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ON THE CURVATURA INTEGRA IN A RIEMANNIAN MANIFOLD

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Introduction

In a previous paper [1] we have given an intrinsic proof of the formula of Allendoerfer-Weil which generalizes to Riemannian manifolds of n dimensions the classical formula of Gauss-Bonnet for $n = 2$. The main idea of the proof is to draw into consideration the manifold of unit tangent vectors which is intrinsically associated to the Riemannian manifold. Denoting by R^n the Riemannian manifold of dimension n and by M^{2n-1} the manifold of dimension $2n - 1$ of its unit tangent vectors, our proof has led, in the case that n is even, to an intrinsic differential form of degree $n - 1$ (which we denoted by Π) in M^{2n-1}. We shall introduce in this paper a differential form of the same nature for both even and odd dimensional Riemannian manifolds. We find that this differential form bears a close relation to the "Curvatura Integra" of a submanifold in a Riemannian manifold, because it will be proved that its integral over a closed submanifold of R^n is equal to the Euler-Poincaré characteristic of the submanifold. The method can be carried over to deduce relations between relative topological invariants of a submanifold of the manifold and differential invariants derived from the imbedding, and some remarks are to be added to this effect.

§1. Definition of the Intrinsic Differential Form in M^{2n-1}

Let R^n be an orientable Riemannian manifold of dimension n and class $\geqq 3$. For a résumé of the fundamental formulas in Riemannian Geometry we refer to §1 of the paper quoted above.

Let M^{2n-1} be the manifold of the unit tangent vectors of R^n. To a unit tangent vector we attach a frame $P\mathfrak{e}_1 \cdots \mathfrak{e}_n$ such that it is the vector $\mathfrak{e}_n$ through P. The frame $P\mathfrak{e}_1 \cdots \mathfrak{e}_n$ is determined up to the transformation

$$\mathfrak{e}_\alpha^* = \sum_\beta a_{\alpha\beta}\mathfrak{e}_\beta \tag{1}$$

where $(a_{\alpha\beta})$ is a proper orthogonal matrix of order $n - 1$ and where, as well as throughout the whole section, we shall fix the ranges of the indices α, β to be from 1 to $n - 1$. Since the manifold of frames over R^n is locally a topological product, we can, to a region in M^{2n-1} the points of which have their local coordinates expressed as differentiable functions of certain parameters, attach the frames $P\mathfrak{e}_1 \cdots \mathfrak{e}_n$ which depend differentiably (with the same class) on the same parameters. From the family of frames we construct the forms ω_i, $\omega_{ij} = -\omega_{ji}$, Ω_{ij} according to the equations

$$\begin{aligned} dP &= \sum_i \omega_i\mathfrak{e}_i , \\ d\mathfrak{e}_i &= \sum_j \omega_{ij}\mathfrak{e}_j , \\ \Omega_{ij} &= d\omega_{ij} - \sum_k \omega_{ik}\omega_{kj} , \end{aligned} \tag{2}$$

it being agreed that the indices i, j, k range from 1 to n. From the forms ω_i, ω_{ij}, Ω_{ij} we construct by exterior multiplication differential forms of higher degree. If we change the frames $P\mathfrak{e}_1 \cdots \mathfrak{e}_n$ into the frames $P\mathfrak{e}_1^* \cdots \mathfrak{e}_n^* (\mathfrak{e}_n^* = \mathfrak{e}_n)$ according to the equations (1), where $a_{\alpha\beta}$ are differentiable functions of the local parameters, and denote by ω_i^*, ω_{ij}^*, Ω_{ij}^* the forms constructed from $P\mathfrak{e}_1^* \cdots \mathfrak{e}_n^*$ as the same forms without asterisks are constructed from the frames $P\mathfrak{e}_1 \cdots \mathfrak{e}_n$, we shall have

$$\begin{aligned} \omega_{\alpha n}^* &= \sum_\beta a_{\alpha\beta}\,\omega_{\beta n}, \\ \Omega_{\alpha\beta}^* &= \sum_{\rho,\sigma=1}^{n-1} a_{\alpha\rho}\,a_{\beta\sigma}\,\Omega_{\rho\sigma}, \\ \Omega_{\alpha n}^* &= \sum_\beta a_{\alpha\beta}\,\Omega_{\beta n}, \\ \Omega_{nn}^* &= \Omega_{nn}. \end{aligned} \tag{3}$$

A differential form constructed from the frames $P\mathfrak{e}_1 \cdots \mathfrak{e}_n$ will be a differential form in M^{2n-1}, if it remains invariant under the transformations (1), (3).

To apply this remark, let us put

$$\begin{aligned} \Phi_k &= \sum \epsilon_{\alpha_1 \cdots \alpha_{n-1}} \Omega_{\alpha_1\alpha_2} \cdots \Omega_{\alpha_{2k-1}\alpha_{2k}}\,\omega_{\alpha_{2k+1}n} \cdots \omega_{\alpha_{n-1}n}, \\ \Psi_k &= 2(k+1) \sum \epsilon_{\alpha_1 \cdots \alpha_{n-1}} \Omega_{\alpha_1\alpha_2} \cdots \Omega_{\alpha_{2k-1}\alpha_{2k}}\,\Omega_{\alpha_{2k+1}n}\,\omega_{\alpha_{2k+2}n} \cdots \omega_{\alpha_{n-1}n}, \end{aligned} \tag{4}$$

where $\epsilon_{\alpha_1 \cdots \alpha_{n-1}}$ is the Kronecker index which is equal to $+1$ or -1 according as $\alpha_1, \cdots, \alpha_{n-1}$ constitute an even or odd permutation of $1, \cdots, n-1$, and is otherwise zero, and where the summation is extended over all the indices $\alpha_1, \cdots, \alpha_{n-1}$. These forms are defined for $k = 0, 1, \cdots, \left[\frac{n}{2}\right] - 1$, where $\left[\frac{n}{2}\right]$ denotes the largest integer $\leqq \frac{n}{2}$. Furthermore, when n is odd, $\Phi_{[\frac{1}{2}n]}$ is also defined. It will be convenient to define by convention

$$\Psi_{-1} = \Psi_{[\frac{1}{2}n]} = 0. \tag{5}$$

Under the transformations (1), (3) each of the forms in (4) is multiplied by the value of the determinant $|a_{\alpha\beta}|$, which is $+1$. Hence they are differential forms in M^{2n-1}. We remark that Φ_k is of degree $n-1$ and Ψ_k is of degree n. When n is even, they reduce to the forms of the same notation introduced in our previous paper.

The exterior derivative $d\Phi_k$ is a differential form in M^{2n-1}, and is equal to

$$\begin{aligned} d\Phi_k = {} & k \sum \epsilon_{\alpha_1 \cdots \alpha_{n-1}}\, d\Omega_{\alpha_1\alpha_2}\,\Omega_{\alpha_3\alpha_4} \cdots \Omega_{\alpha_{2k-1}\alpha_{2k}}\,\omega_{\alpha_{2k+1}n} \cdots \omega_{\alpha_{n-1}n} \\ & + (n-2k-1) \sum \epsilon_{\alpha_1 \cdots \alpha_{n-1}} \Omega_{\alpha_1\alpha_2} \cdots \Omega_{\alpha_{2k-1}\alpha_{2k}}\, d\omega_{\alpha_{2k+1}n}\,\omega_{\alpha_{2k+2}n} \cdots \omega_{\alpha_{n-1}n}. \end{aligned}$$

In substituting the expressions for $d\Omega_{\alpha_1\alpha_2}$, $d\omega_{\alpha_{2k+1}n}$ into this equation, the terms involving $\omega_{\alpha\beta}$ will cancel each other, because $d\Phi_k$ is a differential form in M^{2n-1}.

Hence we immediately get

$$d\Phi_k = \Psi_{k-1} + \frac{n - 2k - 1}{2(k + 1)} \Psi_k. \tag{6}$$

Solving for Ψ_k, we get

$$\Psi_k = d\Theta_k, \qquad k = 0, 1, \cdots, \left[\frac{n}{2}\right] - 1, \tag{7}$$

where

$$\Theta_k = \sum_{\lambda=0}^{k} (-1)^{k-\lambda} \frac{(2k+2) \cdots (2\lambda+2)}{(n-2\lambda-1) \cdots (n-2k-1)} \Phi_\lambda, \qquad k = 0, 1, \cdots, \left[\frac{n}{2}\right] - 1. \tag{8}$$

If n is even, say $= 2p$, then we have

$$d\Theta_{p-1} = \Psi_{p-1}$$

where

$$\Psi_{p-1} = n \sum \epsilon_{\alpha_1 \cdots \alpha_{n-1}} \Omega_{\alpha_1\alpha_2} \cdots \Omega_{\alpha_{n-1}n} = \sum \epsilon_{i_1 \cdots i_n} \Omega_{i_1 i_2} \cdots \Omega_{i_{n-1} i_n}.$$

If n is odd, say $= 2q + 1$, then

$$d\Theta_{q-1} = \Psi_{q-1}.$$

But in this case we have also

$$d\Phi_q = \Psi_{q-1},$$

so that

$$d(\Theta_{q-1} - \Phi_q) = 0.$$

We define*

$$\Pi = \begin{cases} \dfrac{1}{\pi^p} \displaystyle\sum_{\lambda=0}^{p-1} (-1)^\lambda \frac{1}{1 \cdot 3 \cdots (2p - 2\lambda - 1) \cdot 2^{p+\lambda} \cdot \lambda!} \Phi_\lambda, & \text{if } n = 2p \text{ is even}, \\ \dfrac{1}{2^{2q+1} \pi^q q!} \displaystyle\sum_{\lambda=0}^{q} (-1)^{\lambda+1} \binom{q}{\lambda} \Phi_\lambda, & \text{if } n = 2q + 1 \text{ is odd}, \end{cases} \tag{9}$$

or, for a formula covering both cases,

$$\Pi = \frac{(-1)^n}{2^n \pi^{\frac{1}{2}(n-1)}} \sum_{\lambda=0}^{[\frac{1}{2}(n-1)]} (-1)^\lambda \frac{1}{\lambda! \Gamma(\frac{1}{2}(n - 2\lambda + 1))} \Phi_\lambda, \tag{9a}$$

* Our present form Ω differs, in the case of even n, from the corresponding one in our previous paper by a sign. There are several reasons which indicate that the present choice is the appropriate one.

and

$$(10)\quad \Omega = \begin{cases} (-1)^p \dfrac{1}{2^{2p}\pi^p p!} \sum \epsilon_{i_1 \cdots i_n} \Omega_{i_1 i_2} \cdots \Omega_{i_{n-1} i_n}, & \text{if } n = 2p \text{ is even} \\ 0 & \text{, if } n \text{ is odd.} \end{cases}$$

Our foregoing relations can then be summarized in the formula

$$(11)\qquad -d\Pi = \Omega.$$

We remark that Π is a differential form of degree $n - 1$ in M^{2n-1}.

Over a simplicial chain of dimension $n - 1$ in M^{2n-1} whose simplexes are covered by coordinate neighborhoods of M^{2n-1} the integral of Π is defined.

§2. Remarks on the Formula of Allendoerfer-Weil

As we have shown before, the formula (11) leads immediately to a proof of the formula of Allendoerfer-Weil. We shall, however, add here a few remarks.

Let O be a point of R^n, and let $O\mathfrak{e}_1^0 \cdots \mathfrak{e}_n^0$ be a frame with origin at O. A point P of R^n sufficiently near to O is determined by the direction cosines λ^i (referred to $O\mathfrak{e}_1^0 \cdots \mathfrak{e}_n^0$) of the tangent of the geodesic joining O to P and the geodesic distance $s = OP$. The coordinates x^i of P defined by

$$(12)\qquad x^i = s\lambda^i$$

are called the normal coordinates. In a neighborhood of O defined by $s \leqq R$ we shall employ s, λ^i to be the local coordinates, where

$$(13)\qquad \sum_i (\lambda^i)^2 = 1.$$

As the components of a vector $\mathfrak{v}$ through P we shall take the components referred to $O\mathfrak{e}_1^0 \cdots \mathfrak{e}_n^0$ of the vector at O obtained by transporting $\mathfrak{v}$ parallelly along the geodesic OP.

In the neighborhood $s \leqq R$ of O let a field of unit vectors $\mathfrak{v}$ be given, whose components are differentiable functions of the normal coordinates x^i, except possibly at O. The forms Φ_k, $k \geqq 1$, being at least of degree two in dx^i, there exists a constant M such that

$$\left| \int_S \Phi_k \right| < Ms, \qquad k \geqq 1,$$

where S is the geodesic hypersphere of radius s about O. Let I be the index of the vector field at O, which is possibly a singular point. By Kronecker's formula we have

$$(14)\qquad I = \frac{1}{O_{n-1}} \int_S \omega_{1n} \cdots \omega_{n-1,n},$$

where O_{n-1} denotes the area of the unit hypersphere of dimension $n - 1$ and is given by

$$O_{n-1} = \frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}. \tag{15}$$

It follows that there exists a constant M_1 such that

$$\left| I - (-1)^n \int_S \Pi \right| < M_1 s$$

or that

$$I = (-1)^n \lim_{s \to 0} \int_S \Pi. \tag{16}$$

Let the Riemannian manifold R^n be closed. It is well-known and is also easy to prove directly that it is possible to define in R^n a continuous vector field with a finite number of singular points. Draw about each singular point a small geodesic hypersphere. The vector field at points not belonging to the interior of these geodesic hyperspheres defines a chain in M^{2n-1} over which Ω can be integrated. From (11) and (16) we get, by applying the formula of Stokes,

$$\int_{R^n} \Omega = (-1)^n I, \tag{17}$$

where I is the sum of indices of the vector field. Hence the sum of indices of the singular points of a vector field is independent of the choice of the field, provided that their number is finite. By the construction of a particular vector field, as was done by Stiefel and Whitney [2], we get the formula

$$\int_{R^n} \Omega = (-1)^n I = (-1)^n \chi(R^n), \tag{18}$$

where $\chi(R^n)$ is the Euler-Poincaré characteristic of R^n. In particular, it follows that $\chi(R^n) = 0$ if n is odd.

The same idea can be applied to derive the formula of Allendoerfer-Weil for differentiable polyhedra. Let P^n be a differentiable polyhedron whose boundary ∂P^n is a differentiable submanifold imbedded in R^n. Let ∂P^n be orientable and therefore two-sided. To each point of ∂P^n we attach the inner unit normal vector to ∂P^n, the totality of which defines a submanifold of dimension $n - 1$ in M^{2n-1}. The integral of Π over this submanifold we shall denote simply by $\int_{\partial P^n} \Pi$. Then the formula of Allendoerfer-Weil for a differentiable polyhedron P^n is

$$\int_{P^n} \Omega = -\int_{\partial P^n} \Pi + \chi'(P^n), \tag{19}$$

where $\chi'(P^n)$ is the inner Euler-Poincaré characteristic of P^n.

To prove the formula (19), we notice that the field of unit normal vectors on ∂P^n can be extended continuously into the whole polyhedron P^n, with the possible exception of a finite number of singular points. Application of the formula of Stokes gives then

$$\int_{P^n} \Omega = -\int_{\partial P^n} \Pi + (-1)^n J,$$

where J is the sum of indices at these singular points. That $J = (-1)^n \chi'(P^n)$ follows from a well-known theorem in topology [3]. It would also be possible to deduce this theorem if we carry out the construction of Stiefel-Whitney for polyhedra and verify in an elementary way that $J = (-1)^n \chi'(P^n)$ for a particular vector field.

§3. A New Integral Formula

Let R^m be a closed orientable differentiable (of class $\geqq 3$) submanifold of dimension $m \leqq n - 2$ imbedded in R^n. The unit normal vectors to R^m at a point of R^m depend on $n - m - 1$ parameters and their totality defines a submanifold of dimension $n - 1$ in M^{2n-1}. Denote by $\int_{R^m} \Pi$ the integral of Π over this submanifold. Our formula to be proved is then

$$-\int_{R^m} \Pi = \chi(R^m), \tag{20}$$

where the right-hand member stands for the Euler-Poincaré characteristic of R^m, which is zero if m is odd.

As a preparation to the proof we need the formulas for the differential geometry of R^m imbedded in R^n. At a point P of R^m we choose the frames $P\mathfrak{e}_1 \cdots \mathfrak{e}_n$ such that $\mathfrak{e}_1, \cdots, \mathfrak{e}_m$ are the tangent vectors to R^m. We now restrict ourselves on the submanifold R^m and agree on the following ranges of indices

$$1 \leqq \alpha, \beta \leqq m, \qquad m + 1 \leqq r, s \leqq n, \qquad 1 \leqq A, B \leqq n - 1.$$

By our choice of the frames we have

$$\omega_r = 0,$$

and hence, by exterior differentiation,

$$\sum_\alpha \omega_{r\alpha} \omega_\alpha = 0$$

which allows us to put

$$\omega_{r\alpha} = \sum_\beta A_{r\alpha\beta} \omega_\beta \tag{21}$$

with

$$A_{r\alpha\beta} = A_{r\beta\alpha}. \tag{22}$$

Consequently, the fundamental formulas for the Riemannian Geometry on R^m, as induced by the Riemannian metric of R^n, are

$$(23)\qquad \begin{aligned} d\omega_\alpha &= \sum_\beta \omega_\beta\,\omega_{\beta\alpha}\,, \\ d\omega_{\alpha\beta} &= \sum_{\gamma=1}^{m} \omega_{\alpha\gamma}\,\omega_{\gamma\beta} + \tilde{\Omega}_{\alpha\beta}\,, \end{aligned}$$

where

$$(24)\qquad \tilde{\Omega}_{\alpha\beta} = \Omega_{\alpha\beta} + \sum_r \omega_{\alpha r}\,\omega_{r\beta}$$

To evaluate the integral on the left-hand side of (20) we introduce a differentiable family of frames $P\mathfrak{a}_1 \cdots \mathfrak{a}_n$ in a neighborhood of R^m, satisfying the condition that $\mathfrak{a}_\alpha = \mathfrak{e}_\alpha$ and that exactly one of the frames has the origin P. The relation between the vectors $\mathfrak{a}_{m+1}, \cdots, \mathfrak{a}_n$ and $\mathfrak{e}_{m+1}, \cdots, \mathfrak{e}_n$ is then given by the equations

$$(25)\qquad \mathfrak{e}_r = \sum_s u_{rs}\,\mathfrak{a}_s\,,$$

where u_{rs} are the elements of a proper orthogonal matrix. In particular, the quantities $u_{nr} = u_r$ may be regarded as local coordinates of the vector $\mathfrak{e}_n$ with respect to this family of frames. We now get all the normal vectors to R^m at P by letting u_r vary over all values such that $\sum_r (u_r)^2 = 1$. The forms $\omega_{n\alpha}$, ω_{nr} which occur in Π can be calculated according to the formulas

$$(26)\qquad \begin{aligned} \omega_{n\alpha} &= d\mathfrak{e}_n\cdot\mathfrak{e}_\alpha = \sum u_r\,\theta_{r\alpha}\,, \\ \omega_{nr} &= d\mathfrak{e}_n\cdot\mathfrak{e}_r = \sum_s du_s\cdot u_{rs} + \sum_{s,t=m+1}^{n} u_s\,u_{rt}\,\theta_{st}\,, \end{aligned}$$

where the product of vectors is the scalar product and where we define

$$(27)\qquad \theta_{ij} = d\mathfrak{a}_i\cdot\mathfrak{a}_j\,.$$

It is evident that

$$\Phi_k = 0, \qquad 2k > m.$$

For $k \leqq m/2$ we have by definition

$$\Phi_k = \sum \epsilon_{A_1\cdots A_{n-1}}\Omega_{A_1A_2}\cdots\Omega_{A_{2k-1}A_{2k}}\,\omega_{A_{2k+1}n}\cdots\omega_{A_{n-1}n}\,.$$

Each term of this sum is of degree m in the differentials of the local coordinates on R^m and of degree $n - m - 1$ in the differentials du_r. It follows that the non-vanishing terms are the terms where the indices $m + 1, \cdots, n - 1$ occur among $A_{2k+1}, \cdots, A_{n-1}$. We can therefore write

$$\begin{aligned} \Phi_k &= (-1)^{n-1}\frac{(n-2k-1)!}{(m-2k)!}\sum \epsilon_{\alpha_1\cdots\alpha_m}\Omega_{\alpha_1\alpha_2}\cdots\Omega_{\alpha_{2k-1}\alpha_{2k}}\,\omega_{n\alpha_{2k+1}}\cdots \\ &\qquad\qquad \omega_{n\alpha_m}\,\omega_{n,m+1}\cdots\omega_{n,n-1} \\ &= (-1)^{n-m-1}\frac{(n-2k-1)!}{(m-2k)!}\sum \epsilon_{\alpha_1\cdots\alpha_m}\Omega_{\alpha_1\alpha_2}\cdots\Omega_{\alpha_{2k-1}\alpha_{2k}}\Big(\sum_r u_r\theta_{\alpha_{2k+1}r}\Big)\cdots \\ &\qquad\qquad \Big(\sum_r u_r\,\theta_{\alpha_m r}\Big)\,\Lambda_{n-m-1}\,, \end{aligned}$$

where Λ_{n-m-1} is the surface element of a unit hypersphere of dimension $n - m - 1$.

The integration of Φ_k over R^m is then carried out by iteration. In fact, we shall keep a point of R^m fixed and integrate over all the unit normal vectors through that point. This leads us to the consideration of integrals of the form

$$\int u_{m+1}^{\lambda_{m+1}} \cdots u_n^{\lambda_n} \Lambda_{n-m-1}$$

over the unit hypersphere of dimension $n - m - 1$. It is clear that the integral is not zero, only when all the exponents $\lambda_{m+1}, \cdots, \lambda_n$ are even. But for the integrals obtained from Φ_k we have $\sum \lambda_r = m - 2k$. It follows that, if m is odd, we shall have

$$\int_{R^m} \Phi_k = 0, \qquad 0 \leqq k \leqq \frac{m}{2},$$

and hence

$$\int_{R^m} \Pi = 0.$$

This proves the formula (20) for the case that m is odd.

More interesting is naturally the case that m is even, which we are going to suppose from now on. It was proved that [4]

$$(28) \qquad \int u_{m+1}^{2\lambda_{m+1}} \cdots u_n^{2\lambda_n} \Lambda_{n-m-1} = \frac{2\lambda_{m+1}) \cdots 2\lambda_n) O_{n-m-1}}{(n-m)(n-m+2) \cdots (n-m+2\lambda_{m+1}+\cdots+2\lambda_n-2)},$$

where the symbol in the numerator is defined by

$$(29) \qquad 0) = 1, \quad 2\lambda) = 1.3 \cdots (2\lambda - 1).$$

To evaluate the integral of Φ_k over R^m we have to expand the product

$$\Big(\sum_r u_r \theta_{\alpha_{2k+1}r}\Big) \cdots \Big(\sum_r u_r \theta_{\alpha_m r}\Big).$$

We introduce the notation

$$(30) \qquad \Delta(k; \lambda_{m+1}, \cdots, \lambda_n) = \sum \epsilon_{\alpha_1 \cdots \alpha_m} \Omega_{\alpha_1 \alpha_2} \cdots \Omega_{\alpha_{2k-1} \alpha_{2k}} \bigstar$$

where the last symbol stands for a product of θ's, whose first indices are $\alpha_{2k+1}, \cdots, \alpha_m$ respectively and whose second indices are respectively $2\lambda_{m+1} (m+1)$'s, $2\lambda_{m+2} (m+2)$'s, and finally $2\lambda_n$ n's. Let it be remembered that $\Delta(k; \lambda_{m+1}, \cdots, \lambda_n)$ is a differential form of degree m in R^m. Expanding Φ_k and using (28), we shall get

$$(31)\quad \int_{R^m} \Phi_k = (-1)^{n-1} \frac{(n-2k-1)!\,O_{n-m-1}}{2^{\frac{1}{2}m-k}(n-m)(n-m+2)\cdots(n-2k-2)} \sum_{\lambda_{m+1}+\cdots+\lambda_n=\frac{1}{2}m-k} \frac{1}{\lambda_{m+1}!\cdots\lambda_n!} \int_{R^m} \Delta(k;\lambda_{m+1},\cdots,\lambda_n),$$

where the summation is extended over all $\lambda_r \geqq 0$, whose sum is $\frac{m}{2} - k$.

It is now to be remarked that for the curvature forms $\tilde{\Omega}_{\alpha\beta}$ of the Riemannian metric on R^m we have to substitute $\theta_{\alpha r}$ for $\omega_{\alpha r}$ in the expressions (24). $\tilde{\Omega}$ being the form on R^m whose integral over R^m is equal to the Euler-Poincaré characteristic $\chi(R^m)$ by the Allendoerfer-Weil formula, we have

$$\tilde{\Omega} = (-1)^{\frac{1}{2}m} \frac{1}{2^m \pi^{\frac{1}{2}m}(\frac{1}{2}m)!} \sum \epsilon_{\alpha_1\cdots\alpha_m}(\Omega_{\alpha_1\alpha_2} - \sum \theta_{\alpha_1 r}\theta_{\alpha_2 r}) \cdots (\Omega_{\alpha_{m-1}\alpha_m} - \sum \theta_{\alpha_{m-1} r}\theta_{\alpha_m r})$$

or, by expansion,

$$(32)\quad \tilde{\Omega} = \frac{1}{2^m \pi^{\frac{1}{2}m}} \sum_{k=0}^{\frac{1}{2}m} (-1)^k \frac{1}{k!} \sum_{\lambda_{m+1}+\cdots+\lambda_n=\frac{1}{2}m-k} \frac{1}{\lambda_{m+1}!\cdots\lambda_n!} \Delta(k;\lambda_{m+1},\cdots,\lambda_n).$$

By a straightforward calculation which we shall omit here, we get from (9), (31), (32), and (18) the desired formula (20).

So far we have assumed that $m \leqq n - 2$, that is, that R^m is not a hypersurface of R^n. In case $m = n - 1$ the unit normal vectors of $R^{n-1} = R^m$ in R^n are, under our present assumptions concerning orientability, divided into two disjoint families. It is possible to maintain the formula (20) by making suitable conventions. In fact, we suppose that the integrals $\int_{R^{n-1}} \Pi$ over the families of inward and outward unit normal vectors are taken over the oppositely oriented manifold R^{n-1}. Then we have

$$\int_{(R^{n-1})^-} \Pi = (-1)^{n-1} \int_{(R^{n-1})^+} \Pi,$$

where the integrals at the left and right hand sides are over the families of inward and outward normals respectively. If n is even, we have

$$\int_{R^{n-1}} \Pi = \int_{(R^{n-1})^-} \Pi + \int_{(R^{n-1})^+} \Pi = 0.$$

If n is odd, we have

$$\int_{R^{n-1}} \Pi = 2\int_{(R^{n-1})^-} \Pi = -\chi(R^{n-1}).$$

Both cases can be considered as included in the formula (20). In particular, if n is odd and if R^{n-1} is the boundary ∂P^n of a polyhedron P^n, we have also, by (19),

$$\int_{(R^{n-1})^-} \Pi = \chi'(P^n).$$

Comparing the two equations, we get

$$-\chi(\partial P^n) = 2\chi'(P^n),$$

which asserts that the inner Euler-Poincaré characteristic of a polyhedron in an odd-dimensional manifold is $-\frac{1}{2}$ times the Euler-Poincaré characteristic of its boundary, a well-known result in the topology of odd-dimensional manifolds.

It is interesting to remark in passing that, so far as the writer is aware, the formula (20) seems not known even for the Euclidean space.

§4. Fields of Normal Vectors

We consider the case that R^{2n} is an even-dimensional orientable Riemannian manifold of class $\geqq 3$ and R^n a closed orientable submanifold of the same class imbedded in R^{2n}. By considering normal vector fields over R^n, Whitney [5] has defined a topological invariant of R^n in R^{2n}, which is the sum of indices at the singular points of a normal vector field (with a finite number of singular points) over R^n. Let us denote by ψ this invariant of Whitney.

To prepare for the study of this invariant we make use of the discussions at the beginning of §3. To each point P of R^n we attach the frames $P\mathfrak{e}_1 \cdots \mathfrak{e}_{2n}$ such that $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ are tangent vectors to R^n at P. Then we have, in particular,

$$d\omega_{ij} = \sum \omega_{ik}\omega_{kj} + \Theta_{ij}, \tag{33}$$

where

$$\Theta_{ij} = \Omega_{ij} - \sum_{\alpha=1}^{n} \omega_{i\alpha}\,\omega_{j\alpha} \tag{34}$$

the indices i, j running from $n + 1$ to $2n$. The differential forms Θ_{ij} are exterior quadratic differential forms depending on the imbedding of R^n in R^{2n}. They give what is essentially known as the Gaussian torsion of R^n in R^{2n}. We put, similar to (10),

$$\Theta = \begin{cases} (-1)^p \dfrac{1}{2^{2p}\,\pi^p\,p!} \sum \epsilon_{i_1\cdots i_n}\Theta_{i_1 i_2} \cdots \Theta_{i_{n-1}i_n}, & \text{if } n = 2p \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{35}$$

With these preparations we are able to state the following theorems:

1. *If R^n is a closed orientable submanifold imbedded in an orientable Riemannian manifold R^{2n}, the Whitney invariant ψ is given by*

$$\psi = \int_{R^n} \Theta. \tag{36}$$

2. *It is always possible to define a continuous normal vector field over a closed orientable odd-dimensional differentiable submanifold* (*of class* $\geqq 3$) *imbedded in an orientable differentiable manifold of twice its dimension.*

The first theorem can be proved in the same way as the formula of Allendoerfer-Weil. We shall give a proof of the second theorem.

For this purpose we take a simplicial decomposition of our submanifold R^n and denote its simplexes by σ_i^n, $i = 1, \cdots, m$. We assume the decomposition to be so fine that each σ_i^n lies in a coordinate neighborhood of R^n. According to a known property on the decomposition of a pseudo-manifold [6], the simplexes σ_i^n can be arranged in an order, say $\sigma_1^n, \cdots, \sigma_m^n$, such that σ_k^n, $k < m$, contains at least an $(n - 1)$-dimensional side which is not incident to $\sigma_1^n, \cdots, \sigma_{k-1}^n$. We then define a continuous normal vector field by induction on k. It is obviously possible to define a continuous normal vector field over σ_1^n. Suppose that such a field is defined over $\sigma_1^n + \cdots + \sigma_{k-1}^n$. The simplex σ_k^n has in common with $\sigma_1^n + \cdots + \sigma_{k-1}^n$ at most simplexes of dimension $n - 1$ and there exists, when $k < m$, at least one boundary simplex of dimension $n - 1$ of σ_k^n which does not belong to $\sigma_1^n + \cdots + \sigma_{k-1}^n$. It follows that the subset of σ_k^n at which the vector field is defined is contractible to a point in σ_k^n. By a well-known extension theorem [7], the vector field can be extended throughout σ_k^n, $k < m$. In the final step $k = m$ the extension of the vector field throughout σ_m^n will lead possibly to a singular point in σ_m^n. Hence it is possible to define a continuous normal vector field over R^n with exactly one singular point, the index at which is equal to the Whitney invariant ψ. If n is odd, we have, by (36), $\psi = 0$, and the singular point can be removed. This proves our theorem.

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