$$x = \frac{z}{z - \frac{dz}{dT} + \frac{1}{2}\frac{d^2z}{dT^2}} = 1 + \frac{1}{z}\frac{dz}{dT} - \frac{1}{2z}\frac{d^2z}{dT^2} + \frac{1}{z^2}\left(\frac{dz}{dT}\right)^2$$

so that if $dz/dT = 0$ when $z = z_0$ we obtain

$$x = 1 - \frac{1}{2z_0}\frac{d^2z}{dT^2}\bigg|_0 \text{ and from } d^2u\,\frac{d^2u}{dT^2} = -e^u = -z$$

to which he comes, after his various approximations, we find $x = 1 + z_0/2$ as presumably a better result, and this checks with (9). This would make

$$x = 1 + \frac{1}{2}z_0 - \sqrt{2z_0}\tanh\sqrt{z_0/2}\,T$$

with the initial and final values of x, $x_0 = 1 + \frac{1}{2}z_0 + \sqrt{2z_0}$, $x_E = 1 + \frac{1}{2}z_0 - \sqrt{2z_0}$ and with the relationship $x_0 + x_E = 2 + z_0$ instead of $x_0 + x_E = 2$.

[5] These PROCEEDINGS, **28**, 361–367 (1942).

[6] The slight numerical differences between the second and fourth columns in table 2 are due to differences in the rejection of fractional cases.

[7] Brownlee, J., *Proc. Roy. Soc. Med., London*, **2**, Pt. 2, 243–258 (1909). See page 244. This is quoted (on page 206) by Sir Ronald Ross, *Proc. Roy. Soc. London*; **A92**, 204–230 (1915), and by C. O. Stallybrass on p. 510 of his *Principles of Epidemiology*, Chap. 14, 1931.

INTEGRAL FORMULAS FOR THE CHARACTERISTIC CLASSES OF SPHERE BUNDLES

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Communicated July 13, 1944

The notion of sphere bundles[1] of H. Whitney has proved to be very fruitful in the applications of topology. So far in this theory an important rôle has been played by a class of topological invariants in the base space, called the characteristic cohomology classes and which can be regarded as generalizations of the Euler-Poincaré characteristic. By assuming the base space to be a differentiable manifold with a Riemannian metric, on which the bundles of vectors and p-vectors are defined in a natural way, we shall give in this note various relations which exist between these topological invariants and certain differential invariants of the Riemannian manifold. In fact, the relations are such that the former are expressed as integrals of the latter. In the formulae the generalized Gauss-Bonnet formula due to Allendoerfer and Weil[2] is included as a particular case.

1. At the basis of our considerations is a closed orientable n-dimensional Riemannian manifold of class C^m, $m \geqq 4$, which we denote by R^n. In R^n

we consider the frames $P\,e_1 \ldots e_n$, each of which consists of a point P and n mutually perpendicular unit vectors $e_1, \ldots, e_n$ through P, arranged in an order coherent with the orientation of R^n. Using the notations of E. Cartan,[3] the parallelism of Levi-Civita is defined by equations of the form

$$dP = \omega_i e_i, \qquad de_i = \omega_{ij} e_j, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{1}$$

where the ω's are Pfaffian forms. In terms of these Pfaffian forms the equations of structure of the space are

$$d\omega_i = \omega_j \omega_{ji}, \qquad d\omega_{ij} = -\omega_{ik}\omega_{jk} + \Omega_{ij}, \tag{2}$$

and the identities of Bianchi are

$$\omega_j \Omega_{ji} = 0, \qquad d\Omega_{ij} = -\omega_{ik}\Omega_{jk} + \omega_{jk}\Omega_{ik}. \tag{3}$$

It is to be mentioned that Ω_{ij} are exterior quadratic differential forms which give the curvature properties of R^n.

Consider now the tangent sphere bundle of R^n in the sense of Whitney and, in particular, the problem of defining over R^n a field of p-vectors, $1 \leqq p \leqq n$, where by a p-vector is meant an ordered set of p vectors (and not the figure determined by the Grassmannian coördinates). As is well known, the characteristic classes arise from the points of R^n at which the p vectors are linearly dependent. Topologically there is no loss of generality to assume the p vectors, when linearly independent, to be mutually perpendicular unit vectors.

We are thus led to consider in R^n the figures formed by a point P and an ordered set of p mutually perpendicular unit vectors through P. With P fixed, all these figures form, according to a natural topology, a finite manifold $V_{n,p}$ of dimension $p(2n - p - 1)/2$. The Betti group $B^{n-p}(V_{n,p})$ of dimension $n - p$ (with integer coefficients) of $V_{n,p}$ is an infinite cyclic group or a cyclic group of order two, according as one of the following conditions is satisfied or not: $p = 1$ or $n - p$ is even. We consider the sphere bundle $\mathfrak{S}(p)$ with R^n as the base space and the $V_{n,p}$ as fibres and proceed to study the relations between the characteristic classes arising therefrom and certain differential invariants of R^n.

2. With our problem formulated in this way, it is convenient to adopt the language of the method of moving frames. To a figure of the above description we attach a frame $P e_1 \ldots e_n$ such that P is the origin and $e_{n-p+1}, \ldots, e_n$ the p vectors of the figure in question. With the intervention of the family of frames thus attached, Pfaffian forms ω_i, ω_{ij} and differential invariants can be constructed from (1, 2). From them it is possible to construct exterior differential forms of higher degree. Whether they belong to $\mathfrak{S}(p)$ or not can be decided by the following simple criterion: *A differential form* Π *belongs to* $\mathfrak{S}(p)$ *if and only if its exterior derivative* $d\Pi$ *is a form in* ω_i, ω_{iA} $A = n - p + 1, \ldots, n$, *only*. A differential form belonging to $\mathfrak{S}(p)$ is called intrinsic.

3. For our considerations we shall suppose either $p = 1$ or $n - p$ be even. The essential step consists of establishing a formula for differential forms which we shall reproduce here for the case that $n - p$ is even. We consider the following intrinsic differential forms

$$(2k;\ h_1, \ldots, h_p) = \epsilon_{\alpha_1 \ldots \alpha_{n-p}} \Omega_{\alpha_1\alpha_2} \ldots \Omega_{\alpha_{2k-1}\alpha_{2k}} \omega_{\alpha_{2k+1}} \ldots \omega_{\alpha_{2k+h_1}} \ldots \omega_{\alpha_{2k+h_1+\cdots+h_{p-1}+1}} \ldots \omega_{\alpha_{2k+h_1+\cdots+h_p}} \quad (4)$$

where each index α runs from 1 to $n - p = 2k + h_1 + \ldots + h_p$ and where $\epsilon_{\alpha_1 \ldots \alpha_{n-p}}$ denotes the Kronecker symbol which is $+1$ or -1 according as $\epsilon_{\alpha_1, \ldots, \alpha_{n-p}}$ form an even or odd permutation of $1, \ldots, n - p$, and is otherwise zero. We put also

$$\Phi_k = \sum \frac{2.4 \ldots (n - p - 2k)}{(2.4 \ldots 2q_1) \ldots (2.4 \ldots 2q_p)} (2k;\ 2q_1, \ldots, 2q_p) \quad (5)$$

where the summation is extended over all partitions of $\frac{1}{2}(n - p) - k$ as a sum of $q_1, \ldots, q_p$. Further let

$$a_k = (-1)^k \frac{(n - p - 2k + 2) \ldots (n - p)}{2.4 \ldots 2k}, \quad k \geqq 1, \quad (6)$$

$$a_0 = 1.$$

Then we have the formula

$$d\left\{\sum_{k=0}^{1/2(n-p)} a_k \Phi_k\right\} = 0. \quad (7)$$

The proof of this formula can be carried out in a purely algebraic way.

4. Suppose we consider the p-vectors having the same origin, i.e., those belonging to the same $V_{n,p}$. Under this condition we have evidently

$$\omega_1 = \ldots = \omega_n = 0,$$

and formula (7) becomes

$$d\Phi_0 = 0$$

This shows that Φ_0 is a closed intrinsic form in $V_{n,p}$. Noticing that $B^{n-p}(V_{n,p})$ is an infinite cyclic group, it follows from the well-known results of de Rham[4] that the value of the integral

$$\int_Z \Phi_0$$

where Z is an $(n - p)$-dimensional cycle of $V_{n,p}$, will give the homology class to which Z belongs.

The theory of p-fields in R^n leads, when $n - p$ is even, to an integral characteristic class W^{n-p+1}. To derive from it a numerical topological invariant we consider the scalar product $W^{n-p+1}M^{n-p+1}$, where M^{n-p+1} is a submanifold of dimension $n - p + 1$ in R^n, closed or with boundary. When M^{n-p+1} is closed, we derive from (7) and the above remark that

$$W^{n-p+1}.M^{n-p+1} = 0,$$

which corresponds to the known fact that W^{n-p+1} is of finite order.

If, however, M^{n-p+1} has a boundary B, we shall get in this way a formula which links the integral of a differential invariant over B with a topological invariant of M^{n-p+1}. For definiteness we suppose $p = 2$ (so that n is even). B is of dimension $n - 2$ and we suppose for simplicity that B possesses a tangent $(n - 2)$-plane at each of its points. Both M^{n-1} and B are supposed to be orientable, so that M^{n-1} is two-sided in R^n and B two-sided in M^{n-1}. At each point of B we take as e_n the outside (or inside) normal of M^{n-1} and as e_{n-1} the outside (or inside) normal of B in the tangent hyperplane of M^{n-1}. We have thus defined over B a field $\mathfrak{F}$ of two vectors and the form

$$\Pi = \sum_{k=0}^{1/2(n-2)} a_k \Phi_k,$$

being intrinsic in $\mathfrak{S}(2)$, is defined on B. Then we have

$$\int_B \Pi = c.M^{n-1}.LL^{n-1}$$

where c is a numerical constant. This formula is the analog of the Gauss-Bonnet formula for a polyhedron. The corresponding formulation for a general p is without difficulty.

5. As we have shown in another occasion,[5] these discussions lead for $p = 1$ to a proof of the Gauss-Bonnet formula by which it is not necessary to imbed the cells of R^n in a Euclidean space of sufficiently high dimension. There the formula corresponding to (7) is of the form

$$d\Pi = \Omega,$$

where $\Omega = 0$ when n is odd. The differential form Π, of degree $n - 1$, is defined in the space of vectors and has some interest in itself. It is defined, whenever an orientable submanifold M^r, of dimension r, is given in R^n. For, supposing that M^r has a tangent r-plane at each of its points, we attach to a point of M^r all the unit normal vectors at that point if $r \neq n - 1$ and the outside (or inside) unit normal vector at that point if $r = n - 1$. In this way the set of unit vectors attached to the points of M^r forms, according to the natural topology, a manifold of dimension $n - 1$. This process can also be carried out for M^r which ceases to have tangent r-

planes at certain points. Then we attach to each of these points the unit normal vectors belonging to the corresponding dual angle. By such a convention the differential form Π is defined over M^r and generalizes the notion of the element of geodesic curvature for $n = 1$. We can prove that, if r is even,

$$\int_{M^r} \Pi = c\chi(M^r),$$

where $\chi(M^r)$ stands for the Euler-Poincaré characteristic of M^r and c is a constant. The proof of this formula is somewhat complicated.

6. It is evident that the same considerations can be carried out for the normal sphere bundles of a submanifold imbedded in a differentiable manifold. To avoid duplication we shall not give the results here. We only mention that the corresponding differential invariants are those which give the so-called Gaussian torsion of the submanifold.

7. Finally, we wish to make the following remark. The process employed here bears a close resemblance to that used by Gysin[6] in his work on the homology theory of fibre spaces. Our differential forms take the place of cochains and we have taken the group of real numbers as the coefficient group of the homology theory. While we thus lose such topological invariants like torsion, we are compensated by the fact that the differential forms are intrinsic in the sense of differential geometry and have thus arrived at results which connect differential invariants with topological invariants.

[1] Whitney, H., "On the Topology of Differentiable Manifolds," *Lectures in Topology* Michigan, 1941, pp. 101–141. Also: Stiefel, E., "Richtungsfelder und Fernparallelismus in n-dimensionalen Mannigfaltigkeiten," *Comm. Math. Helv.*, **8**, 305–353 (1936).

[2] Allendoerfer, C. B., and Weil, André, "The Gauss-Bonnet Theorem for Riemannian Polyhedra," *Trans. Amer. Math. Soc.*, **53**, 101–129 (1943).

[3] Cartan, E., *Leçons sur la géometrie des espaces de Riemann*, Paris, 1928.

[4] de Rahm, G., "Sur l'analysis situs des variétés a n dimensions," *J. Math. Pures Appl.*, **10** (9), 115–200 (1931).

[5] Chern, S., "A Simple Intrinsic Proof of the Gauss-Bonnet Formula for Closed Riemannian Manifolds," to appear in *Annals of Math.*

[6] Gysin, W., "Zur Homologietheorie der Abbildungen und Faserungen von Mannigfaltigkeiten," *Comm. Math. Helv.*, **14**, 61–122 (1941–1942).