_{Y^\bullet} - (-1)^n d^p_{X^\bullet} f^{p+1}\big)_{p\in\mathbb{Z}}$$

for $(f^p)_{p\in\mathbb{Z}}\in \mathrm{Hom}_R^n(X^\bullet,Y^\bullet)$. For example, if $X\in R$-Mod, then we have

$$\mathrm{Hom}_R^\bullet(X,Y^\bullet) = \big(\mathrm{Hom}_R(X,Y^n), \mathrm{Hom}_R(X,d^n_{Y^\bullet})\big)_{n\in\mathbb{Z}};$$

if $Y\in R$-Mod, then

$$\mathrm{Hom}_R^\bullet(X^\bullet,Y) = \big(\mathrm{Hom}_R(X^{-n},Y), (-1)^{n+1}\mathrm{Hom}_R(d^{-n-1}_{X^\bullet},Y)\big)_{n\in\mathbb{Z}}.$$

For simplicity, we denote $\mathrm{Hom}_R^\bullet(X,Y^\bullet)$ and $\mathrm{Hom}_R^\bullet(X^\bullet,Y)$ by $\mathrm{Hom}_R(X,Y^\bullet)$ and $\mathrm{Hom}_R(X^\bullet,Y)$, respectively. Note that $\mathrm{Hom}_R(X^\bullet,Y)$ is also isomorphic to the complex $\big(\mathrm{Hom}_R(X^{-n},Y), \mathrm{Hom}_R(d^{-n-1}_{X^\bullet},Y)\big)_{n\in\mathbb{Z}}$.

Moreover, it is known that $H^n(\mathrm{Hom}_R^\bullet(X^\bullet,Y^\bullet))\simeq \mathrm{Hom}_{\mathscr{K}(R)}(X^\bullet,Y^\bullet[n])$ for any $n\in\mathbb{Z}$.

Let $Z^\bullet$ be a chain complex over R^{op}-Mod. Then the tensor complex $Z^\bullet\otimes_R^\bullet X^\bullet$ of $Z^\bullet$ and $X^\bullet$ over R is defined to be the complex $\big(Z^\bullet\otimes_R^n X^\bullet, \partial^n_{Z^\bullet,X^\bullet}\big)_{n\in\mathbb{Z}}$ with

$$Z^\bullet\otimes_R^n X^\bullet := \bigoplus_{p\in\mathbb{Z}} Z^p\otimes_R X^{n-p}$$

and the differential $\partial_{Z^\bullet,X^\bullet}$ of degree n given by

$$z\otimes x\mapsto (z)d^p_{Z^\bullet}\otimes x + (-1)^p z\otimes (x)d^{n-p}_{X^\bullet}$$

for $z\in Z^p$ and $x\in X^{n-p}$. For instance, if $X\in R$-Mod, then $Z^\bullet\otimes_R^\bullet X = \big(Z^n\otimes_R X, d^n_{Z^\bullet}\otimes 1\big)_{n\in\mathbb{Z}}$. In this case, we denote $Z^\bullet\otimes_R^\bullet X$ simply by $Z^\bullet\otimes_R X$.

The following result establishes a relationship between Hom-complexes and tensor complexes.

Let S be an arbitrary ring. Suppose that $X^\bullet = (X^n, d^n_{X^\bullet})$ is a bounded complex of R-S-bimodules. If ${}_RX^n$ is finitely generated and projective for all $n\in\mathbb{Z}$, then there is a natural isomorphism of functors:

$$\mathrm{Hom}_R(X^\bullet,R)\otimes_R^\bullet - \xrightarrow{\simeq} \mathrm{Hom}_R^\bullet(X^\bullet,-):\mathscr{C}(R)\to\mathscr{C}(S).$$

To prove this, we note that, for any R-S-bimodule X and any R-module Y, there is a homomorphism of S-modules: $\delta_{X,Y}:\mathrm{Hom}_R(X,R)\otimes_R Y\longrightarrow \mathrm{Hom}_R(X,Y)$ defined by $f\otimes y\mapsto [x\mapsto (x)fy]$ for $f\in\mathrm{Hom}_R(X,R)$,

$y \in Y$ and $x \in X$, which is natural in both X and Y. Moreover, the map $\delta_{X,Y}$ is an isomorphism if ${}_RX$ is finitely generated and projective. For any $Y^\bullet \in \mathscr{C}(R)$ and any $n \in \mathbb{Z}$, it is clear that

$$\mathrm{Hom}_R(X^\bullet, R) \otimes_R^n Y^\bullet = \bigoplus_{p \in \mathbb{Z}} \mathrm{Hom}_R(X^{-p}, R) \otimes_R Y^{n-p} \text{ and } \mathrm{Hom}_R^n(X^\bullet, Y^\bullet) = \bigoplus_{p \in \mathbb{Z}} \mathrm{Hom}_R(X^p, Y^{p+n})$$

since $X^\bullet$ is a bounded complex. Now, we define $\Delta^n_{X^\bullet, Y^\bullet} := \sum_{p \in \mathbb{Z}} (-1)^{p(n-p)} \delta_{X^{-p}, Y^{n-p}}$, which is a homomorphism of S-modules from $\mathrm{Hom}_R(X^\bullet, R) \otimes_R^n Y^\bullet$ to $\mathrm{Hom}_R^n(X^\bullet, Y^\bullet)$. Then, one can check that $\Delta^\bullet_{X^\bullet, Y^\bullet} := \left(\Delta^n_{X^\bullet, Y^\bullet}\right)_{n \in \mathbb{Z}}$ is a chain map from $\mathrm{Hom}_R(X^\bullet, R) \otimes_R^\bullet Y^\bullet$ to $\mathrm{Hom}_R^\bullet(X^\bullet, Y^\bullet)$. Since ${}_RX^{-p}$ is finitely generated and projective for each $p \in \mathbb{Z}$, the map $\delta_{X^{-p}, Y^{n-p}}$ is an isomorphism, and so is the map $\Delta^n_{X^\bullet, Y^\bullet}$. This implies that

$$\Delta^\bullet_{X^\bullet, Y^\bullet} : \mathrm{Hom}_R(X^\bullet, R) \otimes_R^\bullet Y^\bullet \longrightarrow \mathrm{Hom}_R^\bullet(X^\bullet, Y^\bullet)$$

is an isomorphism in $\mathscr{C}(S)$. Since the homomorphism $\delta_{X,Y}$ is natural in the variables X and Y, it can be checked directly that

$$\Delta^\bullet_{X^\bullet, -} : \mathrm{Hom}_R(X^\bullet, R) \otimes_R^\bullet - \longrightarrow \mathrm{Hom}_R^\bullet(X^\bullet, -)$$

defines a natural isomorphism of functors from $\mathscr{C}(R)$ to $\mathscr{C}(S)$.

In the following, we shall recall some definitions and basic facts about derived functors defined on derived module categories. For details and proofs, we refer to [4, 11].

Let $\mathscr{K}(R)_P$ (respectively, $\mathscr{K}(R)_I$) be the smallest full triangulated subcategory of $\mathscr{K}(R)$ which

(i) contains all the bounded above (respectively, bounded below) complexes of projective (respectively, injective) R-modules, and

(ii) is closed under arbitrary direct sums (respectively, direct products).

Note that $\mathscr{K}(R)_P$ is contained in $\mathscr{K}(R\text{-Proj})$, where R-Proj is the full subcategory of R-Mod consisting of all projective R-modules. Moreover, the composition functors

$$\mathscr{K}(R)_P \hookrightarrow \mathscr{K}(R) \to \mathscr{D}(R) \quad \text{and} \quad \mathscr{K}(R)_I \hookrightarrow \mathscr{K}(R) \to \mathscr{D}(R)$$

are equivalences of triangulated categories. This means that, for each complex $X^\bullet$ in $\mathscr{D}(R)$, there exists a complex ${}_pX^\bullet \in \mathscr{K}(R)_P$ together with a quasi-isomorphism ${}_pX^\bullet \to X^\bullet$, as well as a complex ${}_iX^\bullet \in \mathscr{K}(R)_I$ together with a quasi-isomorphism $X^\bullet \to {}_iX^\bullet$. In this sense, we shall simply call ${}_pX^\bullet$ the projective resolution of $X^\bullet$ in $\mathscr{K}(R)$. For example, if X is an R-module, then we can choose ${}_pX$ to be a deleted projective resolution of ${}_RX$.

Furthermore, if either $X^\bullet \in \mathscr{K}(R)_P$ or $Y^\bullet \in \mathscr{K}(R)_I$, then $\mathrm{Hom}_{\mathscr{K}(R)}(X^\bullet, Y^\bullet) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(X^\bullet, Y^\bullet)$, and this isomorphism is induced by the canonical localization functor from $\mathscr{K}(R)$ to $\mathscr{D}(R)$.

For any triangle functor $H : \mathscr{K}(R) \to \mathscr{K}(S)$, there is a total left-derived functor $\mathbb{L}H : \mathscr{D}(R) \to \mathscr{D}(S)$ defined by $X^\bullet \mapsto H({}_pX^\bullet)$, a total right-derived functor $\mathbb{R}H : \mathscr{D}(R) \to \mathscr{D}(S)$ defined by $X^\bullet \mapsto H({}_iX^\bullet)$. Observe that, if H preserves acyclicity, that is, $H(X^\bullet)$ is acyclic whenever $X^\bullet$ is acyclic, then H induces a triangle functor $D(H) : \mathscr{D}(R) \to \mathscr{D}(S)$ defined by $X^\bullet \mapsto H(X^\bullet)$. In this case, we have $\mathbb{L}H = \mathbb{R}H = D(H)$ up to natural isomorphism, and $D(H)$ is then called the derived functor of H.

Let $M^\bullet$ be a complex of R-S-bimodules. Then the functors

$$M^\bullet \otimes_S^\bullet - : \mathscr{K}(S) \to \mathscr{K}(R) \quad \text{and} \quad \mathrm{Hom}_R^\bullet(M^\bullet, -) : \mathscr{K}(R) \to \mathscr{K}(S)$$

form a pair of adjoint triangle functors. Denote by $M^\bullet \otimes_S^{\mathbb{L}} -$ the total left-derived functor of $M^\bullet \otimes_S^\bullet -$, and by $\mathbb{R}\mathrm{Hom}_R(M^\bullet, -)$ the total right-derived functor of $\mathrm{Hom}_R^\bullet(M^\bullet, -)$. It is clear that $\left(M^\bullet \otimes_S^{\mathbb{L}} -, \mathbb{R}\mathrm{Hom}_R(M^\bullet, -)\right)$ is an adjoint pair of triangle functors. Further, the corresponding counit adjunction

$$\varepsilon : M^\bullet \otimes_S^{\mathbb{L}} \mathbb{R}\mathrm{Hom}_R(M^\bullet, -) \longrightarrow Id_{\mathscr{D}(R)}$$

is given by the composite of the following canonical morphisms in $\mathscr{D}(R)$: $M^\bullet\otimes^{\mathbb{L}}_S \mathbb{R}\mathrm{Hom}_R(M^\bullet,X^\bullet)=M^\bullet\otimes^{\mathbb{L}}_S \mathrm{Hom}^\bullet_R(M^\bullet,{}_iX^\bullet)=M^\bullet\otimes^\bullet_S\big({}_p\mathrm{Hom}^\bullet_R(M^\bullet,{}_iX^\bullet)\big)\longrightarrow M^\bullet\otimes^\bullet_S\mathrm{Hom}^\bullet_R(M^\bullet,{}_iX^\bullet)\longrightarrow {}_iX^\bullet\xrightarrow{\simeq}X^\bullet$. Similarly, we have a corresponding unit adjunction $\eta: Id_{\mathscr{D}(S)}\longrightarrow \mathbb{R}\mathrm{Hom}_R(M^\bullet, M^\bullet\otimes^{\mathbb{L}}_S -)$, which is given by the following composites for $Y^\bullet\in\mathscr{D}(S)$: $Y^\bullet\xrightarrow{\simeq}{}_pY^\bullet\longrightarrow \mathrm{Hom}^\bullet_R(M^\bullet,M^\bullet\otimes^\bullet_S({}_pY^\bullet))\longrightarrow \mathrm{Hom}^\bullet_R(M^\bullet,{}_i(M^\bullet\otimes^\bullet_S({}_pY^\bullet)))=\mathbb{R}\mathrm{Hom}_R(M^\bullet,M^\bullet\otimes^\bullet_S({}_pY^\bullet))=\mathbb{R}\mathrm{Hom}_R(M^\bullet,M^\bullet\otimes^{\mathbb{L}}_S Y^\bullet)$.

For $X^\bullet\in\mathscr{D}(R)$ and $n\in\mathbb{Z}$, we have $H^n(\mathbb{R}\mathrm{Hom}_R(M^\bullet,X^\bullet))=H^n(\mathrm{Hom}^\bullet_R(M^\bullet,{}_iX^\bullet))\simeq \mathrm{Hom}_{\mathscr{K}(R)}(M^\bullet,{}_iX^\bullet[n])\simeq \mathrm{Hom}_{\mathscr{D}(R)}(M^\bullet,{}_iX^\bullet[n])\simeq \mathrm{Hom}_{\mathscr{D}(R)}(M^\bullet,X^\bullet[n])$.

2.2 Homological ring epimorphisms and recollements

Let $\lambda: R\to S$ be a homomorphism of rings.

We denote by $\lambda_*: S\text{-Mod}\to R\text{-Mod}$ the restriction functor induced by λ, and by $D(\lambda_*):\mathscr{D}(S)\to\mathscr{D}(R)$ the derived functor of the exact functor λ_*. We say that λ is a *ring epimorphism* if the restriction functor $\lambda_*: S\text{-Mod}\to R\text{-Mod}$ is fully faithful. It is proved that λ is a ring epimorphism if and only if the multiplication map $S\otimes_R S\to S$ is an isomorphism as S-S-bimodules if and only if, for any two homomorphisms $f_1, f_2: S\to T$ of rings, the equality $\lambda f_1=\lambda f_2$ implies that $f_1=f_2$. This means that, for a ring epimorphism, we have $X\otimes_S Y\simeq X\otimes_R Y$ and $\mathrm{Hom}_S(Y,Z)\simeq\mathrm{Hom}_R(Y,Z)$ for all right S-modules X, and for all S-modules Y and Z. Note that, for a ring epimorphism $\lambda: R\to S$, if R is commutative, then so is S.

Following [9], a ring epimorphism $\lambda: R\to S$ is called *homological* if $\mathrm{Tor}^R_i(S,S)=0$ for all $i>0$. Note that a ring epimorphism λ is homological if and only if the derived functor $D(\lambda_*):\mathscr{D}(S)\to\mathscr{D}(R)$ is fully faithful. This is also equivalent to saying that λ induces an isomorphism $S\otimes^{\mathbb{L}}_R S\simeq S$ in $\mathscr{D}(S)$. Moreover, for a homological ring epimorphism, we have $\mathrm{Tor}^R_i(X,Y)\simeq\mathrm{Tor}^S_i(X,Y)$ and $\mathrm{Ext}^i_S(Y,Z)\simeq\mathrm{Ext}^i_R(Y,Z)$ for all $i\geq 0$ and all right S-modules X, and for all S-modules Y and Z (see [9, Theorem 4.4]).

Clearly, if $\lambda: R\to S$ is a ring epimorphism such that either ${}_RS$ or S_R is flat, then λ is homological. In particular, if R is commutative and Φ is a multiplicative subset of R, then the canonical ring homomorphism $R\to\Phi^{-1}R$ is homological, where $\Phi^{-1}R$ stands for the (ordinary) localization of R at Φ.

As a generalization of localizations of commutative rings, universal localizations of arbitrary rings were introduced in [8] (see also [16]) and provide a class of ring epimorphisms with vanishing homology for the first degree. Note that universal localizations were renamed as noncommutative localizations in [12]. Now we mention the following basic fact on universal localizations.

Lemma 2.1. (see [8], [16]) *Let R be a ring and let Σ be a set of homomorphisms between finitely generated projective R-modules. Then there is a ring R_Σ and a homomorphism $\lambda_\Sigma: R\to R_\Sigma$ of rings such that*

(1) *λ_Σ is Σ-inverting, that is, if $\alpha: P\to Q$ belongs to Σ, then $R_\Sigma\otimes_R\alpha: R_\Sigma\otimes_R P\to R_\Sigma\otimes_R Q$ is an isomorphism of R_Σ-modules, and*

(2) *λ_Σ is universal Σ-inverting, that is, if S is a ring such that there exists a Σ-inverting homomorphism $\varphi: R\to S$, then there exists a unique homomorphism $\psi: R_\Sigma\to S$ of rings such that $\varphi=\lambda\psi$.*

(3) *$\lambda_\Sigma: R\to R_\Sigma$ is a ring epimorphism with* $\mathrm{Tor}^R_1(R_\Sigma,R_\Sigma)=0$.

The $\lambda_\Sigma: R\to R_\Sigma$ in Lemma 2.1 is called the *universal localization* of R at Σ. One should be aware that R_Σ may not be flat as a right or left R-module. Even worse, the map λ_Σ in general is not homological (see [13]). Thus it is a fundamental question to find conditions for λ_Σ to be homological. Obviously, if $\mathrm{Tor}^R_i(R_\Sigma,R_\Sigma)=0$ for all $i\geq 2$, then λ_Σ is homological.

Now, we recall the notion of recollements of triangulated categories, which was first defined in [3] to study "exact sequences" of derived categories of coherent sheaves over geometric objects.

Definition 2.2. Let $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories. We say that $\mathcal{D}$ is a *recollement* of $\mathcal{D}'$ and $\mathcal{D}''$ if there are six triangle functors among the three categories:

$$\begin{tikzcd} \mathcal{D}'' \arrow[r, "i_*=i_!"] & \mathcal{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{D}' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

such that

(1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs,

(2) i_*, j_* and $j_!$ are fully faithful functors,

(3) $i^! j_* = 0$ (and thus also $j^! i_! = 0$ and $i^* j_! = 0$), and

(4) for each object $X \in \mathcal{D}$, there are two triangles in $\mathcal{D}$:

$$i_! i^!(X) \longrightarrow X \longrightarrow j_* j^*(X) \longrightarrow i_! i^!(X)[1],$$

$$j_! j^!(X) \longrightarrow X \longrightarrow i_* i^*(X) \longrightarrow j_! j^!(X)[1].$$

By definition, we have the following property of recollements, which will be frequently used in our proofs.

For any objects $X \in \mathcal{D}'$ and $Y \in \mathcal{D}''$, we have

$$\mathrm{Hom}_{\mathcal{D}}\big(j_!(X), i_*(Y)\big) = 0 = \mathrm{Hom}_{\mathcal{D}}\big(i_*(Y), j_*(X)\big).$$

A typical example of recollements of derived module categories is given by triangular matrix rings: Suppose that A and B are rings, and that N is an A-B-bimodule. Let $R = \begin{pmatrix} A & N \\ 0 & B \end{pmatrix}$ be the triangular matrix ring associated with A, B and N. Then there is a recollement of derived module categories:

$$\begin{tikzcd} \mathscr{D}(A) \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}.$$

In this case, the six triangle functors in Definition 2.2 can be described explicitly:

Let $e := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in R$. Then we have

$$j_! = Re \otimes^{\mathbb{L}}_B -,\ j^! = eR \otimes^{\mathbb{L}}_R -,\ j_* = \mathbb{R}\mathrm{Hom}_B(eR, -),\ i^* = A \otimes^{\mathbb{L}}_R -,\ i_* = A \otimes^{\mathbb{L}}_A -,\ i^! = \mathbb{R}\mathrm{Hom}_R(A, -),$$

where A is identified with R/ReR. Note that the canonical surjection $R \to R/ReR$ is always a homological ring epimorphism.

As a further generalization of the above situation, it was shown in [14, Section 4] that, for an arbitrary homological ring epimorphism $\lambda : R \to S$, there is a recollement of triangulated categories:

$$\begin{tikzcd} \mathscr{D}(S) \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathrm{Tria}(Q^\bullet) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

where $Q^\bullet$ is given by the distinguished triangle $R \xrightarrow{\lambda} S \longrightarrow Q^\bullet \longrightarrow R[1]$ in $\mathscr{D}(R)$. In this case, the functor $j_!$ is the canonical embedding and

$$j^! = (Q^\bullet[-1]) \otimes^{\mathbb{L}}_R -,\ i^* = S \otimes^{\mathbb{L}}_R -,\ i_* = S \otimes^{\mathbb{L}}_S -,\ i^! = \mathbb{R}\mathrm{Hom}_R({}_RS, -).$$

Moreover, we have $\mathscr{D}(S) \simeq \mathrm{Ker}\big(\mathrm{Hom}_{\mathscr{D}(R)}(\mathrm{Tria}(Q^\bullet), -)\big) := \{X^\bullet \in \mathscr{D}(R) \mid \mathrm{Hom}_{\mathscr{D}(R)}(Y, X^\bullet) = 0 \text{ for all } Y \in \mathrm{Tria}(\mathrm{Q}^\bullet)\}$. This clearly implies that $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, X^\bullet[n]) = 0$ for all $X^\bullet \in \mathscr{C}(S)$ and $n \in \mathbb{Z}$.

2.3 Coproducts of rings

Next, we recall the definition of coproducts of rings defined by Cohn in [7], and prove some basic properties of coproducts.

Let R_0 be a ring. An R_0-*ring* is a ring R together with a ring homomorphism $\lambda_R : R_0 \to R$. An R_0-*homomorphism* from an R_0-ring R to another R_0-ring S is a ring homomorphism $f : R \to S$ such that $\lambda_S = \lambda_R f$. If R_0 is commutative and the image of $\lambda_R : R_0 \to R$ is contained in the center $Z(R)$ of R, then R is called an R_0-*algebra*.

The *coproduct* of a family $\{R_i \mid i \in I\}$ of R_0-rings with I an index set is defined to be an R_0-ring R together with a family $\{\rho_i : R_i \to R \mid i \in I\}$ of R_0-homomorphisms such that, for any R_0-ring S with a family of R_0-homomorphisms $\{\tau_i : R_i \to S \mid i \in I\}$, there exists a unique R_0-homomorphism $\delta : R \to S$ such that $\tau_i = \rho_i \delta$ for all $i \in I$.

It is well known that the coproduct of a family $\{R_i \mid i \in I\}$ of R_0-rings exists. We denote this coproduct by $\sqcup_{R_0} R_i$. Clearly, $R_0 \sqcup_{R_0} S = S = S \sqcup_{R_0} R_0$ for every R_0-ring S.

In general, the coproduct of two R_0-algebras may not be isomorphic to their tensor product over R_0. For example, given a field k, the coproduct over k of the polynomial rings $k[x]$ and $k[y]$ is the free ring $k\langle x, y\rangle$ in two variables x and y, while the tensor product over k of $k[x]$ and $k[y]$ is the polynomial ring $k[x,y]$. However, under some extra assumptions, coproducts can be interpreted as tensor products of rings.

Lemma 2.3. [5, Lemma 6.3] *Let R_0 be a commutative ring, and let R_i be an R_0-algebra for $i = 1, 2$. If one of the homomorphisms $\lambda_{R_1} : R_0 \to R_1$ and $\lambda_{R_2} : R_0 \to R_2$ is a ring epimorphism, then the coproduct $R_1 \sqcup_{R_0} R_2$ is isomorphic to the tensor product $R_1 \otimes_{R_0} R_2$, that is, the canonical maps $R_1 \to R_1 \otimes_{R_0} R_2$ and $R_2 \to R_1 \otimes_{R_0} R_2$ define the coproduct.*

Another realization of coproducts may be the so-called trivial extensions.

Lemma 2.4. *Suppose that $\lambda : R \to S$ is a ring epimorphism and M is an S-S-bimodule. Let $\widetilde{\lambda} : R \ltimes M \to S \ltimes M$ be the ring homomorphism between trivial extensions induced by λ. Then the coproduct $S \sqcup_R (R \ltimes M)$ is isomorphic to $S \ltimes M$, that is, the inclusion $S \to S \ltimes M$ and $\widetilde{\lambda}$ define the coproduct.*

Proof. Let $\mu : R \to R \ltimes M$ and $\rho : S \to S \ltimes M$ be the inclusions of rings. Note that S and $R \ltimes M$ are R-rings via λ and μ, respectively, and that $\lambda\rho = \mu\widetilde{\lambda} : R \to S \ltimes M$. We claim that $S \ltimes M$, together with ρ and $\widetilde{\lambda}$, is the coproduct of S and $R \ltimes M$ over R. Suppose that $f : R \ltimes M \to \Lambda$ and $g : S \to \Lambda$ are ring homomorphisms such that $\lambda g = \mu f$. In the following, we shall show that there is a unique ring homomorphism $h : S \ltimes M \to \Lambda$ such that $\widetilde{\lambda} h = f$ and $\rho h = g$. Clearly, if such a h exists, then h must be defined by $(s,m) \mapsto (m)f + (s)g$ for $s \in S$ and $m \in M$. This shows the uniqueness of h. So, it remains to show that the above-defined map h is a ring homomorphism. Certainly, h is a homomorphism of abelian groups. We have to show that h preserves multiplication.

Let $s_i \in S$ and $m_i \in M$ for $i = 1, 2$. On the one hand, $\big((s_1,m_1)(s_2,m_2)\big)h = (s_1s_2,\, s_1m_2 + m_1s_2)h = (s_1m_2 + m_1s_2)f + (s_1s_2)g = (s_1m_2)f + (m_1s_2)f + (s_1)g(s_2)g$. On the other hand, $\big((s_1,m_1)\big)h\big((s_2,m_2)\big)h = \big((m_1)f + (s_1)g\big)\big((m_2)f + (s_2)g\big) = (m_1)f(m_2)f + (m_1)f(s_2)g + (s_1)g(m_2)f + (s_1)g(s_2)g = (m_1m_2)f + (m_1)f(s_2)g + (s_1)g(m_2)f + (s_1)g(s_2)g = (m_1)f(s_2)g + (s_1)g(m_2)f + (s_1)g(s_2)g$ since $m_1m_2 = 0$. This implies that if $(s_1m_2)f = (s_1)g(m_2)f$ and $(m_1s_2)f = (m_1)f(s_2)g$, then $\big((s_1,m_1)(s_2,m_2)\big)h = ((s_1,m_1))h((s_2,m_2))h$. So, to prove that h preserves multiplication, we need only to verify these additional conditions under the assumptions of Lemma 2.4.

Now, we show that $(sm)f = (s)g(m)f$ and $(ms)f = (m)f(s)g$ for $s \in S$ and $m \in M$.

To show the former, we first fix an $m \in M$ and define two maps as follows:

$$\varphi : S \to \Lambda,\ s \mapsto (sm)f \ \text{ and } \ \psi : S \to \Lambda,\ s \mapsto (s)g(m)f.$$

One can check that both φ and ψ are homomorphisms of R-modules such that $\lambda\varphi = \lambda\psi$ due to $\lambda g = \mu f$. Note that $\lambda : R \to S$ is a ring epimorphism and Λ is an S-module. This implies that the homomorphism $\mathrm{Hom}_R(\lambda, \Lambda) : \mathrm{Hom}_R(S, \Lambda) \to \mathrm{Hom}_R(R, \Lambda)$ is an isomorphism, and so $\varphi = \psi$. Similarly, we can show that $(ms)f = (m)f(s)g$. Consequently, the map h preserves multiplication and is actually a ring homomorphism. Thus the ring $S \ltimes M$, together with ρ and $\widetilde{\lambda}$, is the coproduct of S and $R \ltimes M$ over R. $\square$

Now we prove a couple of properties on coproducts of rings, which our later proofs will rely on.

Lemma 2.5. *Let R_0 be a ring, and let R_i be an R_0-ring with ring homomorphism $\lambda_{R_i} : R_0 \to R_i$ for $i = 1, 2$.*

(1) *If $\lambda_{R_1} : R_0 \to R_1$ is a ring epimorphism, then so is the canonical homomorphism $\rho_2 : R_2 \to R_1 \sqcup_{R_0} R_2$.*

(2) *Let I be an ideal of R_0, and let J be the ideal of R_2 generated by the image $(I)\lambda_{R_2}$ of I under the map λ_{R_2}. If $R_1 = R_0/I$ and $\lambda_{R_1} : R_0 \to R_1$ is the canonical surjective map, then $R_1 \sqcup_{R_0} R_2 = R_2/J$.*

Proof. (1) It follows from the definition of coproducts of rings that $\lambda_{R_1}\rho_1 = \lambda_{R_2}\rho_2 : R_0 \to R_1 \sqcup_{R_0} R_2$. We point out that ρ_2 is a ring epimorphism. In fact, if $f, g : R_1 \sqcup_{R_0} R_2 \to S$ are two ring homomorphisms such that $\rho_2 f = \rho_2 g$, then $\lambda_{R_2}\rho_2 f = \lambda_{R_2}\rho_2 g$. This means that $\lambda_{R_1}\rho_1 f = \lambda_{R_1}\rho_1 g$, and therefore $\rho_1 f = \rho_1 g$ since λ_{R_1} is a ring epimorphism. By the universal property of coproducts, we have $g = f$. Thus ρ_2 is a ring epimorphism.

(2) Let $\rho_2 : R_2 \to R_2/J$ be the canonical surjection, and let $\rho_1 : R_1 \to R_2/J$ be the ring homomorphism induced by λ_{R_2} since $J = R_2\,(I)\lambda_{R_2} R_2 \supseteq (I)\lambda_{R_2}$. Now, we claim that R_2/J together with ρ_1 and ρ_2 is the coproduct of R_1 and R_2 over R_0. Clearly, we have $\lambda_{R_1}\rho_1 = \lambda_{R_2}\rho_2 : R_0 \to R_2/J$. Further, assume that $\tau_1 : R_1 \to S$ and $\tau_2 : R_2 \to S$ are two ring homomorphisms such that $\lambda_{R_2}\tau_2 = \lambda_{R_1}\tau_1$. Then $(I)\lambda_{R_2}\tau_2 = (I)\lambda_{R_1}\tau_1 = 0$, and therefore $(J)\tau_2 = 0$. This means that there is a unique ring homomorphism $\delta : R_2/J \to S$ such that $\tau_2 = \rho_2\delta$. It follows that $\lambda_{R_1}\tau_1 = \lambda_{R_2}\tau_2 = \lambda_{R_2}\rho_2\delta = \lambda_{R_1}\rho_1\delta$. Since λ_{R_1} is surjective, we have $\tau_1 = \rho_1\delta$. This shows that $R_1 \sqcup_{R_0} R_2 = R_2/J$. $\square$

The next result tells us that universal localizations are preserved by taking coproducts of rings.

Lemma 2.6. [5, Lemma 6.2] *Let R_0 be a ring, Σ a set of homomorphisms between finitely generated projective R_0-modules, and $\lambda_\Sigma : R_0 \to R_1 := (R_0)_\Sigma$ the universal localization of R_0 at Σ. Then, for any R_0-ring R_2, the coproduct $R_1 \sqcup_{R_0} R_2$ is isomorphic to the universal localization $(R_2)_\Delta$ of R_2 at the set $\Delta := \{R_2 \otimes_{R_0} f \mid f \in \Sigma\}$.*

3 Proof of Theorem 1.1

From now on, we keep the notation introduced in Section 1.

Given ring homomorphisms $\lambda : R \to S$ and $\mu : R \to T$, we have defined

$$B := \begin{pmatrix} S & S \otimes_R T \\ 0 & T \end{pmatrix}, \quad C := M_2(S \sqcup_R T) = \begin{pmatrix} S \sqcup_R T & S \sqcup_R T \\ S \sqcup_R T & S \sqcup_R T \end{pmatrix},$$

and a ring homomorphism $\theta : B \longrightarrow C$ in Section 1.

Summing up our notation introduced before, we reach at the following commutative diagram in $\mathscr{K}(R)$ with two rows being distinguished triangles:

$$(*) \qquad \begin{tikzcd}
R \arrow[r, "\lambda"] \arrow[d, "\mu"'] & S \arrow[r, "\pi"] \arrow[d, "\mu'"'] \arrow[ddr, bend left, "\rho"] & Q^\bullet \arrow[r, "\nu"] \arrow[d, "\mu^\bullet"'] & R[1] \arrow[d, "\mu[1]"'] \\
T \arrow[r, "\lambda'"] \arrow[drr, bend right, "\phi"'] & S \otimes_R T \arrow[r] \arrow[dr, dotted, "h"] & Q^\bullet \otimes_R T \arrow[r] & T[1] \\
 & & S \sqcup_R T &
\end{tikzcd}$$

where ρ and ϕ come from the definition of coproducts of R-rings, the map h is defined by $s\otimes t\mapsto (s)\rho(t)\phi$ for $s\in S$ and $t\in T$, and $\mu^\bullet := (\mu^i)_{i\in\mathbb{Z}}$ is the chain map defined by $\mu^{-1}:=\mu$, $\mu^0:=\mu'$ and $\mu^i=0$ for $i\neq -1,0$.

Recall that (λ,μ) is semi-exact if the homomorphism $\begin{pmatrix}\mu'\\ -\lambda'\end{pmatrix}: S\oplus T\to S\otimes_R T$ is surjective. The kernel of this map is denoted by K. It is clear that K is indeed a subring of the direct sum $S\oplus T$ of the rings S and T. If, moreover, the canonical ring homomorphism from R to K is an isomorphism, then (λ,μ) is called an exact pair of ring homomorphisms. Note that (λ,μ) is exact if and only if the mapping cone

$$0\longrightarrow R\xrightarrow{(-\lambda,\mu)} S\oplus\ T\xrightarrow{\binom{\mu'}{\lambda'}} S\otimes_R T\longrightarrow 0$$

of the chain map $\mu^\bullet$ in $\mathscr{C}(R)$ is an exact sequence of R-modules. Clearly, this is equivalent to saying that $\mu^\bullet$ is a quasi-isomorphism in $\mathscr{K}(R)$, that is, the chain map $\mu^\bullet: Q^\bullet\to Q^\bullet\otimes_R T$ is an isomorphism in $\mathscr{D}(R)$.

Set $e_1:=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $e_2:=\begin{pmatrix}0&0\\0&1\end{pmatrix}\in B$, and let $\varphi: Be_1\to Be_2$ be the map sending $\begin{pmatrix}s\\0\end{pmatrix}$ to $\begin{pmatrix}s\otimes 1\\0\end{pmatrix}$ for $s\in S$. Under the isomorphism $\mathrm{Hom}_B(Be_1,Be_2)\simeq S\otimes_R T$, the map φ corresponds to $1\otimes 1$ in $S\otimes_R T$.

Let $P^\bullet$ be the complex $0\to Be_1\xrightarrow{\varphi} Be_2\to 0$ over B with Be_1 and Be_2 in degrees -1 and 0, respectively. Clearly, $P^\bullet$ is a bounded complex over B consisting of finitely generated projective B-modules, and there is a distinguished triangle in $\mathscr{K}(B)$:

$$Be_1\xrightarrow{\varphi} Be_2\longrightarrow P^\bullet\longrightarrow Be_1[1].$$

Note that Be_1 and Be_2 are also right R-modules via λ and μ, respectively, and that the map φ is actually a homomorphism of right R-modules. Hence, we can easily see that the above triangle is also a distinguished triangle in $\mathscr{K}(B\otimes_{\mathbb{Z}} R^{\mathrm{op}})$. In addition, Be_1 and Be_2 can be regarded as a right S-module and a right T-module, respectively.

The map φ will play an important role in our discussion below.

Lemma 3.1. [16, Theorem 4.10, p. 59] *The universal localization B_φ of B at φ coincides with the ring homomorphism θ defined in Introduction. In particular, we have $B_\varphi = C$.*

Combining Lemma 2.1 with Lemma 3.1, the ring homomorphism $\theta: B\to C$ is a ring epimorphism, and therefore the restriction functor $\theta_*: C\text{-Mod}\to B\text{-Mod}$ is fully faithful. Now, we define a full subcategory of $\mathscr{D}(B)$:

$$\mathscr{D}(B)_{C\text{-Mod}} := \{X^\bullet\in\mathscr{D}(B)\mid H^n(X^\bullet)\in C\text{-Mod for all } n\in\mathbb{Z}\}.$$

Clearly, we have $X[n]\in\mathscr{D}(B)_{C\text{-Mod}}$ for all $X\in C$-Mod and all $n\in\mathbb{Z}$. Also, by [5, Proposition 3.3(3)], we have

$$\mathscr{D}(B)_{C\text{-Mod}} = \mathrm{Ker}\big(\mathrm{Hom}_{\mathscr{D}(B)}(\mathrm{Tria}(P^\bullet),-)\big) = \{X^\bullet\in\mathscr{D}(B)\mid \mathrm{Hom}_{\mathscr{D}(B)}(P^\bullet,X^\bullet[n])=0 \text{ for all } n\in\mathbb{Z}\},$$

or equivalently,

$$\mathscr{D}(B)_{C\text{-Mod}} = \{X^\bullet\in\mathscr{D}(B)\mid H^n\big(\mathrm{Hom}^\bullet_B(P^\bullet,X^\bullet)\big)=0 \text{ for all } n\in\mathbb{Z}\}.$$

The following result is taken from [5, Proposition 3.6(a) and (b)(4-5)]. See also [12, Theorem 0.7 and Proposition 5.6].

Lemma 3.2. *Let i_* be the canonical embedding of $\mathscr{D}(B)_{C\text{-Mod}}$ into $\mathscr{D}(B)$. Then there is a recollement*

$$\begin{tikzcd} \mathscr{D}(B)_{C\text{-Mod}} \arrow[r, "i_*"] & \mathscr{D}(B) \arrow[l, bend right, "i^*"'] \arrow[l, bend left] \arrow[r] & \mathrm{Tria}(P^\bullet) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

such that i^* is the left adjoint of i_*. Moreover, the map $\theta: B\to C$ is homological if and only if $H^n\big(i_*i^*(B)\big)=0$ for all $n\neq 0$. In this case, the derived functor $D(\theta_*):\mathscr{D}(C)\to\mathscr{D}(B)_{C-\mathrm{Mod}}$ is an equivalence of triangulated categories.

To realize $\mathrm{Tria}(P^\bullet)$ in Lemma 3.2 by the derived category of a ring, we first establish some connections between semi-exact pairs of ring homomorphisms and self-orthogonal complexes in derived module categories. Recall that a complex $X^\bullet$ in $\mathscr{D}(B)$ is called *self-orthogonal* if $\mathrm{Hom}_{\mathscr{D}(B)}(X^\bullet, X^\bullet[n]) = 0$ for any $n \neq 0$.

Lemma 3.3. (1) $\mathrm{End}_{\mathscr{D}(B)}(P^\bullet) \simeq K$ *as rings.*

(2) *The pair* (λ, μ) *is semi-exact if and only if* $\mathrm{Hom}_{\mathscr{D}(B)}\big(P^\bullet, P^\bullet[n]\big) = 0$ *for any* $n \neq 0$.

Proof. (1) Note that $P^\bullet$ is a bounded complex over B consisting of finitely generated projective B-modules. It follows that $\mathrm{End}_{\mathscr{D}(B)}(P^\bullet) \simeq \mathrm{End}_{\mathscr{K}(B)}(P^\bullet)$ as rings. Since $\mathrm{Hom}_B(Be_2, Be_1) = 0$, we see that $\mathrm{End}_{\mathscr{K}(B)}(P^\bullet) \simeq \mathrm{End}_{\mathscr{C}(B)}(P^\bullet)$. Moreover, if $\mathrm{End}_B(Be_1)$ and $\mathrm{End}_B(Be_2)$ are identified with S and T, respectively, then each chain map in $\mathrm{End}_{\mathscr{C}(B)}(P^\bullet)$ corresponds uniquely to an element of K. It is easy to check that this correspondence is a ring isomorphism. Thus $\mathrm{End}_{\mathscr{D}(B)}(P^\bullet) \simeq K$ as rings.

(2) It is clear that $\mathrm{Hom}_{\mathscr{D}(B)}\big(P^\bullet, P^\bullet[n]\big) \simeq \mathrm{Hom}_{\mathscr{K}(B)}\big(P^\bullet, P^\bullet[n]\big) = 0$ for all $n \in \mathbb{Z}$ with $|n| \geq 2$. Since $\mathrm{Hom}_B(Be_2, Be_1) = 0$, we get $\mathrm{Hom}_{\mathscr{D}(B)}\big(P^\bullet, P^\bullet[-1]\big) = 0$. Observe that $\mathrm{Hom}_{\mathscr{K}(B)}\big(P^\bullet, P^\bullet[1]\big) = 0$ if and only if $\mathrm{Hom}_B(Be_1, Be_2) = \varphi \mathrm{End}_B(Be_2) + \mathrm{End}_B(Be_1)\varphi$. If we identify $\mathrm{Hom}_B(Be_1, Be_2)$, $\mathrm{End}_B(Be_1)$ and $\mathrm{End}_B(Be_2)$ with $S \otimes_R T$, S and T, respectively, then the latter condition is equivalent to that the map

$$\begin{pmatrix} \mu' \\ -\lambda' \end{pmatrix} : S \oplus T \longrightarrow S \otimes_R T, \quad (s,t) \mapsto s \otimes 1 - 1 \otimes t,\ s \in S, t \in T,$$

is surjective, that is, the pair (λ, μ) is semi-exact by definition. This finishes the proof of (2). $\square$

Corollary 3.4. *If* (λ, μ) *is semi-exact, then there is a recollement of derived module categories:*

$$\begin{tikzcd} \mathscr{D}(B)_{C\text{-Mod}} \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(K) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

Moreover, if (λ, μ) *is exact, then the* K *can be replaced by* R *in the recollement.*

Proof. Since (λ, μ) is semi-exact, we see from Lemma 3.3 that the compact complex $P^\bullet$ is self-orthogonal with $\mathrm{End}_{\mathscr{D}(B)}(P^\bullet) \simeq K$. Now, it follows from [11, Corollary 8.4, Theorem 8.5] that $\mathrm{Tria}(P^\bullet)$ is equivalent to $\mathscr{D}(K)$ as triangulated categories. Thus we get the above recollement. Now, the last statement of Corollary 3.4 follows immediately from the definition of exact pairs. This finishes the proof. $\square$

As a consequence of Corollary 3.4 and Lemma 3.2, we get the following important result which will be used in the proof of Theorem 1.1.

Corollary 3.5. *Suppose that* (λ, μ) *is an exact pair. If* $\theta : B \to C$ *is homological, then there exists a recollement of derived module categories:*

$$\begin{tikzcd} \mathscr{D}(S \sqcup_R T) \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

Throughout the rest of this section, **we always assume that** (λ, μ) **is an exact pair.**

Thus, it follows from Corollary 3.4 that there exists a recollement of triangulated categories:

$$(\star): \quad \begin{tikzcd} \mathscr{D}(B)_{C\text{-Mod}} \arrow[r, "i_*"] & \mathscr{D}(B) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!"] & \mathscr{D}(R) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

where i_* is the canonical embedding and the other functors will be specified in the next lemma. In particular, if $\varepsilon : j_! j^! \to Id_{\mathscr{D}(B)}$ is the counit adjunction with respect to the adjoint pair $(j_!, j^!)$, then, for any $X^\bullet \in \mathscr{D}(B)$, there exists a canonical triangle

$$j_! j^!(X^\bullet) \xrightarrow{\varepsilon_{X^\bullet}} X^\bullet \longrightarrow i_* i^*(X^\bullet) \longrightarrow j_! j^!(X^\bullet)[1].$$

Before we state the next lemma, we first define $P^{\bullet *} := \mathrm{Hom}_B(P^\bullet, B)$ which is isomorphic to the complex $0 \longrightarrow e_2 B \xrightarrow{\varphi_*} e_1 B \longrightarrow 0$ over B^{op} with $e_2 B$ and $e_1 B$ in degrees 0 and 1, respectively. Clearly, the latter is a complex of R-B-bimodules. Here, the left R-module structures of $e_1 B$ and $e_2 B$ are given via the maps λ and μ, respectively.

Lemma 3.6. *In the recollement* $(\star)$, *we have*

$$j_! = P^\bullet \otimes^{\mathbb{L}}_R -,\ j^! = \mathrm{Hom}^\bullet_B(P^\bullet, -),\ j_* = \mathbb{R}\mathrm{Hom}_R(P^{\bullet *}, -).$$

Moreover, the functor $j^!$ *induces a triangle equivalence from* $\mathrm{Tria}(P^\bullet)$ *to* $\mathscr{D}(R)$.

Proof. The idea of our proof is motivated by [11]. Since $P^\bullet$ is a complex of B-R-bimodules, the total left-derived functor $P^\bullet \otimes^{\mathbb{L}}_R - : \mathscr{D}(R) \to \mathscr{D}(B)$ and the total right-derived functor $\mathbb{R}\mathrm{Hom}_B(P^\bullet, -) : \mathscr{D}(B) \to \mathscr{D}(R)$ are well defined. Moreover, since $P^\bullet$ is a bounded complex of finitely generated projective B-modules, the functor $\mathrm{Hom}^\bullet_B(P^\bullet, -) : \mathscr{K}(B) \to \mathscr{K}(R)$ preserves acyclicity, that is, $\mathrm{Hom}^\bullet_B(P^\bullet, M^\bullet)$ is acyclic whenever $M^\bullet \in \mathscr{C}(B)$ is acyclic. This automatically induces a derived functor $\mathscr{D}(B) \longrightarrow \mathscr{D}(R)$, which is defined by $M^\bullet \mapsto \mathrm{Hom}^\bullet_B(P^\bullet, M^\bullet)$. Therefore, we can replace $\mathbb{R}\mathrm{Hom}_B(P^\bullet, -)$ with the Hom-functor $\mathrm{Hom}^\bullet_B(P^\bullet, -)$ up to natural isomorphism.

Now, we claim that the functor $P^\bullet \otimes^{\mathbb{L}}_R -$ is fully faithful and induces a triangle equivalence from $\mathscr{D}(R)$ to $\mathrm{Tria}(P^\bullet)$.

To prove this claim, we first show that the functor $P^\bullet \otimes^{\mathbb{L}}_R - : \mathscr{D}(R) \longrightarrow \mathscr{D}(B)$ is fully faithful.

Let

$$\mathscr{Y} := \{Y^\bullet \in \mathscr{D}(R) \mid P^\bullet \otimes^{\mathbb{L}}_R - : \mathrm{Hom}_{\mathscr{D}(R)}(R, Y^\bullet[n]) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{D}(B)}(P^\bullet \otimes^{\mathbb{L}}_R R, P^\bullet \otimes^{\mathbb{L}}_R Y^\bullet[n]) \text{ for all } n \in \mathbb{Z}\}.$$

Clearly, $\mathscr{Y}$ is a full triangulated subcategory of $\mathscr{D}(R)$. Since $P^\bullet \otimes^{\mathbb{L}}_R -$ commutates with arbitrary direct sums and since $P^\bullet$ is compact in $\mathscr{D}(B)$, we know from the property (2) in Section 2.1 that $\mathscr{Y}$ is closed under arbitrary direct sums in $\mathscr{D}(R)$.

In the following, we shall show that $\mathscr{Y}$ contains R. It is sufficient to prove that

(1) $P^\bullet \otimes^{\mathbb{L}}_R -$ induces an isomorphism of rings from $\mathrm{End}_{\mathscr{D}(R)}(R)$ to $\mathrm{End}_{\mathscr{D}(R)}(P^\bullet \otimes^{\mathbb{L}}_R R)$, and

(2) $\mathrm{Hom}_{\mathscr{D}(B)}(P^\bullet \otimes^{\mathbb{L}}_R R, P^\bullet \otimes^{\mathbb{L}}_R R[n]) = 0$ for any $n \neq 0$.

Since $P^\bullet \otimes^{\mathbb{L}}_R R \simeq P^\bullet$ in $\mathscr{D}(B)$, we know that (1) is equivalent to saying that the right multiplication map $R \to \mathrm{End}_{\mathscr{D}(R)}(P^\bullet)$ is an isomorphism of rings, and that (2) is equivalent to $\mathrm{Hom}_{\mathscr{D}(B)}(P^\bullet, P^\bullet[n]) = 0$ for any $n \neq 0$. Actually, since (λ, μ) is an exact pair, (1) and (2) follow directly from Lemma 3.3 (1) and (2), respectively. This shows $R \in \mathscr{Y}$.

Thus we have $\mathscr{Y} = \mathscr{D}(R)$ since $\mathscr{D}(R) = \mathrm{Tria}(R)$. Consequently, for any $Y^\bullet \in \mathscr{D}(R)$, there is the following isomorphism:

$$P^\bullet \otimes^{\mathbb{L}}_R - : \mathrm{Hom}_{\mathscr{D}(R)}(R, Y^\bullet[n]) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{D}(B)}(P^\bullet \otimes^{\mathbb{L}}_R R, P^\bullet \otimes^{\mathbb{L}}_R Y^\bullet[n]) \text{ for all } n \in \mathbb{Z}.$$

Now, fix $N^\bullet \in \mathscr{D}(R)$ and consider

$$\mathscr{X}_{N^\bullet} := \{X^\bullet \in \mathscr{D}(R) \mid P^\bullet \otimes^{\mathbb{L}}_R - : \mathrm{Hom}_{\mathscr{D}(R)}(X^\bullet, N^\bullet[n]) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{D}(B)}(P^\bullet \otimes^{\mathbb{L}}_R X^\bullet, P^\bullet \otimes^{\mathbb{L}}_R N^\bullet[n]) \text{ for all } n \in \mathbb{Z}\}.$$

Then, one can check that $\mathscr{X}_{N^\bullet}$ is a full triangulated subcategory of $\mathscr{D}(R)$, which is closed under arbitrary direct sums in $\mathscr{D}(R)$. Since $R \in \mathscr{X}_{N^\bullet}$ and $\mathscr{D}(R) = \mathrm{Tria}(R)$, we get $\mathscr{X}_{N^\bullet} = \mathscr{D}(R)$. Consequently, for any $M^\bullet \in \mathscr{D}(R)$, we have the following isomorphism:

$$P^\bullet \otimes_R^{\mathbb{L}} - : \mathrm{Hom}_{\mathscr{D}(R)}\big(M^\bullet, N^\bullet[n]\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{D}(B)}\big(P^\bullet \otimes_R^{\mathbb{L}} M^\bullet, P^\bullet \otimes_R^{\mathbb{L}} N^\bullet[n]\big)$$

for all $n \in \mathbb{Z}$. This means that $P^\bullet \otimes_R^{\mathbb{L}} - : \mathscr{D}(R) \to \mathscr{D}(B)$ is fully faithful.

Recall that $\mathrm{Tria}(\mathrm{P}^\bullet)$ is the smallest full triangulated subcategory of $\mathscr{D}(B)$, which contains $P^\bullet$ and is closed under arbitrary direct sums in $\mathscr{D}(B)$. It follows that the image of $\mathscr{D}(R)$ under $P^\bullet \otimes_R^{\mathbb{L}} -$ is $\mathrm{Tria}(P^\bullet)$ (see the property (2) in Section 2.1) and that $P^\bullet \otimes_R^{\mathbb{L}} -$ induces a triangle equivalence from $\mathscr{D}(R)$ to $\mathrm{Tria}(P^\bullet)$.

Note that $\mathrm{Hom}_B^\bullet(P^\bullet, -)$ is a right adjoint of $P^\bullet \otimes_R^{\mathbb{L}} -$. This means that the restriction of the functor $\mathrm{Hom}_B^\bullet(P^\bullet, -)$ to $\mathrm{Tria}(P^\bullet)$ is the quasi-inverse of the functor $P^\bullet \otimes_R^{\mathbb{L}} - : \mathscr{D}(R) \to \mathrm{Tria}(P^\bullet)$. In particular, $\mathrm{Hom}_B^\bullet(P^\bullet, -)$ induces an equivalence of triangulated categories:

$$\mathrm{Tria}(P^\bullet) \xrightarrow{\ \simeq\ } \mathscr{D}(R).$$

Furthermore, it follows from [5, Proposition 3.3(3)] that

$$\mathscr{D}(B)_{C-\mathrm{Mod}} = \{X^\bullet \in \mathscr{D}(B) \mid \mathrm{Hom}_{\mathscr{D}(B)}(P^\bullet, X^\bullet[n]) = 0 \text{ for all } n \in \mathbb{Z}\} = \mathrm{Ker}\big(\mathrm{Hom}_B^\bullet(P^\bullet, -)\big).$$

Therefore, we can choose $j_! = P^\bullet \otimes_R^{\mathbb{L}} -$ and $j^! = \mathrm{Hom}_B^\bullet(P^\bullet, -)$.

Since $P^\bullet$ is a bounded complex of B-R-bimodules with all of its terms being finitely generated and projective as B-modules, there exists a natural isomorphism of functors (see Section 2.1):

$$P^{\bullet *} \otimes_B^\bullet - \xrightarrow{\simeq} \mathrm{Hom}_B^\bullet(P^\bullet, -) : \mathscr{C}(B) \longrightarrow \mathscr{C}(R).$$

This implies that the former functor preserves acyclicity, since the latter always admits this property. It follows that the functors $P^{\bullet *} \otimes_B^{\mathbb{L}} -$ and $P^{\bullet *} \otimes_B^\bullet - : \mathscr{D}(B) \to \mathscr{D}(R)$ are naturally isomorphic, and therefore $j^! \simeq P^{\bullet *} \otimes_B^{\mathbb{L}} -$. Clearly, the functor $P^{\bullet *} \otimes_B^{\mathbb{L}} -$ has a right adjoint $\mathbb{R}\mathrm{Hom}_R(P^{\bullet *}, -)$. This means that the functor $j^!$ can also have $\mathbb{R}\mathrm{Hom}_R(P^{\bullet *}, -)$ as a right adjoint functor (up to natural isomorphism). However, by the uniqueness of adjoint functors in the recollement, we see that j_* is naturally isomorphic to $\mathbb{R}\mathrm{Hom}_R(P^{\bullet *}, -)$. Thus, we can choose $j_* = \mathbb{R}\mathrm{Hom}_R(P^{\bullet *}, -)$. This finishes the proof of Lemma 3.6. $\square$

Now we consider θ as a homomorphism of B-B-bimodules, and denote its mapping cone by $W^\bullet[1]$. Then we have a distinguished triangle

$$W^\bullet \xrightarrow{\xi} B \xrightarrow{\theta} C \longrightarrow W^\bullet[1]$$

in $\mathscr{K}(B \otimes_{\mathbb{Z}} B^{op})$. This yields two relevant triangles

$$W^\bullet e_i \xrightarrow{\xi_i} Be_i \xrightarrow{\theta_i} Ce_i \longrightarrow W^\bullet e_i[1]$$

in $\mathscr{K}(B)$ for $i = 1, 2$. Note that $Ce_1 \simeq Ce_2$ as B-modules.

Lemma 3.7. (1) *There is a triangle* $W^\bullet e_1 \longrightarrow W^\bullet e_2 \longrightarrow P^\bullet \longrightarrow W^\bullet e_1[1]$ *in* $\mathscr{D}(B)$.

(2) $j^!(W^\bullet e_1) \simeq j^!(Be_1) \simeq S[-1]$ *and* $j^!(W^\bullet e_2) \simeq j^!(Be_2) \simeq (Q^\bullet \otimes_R T)[-1] \simeq Q^\bullet[-1]$ *in* $\mathscr{D}(R)$.

(3) $i_* i^*(Be_1) \simeq i_* i^*(Be_2)$ *in* $\mathscr{D}(B)$.

Proof. (1) Let $f^0 = \varphi$, and let $f^1 : Ce_1 \to Ce_2$ be the right multiplication map induced by $e_{12} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in C$. Then we can construct the following commutative diagram in B-Mod:

$$\begin{tikzcd}
W^\bullet e_1 : \arrow[d, "f^\bullet"] & 0 \arrow[r] \arrow[d] & Be_1 \arrow[r, "\theta_1"] \arrow[d, "\varphi"'] & Ce_1 \arrow[d, "f^1"', "\simeq"] \\
W^\bullet e_2 : \arrow[d] & 0 \arrow[r] \arrow[d] & Be_2 \arrow[r, "\theta_2"] \arrow[d, "{(0,1)}"'] & Ce_2 \arrow[d, equal] \\
\mathrm{Cone}(f^\bullet) : & Be_1 \arrow[r, "{(-\theta_1, \varphi)}"] & Ce_1 \oplus Be_2 \arrow[r, "{\binom{f^1}{\theta_2}}"] & Ce_2
\end{tikzcd}$$

Consequently, we get a triangle $W^\bullet e_1 \xrightarrow{f^\bullet} W^\bullet e_2 \longrightarrow \mathrm{Cone}(f^\bullet) \longrightarrow W^\bullet e_1[1]$ in $\mathscr{D}(B)$. Since the map f^1 is an isomorphism, one can check that $\mathrm{Cone}(f^\bullet) \simeq P^\bullet$ in $\mathscr{D}(B)$. This proves (1).

(2) Since $\mathscr{D}(B)_{C-\mathrm{Mod}} = \mathrm{Ker}(j^!)$, it follows from $C \in \mathscr{D}(B)_{C-\mathrm{Mod}}$ that $j^!({}_BC) = 0$, and therefore $j^!(W^\bullet e_1) \simeq j^!(Be_1)$ and $j^!(W^\bullet e_2) \simeq j^!(Be_2)$ in $\mathscr{D}(R)$. Note that the complex $j^!(Be_1)$ is of the form

$$\cdots \longrightarrow 0 \longrightarrow \mathrm{Hom}_B(Be_2, Be_1) \xrightarrow{\varphi_*} \mathrm{Hom}_B(Be_1, Be_1) \longrightarrow 0 \longrightarrow \cdots$$

which is isomorphic to $S[-1]$ as complexes. Similarly, one can show that $j^!(Be_2)$ is isomorphic in $\mathscr{D}(R)$ to the complex $0 \longrightarrow T \xrightarrow{\lambda'} S \otimes_R T \longrightarrow 0$ over R with T in degree 0. Recall that the latter complex is isomorphic to $Q^\bullet[-1]$ in $\mathscr{D}(R)$ and that $Q^\bullet$ is isomorphic to $Q^\bullet \otimes_R T$ in $\mathscr{D}(R)$, since the pair (λ, μ) is exact (see the diagram $(*)$). Thus $j^!(Be_2) \simeq Q^\bullet[-1] \simeq (Q^\bullet \otimes_R T)[-1]$ in $\mathscr{D}(R)$. This completes the proof of (2).

(3) Since $\mathrm{Hom}^\bullet_B(P^\bullet, -) : \mathrm{Tria}(P^\bullet) \longrightarrow \mathscr{D}(R)$ is an equivalence by Lemma 3.6, we see that the morphism $\varepsilon_{P^\bullet} : j_! j^!(P^\bullet) \to P^\bullet$ is an isomorphism in $\mathscr{D}(B)$. Hence $i_* i^*(P^\bullet) = 0$ in $\mathscr{D}(B)$. Then, it follows from the following commutative diagram with all rows and columns being triangles in $\mathscr{D}(B)$:

$$\begin{tikzcd}
j_! j^!(P^\bullet[-1]) \arrow[r] \arrow[d, "\simeq"'] & j_! j^!(Be_1) \arrow[r, "j_! j^!(\varphi)"] \arrow[d, "\varepsilon_{Be_1}"'] & j_! j^!(Be_2) \arrow[r] \arrow[d, "\varepsilon_{Be_2}"'] & j_! j^!(P^\bullet) \arrow[d, "\varepsilon_{P^\bullet}"', "\simeq"] \\
P^\bullet[-1] \arrow[r] \arrow[d] & Be_1 \arrow[r, "\varphi"] \arrow[d] & Be_2 \arrow[r] \arrow[d] & P^\bullet \arrow[d] \\
i_* i^*(P^\bullet[-1]) \arrow[r] \arrow[d] & i_* i^*(Be_1) \arrow[r, "i_* i^*(\varphi)"] \arrow[d] & i_* i^*(Be_2) \arrow[r] \arrow[d] & i_* i^*(P^\bullet) \arrow[d] \\
j_! j^!(P^\bullet) \arrow[r] & j_! j^!(Be_1)[1] \arrow[r] & j_! j^!(Be_2)[1] \arrow[r] & j_! j^!(P^\bullet)[1]
\end{tikzcd}$$

that $i_* i^*(\varphi) : i_* i^*(Be_1) \longrightarrow i_* i^*(Be_2)$ is an isomorphism in $\mathscr{D}(B)$. This proves (3). $\square$

Lemma 3.8. *If $\lambda : R \to S$ is homological, then*

(1) *$i_* i^*(Be_1) \simeq Be_2 \otimes^{\mathbb{L}}_R S$ in $\mathscr{D}(B)$. In particular,* $H^j\big(i_* i^*(Be_1)\big) \simeq \begin{cases} 0 & \text{if } j > 0, \\ \mathrm{Tor}^R_{-j}(Be_2, S) & \text{if } j \leq 0. \end{cases}$

(2) *The homomorphism $T \otimes_R S \to S \sqcup_R T$, defined by $t \otimes s \mapsto (t)\phi(s)\rho$ for $t \in T$ and $s \in S$, induces an isomorphism of T-modules. Moreover, $S \sqcup_R T \simeq \mathrm{End}_T(T \otimes_R S)$ as rings.*

Proof. Set $\Gamma := S \sqcup_R T$. We define four homomorphisms:

$$m : S \otimes_R S \to S, \quad s_1 \otimes s_2 \mapsto s_1 s_2, \qquad \varphi_1 : S \otimes_R S \to S \otimes_R T \otimes_R S, \quad s_1 \otimes s_2 \mapsto s_1 \otimes 1 \otimes s_2,$$

$$\varphi_2 : S \otimes_R T \otimes_R S \to \Gamma, \quad s_1 \otimes t \otimes s_2 \mapsto (s_1)\rho(t)\phi(s_2)\rho, \qquad \varphi_3 : T \otimes_R S \to \Gamma, \quad t \otimes s_1 \mapsto (t)\phi(s_1)\rho$$

for $s_1, s_2 \in S$ and $t \in T$. Note that they are all well defined. Moreover, we identify $Be_1 \otimes_R S$ and $Be_2 \otimes_R S$ with $\left(\begin{smallmatrix} S\otimes_R S \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} S\otimes_R T\otimes_R S \\ T\otimes_R S \end{smallmatrix}\right)$ as B-modules, respectively. Then there are two chain maps in $\mathscr{C}(B)$:

$$\begin{tikzcd}
P^\bullet \otimes_R (S[-1]) : \arrow[d, "f^\bullet"] & 0 \arrow[r] & \left(\begin{smallmatrix} S\otimes_R S \\ 0 \end{smallmatrix}\right) \arrow[r, "\left(\begin{smallmatrix} \varphi_1 \\ 0 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} m \\ 0 \end{smallmatrix}\right)", "\simeq"'] & \left(\begin{smallmatrix} S\otimes_R T\otimes_R S \\ T\otimes_R S \end{smallmatrix}\right) \arrow[r] \arrow[d, "\left(\begin{smallmatrix} \varphi_2 \\ \varphi_3 \end{smallmatrix}\right)"] & 0 \\
W^\bullet e_1 : \arrow[d, "\xi_1"] & 0 \arrow[r] & \left(\begin{smallmatrix} S \\ 0 \end{smallmatrix}\right) \arrow[r, "\left(\begin{smallmatrix} \rho \\ 0 \end{smallmatrix}\right)"] \arrow[d, equal] & \left(\begin{smallmatrix} \Gamma \\ \Gamma \end{smallmatrix}\right) \arrow[r] \arrow[d, "0"] & 0 \\
Be_1 : & 0 \arrow[r] & \left(\begin{smallmatrix} S \\ 0 \end{smallmatrix}\right) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

(1) Let ${}_p\mathrm{Hom}_B(P^\bullet, Be_1)$ be a projective resolution of the complex $\mathrm{Hom}_B(P^\bullet, Be_1)$ in $\mathscr{D}(R)$ with

$$\tau : {}_p\mathrm{Hom}_B(P^\bullet, Be_1) \longrightarrow \mathrm{Hom}_B(P^\bullet, Be_1)$$

a quasi-isomorphism (see Section 2). Note that the left R-module structure of $\mathrm{Hom}_B(P^\bullet, Be_1)$ is induced from the right R-structure of $P^\bullet$ and that $\mathrm{Hom}_B(P^\bullet, Be_1) \simeq S[-1]$ as complexes of R-modules. In the following, we always identify $\mathrm{Hom}_B(P^\bullet, Be_1)$ with $S[-1]$. Then one can check directly that the counit $\varepsilon_{Be_1} : j_! j^!(Be_1) \longrightarrow Be_1$ is just the composite of the following canonical morphisms:

$$j_! j^!(Be_1) = P^\bullet \otimes_R^\bullet \left({}_p\mathrm{Hom}_B(P^\bullet, Be_1)\right) \xrightarrow{P^\bullet \otimes \tau} P^\bullet \otimes_R^\bullet \mathrm{Hom}_B(P^\bullet, Be_1) = P^\bullet \otimes_R^\bullet (S[-1]) \xrightarrow{f^\bullet \xi_1} Be_1.$$

Now we apply the triangle functor $- \otimes_R^{\mathbb{L}} \mathrm{Hom}_B(P^\bullet, Be_1)$ to the distinguished triangle

$$Be_2 \longrightarrow P^\bullet \longrightarrow Be_1[1] \xrightarrow{\varphi[1]} Be_2[1]$$

in $\mathscr{K}(B \otimes_{\mathbb{Z}} R^{\mathrm{op}})$, and establish easily the following commutative diagram with all rows being distinguished triangles in $\mathscr{D}(B)$:

$$\begin{tikzcd}
Be_2 \otimes_R^{\mathbb{L}} \mathrm{Hom}_B(P^\bullet, Be_1) \arrow[r] \arrow[d] & P^\bullet \otimes_R^{\mathbb{L}} \mathrm{Hom}_B(P^\bullet, Be_1) \arrow[r] \arrow[d, "P^\bullet \otimes \tau"'] \arrow[dddr, "\varepsilon_{Be_1}"] & Be_1 \otimes_R^{\mathbb{L}} \mathrm{Hom}_B(P^\bullet, Be_1)[1] \arrow[r, "\varphi \otimes_R^{\mathbb{L}} 1"] \arrow[d, "Be_1 \otimes \tau"', "\simeq"] & Be_2 \otimes_R^{\mathbb{L}} \mathrm{Hom}_B(P^\bullet, Be_1)[1] \arrow[d, "Be_2 \otimes \tau[1]"] \\
Be_2 \otimes_R \mathrm{Hom}_B(P^\bullet, Be_1) \arrow[r] \arrow[d, equal] & P^\bullet \otimes_R^\bullet \mathrm{Hom}_B(P^\bullet, Be_1) \arrow[r] \arrow[d, equal] & Be_1 \otimes_R \mathrm{Hom}_B(P^\bullet, Be_1)[1] \arrow[r, "\varphi \otimes 1"] \arrow[d, equal] & Be_2 \otimes_R \mathrm{Hom}_B(P^\bullet, Be_1)[1] \arrow[d, equal] \\
Be_2 \otimes_R (S[-1]) \arrow[r] \arrow[d] & P^\bullet \otimes_R^\bullet (S[-1]) \arrow[r] \arrow[d, "f^\bullet"] & Be_1 \otimes_R S \arrow[r, "\varphi \otimes 1"] \arrow[d, "\left(\begin{smallmatrix} m \\ 0 \end{smallmatrix}\right)", "\simeq"'] & Be_2 \otimes_R S \arrow[d, "\left(\begin{smallmatrix} \varphi_2 \\ \varphi_3 \end{smallmatrix}\right)"] \\
\left(\begin{smallmatrix} \Gamma \\ \Gamma \end{smallmatrix}\right)[-1] \arrow[r] & W^\bullet e_1 \arrow[r, "\xi_1"] & Be_1 = \left(\begin{smallmatrix} S \\ 0 \end{smallmatrix}\right) \arrow[r, "\theta_1"] \arrow[dr, "\eta_{Be_1}"] & Ce_1 = \left(\begin{smallmatrix} \Gamma \\ \Gamma \end{smallmatrix}\right) \\
 & & & i_* i^*(Be_1)
\end{tikzcd}$$

where η_{Be_1} is the unit adjunction of the adjoint pair (i^*, i_*), and where the first and third isomorphisms in the third column follow from the fact that $\lambda : R \to S$ is homological. This implies that there is an isomorphism

$\beta: Be_2\otimes^{\mathbb{L}}_R \mathrm{Hom}_B(P^\bullet, Be_1)[1] \longrightarrow i_*i^*(Be_1)$ in $\mathscr{D}(B)$ such that the following diagram commutes:

$$\begin{tikzcd}
P^\bullet\otimes^{\mathbb{L}}_R \mathrm{Hom}_B(P^\bullet,Be_1) \arrow[r] \arrow[d, equal] & Be_1\otimes^{\mathbb{L}}_R \mathrm{Hom}_B(P^\bullet,Be_1)[1] \arrow[r, "\varphi\otimes^{\mathbb{L}} 1"] \arrow[d, "(Be_1\otimes\tau)\left(\begin{smallmatrix} m\\ 0\end{smallmatrix}\right)"', "\simeq"] & Be_2\otimes^{\mathbb{L}}_R \mathrm{Hom}_B(P^\bullet,Be_1)[1] \arrow[d, dashed, "\beta"', "\simeq"] \\
P^\bullet\otimes^{\mathbb{L}}_R \mathrm{Hom}_B(P^\bullet,Be_1) \arrow[r, "\varepsilon_{Be_1}"] & Be_1=\begin{pmatrix} S\\ 0\end{pmatrix} \arrow[r, "\eta_{Be_1}"] & i_*i^*(Be_1)
\end{tikzcd}.$$

As a result, we have $Be_2\otimes^{\mathbb{L}}_R S\simeq i_*i^*(Be_1)$ in $\mathscr{D}(B)$. It is clear that $H^j\big(Be_2\otimes^{\mathbb{L}}_R S\big)\simeq \mathrm{Tor}^R_{-j}(Be_2,S)$ for any $j\in\mathbb{Z}$. Since isomorphic objects in $\mathscr{D}(B)$ have the isomorphic cohomology groups in each degree, (1) follows. This finishes the proof of (1).

(2) Define

$$\sigma=\beta^{-1}\,(Be_2\otimes\tau[1]): i_*i^*(Be_1)\longrightarrow Be_2\otimes_R S,$$

$$\gamma=\begin{pmatrix}\varphi_2\\ \varphi_3\end{pmatrix}: Be_2\otimes_R S\longrightarrow Ce_1 \quad\text{and}\quad \omega=\eta_{Be_1}\sigma: Be_1\longrightarrow Be_2\otimes_R S.$$

Then, it follows from the above two commutative diagrams that $\omega=\left(\begin{smallmatrix} m\\ 0\end{smallmatrix}\right)^{-1}(\varphi\otimes 1)$. This means that $\theta_1=\omega\gamma$.

Now, we claim that the map γ is an isomorphism in B-Mod. In order to show this, it is sufficient to prove that $Be_2\otimes_R S\in C$-Mod and that the induced map

$$\mathrm{Hom}_B(\gamma,M):\ \mathrm{Hom}_B(Ce_1,M)\longrightarrow \mathrm{Hom}_B(Be_2\otimes_R S,M)$$

is bijective for every C-module M.

In fact, by (1), we know that $H^0(i_*i^*(Be_1))\simeq Be_2\otimes_R S$. In particular, $Be_2\otimes_R S\in C$-Mod because $i_*i^*(Be_1)\in\mathscr{D}(B)_{C-\mathrm{Mod}}$. Moreover, since $\theta: B\to C$ is a ring epimorphism by Lemma 3.1, the map θ_1 always induces a bijection

$$\mathrm{Hom}_B(\theta_1,M):\ \mathrm{Hom}_B(Ce_1,M)\xrightarrow{\simeq}\mathrm{Hom}_B(Be_1,M).$$

Then, it follows from $\theta_1=\omega\gamma$ that $\mathrm{Hom}_B(\theta_1,M)=\mathrm{Hom}_B(\gamma,M)\mathrm{Hom}_B(\omega,M)$. This means that, to verify the bijection of $\mathrm{Hom}_B(\gamma,M)$, it suffices to show that

$$\mathrm{Hom}_B(\omega,M):\ \mathrm{Hom}_B(Be_2\otimes_R S,M)\longrightarrow\mathrm{Hom}_B(Be_1,M)$$

is bijective. This is equivalent to verifying that both $\mathrm{Hom}_{\mathscr{D}(B)}(\eta_{Be_1},M))$ and $\mathrm{Hom}_{\mathscr{D}(B)}(\sigma,M)$ are bijective.

On the one hand, since $M\in C$-Mod and $j_!j^!(Be_1)\in\mathrm{Tria}(P^\bullet)$ by Lemma 3.6, we have $\mathrm{Hom}_{\mathscr{D}(B)}(j_!j^!(Be_1),M[n])$ = 0 for any $n\in\mathbb{Z}$. Applying $\mathrm{Hom}_{\mathscr{D}(B)}(-,M)$ to the triangle

$$j_!j^!(Be_1)\xrightarrow{\varepsilon_{Be_1}} Be_1\xrightarrow{\eta_{Be_1}} i_*i^*(Be_1)\longrightarrow j_!j^!(Be_1)[1],$$

we infer that

$$\mathrm{Hom}_{\mathscr{D}(B)}(\eta_{Be_1},M):\mathrm{Hom}_{\mathscr{D}(B)}(i_*i^*(Be_1),\,M)\longrightarrow\mathrm{Hom}_B(Be_1,M)$$

is bijective. On the other hand, since $i_*i^*(Be_1)\simeq Be_2\otimes^{\mathbb{L}}_R S$ in $\mathscr{D}(B)$ by (1), we know that $H^j(i_*i^*(Be_1))=0$ for any $j>0$ and $H^0(\sigma)=H^0(\beta)^{-1}H^0(Be_2\otimes\tau[1]): H^0(i_*i^*(Be_1))\xrightarrow{\simeq} Be_2\otimes_R S$. Now, we apply the cohomology functor $H^m(-)$ to the triangle

$$(\dagger)\qquad U^\bullet\longrightarrow i_*i^*(Be_1)\xrightarrow{\sigma} Be_2\otimes_R S\longrightarrow U^\bullet[1]$$

in $\mathscr{D}(B)$ induced from σ, and get $H^m(U^\bullet)=0$ for any $m\geq 0$. Let $U^\bullet=(U^i,d^i)_{i\in\mathbb{Z}}$ and $V^\bullet$ be the complex

$$\cdots\longrightarrow U^{-3}\xrightarrow{d^{-3}} U^{-2}\xrightarrow{d^{-2}}\mathrm{Ker}(d^{-1})\longrightarrow 0\longrightarrow\cdots.$$

Then $V^\bullet \simeq U^\bullet$ in $\mathscr{D}(B)$, and therefore $\mathrm{Hom}_{\mathscr{D}(B)}(U^\bullet, M) \simeq \mathrm{Hom}_{\mathscr{D}(B)}(V^\bullet, {}_iM) \simeq \mathrm{Hom}_{\mathscr{K}(B)}(V^\bullet, {}_iM) = 0$. Similarly, we can show that $\mathrm{Hom}_{\mathscr{D}(B)}(U^\bullet[1], M) = 0$. Applying $\mathrm{Hom}_{\mathscr{D}(B)}(-, M)$ to the triangle $(\dagger)$, we conclude that

$$\mathrm{Hom}_{\mathscr{D}(B)}(\sigma, M):\ \mathrm{Hom}_B(Be_2 \otimes_R S, M) \longrightarrow \mathrm{Hom}_B(i_* i^*(Be_1), M)$$

is bijective. Thus, $\mathrm{Hom}_B(\omega, M)$ is bijective and γ is an isomorphism of B-modules.

Now, it follows from $\gamma = \begin{pmatrix} \varphi_2 \\ \varphi_3 \end{pmatrix}$ that $\varphi_3 : T \otimes_R S \to \Gamma$ is an isomorphism of T-modules. Since $\lambda : R \to S$ is a ring epimorphism, we deduce from Lemma 2.5(1) that $\phi : T \to \Gamma$ is also a ring epimorphism, and therefore $\Gamma \simeq \mathrm{End}_\Gamma(\Gamma) \simeq \mathrm{End}_T(\Gamma) \simeq \mathrm{End}_T(T \otimes_R S)$ as rings. This completes the proof of (2). $\square$

To prove Theorem 1.1, we need to establish the following two important lemmas.

Lemma 3.9. *If $\lambda : R \to S$ is homological, then $\mathrm{Tor}_i^R(S, T) = 0$ for any $i > 0$.*

Proof. Recall that we have a distinguished triangle $R \xrightarrow{\lambda} S \to Q^\bullet \to R[1]$ in $\mathscr{D}(R)$. Since λ is homological, it follows from [9, Theorem 4.4] that λ induces the following isomorphisms $S \xrightarrow{\simeq} S \otimes_R^{\mathbb{L}} R \xrightarrow{S \otimes_R^{\mathbb{L}} \lambda} S \otimes_R^{\mathbb{L}} S$ in $\mathscr{D}(S)$. This clearly implies that $S \otimes_R^{\mathbb{L}} Q^\bullet = 0$ in $\mathscr{D}(S)$, and therefore $S \otimes_R^{\mathbb{L}} Q^\bullet = 0$ in $\mathscr{D}(R)$. Since (λ, μ) is an exact pair, we have seen that $\mu^\bullet : Q^\bullet \to Q^\bullet \otimes_R T$ is an isomorphism in $\mathscr{D}(R)$ (see the diagram $(*)$). As a result, we have $S \otimes_R^{\mathbb{L}} \left(Q^\bullet \otimes_R T\right) \simeq S \otimes_R^{\mathbb{L}} Q^\bullet = 0$ in $\mathscr{D}(S)$. By applying $S \otimes_R^{\mathbb{L}} -$ to the triangle $T \xrightarrow{\lambda'} S \otimes_R T \longrightarrow Q^\bullet \otimes_R T \longrightarrow T[1]$, we obtain $S \otimes_R^{\mathbb{L}} T \simeq S \otimes_R^{\mathbb{L}} \left(S \otimes_R T\right)$ in $\mathscr{D}(S)$ (and also in $\mathscr{D}(R)$). This yields that $\mathrm{Tor}_i^R(S, T) \simeq \mathrm{Tor}_i^R(S, S \otimes_R T)$ for any $i \geq 0$. As $S \otimes_R T$ is a left S-module and λ is homological, it follows that $\mathrm{Tor}_i^R(S, S \otimes_R T) = \mathrm{Tor}_i^S(S, S \otimes_R T) = 0$ for any $i > 0$, and therefore $\mathrm{Tor}_i^R(S, T) = 0$. This finishes the proof. $\square$

Lemma 3.10. *Given a commutative diagram of ring homomorphisms:*

$$\begin{array}{ccc} R & \xrightarrow{\lambda} & S \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & \Gamma, \end{array}$$

if λ is homological and (λ, μ) is exact, then the following statements are equivalent:

(1) *The ring homomorphism $g : T \to \Gamma$ is homological.*

(2) *The ring homomorphism*

$$\theta_{f,g} : B \longrightarrow M_2(\Gamma), \quad \begin{pmatrix} s_1 & s_2 \otimes t_2 \\ 0 & t_1 \end{pmatrix} \mapsto \begin{pmatrix} (s_1)f & (s_2)f(t_2)g \\ 0 & (t_1)g \end{pmatrix}, s_i \in S, t_i \in T, i = 1, 2$$

is homological.

Proof. Set $\Lambda := M_2(\Gamma)$. Let $e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in B$, and let $e := (e_2)\theta_{f,g} \in \Lambda$. Then we have $e = e^2$, $\mathrm{End}_\Lambda(\Lambda e) \simeq \Gamma$ and $\mathrm{End}_B(Be_2) \simeq T$. Observe that Λe is a projective generator for Λ-Mod. Then, by Morita theory, the tensor functor $e\Lambda \otimes_\Lambda - : \Lambda\text{-Mod} \longrightarrow \Gamma\text{-Mod}$ is an equivalence of module categories, which can be canonically extended to a triangle equivalence $D(e\Lambda \otimes_\Lambda -) : \mathscr{D}(\Lambda) \to \mathscr{D}(\Gamma)$.

It is clear that $e_2 B \otimes_B \Lambda \simeq e_2 \cdot \Lambda = e\Lambda$ as T-Λ-bimodules, where the left T-module structure of $e\Lambda$ is induced by $g : T \to \Gamma$. Thus the following diagram of functors between module categories

$$\begin{array}{ccc} \Lambda\text{-Mod} & \xrightarrow{e\Lambda \otimes_\Lambda -} & \Gamma\text{-Mod} \\ {\scriptstyle (\theta_{f,g})_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ B\text{-Mod} & \xrightarrow{e_2 B \otimes_B -} & T\text{-Mod} \end{array}$$

is commutative, where $\big(\theta_{f,g}\big)_*$ and g_* stand for the restriction functors induced by the ring homomorphisms $\theta_{f,g}$ and g, respectively. Since all of the functors appearing in the diagram are exact, we can pass to derived module categories and get the following commutative diagram of functors between derived module categories:

$$(\dagger)\qquad \begin{tikzcd} \mathscr{D}(\Lambda) \arrow[r, "D\left(e\Lambda\otimes_\Lambda -\right)"] \arrow[d, "D\left((\theta_{f,g})_*\right)"'] & \mathscr{D}(\Gamma) \arrow[d, "D(g_*)"] \\ \mathscr{D}(B) \arrow[r, "D\left(e_2B\otimes_B -\right)"] & \mathscr{D}(T) \end{tikzcd}$$

where the functor $D\big(e\Lambda\otimes_\Lambda -\big)$ in the upper row is a triangle equivalence.

Note that $\theta_{f,g}: B\to\Lambda$ (respectively, $g: T\to\Gamma$) is homological if and only if the functor $D\big((\theta_{f,g})_*\big)$ (respectively, $D(g_*)$) is fully faithful. This means that, to prove that (1) and (2) are equivalent, it is necessary to establish some further connection between $D\big((\theta_{f,g})_*\big)$ and $D(g_*)$ in the diagram ($\dagger$).

Actually, the triangle functor $D(e_2B\otimes_B -)$ induces a triangle equivalence from $\mathrm{Tria}(Be_2)$ to $\mathscr{D}(T)$. This can be obtained from the following classical recollement of derived module categories:

$$\begin{tikzcd} \mathscr{D}(S) \arrow[r, "S\otimes^{\mathbb{L}}_S -"] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "D(e_2B\otimes_B -)"] & \mathscr{D}(T) \arrow[l, bend right, "Be_2\otimes^{\mathbb{L}}_T -"'] \arrow[l, bend left] \end{tikzcd}$$

which arises form the triangular structure of the ring B.

Suppose that the image $\mathrm{Im}\big(D\big((\theta_{f,g})_*\big)\big)$ of the functor $D\big((\theta_{f,g})_*\big)$ belongs to $\mathrm{Tria}(Be_2)$. Then we can strengthen the diagram ($\dagger$) by the following commutative diagram of functors between triangulated categories:

$$\begin{tikzcd} & \mathscr{D}(\Lambda) \arrow[dl, "D\left((\theta_{f,g})_*\right)"'] \arrow[d, "D\left((\theta_{f,g})_*\right)"] \arrow[r, "D(e\Lambda\otimes_\Lambda -)", "\simeq"'] & \mathscr{D}(\Gamma) \arrow[d, "D(g_*)"] \\ \mathscr{D}(B) & \mathrm{Tria}(Be_2) \arrow[l, hook'] \arrow[r, "D(e_2B\otimes_B -)", "\simeq"'] & \mathscr{D}(T) \end{tikzcd}$$

This implies that $D\big((\theta_{f,g})_*\big)$ is fully faithful if and only if so is $D(g_*)$, and therefore $\theta_{f,g}$ is homological if and only if g is homological.

So, to finish the proof of Lemma 3.10, it suffices to prove that $\mathrm{Im}\big(D\big((\theta_{f,g})_*\big)\big)\subseteq\mathrm{Tria}(Be_2)$. In the following, we shall concentrate on proving this inclusion.

In fact, it is known that $\mathscr{D}(\Lambda)=\mathrm{Tria}(\Lambda e)$ and $D\big((\theta_{f,g})_*\big)$ commutes with small coproducts since it admits a right adjoint. Therefore, according to the property (2) in Section 2.1, in order to check the above inclusion, it is enough to prove $\Lambda e\in\mathrm{Tria}(Be_2)$ when considered as a B-module via $\theta_{f,g}$. If we identify $e_2B\otimes_B -$ with the left multiplication functor by e_2, then $\Lambda e\in\mathrm{Tria}(Be_2)$ if and only if $Be_2\otimes^{\mathbb{L}}_T e_2\cdot(\Lambda e)\xrightarrow{\simeq}\Lambda e$ in $\mathscr{D}(B)$. Clearly, the latter is equivalent to that $\mathrm{Tor}^T_n(Be_2,\, e_2\cdot(\Lambda e))=0$ for any $n>0$ and the canonical multiplication map $Be_2\otimes_T e_2\cdot(\Lambda e)\longrightarrow\Lambda e$ is an isomorphism.

Set $M:=S\otimes_R T$ and write B-modules in the form of triples (X,Y,h) with $X\in T\text{-Mod}$, $Y\in S\text{-Mod}$ and $h: M\otimes_T X\to Y$ a homomorphism of S-modules. The morphisms between two modules (X,Y,h) and (X',Y',h') are pairs of morphisms (α,β), where $\alpha: X\to X'$ and $\beta: Y\to Y'$ are homomorphisms in T-Mod and S-Mod, respectively, such that $h\beta=(M\otimes_T\alpha)h'$.

With these interpretations, we rewrite $\Lambda e=(\Gamma,\Gamma,\delta_\Gamma)\in B\text{-Mod}$, where $\delta_\Gamma: M\otimes_T\Gamma\to\Gamma$ is defined by $(s\otimes t)\otimes\gamma\mapsto(s)f(t)g\gamma$ for $s\in S, t\in T$ and $\gamma\in\Gamma$. Then $e_2\cdot(\Lambda e)=e\Lambda e\simeq\Gamma$ as left T-modules, and $Be_2\simeq M\oplus T$ as right T-modules. Consequently, we have

$$Be_2\otimes_T e_2\cdot(\Lambda e)\simeq Be_2\otimes_T\Gamma\simeq(\Gamma,\, M\otimes_T\Gamma,\, 1)\ \text{ and }\ \mathrm{Tor}^T_n(Be_2,\, e_2\cdot(\Lambda e))\simeq\mathrm{Tor}^T_n(M\oplus T,\Gamma)\simeq\mathrm{Tor}^T_n(M,\Gamma)$$

for any $n > 0$. This implies that the multiplication map $Be_2 \otimes_T e_2 \cdot (\Lambda e) \longrightarrow \Lambda e$ is an isomorphism if and only if so is the map δ_Γ. It follows that $Be_2 \otimes^{\mathbb{L}}_T e_2 \cdot (\Lambda e) \simeq \Lambda e$ in $\mathscr{D}(B)$ if and only if δ_Γ is an isomorphism of S-modules and $\mathrm{Tor}^T_n(M,\Gamma) = 0$ for any $n > 0$.

In order to verify the latter conditions just mentioned, we shall prove the following general result:

For any Γ-module W, if we regard W as a left T-module via g and an S-module via f, then the map $\delta_W : M \otimes_T W \to W$, defined by $(s \otimes t) \otimes w \mapsto (s)f(t)gw$ for $s \in S, t \in T$ and $w \in W$, is an isomorphism of S-modules, and $\mathrm{Tor}^T_i(M,W) = 0$ for any $i > 0$.

To prove this general result, we fix a projective resolution $V^\bullet$ of S_R:

$$\cdots \longrightarrow V^n \longrightarrow V^{n-1} \longrightarrow \cdots \longrightarrow V^1 \longrightarrow V^0 \longrightarrow S_R \longrightarrow 0$$

with V^i a projective right R-module for each i. By Lemma 3.9, we have $\mathrm{Tor}^R_j(S,T) = 0$ for any $j > 0$. It follows that the complex $V^\bullet \otimes_R T$ is a projective resolution of the right T-module M. Thus the following isomorphisms of complexes of abelian groups:

$$\big(V^\bullet \otimes_R T\big) \otimes_T W \simeq V^\bullet \otimes_R \big(T \otimes_T W\big) \simeq V^\bullet \otimes_R W$$

imply that $\mathrm{Tor}^T_i(M,W) \simeq \mathrm{Tor}^R_i(S,W)$ for any $i > 0$. Recall that W admits an S-module structure via the map f. Moreover, it follows from $\lambda f = \mu g$ that the R-module structure of W endowed via the ring homomorphism μg is the same as the one endowed via the ring homomorphism λf. Then, it follows from λ being a homological ring epimorphism that the multiplication map $S \otimes_R W \longrightarrow W$ is an isomorphism of S-modules and that $\mathrm{Tor}^R_i(S,W) = 0$ for all $i > 0$ (see [9, Theorem 4.4]). Therefore, for any $i > 0$, we have $\mathrm{Tor}^T_i(M,W) \simeq \mathrm{Tor}^R_i(S,W) = 0$. Note that

$$M \otimes_T W = (S \otimes_R T) \otimes_T W \simeq S \otimes_R (T \otimes_T W) \simeq S \otimes_R W \simeq W$$

as S-modules. Thus the map δ_W is an isomorphism of S-modules. So the above-mentioned general result follows.

Now, by applying the above general result to Γ, we can show that δ_Γ is an isomorphism and $\mathrm{Tor}^T_n(M,\Gamma) = 0$ for any $n > 0$. This completes the proof of Lemma 3.10. $\square$

With the above preparations, we now give a proof of Theorem 1.1.

Proof of Theorem 1.1. Note that the second part of Theorem 1.1 is a consequence of Corollary 3.5 and Lemma 3.8(2). Moreover, in Lemma 3.10, if we take $\Gamma := S \sqcup_R T$, $f := \rho$ and $g := \phi$, then $\theta_{f,g} = \theta$, and therefore (1) and (2) in the first part of Theorem 1.1 are equivalent.

In the following, we shall prove that (1) and (3) in the first part of Theorem 1.1 are equivalent.

In fact, by Lemma 3.2, the ring homomorphism $\theta : B \to C$ is homological if and only if $H^n(i_*i^*(B)) = 0$ for any $n \neq 0$. This is equivalent to saying that $H^n(i_*i^*(Be_1)) = 0$ for any $n \neq 0$ since $i_*i^*(B) \simeq i_*i^*(Be_1) \oplus i_*i^*(Be_2) \simeq i_*i^*(Be_1) \oplus i_*i^*(Be_1)$ in $\mathscr{D}(B)$ by Lemma 3.7(3). Furthermore, Lemma 3.8 shows

$$H^n\big(i_*i^*(Be_1)\big) \simeq \begin{cases} 0 & \text{if } n > 0, \\ \mathrm{Tor}^R_{-n}(Be_2,S) & \text{if } n \leq 0. \end{cases}$$

This implies that θ is homological if and only if $\mathrm{Tor}^R_{-n}(Be_2,S) = 0$ for any $n < 0$. Note that $Be_2 \simeq T \oplus (S \otimes_R T)$ as right R-modules and that there is an exact sequence of R-R-bimodules:

$$0 \longrightarrow R \xrightarrow{(-\lambda,\mu)} S \oplus T \xrightarrow{\binom{\mu'}{\lambda'}} S \otimes_R T \longrightarrow 0.$$

Since $\lambda : R \to S$ is a homological ring epimorphism, we have $\mathrm{Tor}^R_j(S,S) = 0$ for any $j > 0$, and the map $\lambda \otimes_R S : R \otimes_R S \to S \otimes_R S$ is an isomorphism. It follows that $\mathrm{Tor}^R_j(T,S) \simeq \mathrm{Tor}^R_j(S \otimes_R T, S)$ for $j > 0$ if we apply the functor $-\otimes_R S$ to the above exact sequence. Thus $\mathrm{Tor}_j^R(Be_2,S)\simeq \mathrm{Tor}_j^R(T,S)\oplus\mathrm{Tor}_j^R(T,S)$ for all $j>0$. Consequently, the map θ is homological if and only if $\mathrm{Tor}_n^R(T,S)=0$ for any $n>0$. This shows that (1) and (3) are equivalent. Thus, we have verified that all the assertions in the first part of Theorem 1.1 are equivalent. $\square$

Now, let us illustrate Theorem 1.1 visually by the following diagram which indicates explicitly the relationship among all the assertions in Theorem 1.1. For convenience of the reader, we state it as a corollary.

Corollary 3.11. *Let $\lambda: R\to S$ and $\mu: R\to T$ be ring homomorphisms such that (λ,μ) is an exact pair. Suppose that both $\lambda: R\to S$ and $\phi: T\to S\sqcup_R T$ are homological. Then we can construct the following 'pull-back' of recollements of triangulated categories:*

$$\begin{tikzcd}
 & \mathscr{D}(S) \arrow[r, equal] \arrow[d] & \mathscr{D}(S) \arrow[d, "D(\lambda_*)"] & \\
\mathscr{D}(C) \arrow[r, "D(\theta_*)"] \arrow[d, "e_2\cdot", "\simeq"'] & \mathscr{D}(B) \arrow[l, bend right, "C\otimes_B^{\mathbb{L}}-"'] \arrow[l, bend left] \arrow[u, bend left] \arrow[u, bend right] \arrow[r, "j^!"] \arrow[d, "e_2\cdot"] & \mathscr{D}(R) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \arrow[u, bend left] \arrow[u, bend right] \arrow[dr] & \\
\mathscr{D}(S\sqcup_R T) \arrow[r, "D(\phi_*)"] & \mathscr{D}(T) \arrow[l, bend right] \arrow[l, bend left] \arrow[u, bend left] \arrow[u, bend right] \arrow[dr] & & \mathrm{Tria}(Q^\bullet) \arrow[ul, bend right, "F_1"'] \arrow[ul, bend left] \arrow[dl, "T\otimes_R^{\mathbb{L}}-", "\simeq"'] \\
 & & \mathrm{Tria}(T\otimes_R Q^\bullet) \arrow[ul, bend right, "F_2"'] \arrow[ul, bend left] &
\end{tikzcd}$$

where F_i is the canonical embedding for $i=1,2$, and $T\otimes_R^{\mathbb{L}}-$ induces a triangle equivalence from $\mathrm{Tria}(Q^\bullet)$ *to* $\mathrm{Tria}(T\otimes_R Q^\bullet)$.

Proof. First of all, we point out that, under the assumptions of Corollary 3.11, all the assertions in Theorem 1.1 are true. In particular, the map $\theta: B\to C$ is homological.

Next, we observe the following facts on the above diagram:

(1) The recollement of derived module categories in the second column arises from the triangular structure of the triangular matrix ring B.

(2) The recollement of triangulated categories in the third column follows from the assumption that λ is homological (see the end of Section 2.2).

(3) The recollement of derived module categories in the middle row has been stated in Theorem 1.1, where $\mathrm{End}_T(T\otimes_R S)$ in Theorem 1.1 is isomorphic to the coproduct $S\sqcup_R T$, which is Morita equivalent to C.

(4) The left square in the diagram has been discussed in the proof of Lemma 3.10.

So, to complete the proof of Corollary 3.11, it remains to verify the following two statements:

(5) The recollement of triangulated categories in the third row does exist.

(6) The functor $T\otimes_R^{\mathbb{L}}-: \mathrm{Tria}(Q^\bullet)\longrightarrow \mathrm{Tria}(T\otimes_R Q^\bullet)$ is an equivalence of triangulated categories.

In order to prove (5), we consider the distinguished triangle $T\xrightarrow{\phi} S\sqcup T\longrightarrow V^\bullet\longrightarrow T[1]$ in $\mathscr{K}(T)$, where $V^\bullet$ is the mapping cone of ϕ, and claim that $V^\bullet\simeq T\otimes_R Q^\bullet$ in $\mathscr{D}(T)$. In fact, from the triangle $R\xrightarrow{\lambda} S\xrightarrow{\pi} Q^\bullet\longrightarrow R[1]$ in $\mathscr{K}(R)$, we get the following commutative diagram with all rows being triangles in $\mathscr{D}(T)$:

$$(\lozenge)\qquad \begin{tikzcd}
T\otimes_R^{\mathbb{L}} R \arrow[r, "1\otimes^{\mathbb{L}}\lambda"] \arrow[d, "\simeq"'] & T\otimes_R^{\mathbb{L}} S \arrow[r, "1\otimes^{\mathbb{L}}\pi"] \arrow[d, "\simeq"] & T\otimes_R^{\mathbb{L}} Q^\bullet \arrow[r] \arrow[d] & T\otimes_R^{\mathbb{L}} R[1] \arrow[d, "\simeq"] \\
T\otimes_R R \arrow[r, "1\otimes\lambda"] \arrow[d, "\simeq"'] & T\otimes_R S \arrow[r, "1\otimes\pi"] \arrow[d, "\simeq"] & T\otimes_R Q^\bullet \arrow[r] \arrow[d] & T\otimes_R R[1] \arrow[d, "\simeq"] \\
T \arrow[r, "\phi"] & S\sqcup T \arrow[r] & V^\bullet \arrow[r] & T[1]
\end{tikzcd}$$

where the first and second isomorphisms in the second column follow from $\mathrm{Tor}_i^R(T,S)=0$ for each $i\geq 1$ and Lemma 3.8(2), respectively. Thus $T\otimes_R^{\mathbb{L}}Q^\bullet\simeq T\otimes_R Q^\bullet\simeq V^\bullet$ in $\mathscr{D}(T)$. Since ϕ is homological, the recollement in the third row of the diagram does exist (see Section 2.2). So (5) follows.

Finally, we prove (6). By the proof of (5), we see that $T\otimes_R^{\mathbb{L}}Q^\bullet\simeq T\otimes_R Q^\bullet$. Hence the image of $\mathrm{Tria}(Q^\bullet)$ under the functor $T\otimes_R^{\mathbb{L}}-$ lies in $\mathrm{Tria}(T\otimes_R Q^\bullet)$. This follows from the property (2) in Section 2.1.

To prove that the restriction functor $T\otimes_R^{\mathbb{L}}-:\mathrm{Tria}(Q^\bullet)\longrightarrow\mathrm{Tria}(T\otimes_R Q^\bullet)$ is fully faithful, we shall show that the following full subcategory

$$\mathscr{Y}:=\{Y^\bullet\in\mathscr{D}(R)\mid T\otimes_R^{\mathbb{L}}-:\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,Y^\bullet[n])\xrightarrow{\simeq}\mathrm{Hom}_{\mathscr{D}(T)}(T\otimes_R^{\mathbb{L}}Q^\bullet,T\otimes_R^{\mathbb{L}}Y^\bullet[n])\ \text{for all}\ n\in\mathbb{Z}\}$$

contains $Q^\bullet$ and is closed under arbitrary direct sums in $\mathscr{D}(R)$.

Indeed, let $\eta: Id_{\mathscr{D}(R)}\longrightarrow D(\mu_*)(T\otimes_R^{\mathbb{L}}-)$ be the unit adjunction with respect to the adjoint pair $(T\otimes_R^{\mathbb{L}}-,D(\mu_*))$, where $D(\mu_*):\mathscr{D}(T)\to\mathscr{D}(R)$ is the derived functor induced from the functor $\mu_*:T\text{-Mod}\to R\text{-Mod}$. Then, for each $Y^\bullet\in\mathscr{D}(R)$, there is a unique triangle (up to isomorphism) in $\mathscr{D}(R)$:

$$C_{Y^\bullet}[-1]\longrightarrow Y^\bullet\xrightarrow{\eta_{Y^\bullet}}{}_RT\otimes_R^{\mathbb{L}}Y^\bullet\longrightarrow C_{Y^\bullet}$$

where $C_{Y^\bullet}$ is an object in $\mathscr{D}(R)$ uniquely determined by the morphism $\eta_{Y^\bullet}$ (up to isomorphism). Since $(T\otimes_R^{\mathbb{L}}-,D(\mu_*))$ is an adjoint pair, one can further prove that $\mathscr{Y}$ coincides with

$$\{Y^\bullet\in\mathscr{D}(R)\mid(\eta_{Y^\bullet})^*:\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,Y^\bullet[n])\xrightarrow{\simeq}\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,{}_RT\otimes_R^{\mathbb{L}}Y^\bullet[n])\ \text{for all}\ n\in\mathbb{Z}\}.$$

Thus we have

$$\mathscr{Y}=\{Y^\bullet\in\mathscr{D}(R)\mid\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,C_{Y^\bullet}[n])=0\ \text{for all}\ n\in\mathbb{Z}\}.$$

Before giving a further description of $\mathscr{Y}$ in terms of $\mathscr{D}(S)$, we mention the following general fact:

For any $X^\bullet\in\mathscr{D}(R)$, we define $\mathcal{S}_{X^\bullet}:=\{X^\bullet[n]\mid n\in\mathbb{Z}\}$. Then $\mathrm{Ker}\big(\mathrm{Hom}_{\mathscr{D}(R)}(-,\mathcal{S}_{X^\bullet})\big)$ is a full triangulated subcategory of $\mathscr{D}(R)$ closed under arbitrary direct sums. Dually, $\mathrm{Ker}\big(\mathrm{Hom}_{\mathscr{D}(R)}(\mathcal{S}_{X^\bullet},-)\big)$ is a full triangulated subcategory of $\mathscr{D}(R)$ closed under arbitrary direct products.

From this general fact, we deduce that

$$\mathrm{Ker}\big(\mathrm{Hom}_{\mathscr{D}(R)}(\mathrm{Tria}(Q^\bullet),-)\big)=\{Y^\bullet\in\mathscr{D}(R)\mid\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,Y^\bullet[n])=0\ \text{for all}\ n\in\mathbb{Z}\}.$$

Further, it follows from the recollement in the third 'tilted' column that $\mathscr{D}(S)=\mathrm{Ker}\big(\mathrm{Hom}_{\mathscr{D}(R)}(\mathrm{Tria}(Q^\bullet),-)\big)$. This implies that $\mathscr{Y}=\{Y^\bullet\in\mathscr{D}(R)\mid C_{Y^\bullet}\in\mathscr{D}(S)\}$. Here, we consider $\mathscr{D}(S)$ as a full triangulated subcategory of $\mathscr{D}(R)$.

Note that $T\otimes_R^{\mathbb{L}}-$ commutes with arbitrary direct sums and that $\mathscr{D}(S)$ is a triangulated subcategory of $\mathscr{D}(R)$ closed under arbitrary direct sums. Consequently, the subcategory $\mathscr{Y}$ is also closed under arbitrary direct sums in $\mathscr{D}(R)$.

To prove $Q^\bullet\in\mathscr{Y}$, we use the diagram $(\Diamond)$ and form another commutative diagram in $\mathscr{D}(R)$:

$$\begin{tikzcd}
R \arrow[r, "\lambda"] \arrow[d, "\eta_R"'] & S \arrow[r, "\pi"] \arrow[d, "\eta_S"] & Q^\bullet \arrow[r, "\nu"] \arrow[d, "\eta_{Q^\bullet}"] & R[1] \arrow[d] \\
T\otimes_R^{\mathbb{L}}R \arrow[r, "1\otimes^{\mathbb{L}}\lambda"] \arrow[d, "\simeq"'] & T\otimes_R^{\mathbb{L}}S \arrow[r, "1\otimes^{\mathbb{L}}\pi"] \arrow[d, "\simeq"] & T\otimes_R^{\mathbb{L}}Q^\bullet \arrow[r, "1\otimes^{\mathbb{L}}\nu"] \arrow[d, "\delta"', "\simeq"] & T\otimes_R R[1] \arrow[d, "\simeq"] \\
T \arrow[r, "\phi"] & S\sqcup T \arrow[r] & V^\bullet \arrow[r] & T[1],
\end{tikzcd}$$

where the composites of the two morphisms in the first and second columns are equal to μ and ρ, respectively. Let $f^\bullet:=(f^i)_{i\in\mathbb{Z}}$ be the chain map defined by $f^{-1}:=\mu$, $f^0:=\rho$ and $f^i=0$ for $i\neq-1,0$. Since λ is homological and since $S \sqcup T$ is an S-module, we have $\operatorname{Hom}_{\mathscr{D}(R)}(Q^\bullet, S \sqcup T) = 0$. This means that there is a unique morphism $\gamma\colon Q^\bullet \to V^\bullet$ such that the following diagram is commutative:

$$\begin{tikzcd}
R \arrow[r, "\lambda"] \arrow[d, "\mu"'] & S \arrow[r, "\pi"] \arrow[d, "\rho"] & Q^\bullet \arrow[r, "\nu"] \arrow[d, "\gamma"] & R[1] \arrow[d, "\mu[1]"] \\
T \arrow[r, "\phi"] & S \sqcup T \arrow[r] & V^\bullet \arrow[r] & T[1]
\end{tikzcd}$$

Since both $f^\bullet$ and $\eta_{Q^\bullet}\delta$ make the diagram commutative, we have $f^\bullet = \eta_{Q^\bullet}\delta$ in $\mathscr{D}(R)$. Since δ is an isomorphism in $\mathscr{D}(R)$, we have $\operatorname{Cone}(f^\bullet) \simeq C_{Q^\bullet}$ in $\mathscr{D}(R)$. Note that $\operatorname{Cone}(f^\bullet)$ is of the form:

$$0 \longrightarrow R \xrightarrow{(-\lambda,\mu)} S \oplus T \xrightarrow{\binom{\rho}{\phi}} S \sqcup_R T \longrightarrow 0$$

with $S \sqcup_R T$ in degree 0, where h is given by $s \otimes t \mapsto (s)\rho(t)\phi$ for $s \in S$ and $t \in T$ (see the diagram $(*)$ at the beginning of Section 3). Let $Z^\bullet$ be the following complex of S-modules:

$$0 \longrightarrow S \otimes_R T \xrightarrow{h} S \sqcup_R T \longrightarrow 0$$

which can be considered as an object in $\mathscr{D}(R)$, and let $\nu^\bullet : \operatorname{Cone}(f^\bullet) \to Z^\bullet$ be the chain map defined by $\nu^{-1} = \binom{\mu'}{\lambda'}$, $\nu^0 = 1_{S \sqcup_R T}$ and $\nu^i = 0$ for $i \neq 0, -1$. Since (λ, μ) is an exact pair, we infer that $\operatorname{Cone}(\nu^\bullet)$ is acyclic and that $\operatorname{Cone}(f^\bullet)$ is isomorphic to $Z^\bullet$ in $\mathscr{D}(R)$. Clearly, $Z^\bullet$ lies in $\mathscr{D}(S)$, and therefore $C_{Q^\bullet} \in \mathscr{D}(S)$ and $Q^\bullet \in \mathscr{Y}$.

Recall that $\operatorname{Tria}(Q^\bullet)$ is the smallest full triangulated subcategory of $\mathscr{D}(R)$, which contains $Q^\bullet$ and is closed under arbitrary direct sums. Consequently, the category $\mathscr{Y}$ contains $\operatorname{Tria}(Q^\bullet)$. This means that, for any $Y^\bullet \in \operatorname{Tria}(Q^\bullet)$, we have

$$\operatorname{Hom}_{\mathscr{D}(R)}(Q^\bullet, Y^\bullet[n]) \xrightarrow{\simeq} \operatorname{Hom}_{\mathscr{D}(T)}(T \otimes_R^{\mathbb{L}} Q^\bullet, T \otimes_R^{\mathbb{L}} Y^\bullet[n]) \text{ for all } n \in \mathbb{Z}.$$

Now, fix $N^\bullet \in \operatorname{Tria}(Q^\bullet)$ and consider the following full subcategory of $\mathscr{D}(R)$:

$$\mathscr{X}_{N^\bullet} := \{X^\bullet \in \mathscr{D}(R) \mid T \otimes_R^{\mathbb{L}} - : \operatorname{Hom}_{\mathscr{D}(R)}(X^\bullet, N^\bullet[n]) \xrightarrow{\simeq} \operatorname{Hom}_{\mathscr{D}(T)}(T \otimes_R^{\mathbb{L}} X^\bullet, T \otimes_R^{\mathbb{L}} N^\bullet[n]) \text{ for all } n \in \mathbb{Z}\}.$$

Clearly, $Q^\bullet \in \mathscr{X}_{N^\bullet}$. Furthermore, one can verify that $\mathscr{X}_{N^\bullet}$ is a full triangulated subcategory of $\mathscr{D}(R)$, which is closed under arbitrary direct sums in $\mathscr{D}(R)$. This implies that $\mathscr{X}_{N^\bullet}$ contains $\operatorname{Tria}(Q^\bullet)$. As a result, for any $M^\bullet \in \operatorname{Tria}(Q^\bullet)$, we have an isomorphism

$$T \otimes_R^{\mathbb{L}} - : \operatorname{Hom}_{\mathscr{D}(R)}(M^\bullet, N^\bullet) \xrightarrow{\simeq} \operatorname{Hom}_{\mathscr{D}(T)}(T \otimes_R^{\mathbb{L}} M^\bullet, T \otimes_R^{\mathbb{L}} N^\bullet).$$

This shows that the restriction of $T \otimes_R^{\mathbb{L}} -$ to $\operatorname{Tria}(Q^\bullet)$ is fully faithful. Further, since $T \otimes_R^{\mathbb{L}} -$ commutes with arbitrary direct sums and $T \otimes_R^{\mathbb{L}} Q^\bullet \simeq T \otimes_R Q^\bullet$ in $\mathscr{D}(T)$, we can infer from the property (2) in Section 2.1 that the restriction functor $T \otimes_R^{\mathbb{L}} -\colon \operatorname{Tria}(Q^\bullet) \to \operatorname{Tria}(T \otimes_R Q^\bullet)$ is a triangle equivalence. This completes the proof of Corollary 3.11. $\square$.

4 Proofs of Corollaries

In this section, we shall prove all corollaries of Theorem 1.1, which were mentioned in Introduction.

We preserve all notation introduced in the previous sections.

4.1 Proof of Corollary 1.2

As a preparation for the proof of Corollary 1.2, we obtain the following consequence of Theorem 1.1, which produces homological ring epimorphisms for quotient rings from those between given rings.

Corollary 4.1. *Let $\lambda : R \to S$ be a homological ring epimorphism. Suppose that I is an ideal of R such that the image J' of I under λ is a left ideal in S and that the restriction of λ to I is injective. Let J be the ideal of S generated by J', and $B := \begin{pmatrix} S & S/J' \\ 0 & R/I \end{pmatrix}$. Then the following statements are equivalent:*

(1) *The homomorphism $R/I \to S/J$ induced from λ is homological.*

(2) $\mathrm{Tor}_i^{R/I}\big(J/J', S/J\big) = 0$ *for all* $i \geq 1$.

(3) *The multiplication map $I \otimes_R S \to J$ is an isomorphism and* $\mathrm{Tor}_j^R(I, S) = 0$ *for all* $j \geq 1$.

If one of the above statements holds true, then there is a recollement of derived module categories:

$$\begin{tikzcd} \mathscr{D}(S/J) \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

Proof. In Theorem 1.1, we take $T := R/I$ and choose $\mu : R \to R/I$ to be the canonical surjective homomorphism of rings. Since J' is a left ideal of S, we have $S \otimes_R T = S \otimes_R (R/I) \simeq S/(S \cdot I) = S/J'$. This means that B in Corollary 4.1 coincides with the one in Theorem 1.1. Moreover, one can verify that the pair (λ, μ) is exact if and only if $\lambda|_I : I \to J'$ is an isomorphism.

By Lemma 2.5(2), we see that $S \sqcup_R T = S \sqcup (R/I) = S/J$ with $J := J'S$ and that the ring homomorphism $\phi : T \to S \sqcup_R T$ in Theorem 1.1 can be chosen to be the canonical map $\widetilde{\lambda} : R/I \to S/J$ induced from λ. Therefore, by Theorem 1.1, if $\widetilde{\lambda}$ is homological, then the recollement of derived module categories in Corollary 4.1 does exist. This finishes the proof of the second part of Corollary 4.1.

To prove the first part of Corollary 4.1, we shall show that (1) is equivalent to (3) and (2), respectively.

In fact, due to Theorem 1.1, we can see that (1) is equivalent to $\mathrm{Tor}_j^R(R/I, S) = 0$ for all $j \geq 1$. To verify the latter condition, we apply $- \otimes_R S$ to the sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$, and then get $\mathrm{Tor}_{j+1}^R(R/I, S) \simeq \mathrm{Tor}_j^R(I, S)$ and $\mathrm{Tor}_1^R(R/I, S) \simeq \mathrm{Ker}(\delta)$, where $\delta : I \otimes_R S \to J$ is the multiplication map defined by $x \otimes s \mapsto (x)\lambda s$ for $x \in I$ and $s \in S$. Clearly, this implies that (1) is equivalent to (3).

Now we show that (1) and (2) are equivalent.

According to Lemma 2.5(1) and the fact that λ is a ring epimorphism, $\widetilde{\lambda}$ is a ring epimorphism. By assumption, J' is a left ideal of S, and therefore $S \otimes_R (R/I) \simeq S/(S \cdot I) = S/J'$. Thanks to the general result proved in the last part of the proof of Lemma 3.10, we see that

$$\mathrm{Tor}_i^{R/I}(S/J', W) \simeq \mathrm{Tor}_i^{R/I}(S \otimes_R (R/I), W) = 0$$

for all $i \geq 1$ and all S/J-modules W. It follows then that $\mathrm{Tor}_i^{R/I}(S/J', S/J) = 0$ for all $i \geq 1$. Consider the short exact sequence of right R/I-modules:

$$0 \longrightarrow J/J' \longrightarrow S/J' \longrightarrow S/J \longrightarrow 0.$$

If we apply the functor $- \otimes_{R/I} (S/J)$ to this sequence, then we can check that $\mathrm{Tor}_i^{R/I}(J/J', S/J) \simeq \mathrm{Tor}_{i+1}^{R/I}(S/J, S/J)$ for all $i \geq 1$ and that the connecting homomorphism $\mathrm{Tor}_1^{R/I}(S/J, S/J) \to (J/J') \otimes_{R/I} (S/J)$ is injective.

Clearly, if $\mathrm{Tor}_1^{R/I}(S/J, S/J) = 0$, then $\mathrm{Tor}_j^{R/I}(S/J, S/J) = 0$ for all $j \geq 1$ if and only if $\mathrm{Tor}_i^{R/I}(J/J', S/J) = 0$ for all $i \geq 1$. This implies that the statements (1) and (2) in Corollary 4.1 are equivalent.

Now, we claim that $\mathrm{Tor}_1^{R/I}(S/J, S/J) = 0$ always holds under the assumptions of Corollary 4.1. To show this claim, it is enough to prove that $(J/J') \otimes_{R/I} (S/J) = 0$. Note that if $C \to D$ is a ring epimorphism, then

$D \otimes_C X \simeq X$ as D-modules for any D-module X, and $Y \otimes_C D \simeq Y$ as right D-modules for any right D-module Y. This fact together with properties of ring epimorphisms implies the following isomorphisms:

$$(J/J') \otimes_{R/I} (S/J) \simeq (J/J') \otimes_R (S/J) \simeq (J/J') \otimes_R \big(S \otimes_R (S/J)\big) \simeq \big((J/J') \otimes_R S\big) \otimes_R (S/J).$$

Since $SJ' = J'$ and $JJ' \subseteq J'$, we deduce that $\big((J/J') \otimes_R S\big)J' = 0$. This means that $(J/J') \otimes_R S$ is a right S/J-module. Clearly, the composite of the two ring epimorphisms $R \to S$ and $S \to S/J$ is again a ring epimorphism. It follows that $\big((J/J') \otimes_R S\big) \otimes_R (S/J) \simeq (J/J') \otimes_R S$ as right S/J-modules.

In the following, we shall show that $(J/J') \otimes_R S = 0$. Actually, applying the functor $- \otimes_R S$ to the exact sequence

$$0 \longrightarrow J' \xrightarrow{\alpha} J \longrightarrow J/J' \longrightarrow 0$$

of right R-modules, we get an exact sequence

$$J' \otimes_R S \xrightarrow{\alpha \otimes_R S} J \otimes_R S \longrightarrow (J/J') \otimes_R S \longrightarrow 0$$

of right S-modules. Since J is a right S-module and $\lambda : R \to S$ is a ring epimorphism, the multiplication map $\psi : J \otimes_R S \to J$, defined by $x \otimes s \mapsto xs$ for $x \in J$ and $s \in S$, is an isomorphism. Note that the map $(\alpha \otimes_R S)\psi : J' \otimes_R S \to J$ is surjective. This yields that $\alpha \otimes_R S$ is surjective and that $(J/J') \otimes_R S = 0$. Hence $\mathrm{Tor}_1^{R/I}(S/J, S/J) = 0$.

Thus, we have shown that the three statements in Corollary 4.1 are equivalent. This finishes the proof. $\square$

Remark. There is a connection between exact pairs and ring homomorphisms described in Corollary 4.1. Indeed, the proof of Corollary 4.1 shows that each ring homomorphism $\lambda : R \to S$ together with an ideal I of R satisfying the assumption in Corollary 4.1 provides us an exact pair (λ, π), where $\pi : R \to R/I$ is the canonical surjection. Conversely, for any exact pair (λ, μ) of ring homomorphisms $\lambda : R \to S$ and $\mu : R \to T$, the ideal $I := \mathrm{Ker}(\mu)$ of R satisfies the assumption in Corollary 4.1.

Proof of Corollary 1.2.

(1) In Corollary 4.1, we take $S := R/I_1$ and $I := I_2$, and let $\lambda : R \to S$ be the canonical surjective ring homomorphism. Then $J' = (I)\lambda = (I_2 + I_1)/I_1 = J$, which is an ideal of S. In particular, $\mathrm{Tor}_i^{R/J}\big(J/J', S/J\big) = 0$ for all $i > 0$. Furthermore, $S/J' \simeq R/(I_1 + I_2)$, and the map $\lambda|_I : I \to (I)\lambda$ is an isomorphism if and only if $I_1 \cap I_2 = 0$. Hence Corollary 1.2(1) follows from Corollary 4.1.

(2) Suppose that $\lambda : R \to S$ is homological. In Theorem 1.1, we take $T := R \ltimes M$ and define $\mu : R \to T$ to be the inclusion from R into T. By Lemma 2.4, the ring $S \ltimes M$, together with the inclusion $\rho : S \to S \ltimes M$ and $\widetilde{\lambda} : T \to S \ltimes M$, is the coproduct of S and T over R. In particular, we can take $\phi = \widetilde{\lambda}$ in Theorem 1.1.

First of all, we claim that (λ, μ) is an exact pair. Actually, it follows from the split exact sequence $0 \longrightarrow R \xrightarrow{\mu} T \longrightarrow M \longrightarrow 0$ of R-R-bimodules that ${}_RT_R \simeq R \oplus M$ as R-R-bimodules. Since λ is a ring epimorphism and M is an S-S-bimodule, the map

$$S \otimes_R T \longrightarrow S \ltimes M,\ s \otimes (r, m) \mapsto (sr, sm)$$

for $s \in S$ and $m \in M$, is an isomorphism of S-T-bimodules. Under this isomorphism, we can identify the map $\mu' : S \to S \otimes T$ (see Introduction) with ρ, and the ring B in Theorem 1.1 with the one defined in Corollary 1.2(2). Note that $0 \longrightarrow S \xrightarrow{\rho} S \ltimes M \longrightarrow M \longrightarrow 0$ is also a split exact sequence of S-S-bimodules. It follows that $\mathrm{Coker}(\mu) \simeq \mathrm{Coker}(\mu') \simeq M$ as R-R-bimodules, and therefore the pair (λ, μ) is exact.

Next, we shall show that the assertion (3) in Theorem 1.1 holds for the pair (λ, μ). In fact, for each $i \geq 1$, we have $\mathrm{Tor}_i^R(T, S) \simeq \mathrm{Tor}_i^R(R \oplus M, S) \simeq \mathrm{Tor}_i^R(M, S) \simeq \mathrm{Tor}_i^S(M, S) = 0$, where the third isomorphism follows from the fact that λ is homological and M is a right S-module. Now, the necessity condition of Corollary 1.2(2) follows immediately from Theorem 1.1.

To see the sufficiency condition of Corollary 1.2(2), we suppose that $\widetilde{\lambda}$ is homological. Then we may apply Corollary 4.1 to see that λ is homological. This finishes the proof. $\square$

As an easy application of Corollary 4.1, we obtain the following interesting result in which the left-hand side of the recollement is the derived module category $\mathscr{D}(S/I)$ instead of $\mathscr{D}(S)$.

Corollary 4.2. *Let $R \subseteq S$ be an extension of rings. Suppose that I is an ideal of S with $I \subseteq R$. Define $B := \begin{pmatrix} S & S/I \\ 0 & R/I \end{pmatrix}$. If the inclusion $R \to S$ is a homological ring epimorphism, then so is the homomorphism $R/I \to S/I$ induced from λ. In this case, there is a recollement of derived module categories:*

$$\begin{tikzcd} \mathscr{D}(S/I) \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

Remark. Note that, in Corollary 4.2, the ring B is derived equivalent to the ring $T := \begin{pmatrix} S & I \\ S & R \end{pmatrix}$. This is an obvious consequence of [19, Lemma 3.4]. Thus, the algebraic K-theory of T is isomorphic to that of B. For further discussions on calculating higher algebraic K-groups using derived equivalences and $\mathcal{D}$-split sequences, we refer the reader to [19].

Finally, we point out a possible choice for the ideal I in Corollary 4.1.

Lemma 4.3. *Let $\lambda : R \to S$ be a ring epimorphism such that $\mathrm{Hom}_R(S,\lambda) : \mathrm{Hom}_R({}_RS, {}_RR) \to \mathrm{Hom}_R({}_RS, {}_RS)$ is injective. Define $I := \{(1)f \mid f \in \mathrm{Hom}_R(S,R)\}$. Then I is an ideal of R such that the image $(I)\lambda$ of I under λ is a left ideal of S and that the restriction map $\lambda|_I : I \to (I)\lambda$ is an isomorphism.*

Proof. Since λ is a ring epimorphism, we have $\mathrm{Hom}_R({}_RS, {}_RS) = \mathrm{Hom}_S({}_SS, {}_SS)$. By identifying R and S with $\mathrm{Hom}_R(R,R)$ and $\mathrm{Hom}_S(S,S)$ through the right multiplication, respectively, we can re-write $\mathrm{Hom}_R(\lambda,R)$: $\mathrm{Hom}_R(S,R) \to R$ by $f \mapsto (1)f$, and $\mathrm{Hom}_R(S,\lambda) : \mathrm{Hom}_R(S,R) \to S$ by $f \mapsto (1)f\lambda$ for each $f \in \mathrm{Hom}_R(S,R)$. It follows that $\mathrm{Hom}_R(S,\lambda) = \mathrm{Hom}_R(\lambda,R)\lambda$, and therefore $\mathrm{Hom}_R(\lambda,R)$ is injective. This also implies that $\lambda|_I : I \to (I)\lambda$ is an isomorphism. Since $\mathrm{Hom}_R(S,R)$ is an S-R-bimodule, we know that I is an ideal of R such that $(I)\lambda$ is a left ideal of S. This finishes the proof. $\square$

In the next section, we shall consider ring epimorphisms λ with the property mentioned in Lemma 4.3 in detail.

4.2 Proofs of Corollaries 1.3–1.6

In this section, we follow again the notation introduced in Section 1. Fix a ring homomorphism $\lambda : R \to S$, and let

$$(**) \quad R \xrightarrow{\lambda} S \xrightarrow{\pi} Q^\bullet \xrightarrow{\nu} R[1]$$

be the distinguished triangle in the homotopy category $\mathscr{K}(R)$ of R, where the complex $Q^\bullet$ stands for the mapping cone of λ.

Now, we set $S' := \mathrm{End}_{\mathscr{D}(R)}(Q^\bullet)$, and define $\overline{\mu} : R \to S'$ by $r \mapsto f^\bullet$ for $r \in R$, where $f^\bullet$ is the chain map with $f^{-1} := \cdot r$, $f^0 := \cdot(r)\lambda$ and $f^i = 0$ for $i \neq 0, -1$. Here, $\cdot r$ and $\cdot(r)\lambda$ stand for the right multiplication maps by r and $(r)\lambda$, respectively. These data can be recorded in the following diagram:

$$\begin{tikzcd} R \arrow[r, "\lambda"] \arrow[d, "\cdot r"] & S \arrow[r, "\pi"] \arrow[d, "\cdot(r)\lambda"] & Q^\bullet \arrow[r, "\nu"] \arrow[d, "f^\bullet"] & R[1] \arrow[d, "(\cdot r)[1]"] \\ R \arrow[r, "\lambda"] & S \arrow[r, "\pi"] & Q^\bullet \arrow[r, "\nu"] & R[1] \end{tikzcd}$$

The map $\overline{\mu}$ is called the ring homomorphism *associated to* λ. Note that if λ is injective, then $Q^\bullet$ can be identified with the R-module S/R in $\mathscr{D}(R)$, and, under this identification, the map $\overline{\mu}$ coincides with the induced map from R to $\mathrm{End}_R(S/R)$ by the right multiplication. In this case, we shall replace the complex $Q^\bullet$ with the R-module S/R.

Recall that Λ denotes the ring $\mathrm{End}_{\mathscr{D}(R)}\big(S\oplus Q^\bullet\big)$ and that π^* is the induced map

$$\mathrm{Hom}_{\mathscr{D}(R)}(S\oplus Q^\bullet,\pi):\mathrm{Hom}_{\mathscr{D}(R)}(S\oplus Q^\bullet,S)\longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(S\oplus Q^\bullet,Q^\bullet).$$

Let $\lambda_{\pi^*}:\Lambda\to\Lambda_{\pi^*}$ stand for the universal localization of Λ at π^*.

In [5, Lemma 6.5], we proved that if λ is an injective ring epimorphism with $\mathrm{Tor}_1^R(S,S)=0$, then the pair $(\lambda,\overline{\mu})$ is exact. As a generalization of this result, we shall show, in this section, that if λ is a ring epimorphism such that $\mathrm{Tor}_1^R(S,S)=0=\mathrm{Hom}_R\big(S,\mathrm{Ker}(\lambda)\big)$, then $(\lambda,\overline{\mu})$ is exact. In this general case, the complex $Q^\bullet$ may have two terms of non-zero cohomologies.

If λ is a ring epimorphism, then $S\simeq \mathrm{End}_R(S)$ as rings, and therefore $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,S)=0$. Moreover, there is a canonical homomorphism $\tau:S\otimes_R S'\to \mathrm{Hom}_{\mathscr{D}(R)}(S,Q^\bullet)$ of S-S'-bimodules, defined by $s\otimes f\mapsto s\cdot(\pi f)$ for $s\in S$ and $f\in S'$. In this case, we obtain a relevant ring homomorphism:

$$\widetilde{\tau}:\begin{pmatrix} S & S\otimes_R S' \\ 0 & S'\end{pmatrix}\longrightarrow \begin{pmatrix} S & \mathrm{Hom}_{\mathscr{D}(R)}(S,Q^\bullet) \\ 0 & S'\end{pmatrix}=\Lambda.$$

In the following lemma, we shall provide a sufficient condition to ensure that the ring homomorphism $\tilde{\tau}$ is an isomorphism. This generalizes some known facts in [5, Lemmas 6.4 and 6.5] on injective ring epimorphisms.

Lemma 4.4. *Suppose that* $\lambda:R\to S$ *is a ring epimorphism with* $\mathrm{Tor}_1^R(S,S)=0$ *such that the map* $\mathrm{Hom}_R(\lambda,R)$: $\mathrm{Hom}_R(S,R)\to\mathrm{Hom}_R(R,R)$ *is injective. Then the following holds:*

(1) $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,S)=0=\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,S[1])$.

(2) $\mathrm{Ker}(\overline{\mu})\simeq\mathrm{Hom}_R(S,R)$ *and* $\mathrm{Coker}(\overline{\mu})\simeq\mathrm{Ext}_R^1(S,R)$ *as R-R-bimodules. In particular, if* $\mathrm{Ext}_R^1(S,R)=0$, *then $S'\simeq R/I$ as rings, where $I:=\{(1)h\mid h\in\mathrm{Hom}_R(S,R)\}$ is an ideal of R.*

(3) *The canonical homomorphism* $\tau:S\otimes_R S'\longrightarrow\mathrm{Hom}_{\mathscr{D}(R)}(S,Q^\bullet)$ *is an isomorphism of S-S'-bimodules such that* $1\otimes 1$ *is mapped to* π. *In particular,*

$$\Lambda\simeq\begin{pmatrix} S & S\otimes_R S' \\ 0 & S'\end{pmatrix}.$$

(4) *Suppose that λ is injective. If R is commutative, then so is S'.*

Proof. (1) We claim that if λ is an arbitrary homomorphism of rings, then $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,S[i])\simeq\mathrm{Ext}_R^i(S,S)$ for any $i\in\mathbb{Z}\setminus\{0\}$, and $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,S)\simeq\mathrm{Ker}\big(\mathrm{Hom}_R(\lambda,S)\big)$.

In fact, applying $\mathrm{Hom}_{\mathscr{D}(R)}(-,S[j])$ to the triangle $(**)$, we get the following long exact sequence:

$$\mathrm{Hom}_{\mathscr{D}(R)}(R,S[j-1])\longrightarrow\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,S[j])\longrightarrow\mathrm{Hom}_{\mathscr{D}(R)}(S,S[j])\xrightarrow{\phi_j}\mathrm{Hom}_{\mathscr{D}(R)}(R,S[j])$$

for each $j\in\mathbb{Z}$, where $\phi_j:=\mathrm{Hom}_{\mathscr{D}(R)}(\lambda,S[j])$. Since $\mathrm{Hom}_{\mathscr{D}(R)}(R,S[k])=0$ for any $k\neq 0$ and since ϕ_0 : $\mathrm{Hom}_R(S,S)\to\mathrm{Hom}_R(R,S)$ is surjective, it is easy to verify that $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,S[i])\simeq\mathrm{Hom}_{\mathscr{D}(R)}(S,S[i])$ for any $0\neq i\in\mathbb{Z}$. We leave the details to the reader. Note that this claim implies that $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,S[i])=0$ for any $i<0$.

Since λ is a ring epimorphism with $\mathrm{Tor}_1^R(S,S)=0$, it follows from [16, Theorem 4.8] that ϕ_0 is an isomorphism and $\mathrm{Ext}_R^1(S,S)\simeq\mathrm{Ext}_S^1(S,S)=0$. We have seen that $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,S)=0$. This proves (1).

(2) First, we point out that $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, R) = 0$. Actually, this can be concluded from the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, R) \longrightarrow \mathrm{Hom}_R(S,R) \xrightarrow{\mathrm{Hom}_R(\lambda,R)} \mathrm{Hom}_R(R,R)$$

together with the assumption on $\mathrm{Hom}_R(\lambda, R)$.

Recall that we have the following commutative diagram in $\mathscr{K}(R)$ for each $r \in R$:

$$\begin{tikzcd}
R \arrow[r, "\lambda"] \arrow[d, "\cdot r"'] & S \arrow[r, "\pi"] \arrow[d, "\cdot(r)\lambda"'] & Q^\bullet \arrow[r, "\nu"] \arrow[d, "(r)\overline{\mu}"'] & R[1] \arrow[d, "(\cdot r)[1]"'] \\
R \arrow[r, "\lambda"] & S \arrow[r, "\pi"] & Q^\bullet \arrow[r, "\nu"] & R[1]
\end{tikzcd}$$

Note that if the above diagram is considered in $\mathscr{D}(R)$, then $(r)\overline{\mu}$ is the unique morphism in S' such that the above diagram is commutative. In fact, if there exists another $g \in S'$ such that this diagram commutes, then $g\nu = \nu(\cdot r)[1] = (r)\overline{\mu}\nu$. This implies that $g - (r)\overline{\mu} = g'\pi$ for some $g' \in \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, S)$. Since $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, S) = 0$ by (1), we have $g' = 0$ and $g = (r)\overline{\mu}$. Thus, in $\mathscr{D}(R)$, the map $(r)\overline{\mu}$ is uniquely determined by the triangle $(**)$ and by the maps $\cdot r$ and $\cdot(r)\lambda$.

Next, we calculate the kernel and cokernel of $\overline{\mu}$. In fact, since $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, R) = 0$, we can easily form the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(S,R) \longrightarrow \mathrm{Hom}_R(R,R) \xrightarrow{\xi} \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet[-1], R) \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(S[-1], R) \longrightarrow 0$$

where $\xi := \mathrm{Hom}_{\mathscr{D}(R)}\big(\nu[-1], R\big)$. Moreover, since $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, S) = 0 = \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, S[1])$ by (1), we know that

$$\psi := \mathrm{Hom}_{\mathscr{D}(R)}\big(Q^\bullet[-1], \nu[-1]\big) : \mathrm{Hom}_{\mathscr{D}(R)}\big(Q^\bullet[-1], Q^\bullet[-1]\big) \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}\big(Q^\bullet[-1], R\big)$$

is an isomorphism. Certainly, the shift functor $[1]$ induces an isomorphism of rings from $\mathrm{End}_{\mathscr{D}(R)}\big(Q^\bullet[-1]\big)$ to S'. Consider the composite $\xi\psi^{-1}[1] : \mathrm{End}_R(R) \to S'$. One can check directly that this map coincides with $\overline{\mu}$ if R is identified with $\mathrm{End}_R(R)$ by the right multiplication. Thus $\mathrm{Ker}(\overline{\mu}) \simeq \mathrm{Ker}(\xi)$ and $\mathrm{Coker}(\overline{\mu}) \simeq \mathrm{Coker}(\xi)$ since ψ^{-1} is bijective, and therefore $\mathrm{Ker}(\overline{\mu}) \simeq \mathrm{Hom}_R(S,R)$ and $\mathrm{Coker}(\overline{\mu}) \simeq \mathrm{Ext}^1_R(S,R)$ as R-R-bimodules. For the last statement of Lemma 4.4(2), we observe that if $\mathrm{Ext}^1_R(S,R) = 0$, then $\overline{\mu}$ is surjective with $\mathrm{Ker}(\overline{\mu}) = I$.

(3) We first prove that $S \otimes_R \mathrm{Coker}(\lambda) = 0 = \mathrm{Tor}^R_1(S, \mathrm{Coker}(\lambda))$. Indeed, by applying $S \otimes_R -$ to the exact sequence

$$0 \longrightarrow \mathrm{Ker}(\lambda) \xrightarrow{\omega} R \xrightarrow{\lambda} S \longrightarrow \mathrm{Coker}(\lambda) \longrightarrow 0,$$

we get the following two relevant exact sequences:

$$S \otimes_R \mathrm{Ker}(\lambda) \xrightarrow{S \otimes \omega} S \otimes_R R \longrightarrow S \otimes_R \mathrm{Im}(\lambda) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Tor}^R_1(S, \mathrm{Coker}(\lambda)) \longrightarrow S \otimes_R \mathrm{Im}(\lambda) \longrightarrow S \otimes_R S \longrightarrow S \otimes_R \mathrm{Coker}(\lambda) \longrightarrow 0.$$

Since λ is a ring epimorphism, the map $S \otimes_R \lambda : S \otimes_R R \longrightarrow S \otimes_R S$ is an isomorphism. Consequently, we get $S \otimes_R R \simeq S \otimes_R \mathrm{Im}(\lambda) \simeq S \otimes_R S$. This means that $S \otimes \omega = 0$ and $S \otimes_R \mathrm{Coker}(\lambda) = 0 = \mathrm{Tor}^R_1(S, \mathrm{Coker}(\lambda))$.

Next, we show that $\mathrm{Hom}_{\mathscr{D}(R)}(\lambda, Q^\bullet) : \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(R, Q^\bullet)$ is surjective. In fact, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet) \arrow[r, "\lambda_*"] & \mathrm{Hom}_{\mathscr{D}(R)}(R, Q^\bullet) \\
\mathrm{Hom}_{\mathscr{K}(R)}(S, Q^\bullet) \arrow[r, "\lambda_*"] \arrow[u] & \mathrm{Hom}_{\mathscr{K}(R)}(R, Q^\bullet) \arrow[u, "\simeq"']
\end{tikzcd}$$

where the vertical maps are the canonical localization maps from the homotopy category to its derived category. One can check that the λ_* in the bottom row is surjective. This implies that the map on the top is surjective, as desired. We should remark that $\mathrm{Hom}_{\mathscr{D}(R)}(\lambda, Q^\bullet)$ is surjective for any ring homomorphism λ because our proof of this fact does not relay on any additional conditions on λ.

Now, by applying $\mathrm{Hom}_{\mathscr{D}(R)}(-, Q^\bullet)$ to the triangle $(**)$, we can construct the following long exact sequence of R-modules:

$$\mathrm{Hom}_{\mathscr{D}(R)}(R[1], Q^\bullet) \xrightarrow{\nu_*} \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, Q^\bullet) \xrightarrow{\pi_*} \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(R, Q^\bullet) \longrightarrow 0.$$

Write $L := \mathrm{Im}(\pi_*)$. Since $\mathrm{Hom}_{\mathscr{D}(R)}(R, Q^\bullet) \simeq H^0(Q^\bullet) \simeq \mathrm{Coker}(\lambda)$, it follows from $S \otimes_R \mathrm{Coker}(\lambda) = 0 = \mathrm{Tor}_1^R(S, \mathrm{Coker}(\lambda))$ that $S \otimes_R L \simeq S \otimes_R \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet)$. Clearly, the sequence

$$S \otimes_R \mathrm{Hom}_{\mathscr{D}(R)}(R[1], Q^\bullet) \xrightarrow{S \otimes (\nu_*)} S \otimes_R \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, Q^\bullet) \longrightarrow S \otimes_R L \longrightarrow 0$$

is exact. Note that $\mathrm{Hom}_{\mathscr{D}(R)}(R[1], Q^\bullet) \simeq H^{-1}(Q^\bullet) \simeq \mathrm{Ker}(\lambda)$. Under these identifications, one can check step by step that $\nu_* : \mathrm{Ker}(\lambda) \to S'$ is just the composite of $\omega : \mathrm{Ker}(\lambda) \to R$ and $\overline{\mu} : R \to S'$. Since $S \otimes \omega = 0$, we infer that $S \otimes (\nu_*) = 0$, and therefore $S \otimes_R S' \simeq S \otimes_R L \simeq S \otimes_R \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet)$ as S-S'-bimodules. Since λ is a ring epimorphism, we know that $S \otimes_R \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet)$ as S-S'-bimodules. It follows that $S \otimes_R S' \simeq \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet)$ as S-S'-bimodules. Clearly, under this isomorphism, one can verify that the element $1 \otimes 1$ in $S \otimes_R S'$ is sent to π. This finishes the proof of the first part of (3).

By the first part of (3) and the fact that $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, S) = 0$, we obtain the second part of (3).

(4) This was proved in [5, Lemma 6.5(5)] under the identification of $Q^\bullet$ with S/R. $\square$

Let us remark that if λ is a ring epimorphism such that $\mathrm{Hom}_R(S, \lambda) : \mathrm{Hom}_R(S, R) \to \mathrm{Hom}_R(S, S)$ is injective, or equivalently, $\mathrm{Hom}_R(S, \mathrm{Ker}(\lambda)) = 0$, then $\mathrm{Hom}_R(\lambda, R) : \mathrm{Hom}_R(S, R) \to \mathrm{Hom}_R(R, R)$ is also injective (see the proof of Lemma 4.3). Clearly, if λ is injective, then so is $\mathrm{Hom}_R(S, \lambda)$.

As a consequence of Lemma 4.4, we have the following conclusion which will be used in the proof of Corollary 1.3.

Corollary 4.5. *If $\lambda : R \to S$ is a ring epimorphism such that $\mathrm{Tor}_1^R(S, S) = 0 = \mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)\big)$, then the pair $(\lambda, \overline{\mu})$ is exact, where $\overline{\mu}$ is the map associated to λ.*

Proof. On the one hand, since $\mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)\big) = 0$, the map $\mathrm{Hom}_R(S, \lambda)$ is injective, and therefore $\mathrm{Hom}_{\mathscr{D}(R)}(S[1], Q^\bullet) = 0$ by applying $\mathrm{Hom}_{\mathscr{D}(R)}(S[1], -)$ to the triangle $(**)$ and by observing the fact that $\mathrm{Hom}_{\mathscr{D}(R)}(S[1], S) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(S, S[-1]) \simeq \mathrm{Ext}_R^{-1}(S, S) = 0$. On the other hand, we have a surjective map $\mathrm{Hom}_{\mathscr{D}(R)}(\lambda, Q^\bullet) : \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(R, Q^\bullet)$ (see the proof of Lemma 4.4(3)). Thus, by combining the diagram $(*)$ at the beginning of Section 3 with Lemma 4.4(3), we can construct the following commutative diagram in $\mathscr{C}(R)$ with two exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(\lambda) \arrow[r] \arrow[dd, "\simeq"'] & R \arrow[r, "\lambda"] \arrow[d, "\overline{\mu}"'] & S \arrow[r] \arrow[d, "\overline{\mu}'"'] & \mathrm{Coker}(\lambda) \arrow[r] \arrow[dd, "\simeq"'] & 0 \\
 & & S' \arrow[r, "\lambda'"] \arrow[d, equal] & S \otimes_R S' \arrow[d, "\simeq"'] & & \\
0 \arrow[r] & \mathrm{Hom}_{\mathscr{D}(R)}(R[1], Q^\bullet) \arrow[r, "\nu_*"] & \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, Q^\bullet) \arrow[r, "\pi_*"] & \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet) \arrow[r, "\lambda_*"] & \mathrm{Hom}_{\mathscr{D}(R)}(R, Q^\bullet) \arrow[r] & 0
\end{tikzcd}$$

By calculating cohomology groups from this diagram, we see that $\overline{\mu}^\bullet : Q^\bullet \longrightarrow Q^\bullet \otimes_R S'$ is a quasi-isomorphism in $\mathscr{C}(R)$. According to the equivalent conditions mentioned at the beginning of Section 3, the pair $(\lambda, \overline{\mu})$ is exact. $\square$

Proof of Corollary 1.3.

By Lemma 4.4(3), there are isomorphisms of rings

$$\Lambda := \mathrm{End}_{\mathscr{D}(R)}\big(S\oplus Q^\bullet\big) \simeq \begin{pmatrix} S & \mathrm{Hom}_{\mathscr{D}(R)}(S,Q^\bullet) \\ 0 & S' \end{pmatrix} \simeq B := \begin{pmatrix} S & S\otimes_R S' \\ 0 & S' \end{pmatrix}$$

where the second isomorphism sends $\begin{pmatrix} 0 & \pi \\ 0 & 0 \end{pmatrix}$ to $\begin{pmatrix} 0 & 1\otimes 1 \\ 0 & 0 \end{pmatrix}$. Set $e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in B$. Let $\varphi : Be_1 \to Be_2$ be the map sending $\begin{pmatrix} s \\ 0 \end{pmatrix}$ to $\begin{pmatrix} s\otimes 1 \\ 0 \end{pmatrix}$ for $s\in S$. Then π^* corresponds to φ under the isomorphism $\Lambda\simeq B$, and therefore $\lambda_{\pi^*} : \Lambda \to \Lambda_{\pi^*}$ is equivalent to the universal localization $B\to B_\varphi$ of B at φ. Note that the latter map coincides with $\theta : B \to C := M_2(S\sqcup_R S')$ given in Theorem 1.1 (see also Lemma 3.1). This means that λ_{π^*} is homological if and only if θ is homological.

By Corollary 4.5, the pair $(\lambda,\overline{\mu})$ is exact. Since λ is homological, Corollary 1.3 follows immediately from Theorem 1.1. $\square$

Combining Corollary 1.3 with Lemma 2.6, we get the following criterion for λ_{π^*} to be homological.

Corollary 4.6. *Let Σ be a set of homomorphisms between finitely generated projective R-modules. Suppose that $\lambda_\Sigma : R\to R_\Sigma$ is homological such that $\mathrm{Hom}_R\big(R_\Sigma, \mathrm{Ker}(\lambda_\Sigma)\big) = 0$. Set $S := R_\Sigma$, $\lambda := \lambda_\Sigma$ and $\Phi := \{S'\otimes_R f \mid f\in\Sigma\}$. Then the universal localization $\lambda_{\pi^*} : \Lambda\to\Lambda_{\pi^*}$ of Λ at π^* is homological if and only if the universal localization $\lambda_\Phi : S'\to S'_\Phi$ of S' at Φ is homological. In particular, if one of the above equivalent conditions holds , then there is a recollement of derived module categories:*

$$\begin{tikzcd} \mathscr{D}(S'_\Phi) \arrow[r] & \mathscr{D}(\Lambda) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

As a consequence of Corollary 4.6, we obtain the following result which can be used to adjudge whether a universal localizations of the form $\lambda_{\pi^*} : \Lambda\to\Lambda_{\pi^*}$ is homological or not.

Corollary 4.7. *Let $C\subseteq D$ be an arbitrary extension of rings, that is, C is a subring of the ring D with the same identity. Let $\omega : D\to D/C$ be the canonical surjection of C-modules. Set $R := \begin{pmatrix} D & D \\ 0 & C \end{pmatrix}$ and $S := M_2(D)$. Let $\lambda : R\to S$ be the canonical inclusion, and let $\pi : S\to S/R$ be the canonical surjective homomorphism of R-modules. Then the universal localization $\lambda_{\pi^*} : \Lambda\to\Lambda_{\pi^*}$ of Λ at π^* is homological if and only if the universal localization $\lambda_{\omega^*} : E\to E_{\omega^*}$ of E at ω^* is homological, where $E := \mathrm{End}_C(D\oplus D/C)$, and $\omega^* : \mathrm{Hom}_C(D\oplus D/C, D)\to \mathrm{Hom}_C(D\oplus D/C, D/C)$ is the homomorphism of E-modules induced by ω.*

Proof. Since $Q^\bullet$ can be identified with S/R in $\mathscr{D}(R)$, we have $S' = \mathrm{End}_R(S/R)$. Thus the map $\overline{\mu} : R\to S'$ is given by the right multiplication. Set $e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $e_{12} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$. Furthermore, let $\varphi : Re_1\to Re_2$ and $\varphi' : S'(e_1)\overline{\mu}\to S'(e_2)\overline{\mu}$ be the right multiplication maps of e_{12} and $(e_{12})\overline{\mu}$, respectively.

It follows from Lemma 3.1 and $D\sqcup_C C = D$ that $\lambda : R\to S$ is the universal localization of R at φ. In particular, λ is a ring epimorphism. Since $S\simeq e_1R\oplus e_1R$ as right R-modules, the embedding λ is even homological. Note that $S'\otimes_R\varphi$ can be identified with φ'. By Corollary 4.6, the map $\lambda_{\pi^*} : \Lambda\to\Lambda_{\pi^*}$ is homological if and only if the map $\lambda_{\varphi'} : S'\to S'_{\varphi'}$ is homological.

Clearly, $R/Re_1R\simeq C$ as rings. So, every C-module can be regarded as an R-module. In particular, $D\oplus D/C$ can be seen as an R-module. Further, one can check that the map

$$\alpha :\ D\oplus D/C \to S/R, \quad (d, t+C)\mapsto \begin{pmatrix} 0 & 0 \\ d & t \end{pmatrix} + R$$

for $d,t \in D$, is an isomorphism of R-modules. Thus $S' \simeq E$, φ' corresponds to ω^* under this isomorphism, and $S'_{\varphi'} \simeq E_{\omega^*}$. It follows that $\lambda_{\varphi'} : S' \to S'_{\varphi'}$ is homological if and only if so is $\lambda_{\omega^*} : E \to E_{\omega^*}$. This finishes the proof. $\square$

Before starting with the proof of Corollary 1.4, we first introduce a couple of more definitions and notation.

Recall that a complex $U^\bullet$ in $\mathscr{D}(R)$ is called a *tilting* complex if $U^\bullet$ is self-orthogonal, isomorphic in $\mathscr{D}(R)$ to a bounded complex of finitely generated projective R-modules, and $\mathrm{Tria}(U^\bullet) = \mathscr{D}(R)$. It is well known that if $U^\bullet$ is a tilting complex over R, then $\mathscr{D}(R)$ is equivalent to $\mathscr{D}(\mathrm{End}_{\mathscr{D}(R)}(U^\bullet))$ as triangulated categories (see [15, Theorem 6.4]). In this case, R and $\mathrm{End}_{\mathscr{D}(R)}(U^\bullet)$ are called *derived equivalent*.

If I is an index set, we denote by $U^{\bullet\,(I)}$ the direct sum of I copies of $U^\bullet$ in $\mathscr{D}(R)$, and by $\mathrm{Add}(U^\bullet)$ the full subcategory of $\mathscr{D}(R)$ consisting of all direct summands of arbitrary direct sums of copies of $U^\bullet$.

The following result generalizes some known results in the literature. See, for example, [9, Theorem 4.14], [2, Theorem 3.5(5)] and [19, Lemma 3.1(3)], where the ring homomorphism $\lambda : R \to S$ is required to be injective. We shall use this generalization to prove Corollary 1.4.

Lemma 4.8. *Let $\lambda : R \to S$ be a ring homomorphism, and let I be an arbitrary nonempty set. Define $U^\bullet := S \oplus Q^\bullet$. Then $\mathrm{Hom}_{\mathscr{D}(R)}(U^\bullet, U^{\bullet\,(I)}[n]) = 0$ for any $0 \neq n \in \mathbb{Z}$ if and only if the following conditions hold:*

(1) $\mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)\big) = 0$ *and*

(2) $\mathrm{Ext}^i_R(S, S^{(I)}) = 0 = \mathrm{Ext}^{i+1}_R(S, R^{(I)})$ *for any $i \geq 1$.*

In particular, the complex $U^\bullet$ is a tilting complex in $\mathscr{D}(R)$ if and only if $\mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)\big) = 0$, $\mathrm{Ext}^1_R(S,S) = 0$ and there is an exact sequence: $0 \to P_1 \to P_0 \to {}_RS \to 0$ of R-modules, such that P_i is finitely generated and projective for $i = 0,1$.

Proof. Recall that we have a distinguished triangle

$$(**) \quad R \xrightarrow{\lambda} S \xrightarrow{\pi} Q^\bullet \xrightarrow{\nu} R[1]$$

in $\mathscr{K}(R)$.

First of all, we mention two general facts: Let I be an arbitrary nonempty set.

(a) By applying $\mathrm{Hom}_{\mathscr{D}(R)}(-, S^{(I)})$ to $(**)$, one can prove that

$$\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, S^{(I)}[i]) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(S, S^{(I)}[i]) \text{ for } i \in \mathbb{Z} \setminus \{0\} \text{ and } \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, S^{(I)}) \simeq \mathrm{Ker}\big(\mathrm{Hom}_R(\lambda, S^{(I)})\big).$$

(b) By applying $\mathrm{Hom}_{\mathscr{D}(R)}(-, R^{(I)})$ to $(**)$, one can show that

$$\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, R^{(I)}[j]) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(S, R^{(I)}[j]) \text{ for } j \in \mathbb{Z} \setminus \{0,1\}.$$

Next, we show the necessity of the first part of Lemma 4.8.

Suppose that $\mathrm{Hom}_{\mathscr{D}(R)}(U^\bullet, U^{\bullet\,(I)}[n]) = 0$ for any $n \neq 0$. Then $\mathrm{Ext}^i_R(S, S^{(I)}) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, S^{(I)}[i]) = 0$ for any $i \geq 1$, and $\mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet[-1]) = 0$. Consequently, the map $\mathrm{Hom}_R(S, \lambda) : \mathrm{Hom}_R(S,R) \to \mathrm{Hom}_R(S,S)$ is injective. This means that the condition (1) holds. Further, applying $\mathrm{Hom}_{\mathscr{D}(R)}(S, -)$ to the triangle $R^{(I)} \xrightarrow{\lambda^{(I)}} S^{(I)} \xrightarrow{\pi^{(I)}} Q^{\bullet (I)} \longrightarrow R^{(I)}[1]$, we get $\mathrm{Ext}^{i+1}_R(S, R^{(I)}) \simeq \mathrm{Hom}_{\mathscr{D}(R)}\big(S, Q^{\bullet (I)}[i]\big) = 0$. Thus, the conditions (1) and (2) in Lemma 4.8 are satisfied.

In the following, we shall show the sufficiency of the first part of Lemma 4.8.

Assume that the conditions (1) and (2) in Lemma 4.8 hold true. Then, it follows from (a) and (b) that

$$\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, S^{(I)}[n]) = 0 = \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, R^{(I)}[m+1])$$

for $n\in\mathbb{Z}\setminus\{0\}$ and $m\in\mathbb{Z}\setminus\{-1,0\}$. Applying $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet-)$ to the triangle $R^{(I)}\xrightarrow{\lambda^{(I)}}S^{(I)}\xrightarrow{\pi^{(I)}}Q^{\bullet(I)}\longrightarrow R^{(I)}[1]$, one can show that $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,Q^{\bullet\,(I)}[m])=0$ for $m\in\mathbb{Z}\setminus\{-1,0\}$. Furthermore, we shall show that the condition (1) in Lemma 4.8 implies also that $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,Q^{\bullet\,(I)}[-1])=0$: Clearly, $\mathrm{Hom}_R\big(S,\mathrm{Ker}(\lambda)^I\big)\simeq\mathrm{Hom}_R\big(S,\mathrm{Ker}(\lambda)\big)^I=0$, where $\mathrm{Ker}(\lambda)^I$ stands for the direct product of I copies of $\mathrm{Ker}(\lambda)$. Since $\mathrm{Ker}(\lambda)^I$ contains $\mathrm{Ker}(\lambda)^{(I)}$ as a submodule, we get $\mathrm{Hom}_R\big(S,\mathrm{Ker}(\lambda)^{(I)}\big)=0$ and $\mathrm{Ker}\big(\mathrm{Hom}_R(S,\lambda^{(I)})\big)\simeq\mathrm{Hom}_R\big(S,\mathrm{Ker}(\lambda)^{(}I)\big)$ $=0$. Now, it follows from the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,Q^{\bullet\,(I)}[-1]) \arrow[r, "(\nu[-1])^*"] \arrow[d, hook, dashed] & \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,R^{(I)}) \arrow[r, "(\lambda^{(I)})^*"] \arrow[d, hook, "\pi_*"] & \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,S^{(I)}) \arrow[d, hook, "\pi_*"] \\
0 \arrow[r] & \mathrm{Ker}\big(\mathrm{Hom}_R(S,\lambda^{(I)})\big) \arrow[r] & \mathrm{Hom}_R(S,R^{(I)}) \arrow[r, "(\lambda^{(I)})^*"] & \mathrm{Hom}_R(S,S^{(I)})
\end{tikzcd}$$

that $\mathrm{Ker}\big(\mathrm{Hom}_R(S,\lambda^{(I)})\big)\simeq\mathrm{Hom}_R\big(S,\mathrm{Ker}(\lambda)^{(I)}\big)=0$, and therefore $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,Q^{\bullet\,(I)}[-1])=0$. Thus,

$$\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet,Q^{\bullet\,(I)}[n])=0\ \text{ for }\ n\neq 0.$$

It remains to prove $\mathrm{Hom}_{\mathscr{D}(R)}(S,Q^{\bullet\,(I)}[n])=0$ for $n\neq 0$. Actually, applying $\mathrm{Hom}_{\mathscr{D}(R)}(S,-)$ to the triangle $R^{(I)}\xrightarrow{\lambda^{(I)}}S^{(I)}\xrightarrow{\pi^{(I)}}Q^{\bullet(I)}\longrightarrow R^{(I)}[1]$, we have the following long exact sequence:

$$\cdots\to\mathrm{Hom}_{\mathscr{D}(R)}(S,S^{(I)}[j])\longrightarrow\mathrm{Hom}_{\mathscr{D}(R)}(S,Q^{\bullet(I)}[j])\longrightarrow\mathrm{Hom}_{\mathscr{D}(R)}(S,R^{(I)}[j+1])\xrightarrow{(\lambda^{(I)})^*}\mathrm{Hom}_{\mathscr{D}(R)}(S,S^{(I)}[j+1])\to\cdots$$

for $j\in\mathbb{Z}$. Since $\mathrm{Hom}_{\mathscr{D}(R)}(S,S^{(I)}[r])=0$ for $0\neq r\in\mathbb{Z}$ and $\mathrm{Hom}_{\mathscr{D}(R)}(S,R^{(I)}[t])=0$ for $t\in\mathbb{Z}\setminus\{0,1\}$, we see that $\mathrm{Hom}_{\mathscr{D}(R)}(S,Q^{\bullet(I)}[j])=0$ for $j\in\mathbb{Z}\setminus\{-1,0\}$ and that $\mathrm{Hom}_{\mathscr{D}(R)}(S,Q^{\bullet(I)}[-1])\simeq\mathrm{Ker}\big(\mathrm{Hom}_R(S,\lambda^{(I)})\big)=0$. It follows that $\mathrm{Hom}_{\mathscr{D}(R)}(S,Q^{\bullet\,(I)}[n])=0$ for $n\neq 0$. Hence $\mathrm{Hom}_{\mathscr{D}(R)}(U^\bullet,U^{\bullet\,(I)}[n])=0$ for any $n\neq 0$. This finishes the proof of the sufficiency.

As to the second part of Lemma 4.8, we observe the following: The complex $U^\bullet$ over R is a generator of $\mathscr{D}(R)$, that is, $\mathrm{Tria}(U^\bullet)=\mathscr{D}(R)$, since $R\in\mathrm{Tria}(U^\bullet)$ by the triangle $(**)$. Moreover, the complex $U^\bullet$ is a tilting complex in $\mathscr{D}(R)$ if and only if it is self-orthogonal, and ${}_RS$ has a projective resolution of finite length consisting of finitely generated projective R-modules. Furthermore, if ${}_RS$ has finite projective dimension and $\mathrm{Ext}_R^{i+1}(S,R^{(I)})=0$ for any $i\geq 1$, then ${}_RS$ does have projective dimension at most 1. Now, combining these observations with the first part of Lemma 4.8, we can show the second part of Lemma 4.8. $\square$

Proof of Corollary 1.4.

(1) Here, we follow the notation introduced in Section 3. Let

$$T:=S',\quad \mu:=\overline{\mu},\quad B:=\begin{pmatrix} S & S\otimes_R S' \\ 0 & S'\end{pmatrix}.$$

Since λ is homological and $\mathrm{Hom}_R\big(S,\mathrm{Ker}(\lambda)\big)=0$ by assumption, the pair $(\lambda,\overline{\mu})$ is exact by Corollary 4.5.

Now, we assume that ${}_RS$ has projective dimension at most 1. Let

$$0\longrightarrow P^{-1}\xrightarrow{\delta}P^0\longrightarrow {}_RS\longrightarrow 0$$

be a projective resolution of ${}_RS$ with all P^j projective R-modules. This exact sequence gives rise to a triangle $P^{-1}\to P^0\to S\to P^{-1}[1]$ in $\mathscr{D}(R)$.

By Theorem 1.1, the map $\lambda_{\pi^*}:\Lambda\to\Lambda_{\pi^*}$ is homological if and only if $\mathrm{Tor}_j^R(S',S)=0$ for all $j\geq 1$. Let us check the latter condition.

First of all, by the assumption on ${}_RS$, we have $\mathrm{Tor}_i^R(S',S)=0$ for all $i\geq 2$. So, it remains to show $\mathrm{Tor}_1^R(S',S)=0$. Further, since $Be_2=S'\oplus S\otimes_R S'$ as right R-modules, it is enough to show $\mathrm{Tor}_1^R(Be_2,S)=0$.

From the proof of Lemma 3.6, we obtain a triple $(j_!,\, j^!,\, j_*)$ of adjoint triangle functors. Let $\eta: Id_{\mathscr{D}(B)}\to j_*j^!$ be the unit adjunction with respect to the adjoint pair $(j^!,\, j_*)$. Then we have the following fact:

For any $X^\bullet\in\mathscr{D}(B)$, there exists a canonical triangle in $\mathscr{D}(B)$:

$$i_*i^!(X^\bullet)\longrightarrow X^\bullet\xrightarrow{\eta_{X^\bullet}} j_*j^!(X^\bullet)\longrightarrow i_*i^!(X^\bullet)[1],$$

where $j_*j^!(X^\bullet)=\mathbb{R}\mathrm{Hom}_R(P^{\bullet *},\mathrm{Hom}_B^\bullet(P^\bullet,X^\bullet))$. For the other triple $(i^*,\, i_*,\, i^!)$ of adjoint triangle functors, we refer the reader to the diagram $(\star)$ in Section 3.

By applying this fact to each term of the triangle

$$P^{-1}\to P^0\to S\to P^{-1}[1]$$

in $\mathscr{D}(R)$, it follows from the recollement $(\star)$ (see Section 3) that there is the following exact commutative diagram:

$$\begin{tikzcd}
i_*i^!(Be_2\otimes_R P^{-1}) \arrow[r]\arrow[d] & Be_2\otimes_R P^{-1} \arrow[r, "\eta_{Be_2\otimes_R P^{-1}}"]\arrow[d, "1\otimes\delta"'] & j_*j^!(Be_2\otimes_R P^{-1}) \arrow[r]\arrow[d, "j_*j^!(1\otimes\delta)"'] & i_*i^!(Be_2\otimes_R P^{-1})[1]\arrow[d] \\
i_*i^!(Be_2\otimes_R P^{0}) \arrow[r]\arrow[d] & Be_2\otimes_R P^{0} \arrow[r, "\eta_{Be_2\otimes_R P^{0}}"]\arrow[d] & j_*j^!(Be_2\otimes_R P^{0}) \arrow[r]\arrow[d] & i_*i^!(Be_2\otimes_R P^{0})[1]\arrow[d] \\
i_*i^!(Be_2\otimes_R^{\mathbb{L}} S) \arrow[r]\arrow[d] & Be_2\otimes_R^{\mathbb{L}} S \arrow[r, "\eta_{Be_2\otimes_R^{\mathbb{L}} S}"]\arrow[d] & j_*j^!(Be_2\otimes_R^{\mathbb{L}} S) \arrow[r]\arrow[d] & i_*i^!(Be_2\otimes_R^{\mathbb{L}} S)[1]\arrow[d] \\
i_*i^!(Be_2\otimes_R P^{-1})[1] \arrow[r] & Be_2\otimes_R P^{-1}[1] \arrow[r] & j_*j^!(Be_2\otimes_R P^{-1})[1] \arrow[r] & i_*i^!(Be_2\otimes_R P^{-1})[2]
\end{tikzcd}$$

Since $i_*i^*(Be_1)\simeq Be_2\otimes_R^{\mathbb{L}} S$ in $\mathscr{D}(B)$ by Lemma 3.8(1), we know that $j_*j^!(Be_2\otimes_R^{\mathbb{L}} S)\simeq j_*j^!i_*i^*(Be_1)=0$, due to $j^!i_*=0$ in the recollement $(\star)$. It follows that $j_*j^!(1\otimes\delta)$ is an isomorphism, and so is $H^0(j_*j^!(1\otimes\delta))$.

Suppose that $H^0(\eta_P): P\to H^0\big(j_*j^!(P)\big)$ is injective for any projective B-module P. Then $H^0(\eta_{Be_2\otimes_R P^{-1}})$ is injective since ${}_RP^{-1}$ is projective. It follows from the isomorphism $H^0(j_*j^!(1\otimes\delta))$ that the map $1\otimes\delta: Be_2\otimes_R P^{-1}\to Be_2\otimes_R P^0$ is injective. This implies that $\mathrm{Tor}_1^R(Be_2,S)=0$, as desired.

Thus, in the following, we shall prove that $H^0(\eta_P): P\to H^0\big(j_*j^!(P)\big)$ is injective for any projective B-module P.

First, we point out that $H^0(\eta_P)$ is injective if and only if $\mathrm{Hom}_{\mathscr{D}(B)}(B,P)\xrightarrow{j^!}\mathrm{Hom}_{\mathscr{D}(R)}\big(j^!(B),j^!(P)\big)$ is injective. To see this, we consider the the following composite of maps:

$$\omega_{X^\bullet}^n:\mathrm{Hom}_{\mathscr{D}(B)}(B,X^\bullet[n])\xrightarrow{j^!}\mathrm{Hom}_{\mathscr{D}(R)}\big(j^!(B),j^!(X^\bullet)[n]\big)\xrightarrow{\simeq}\mathrm{Hom}_{\mathscr{D}(B)}(B,j_*j^!(X^\bullet)[n])$$

for each $n\in\mathbb{Z}$, where the second map is an isomorphism induced by the adjoint pair $(j^!,j_*)$. Then, one can check directly that $\omega_{X^\bullet}^n=\mathrm{Hom}_{\mathscr{D}(B)}(B,\eta_{X^\bullet[n]})$. It is known that the n-th cohomology functor $H^n(-):\mathscr{D}(B)\to B\text{-Mod}$ is naturally isomorphic to the Hom-functor $\mathrm{Hom}_{\mathscr{D}(B)}(B,-[n])$. So, under this identification, the map $\omega_{X^\bullet}^n$ coincides with $H^n(\eta_{X^\bullet}):H^n(X^\bullet)\to H^n(j_*j^!(X^\bullet))$. It follows that $H^0(\eta_P)$ is injective if and only if so is the map $\mathrm{Hom}_{\mathscr{D}(B)}(B,P)\xrightarrow{j^!}\mathrm{Hom}_{\mathscr{D}(R)}\big(j^!(B),j^!(P)\big)$.

Second, we claim that if $\mathrm{Hom}_{\mathscr{D}(B)}(i_*i^*(B),P)=0$, then $\mathrm{Hom}_{\mathscr{D}(B)}(B,P)\xrightarrow{j^!}\mathrm{Hom}_{\mathscr{D}(R)}\big(j^!(B),j^!(P)\big)$ is injective.

Let $\varepsilon : j_! j^! \to Id_{\mathscr{D}(B)}$ be the counit adjunction with respect to the adjoint pair $(j_!, j^!)$. Then, for each $X^\bullet \in \mathscr{D}(B)$, there exists a canonical triangle in $\mathscr{D}(B)$:

$$j_! j^!(X^\bullet) \xrightarrow{\varepsilon_{X^\bullet}} X^\bullet \longrightarrow i_* i^*(X^\bullet) \longrightarrow j_! j^!(X^\bullet)[1].$$

Now, we consider the following morphisms:

$$\mathrm{Hom}_{\mathscr{D}(B)}(B, X^\bullet[m]) \xrightarrow{j^!} \mathrm{Hom}_{\mathscr{D}(R)}\big(j^!(B), j^!(X^\bullet)[m]\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{D}(B)}\big(j_! j^!(B), X^\bullet[m]\big)$$

for any $m \in \mathbb{Z}$, where the last map is an isomorphism given by the adjoint pair $(j_!, j^!)$. One can check that the composite of the above two morphisms is the map $\mathrm{Hom}_{\mathscr{D}(B)}(\varepsilon_B, X^\bullet[m])$. This means that, to show that $\mathrm{Hom}_{\mathscr{D}(B)}(B,P) \xrightarrow{j^!} \mathrm{Hom}_{\mathscr{D}(R)}\big(j^!(B), j^!(P)\big)$ is injective, it suffices to show that $\mathrm{Hom}_{\mathscr{D}(B)}(\varepsilon_B, P)$ is injective. For this aim, we apply $\mathrm{Hom}_{\mathscr{D}(B)}(-,P)$ to the triangle $j_! j^!(B) \xrightarrow{\varepsilon_B} B \longrightarrow i_* i^*(B) \longrightarrow j_! j^!(B)[1]$, and get the following exact sequence of abelian groups:

$$\mathrm{Hom}_{\mathscr{D}(B)}(i_* i^*(B), P) \longrightarrow \mathrm{Hom}_{\mathscr{D}(B)}(B,P) \xrightarrow{\mathrm{Hom}_{\mathscr{D}(B)}(\varepsilon_B,P)} \mathrm{Hom}_{\mathscr{D}(B)}\big(j_! j^!(B), P\big)\ .$$

Clearly, if $\mathrm{Hom}_{\mathscr{D}(B)}(i_* i^*(B), P) = 0$, then $\mathrm{Hom}_{\mathscr{D}(B)}(\varepsilon_B, P)$ is injective, and therefore the map $j^! : \mathrm{Hom}_{\mathscr{D}(B)}(B,P) \xrightarrow{j^!} \mathrm{Hom}_{\mathscr{D}(R)}\big(j^!(B), j^!(P)\big)$ is injective, as desired.

Third, we show that if $\mathrm{Hom}_R(S,S') = 0$, then $\mathrm{Hom}_{\mathscr{D}(B)}(i_* i^*(B), P) = 0$ for any projective B-module P.

In fact, by Lemmas 3.7(3) and 3.8(1), we have $i_* i^*(Be_2) \simeq i_* i^*(Be_1) \simeq Be_2 \otimes^{\mathbb{L}}_R S$ in $\mathscr{D}(B)$. This implies that $\mathrm{Hom}_{\mathscr{D}(B)}(i_* i^*(B), P) = 0$ if and only if $\mathrm{Hom}_{\mathscr{D}(B)}(Be_2 \otimes^{\mathbb{L}}_R S, P) = 0$. Consider the following isomorphisms

$$\mathrm{Hom}_{\mathscr{D}(B)}(Be_2 \otimes^{\mathbb{L}}_R S, P) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(S, \mathbb{R}\mathrm{Hom}_B(Be_2, P)) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(S, e_2P) \simeq \mathrm{Hom}_R(S, e_2P).$$

Since $e_2B \simeq S'$ as R-modules, we have $\mathrm{Hom}_R(S, e_2B) \simeq \mathrm{Hom}_R(S,S') = 0$. Note that $P \in \mathrm{Add}({}_BB)$ and $e_2P \in \mathrm{Add}({}_RS')$. Thus there is an index set I such that e_2P is a direct summand of $(S')^{(I)}$. Since $(S')^{(I)}$ is a submodule of the product $(S')^I$ of S', it follows that $\mathrm{Hom}_R(S,(S')^{(I)})$ is a subgroup of $\mathrm{Hom}_R(S,(S')^I)$ which is isomorphic to $\mathrm{Hom}_R(S,S')^I$. Hence $\mathrm{Hom}_R(S,(S')^{(I)}) = 0$, $\mathrm{Hom}_R(S, e_2P) = 0$ and $\mathrm{Hom}_{\mathscr{D}(B)}(i_* i^*(B), P) = 0$, as desired.

Now, it remains to show that $\mathrm{Hom}_R(S,S') = 0$. In the following, we shall prove a stronger statement, namely, $\mathrm{Hom}_{\mathscr{D}(R)}(S, S'[n]) = 0$ for any $n \in \mathbb{Z}$.

Since λ is a ring epimorphism with $\mathrm{Tor}^R_1(S,S) = 0$, we know from [16, Theorem 4.8] that

$$\mathrm{Ext}^1_R(S, S^{(I)}) \simeq \mathrm{Ext}^1_S(S, S^{(I)}) = 0$$

for any set I. As ${}_RS$ is of projective dimension at most 1, we can apply Lemma 4.8 to the complex $U^\bullet := S \oplus Q^\bullet$, and get $\mathrm{Hom}_{\mathscr{D}(R)}(U^\bullet, U^\bullet[m]) = 0$ for $m \neq 0$. This implies that $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, Q^\bullet[m]) = 0$ for $m \neq 0$, and that

$$H^m(\mathbb{R}\mathrm{Hom}_R(Q^\bullet, Q^\bullet)) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, Q^\bullet[m]) = \begin{cases} 0 & \text{if } m \neq 0, \\ S' & \text{if } \ m = 0. \end{cases}$$

Thus the complex $\mathbb{R}\mathrm{Hom}_R(Q^\bullet, Q^\bullet)$ is isomorphic in $\mathscr{D}(R)$ to the stalk complex S'. On the one hand, by the adjoint pair $\big(Q^\bullet \otimes^{\mathbb{L}}_R -, \mathbb{R}\mathrm{Hom}_R(Q^\bullet, -)\big)$ of the triangle functors, we have

$$\mathrm{Hom}_{\mathscr{D}(R)}(S, S'[n]) \simeq \mathrm{Hom}_{\mathscr{D}(R)}\big(S, \mathbb{R}\mathrm{Hom}_R(Q^\bullet, Q^\bullet)[n]\big) \simeq \mathrm{Hom}_{\mathscr{D}(R)}\big(S, \mathbb{R}\mathrm{Hom}_R(Q^\bullet, Q^\bullet[n])\big) \simeq \mathbb{R}\mathrm{Hom}_R(Q^\bullet \otimes^{\mathbb{L}}_R S, Q^\bullet[n])$$

for any $n \in \mathbb{Z}$. On the other hand, since λ is homological by assumption, the homomorphism $\lambda \otimes^{\mathbb{L}}_R S : R \otimes^{\mathbb{L}}_R S \longrightarrow S \otimes^{\mathbb{L}}_R S$ is an isomorphism in $\mathscr{D}(R)$. It follows from the triangle

$$R \otimes^{\mathbb{L}}_R S \xrightarrow{\lambda \otimes^{\mathbb{L}}_R S} S \otimes^{\mathbb{L}}_R S \longrightarrow Q^\bullet \otimes^{\mathbb{L}}_R S \longrightarrow R \otimes^{\mathbb{L}}_R S[1]$$

that $Q^\bullet \otimes^{\mathbb{L}}_R S = 0$. Hence $\mathrm{Hom}_{\mathscr{D}(R)}(S, S'[n]) \simeq \mathbb{R}\mathrm{Hom}_R(Q^\bullet \otimes^{\mathbb{L}}_R S, Q^\bullet[n]) = 0$ for any $n \in \mathbb{Z}$.

Thus, we have proved that, for any projective B-module P, the homomorphism $H^0(\eta_P) : P \to H^0(j_* j^!(P))$ is injective in B-Mod. This finishes the proof of Corollary 1.4(1).

(2) Combining Corollary 3.5 with Lemma 3.6, we see that the ring Λ_{π^*} is zero if and only if the functor $j^!$ induces a triangle equivalence from $\mathscr{D}(B)$ to $\mathscr{D}(R)$. This is equivalent to the statement that $j^!(B)$ is a tilting complex over R. Note that $j^!(B) \simeq U^\bullet[-1]$ by Lemma 3.7(2). Thus, the ring Λ_{π^*} is zero if and only if $U^\bullet$ is a tilting complex over R. Now, the second part of Corollary 1.4 follows directly from Lemma 4.8. $\square$.

Proof of Corollary 1.5.

Let us consider the pair $(\lambda, \overline{\mu})$ of ring homomorphisms λ and $\overline{\mu}$, where $\overline{\mu}$ is associated to λ. By Corollary 4.5, the pair $(\lambda, \overline{\mu})$ is exact. It follows from Lemma 4.4(4) that S' is a commutative ring since λ is an injective homological ring epimorphism and R is a commutative ring. This means that the tensor product $S \otimes_R S'$ of S and S' over R is a commutative ring. Moreover, the map $\lambda' : S' \to S \otimes_R S'$ and $\overline{\mu}' : S \to S \otimes_R S'$ are ring homomorphisms. So, $S \otimes_R S'$ is an S'-S'-bimodule via λ'.

By Lemma 3.9, we know that $\mathrm{Tor}^R_i(S, S') = 0$ for any $i > 0$. Since R, S and S' are commutative rings, it follows that $\mathrm{Tor}^R_i(S', S) \simeq \mathrm{Tor}^R_i(S, S') = 0$. Note that λ is a ring epimorphism, and so is $\lambda' : S' \to S \otimes_R S'$ by Lemmas 2.3 and 2.5(1). Thus $\mathrm{End}_{S'}(S' \otimes_R S) \simeq \mathrm{End}_{S'}(S \otimes_R S') \simeq \mathrm{End}_{S \otimes_R S'}(S \otimes_R S') \simeq S \otimes_R S'$ as rings. Now, Corollary 1.5 is an immediate consequence of Corollary 1.3. $\square$

Proof of Corollary 1.6.

For a commutative ring R and a multiplicative set Φ of R, the localization map $R \to S := \Phi^{-1}R$ is always homological since ${}_RS$ is flat. Therefore, by Corollary 1.5, it suffices to show that $S \otimes_R S'$ is isomorphic to $\Psi^{-1}S'$. In fact, one can check that the well defined map $\alpha : \Phi^{-1}R \otimes_R S' \longrightarrow \Psi^{-1}S'$, given by

$$\frac{r}{x} \otimes y \mapsto \frac{(r)\overline{\mu}\, y}{(x)\overline{\mu}}$$

for $r \in R$, $x \in \Phi$ and $y \in S'$, is an isomorphism of rings, where $\overline{\mu} : R \to S'$ is the ring homomorphism associated to λ. Clearly, this map is surjective. To see that this map is injective, we note that the map $\beta : \Psi^{-1}S' \longrightarrow \Phi^{-1}R \otimes_R S'$, defined by $\frac{y}{(x)\overline{\mu}} \mapsto \frac{1}{x} \otimes y$ for $x \in \Phi$ and $y \in S'$, is a well defined ring homomorphism with $\alpha\beta = 1$. Observe that α preserves the multiplication of $S \otimes_R S'$. This finishes the proof of Corollary 1.6. $\square$

Finally, we mention a relationship between the results in Section 3 and the ones in Section 4.

Recall that we have defined the ring homomorphism $\overline{\mu} : R \to S'$ associated to a ring homomorphism λ at the beginning of Section 4.2. There is a connection between this homomorphism $\overline{\mu}$ and the $\mu : R \to T$ in an arbitrary exact pair (λ, μ) of ring homomorphisms. This connection is revealed by the following result which not only establishes a relationship between the results in Section 3 and those in Section 4, but also demonstrates a "maximality" property of $\overline{\mu}$.

Let T be an arbitrary ring and $\mu : R \to T$ a homomorphism of rings. If the pair (λ, μ) is exact, then there exists a ring homomorphism $\eta : T \to S'$ such that $\overline{\mu} = \mu\eta$.

Proof. We keep the notation introduced in Section 3. Recall that the complex $Q^\bullet \otimes_R T$ is of the form $0 \longrightarrow T \xrightarrow{\lambda'} S \otimes_R T \longrightarrow 0$ with T in degree -1, and isomorphic to $Q^\bullet$ in $\mathscr{D}(R)$ via the quasi-isomorphism $\mu^\bullet : Q^\bullet \to Q^\bullet \otimes_R T$ (see the diagram $(*)$ in Section 3).

We define $\omega : T \to \mathrm{End}_{\mathscr{D}(R)}(Q^\bullet \otimes_R T)$ by $t \mapsto g^\bullet$ for $t \in T$, where $g^\bullet : Q^\bullet \otimes_R T \to Q^\bullet \otimes_R T$ is the chain map with $g^{-1} := \cdot t$, $g^0 := \cdot t$ and $g^i = 0$ for $i \neq 0, -1$, which can be described by the following diagram:

$$\begin{tikzcd}
T \arrow[r, "\lambda'"] \arrow[d, "\cdot t"] & S \otimes_R T \arrow[r] \arrow[d, "\cdot t"] & Q^\bullet \otimes_R T \arrow[r] \arrow[d, "g^\bullet"] & T[1] \arrow[d, "(\cdot t)[1]"] \\
T \arrow[r, "\lambda'"] & S \otimes_R T \arrow[r] & Q^\bullet \otimes_R T \arrow[r] & T[1]
\end{tikzcd}$$

where $\cdot t$ stands for the right multiplication map by t. Then, one can check that ω is a ring homomorphism and that
$$(r)\overline{\mu}\,\mu^{\bullet} = \mu^{\bullet}\big((r)\mu\big)\omega : Q^{\bullet} \longrightarrow Q^{\bullet} \otimes_R T$$
as chain maps for any $r \in R$. This implies that if we define $\eta : T \to S'$ by $t \mapsto \mu^{\bullet} g^{\bullet} (\mu^{\bullet})^{-1}$ for $t \in T$, then $\overline{\mu} = \mu\eta$. $\square$

5 Examples

Now we present a few examples to show that some conditions in our results cannot be dropped or weakened.

(1) The condition that $\lambda : R \to S$ is a homological ring epimorphism in Corollary 1.3 cannot be weakened to that $\lambda : R \to S$ is a ring epimorphism.

Let $R = \begin{pmatrix} k & 0 & 0 \\ k[x]/(x^2) & k & 0 \\ k[x]/(x^2) & k[x]/(x^2) & k \end{pmatrix}$, where k is a field and $k[x]$ is the polynomial algebra over k in one variable x. Let S be the 3 by 3 matrix ring $M_3(k[x]/(x^2))$. Then the inclusion λ of R into S is a universal localization of R, and therefore a ring epimorphism. Further, we have $\mathrm{Tor}_1^R(S,S) = 0 \neq \mathrm{Tor}_2^R(S,S)$ (see [13]). Thus λ is not homological. So, ${}_RS$ cannot have projective dimension less than or equal to 1. Moreover, one can check that the ring homomorphism $\overline{\mu} : R \to S'$ associated to λ is an isomorphism of rings. In this case, we have $S \sqcup_R S' = S$ and $\phi = (\overline{\mu})^{-1}\lambda : S' \to S$ in Corollary 1.3. Consequently, ϕ is not homological. However, we shall show that the map λ_{π^*} is homological. Hence, without the 'homological' assumption on λ, the conditions (1) and (2) in Corollary 1.3 are not equivalent.

In the following, we prove that λ_{π^*} is always homological, even though the ring epimorphism $\lambda : R \to S$ may not be homological.

Let $\lambda : R \to S$ be a ring epimorphism such that $\mathrm{Tor}_1^R(S,S) = 0 = \mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)\big)$. If the ring homomorphism $\overline{\mu} : R \to S'$ associated to λ is an isomorphism of rings, then the universal localization $\lambda_{\pi^*} : \Lambda \to \Lambda_{\pi^*}$ of Λ at π^* is always homological.

Proof. It follows from Lemmas 3.1 and 4.4(3) that $\Lambda \simeq \begin{pmatrix} S & S \\ 0 & R \end{pmatrix}$ and $\Lambda_{\pi^*} \simeq M_2(S)$ as rings. Under these isomorphisms, the universal localization $\lambda_{\pi^*} : \Lambda \to \Lambda_{\pi^*}$ is equivalent to the canonical ring homomorphism
$$\theta : B = \begin{pmatrix} S & S \\ 0 & R \end{pmatrix} \longrightarrow M_2(S) = \begin{pmatrix} S & S \\ S & S \end{pmatrix}$$
induced by the ring homomorphism λ. Clearly, θ is an ring epimorphism. Moreover, $M_2(S)$ is projective as a right B-module. Thus θ is homological, and consequently, λ_{π^*} is homological. $\square$

(2) That λ is homological does not guarantee that the universal localization $\lambda_{\pi^*} : \Lambda \to \Lambda_{\pi^*}$ of Λ at π^* in Corollary 1.3 is always homological.

In the following, we shall use Corollary 4.7 to give a counterexample.

Now, take $C = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \mid a, b \in k \right\}$ and $D = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$ with k a field. Then one can verify that the extension $\lambda : R \to S$, defined in Corollary 4.7, is homological, and that the canonical map $\omega : D \to D/C$ is a split epimorphism in C-Mod, and therefore ${}_CD \simeq C \oplus D/C$. Let e be the idempotent of E corresponding the direct summand C of the C-module $D \oplus D/C$. Then $E_{\omega^*} \simeq E/EeE \simeq M_2(k)$. Furthermore, the universal localization $\lambda_{\omega^*} : E \to E_{\omega^*}$ of E at ω^* is equivalent to the canonical projection $\tau : E \to E/EeE$. Since $\mathrm{Ext}_E^2(E/EeE, E/EeE) \neq 0$, we see that τ is not homological. This implies that λ_{ω^*} is not homological, too. Thus $\lambda_{\pi^*} : \Lambda \to \Lambda_{\pi^*}$ is not homological by Corollary 4.7, that is, the derived functor $D\big((\lambda_{\pi^*})_*\big) : \mathscr{D}(\Lambda_{\pi^*}) \to$

$\mathscr{D}(\Lambda)$ is not fully faithful. In addition, one can check that, for this extension, the R-module ${}_RS$ has infinite projective dimension.

(3) In Corollary 1.4(1), we assume that the projective dimension of ${}_RS$ is at most 1. But there does exist an injective homological ring epimorphism $\lambda : R \to S$ such that the projective dimension of ${}_RS$ is greater than 1 and that $\lambda_{\pi^*} : \Lambda \to \Lambda_{\pi^*}$ is homological.

Let R be a Prüfer domain which is not a Matlis domain. Recall that a Matlis domain is a domain R for which the projective dimension of the fractional field Q of R as an R-module is at most 1. In this case, the inclusion $\lambda : R \to Q$ is an injective homological ring epimorphism. By Corollary 1.5, the map $\lambda_{\pi^*} : \Lambda \to \Lambda_{\pi^*}$ is homological.

(4) Now we display a concrete example which satisfies the conditions in Corollary 1.2(2).

We fix a field k. Let R and S be the k-algebras given by the following quivers with relations, respectively:

$$\begin{tikzcd} 1\bullet \arrow[r, bend left, "\alpha"] & \bullet 2 \arrow[l, bend left, "\beta"] \end{tikzcd} \qquad \alpha\beta = \beta\alpha = 0; \qquad\qquad \begin{tikzcd} 1\bullet \arrow[r, bend left, "\alpha"] & \bullet 2 \arrow[l, bend left, "\alpha^{-1}"] \end{tikzcd} \qquad \alpha\alpha^{-1} = e_1,\ \alpha^{-1}\alpha = e_2.$$

Let $\lambda : R \to S$ be the map defined by $e_i \mapsto e_i,\ \alpha \mapsto \alpha,\ \beta \mapsto 0$.Then λ is the universal localization of R at the map $Re_2 \to Re_1$ induced from α. Since ${}_RS \simeq Re_1 \oplus Re_1$, the R-module S is projective. Hence λ is homological. By calculation, the trivial extension $R \ltimes S$ of R by the R-R-bimodule ${}_RS_R$ is the algebra given by the following quiver with relations:

$$\begin{tikzcd} 1\bullet \arrow[r, bend left, "\alpha"] & \bullet 2 \arrow[l, "\gamma"] \arrow[l, bend left, "\beta"] \end{tikzcd} \qquad\qquad \alpha\beta = \beta\alpha = \gamma\alpha\gamma = 0.$$

Let $B := \begin{pmatrix} S & S \ltimes S \\ 0 & R \ltimes S \end{pmatrix}$. By Corollary 1.2(2), we have the following recollement:

$$\begin{tikzcd} \mathscr{D}(S \ltimes S) \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

Note that $S \ltimes S$ is isomorphic to $M_2\big(k[X]/(X^2)\big)$.

Finally, we mention an open question related to stratifications and recollements in this paper. We have exhibited counterexamples in [5] (see also [6]) to the Jordan-Hölder Theorem for the stratification of derived module categories of rings by derived module categories of rings. But in these recollements, not all of the rings involved are finite dimensional algebras. So, one may naturally ask the following question:

Question. *If we restrict to derived categories of finite dimensional algebras, can the Jordan-Hölder Theorem be true for stratifications of derived module categories of finite dimensional algebras by derived module categories of finite dimensional algebras (up to derived equivalence)?*

Note that some positive answers to this question are given recently in [1, Theorem 5.7]. Moreover, we do not know any counterexample to this question at moment, and expect the results in this paper, especially Corollary 1.2, could be helpful for understanding this question.

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