

Winding Numbers on Surfaces II. Applications.

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## Winding Numbers on Surfaces. II

### Applications

D. R. J. Chillingworth

#### § 0. Introduction

This is a sequel to the paper [7] in which a theory of *winding numbers* of regular closed curves on surfaces, based on [16, 18], was extended to a theory of winding numbers of homotopy classes of closed curves on surfaces. We give here two applications of the theory.

The main application is to the problem of determining when a given element of the fundamental group of a compact surface can be represented by simple closed curves: we are able to obtain a workable algorithm which uses the formulae for calculating winding numbers given in [7]. Earlier algorithms for solving this problem using non-euclidean geometry have been given by Reinhart [17] and Călugăreanu [3], and a combinatorial algorithm has been given by Zieschang [19, 21].

The second application is to the investigation of the structure of the *homeotopy group* of a closed surface, i.e. the group of homeomorphisms modulo those isotopic to the identity. Although the results produced at this stage are not essentially new, the fact that some concrete results can be obtained does suggest that a fuller understanding of the algebraic properties of winding numbers may lead to more powerful methods for analyzing homeotopy groups.

The definitions and notations used throughout are those of [7]. A *surface* is a connected separable 2-manifold, called a *closed surface* if it is compact and with empty boundary. Any compact orientable or non-orientable surface may be regarded as a 2-sphere with a finite number  $r \geq 0$  of open discs removed and a finite number  $g \geq 0$  of handles attached in the orientable case or a finite number  $n \geq 0$  of Möbius strips attached in the non-orientable case; the *genus* of the surface is  $g$  or  $n$ .

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#### § 1. Geometric Preliminaries

Let  $M$  be a compact surface, and  $x$  a point of  $M$ . In all that follows we shall assume that  $M$  is not a sphere, since the problems with which we are concerned are trivial in that case.

We begin with some technical constructions which provide a machinery for relating the algebraic structure of  $\pi_1(M, x)$  to the geometry of  $M$ . These are taken directly from [19] or [22].

We recall from [7, § 7] (see [19, 22]) that every compact surface  $M$  admits a *canonical curve system*  $\Sigma$  described as follows. If  $M$  is orientable with boundary  $\partial M \neq \emptyset$  then  $\Sigma$  consists of a system of oriented simple closed curves  $\alpha_i, \beta_j$  and  $\sigma_k$  ( $1 \leq i, j \leq g; 1 \leq k \leq r$ ) with the following properties:

- (i) the curves all meet at  $x$  and are otherwise disjoint;
- (ii)  $\sigma_k$  together with a component  $\varrho_k \subset \partial M$  bounds an annulus  $R_k$  whose interior is disjoint from all the other curves of  $\Sigma$ ;
- (iii) cutting along the curves of  $\Sigma$  dissects  $M$  into the disjoint union of the  $R_k$  together with a disc whose boundary runs

$$\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2 \beta_2 \alpha_2^{-1} \beta_2^{-1} \dots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1} \sigma_1 \sigma_2 \dots \sigma_r$$

where " $\alpha_i^{-1}$ " here means " $\alpha_i$  backwards" etc.

The fundamental group  $\pi_1(M, x)$  has the presentation

$$\{a_i, b_j, s_k (1 \leq i, j \leq g; 1 \leq k \leq r) | d = 1\}$$

where  $a_i, b_j, s_k$  denote the homotopy classes of  $\alpha_i, \beta_j, \sigma_k$  respectively, and

$$d = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} s_1 s_2 \dots s_r.$$

When  $\partial M = \emptyset$  we ignore (ii) and dispense with the  $\varrho_k, \sigma_k, R_k$  and  $s_k$ . If  $M$  is non-orientable with  $\partial M \neq \emptyset$  then  $\Sigma$  consists of a system of oriented simple closed curves  $\eta_i, \sigma_k$  ( $1 \leq i \leq n; 1 \leq k \leq r$ ) satisfying (i), (ii) above together with:

- (iii)' cutting along the curves of  $\Sigma$  dissects  $M$  into the disjoint union of the  $R_k$  together with a disc whose boundary runs

$$\eta_1 \eta_1 \eta_2 \eta_2 \dots \eta_n \eta_n \sigma_1 \sigma_2 \dots \sigma_r.$$

In this case  $\pi_1(M, x)$  has the presentation

$$\{u_i, s_k (1 \leq i \leq n; 1 \leq k \leq r) | d' = 1\}$$

where  $u_i, s_k$  denote the homotopy classes of  $\eta_i, \sigma_k$  respectively and

$$d' = u_1^2 u_2^2 \dots u_n^2 s_1 s_2 \dots s_r.$$

When  $\partial M = \emptyset$  we again ignore (ii) and dispense with  $\varrho_k, \sigma_k, R_k$  and  $s_k$ .

The following definition also comes from [19] or [22]:

**(1.1) Definition.** When  $M$  is orientable a *dual dissection*  $\Sigma^*$ , corresponding to the canonical curve-system  $\Sigma$ , consists of a collection of oriented simple closed curves  $\alpha_i^*, \beta_j^*$  ( $1 \leq i, j \leq g$ ) and (if  $\partial M \neq \emptyset$ ) arcs  $\sigma_k^*$  ( $1 \leq k \leq r$ ) with the following properties:

(1) the curves and arcs all have a common base-point  $q$  but are otherwise disjoint;

(2)  $\sigma_k^*$  connects  $q$  to a point of  $\varrho_k$ ;

(3) each  $\alpha_i^*$  (or  $\beta_j^*, \sigma_k^*$ ) meets  $\alpha_i$  (or  $\beta_j, \sigma_k$  respectively) exactly once, and meets none of the other curves of  $\Sigma$ ;

(4) cutting along the curves and arcs of  $\Sigma^*$  dissects  $M$  into a disc  $D$  whose boundary runs

$$\alpha_1^* \beta_1^{*-1} \alpha_1^{*-1} \beta_1^* \alpha_2^* \beta_2^{*-1} \dots \alpha_g^{*-1} \beta_g^* \sigma_1^* \varrho_1 \sigma_1^{*-1} \sigma_2^* \varrho_2 \sigma_2^{*-1} \dots \sigma_r^* \varrho_r \sigma_r^{*-1}.$$

(Note the difference between this and the expression for  $d$ .)

If  $\partial M = \emptyset$  there are no arcs  $\sigma_k^*$ .

When  $M$  is non-orientable a *dual dissection*  $\Sigma^*$  consists of oriented simple closed curves  $\eta_i^*$  ( $1 \leq i \leq n$ ) and arcs  $\sigma_k^*$  ( $1 \leq k \leq r$ ) satisfying (1), (2) above and

(3') each  $\eta_i^*$  or  $\sigma_k^*$  meets  $\eta_i$  or  $\sigma_k$  respectively exactly once, and meets none of the other curves of  $\Sigma$

(4') cutting along the curves and arcs of  $\Sigma^*$  dissects  $M$  into a disc  $D$  whose boundary runs

$$\eta_1^* \eta_1^* \eta_2^* \eta_2^* \dots \eta_n^* \eta_n^* \sigma_1^* \varrho_1 \sigma_1^{*-1} \sigma_2^* \varrho_2 \sigma_2^{*-1} \dots \sigma_r^* \varrho_r \sigma_r^{*-1}.$$

Again if  $\partial M = \emptyset$  there are no arcs  $\sigma_k^*$ .

**(1.2) Definition.** Define an equivalence relation “ $\sim$ ” on the set of all words in a given collection of symbols (“letters”) by  $w_1 \sim w_2$  if and only if  $w_1$  is a cyclic permutation of  $w_2$ . Each equivalence class is called a *cyclic word*, and the equivalence class of  $w$  is denoted by  $(w)$ .

Any word  $w$  in the generators of  $\pi_1(M, x)$  and their inverses represents an element of  $\pi_1(M, x)$  which we denote by  $[w]$ . If  $w_1 \sim w_2$  then  $[w_1]$  and  $[w_2]$  are conjugate elements.

Let  $\gamma$  be an oriented closed curve not passing through  $q$ , and in *general position* with respect to  $\Sigma^*$  (i.e.  $\gamma$  meets  $\Sigma^*$  in at most a finite number of points none of which is a self-intersection point of  $\gamma$ , and at each of which  $\gamma$  crosses over the curve of  $\Sigma^*$ ). Let  $\chi^*$  be any curve or arc of  $\Sigma^*$ , and let  $p$  be a point of  $\gamma \cap \chi^*$ . By assigning to a neighbourhood of  $p$  the orientation induced by the path  $qp$  along  $\chi^*$  (in the positive direction) from some chosen fixed orientation at  $q$ , we may give meaning to the statements “ $\gamma$  cuts  $\chi^*$  from right to left (positively) at  $p$ ” or “ $\gamma$  cuts  $\chi^*$  from left to right (negatively) at  $p$ ”.

**(1.3) Definition.** The *total word* (“Ablesung” in [19, 22]) of  $\gamma$  with respect to  $\Sigma^*$  is a cyclic word in the letters  $a_i^{\pm 1}, b_j^{\pm 1}, s_k^{\pm 1}$  ( $1 \leq i, j \leq g$ ;  $1 \leq k \leq r$ ) if  $M$  is orientable or  $u_i^{\pm 1}, s_k^{\pm 1}$  ( $1 \leq i \leq n$ ;  $1 \leq k \leq r$ ) if  $M$  is non-orientable, obtained from  $\gamma$  as follows. Choose any point  $y$  on  $\gamma$  and not

on any curve or arc of  $\Sigma^*$ , and proceed once round  $\gamma$  in the positive direction. At each point where  $\gamma$  crosses a curve or arc of  $\Sigma^*$  write

$a_i$  if  $\gamma$  crosses  $\alpha_i^*$  positively ,  
 $a_i^{-1}$  if  $\gamma$  crosses  $\alpha_i^*$  negatively,  
 $b_j$  if  $\gamma$  crosses  $\beta_j^*$  positively ,  
 etc.

The sequence of letters so obtained gives a word  $w$ , and the *total word* of  $\gamma$  is then defined to be the cyclic word  $(w)$ , which we denote by  $\text{tw}(\gamma)$ . Clearly, if we had chosen a different starting point on  $\gamma$  we would have obtained a word  $w'$  with  $(w') = (w)$ , and so the total word does not depend on the choice of starting point.

The curves  $\alpha_i, \eta_i$  etc. have total words  $(a_i), (u_i)$  etc., and it is easy to see that if two closed curves have the same total word they are (freely) homotopic.

Given a cyclic word  $(w)$  it is always possible to construct a closed curve  $\gamma$  on  $M$  with  $\text{tw}(\gamma) = (w)$ . However, if  $w$  represents an element of  $\pi_1(M, x)$  which is known to contain simple closed curves, such a  $\gamma$  may not necessarily itself be simple. For example, on a torus a closed curve whose total word is  $(a_1^3 b_1^2)$  cannot be simple although it is homotopic to a simple curve (e.g. with total word  $(a_1^2 b_1 a_1 b_1)$ ). In [19, 22] Zieschang uses a method of “reduction” of a word  $w$  to a word  $W$  with the property that if the element  $[w] \in \pi_1(M, x)$  contains simple closed curves then some element conjugate to  $[w]$  contains a simple closed curve whose total word is  $(W)$ . We follow this method here.

**(1.4) Definition.** A word  $w$  in the  $a_i^{\pm 1}$  etc. (or  $u_i^{\pm 1}$  etc.) is *completely reduced* if it satisfies the following conditions:

(i)  $w$  is cyclically reduced (i.e. no two cyclically adjacent letters in  $w$  are mutual inverses);

(ii) neither  $w$  nor any cyclic permutation of  $w$  contains any subword of the element  $d^{\pm 1}$  (or  $d'^{\pm 1}$ ), or of any cyclic permutation of  $d^{\pm 1}$  (or  $d'^{\pm 1}$ ), whose length is more than half the length of  $d$  (or  $d'$ );

(iii) if  $w$  or any cyclic permutation of  $w$  contains a subword which consists of exactly half of a cyclic permutation of  $d^\varepsilon$  ( $\varepsilon = \pm 1$ ) then this half contains the first element of  $d$  (i.e.  $a_1, u_1$  ( $g, n > 0$ ) or  $s_1$  ( $g, n = 0$ )) when  $\varepsilon = +1$ , or the last element of  $d^{-1}$  (i.e.  $a_1^{-1}, u_1^{-1}$  or  $s_1^{-1}$ ) when  $\varepsilon = -1$ .

If  $w$  is any given word it is easy to see that in general a completely reduced word  $W$  may be obtained from  $w$  by reducing  $w$  cyclically, replacing “long” subwords of  $d^{\pm 1}$  (or  $d'^{\pm 1}$ ) by shorter words, replacing “half-length” subwords if necessary, reducing cyclically again if possible, and so on. We call this process *complete reduction*. Exceptions to this

rule occur when  $M$  is a torus ( $g = 1, r = 0$ ) or a Klein bottle ( $n = 2, r = 0$ ), since for example the word  $a_1 b_1 a_1$  in the first case or the word  $u_1 u_2 u_1 u_2$  in the second case cannot be reduced to words satisfying (iii). However, these cases are treated separately in § 3.

Note that  $[W]$  is conjugate to  $[w]$  in  $\pi_1(M, x)$ .

If  $W$  is completely reduced and  $(W') = (W)$  then  $W'$  is completely reduced. Therefore we also refer to the *cyclic word*  $(W)$  as being *completely reduced*. The element  $1 \in \pi_1(M, x)$  is represented by the empty word which is conventionally regarded as being completely reduced.

The following two lemmas are taken straight from [19] with appropriate changes in notation. The proof of the first is elementary, whereas the second is more difficult and is based on the methods of [8].

**(1.5) Lemma.** *Any simple closed curve  $\gamma$  based at  $x$  on  $M$  may be deformed by an isotopy (which in general moves  $x$ ) to obtain a simple closed curve whose total word is completely reduced.*

**(1.6) Lemma.** *Suppose the number of generators of  $\pi_1(M, x)$  is at least 5. If  $W$  and  $W'$  are completely reduced words such that  $[W]$  and  $[W']$  are conjugate in  $\pi_1(M, x)$  and  $[W]$  contains simple closed curves then  $(W) = (W')$ .*

**(1.7) Lemma.** *If  $c \in \pi_1(M, x)$  contains simple closed curves then any element conjugate to  $c$  contains simple closed curves.*

*Proof.* This follows immediately from the well-known elementary fact that any inner automorphism of  $\pi_1(M, x)$  is induced by a homeomorphism of  $M$  (see e.g. [20]). The inner automorphism defined by an element  $z$  is induced by a homeomorphism of  $M$  which may be described as the result of sliding  $x$  once round a closed curve representing  $z$ .

We shall now assume until § 3 that the number of generators of  $\pi_1(M, x)$  is at least 5: thus every word  $w$  in the generators of  $\pi_1(M, x)$  and their inverses can be reduced completely, and if  $[w]$  contains simple closed curves the resulting completely reduced cyclic word depends only on the conjugacy class of  $[w]$  in  $\pi_1(M, x)$ .

Lemmas (1.5)–(1.7) can be combined to yield the following:

**(1.8) Lemma.** *Let  $c \in \pi_1(M, x)$  be a non-trivial element and let  $w$  be a word representing  $c$ . Let  $W$  be a word obtained by reducing  $w$  completely. Then there exists an oriented closed curve  $\gamma$  in general position with respect to  $\Sigma^*$  such that the total word of  $\gamma$  is  $(W)$ , and  $\gamma$  can be chosen to be simple if and only if  $c$  contains simple closed curves.*

We now turn attention to the geometric situation. Recall that  $\Sigma^*$  dissects  $M$  into a disc  $D$ . Let  $\gamma$  be any closed curve in general position with respect to  $\Sigma^*$  on  $M$ , and not nullhomotopic. On  $D$  the curve becomes a finite collection  $\mathcal{C}$  of oriented paths whose end-points lie on  $\partial D$  and are

identified in pairs to form  $\gamma$  from  $\mathcal{C}$  when segments of  $\partial D$  are identified in pairs to form  $M$  from  $D$ . Let  $D^0$  denote the interior of  $D$  (i.e.  $D^0 = D - \partial D$ ). It is easy to see that  $\gamma$  may be deformed by a homotopy if necessary to ensure that  $\mathcal{C}$  satisfies the following conditions:

- (1) each path of  $\mathcal{C}$  is a simple arc;
- (2) any pair of distinct arcs of  $\mathcal{C}$  are either disjoint or meet at just one point of  $D^0$  where they actually cross;
- (3) no three arcs of  $\mathcal{C}$  have a common point of intersection.

**(1.9) Definition.** If  $\gamma$  is an oriented closed curve as described in Lemma (1.8) and such that  $\mathcal{C}$  satisfies conditions (1), (2) and (3) above, then  $\gamma$  is called an *elementary representative curve* for the conjugacy class of  $c$  in  $\pi_1(M, x)$ .

We recall from [7] that an orientation-preserving closed curve  $\gamma$  in  $M$  is said to be *direct* if there is a covering of  $M$  such that  $\gamma$  lifts to a simple closed curve. (In fact in [7] we choose an infinite cyclic covering, but this restriction is unimportant.) Lemma (2.4) of [7] characterizes these direct curves by the absence of null-homotopic loops. An orientation-reversing closed curve  $\gamma$  is *direct* if  $2\gamma$  (the curve obtained by going twice round  $\gamma$ ) is direct.

**(1.10) Lemma.** *An elementary representative curve is direct.*

*Proof.* If  $\gamma$  is an orientation-preserving elementary representative curve which is not direct then by Lemma (2.4) of [7] it contains a null-homotopic loop. By (1) the loop cannot be disjoint from  $\Sigma^*$ , and so  $\text{tw}(\gamma) = (w)$  for some  $w$  containing a null subword, i.e. a subword representing  $1 \in \pi_1(M, x)$ . However, a word containing a null subword must fail to satisfy (i) or (ii) of Definition (1.4): as  $\pi_1(M, x)$  has at least five generators this fact follows from Reidemeister's solution of the word problem [15, pp. 200–204] or from [22, Satz IV.15]. (See [22, p. 130] or [19, p. 26] for details of the argument when  $\partial M \neq \emptyset$ ; see also [19, p. 20] for other references.) Similarly, if  $\gamma$  is orientation-reversing and not direct then it is easy to see that  $ww$  contains a null subword and that this is impossible when  $(w)$  is completely reduced.

## § 2. Construction of the Algorithm

From now until Lemma (2.8) (excepting Lemma (2.4)) in order to save space we shall only give proofs of lemmas in the case when  $M$  is *orientable*. However, the results remain valid in the non-orientable case, winding numbers (see below) being taken mod 2 where appropriate. The detailed proofs for the general case can be found in [4, pp. 127–147].

Let  $M$  be given some fixed differentiable structure. A closed curve in  $M$  will be called *regular* if it has a continuous non-vanishing tangent: see [7].

### The Simple Case

Suppose  $c \neq 1 \in \pi_1(M, x)$  contains simple closed curves, and let  $\gamma$  be a simple elementary representative curve for the conjugacy class  $C$  of  $c$ . Without loss of generality  $\gamma$  may be chosen to be a regular curve. Since  $\gamma$  is simple the arcs of  $\mathcal{C}$  are mutually disjoint. We assume that  $c$  is not one of the elements  $a_i^{\pm 1}, b_j^{\pm 1}, s_k^{\pm 1}$ , so that  $\mathcal{C}$  contains at least two arcs. Let  $\kappa_1, \kappa_2$  be any two arcs of  $\mathcal{C}$ , and let  $v$  be a smooth arc in  $D^0$  joining a point  $p_1$  on  $\kappa_1$  to a point  $p_2$  on  $\kappa_2$ , otherwise disjoint from  $\kappa_1 \cup \kappa_2$ , and chosen so that  $v$  meets no other arc of  $\mathcal{C}$  more than once. Let  $\xi'_1$  denote the arc in  $D$  obtained by going along  $\kappa_2$  in the negative direction to  $p_2$ , then along  $v^{-1}$  to  $p_1$ , and finally along  $\kappa_1$  in the positive direction; let  $\xi'_2$  be the arc going along  $\kappa_2$  in the positive direction to  $p_2$ , then along  $v^{-1}$  to  $p_1$ , and then along  $\kappa_1$  in the negative direction. Approximate  $\xi'_1, \xi'_2$  by smooth arcs  $\xi_1, \xi_2$  which are the same as  $\xi'_1, \xi'_2$  outside a closed disc  $B$  with  $v \subset B^0 \subset B \subset D^0$  (where  $B^0$  denotes the interior of  $B$ ), and which meet just once, touching or crossing according to whether the orientations of  $\kappa_1, \kappa_2$  at  $p_1, p_2$  are the same or opposite with respect to an orientation of  $v$ . See Fig. 1, which illustrates the two possibilities. Furthermore, assume that the point  $p = \xi_1 \cap \xi_2$  lies on no arc of  $\mathcal{C}$ , and that  $\xi_i$  meets each arc  $\kappa$  of  $\mathcal{C} - (\kappa_1 \cup \kappa_2)$  exactly once or not at all depending on whether the end-points of  $\xi_i$  separate those of  $\kappa$  on  $\partial D$  or not ( $i = 1, 2$ ).

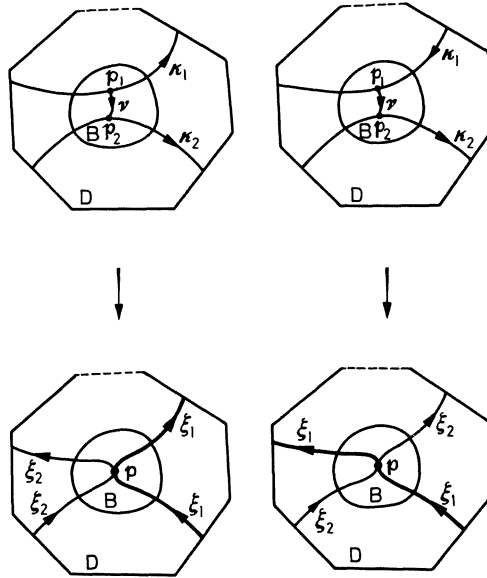


Fig. 1



The arcs  $\xi_1$  and  $\xi_2$  and those of  $\mathcal{C} - (\kappa_1 \cup \kappa_2)$  fit together to give (with a suitable parametrization) a regular curve  $\gamma'$  on  $M$ . Let  $\delta$  denote the curve which starts at  $p$  and proceeds along  $\xi_1$  in the positive direction and then around the rest of  $\gamma'$  until first returning to  $p$  (along  $\xi_2$ ), and let  $\varepsilon$  be the curve which starts at  $p$  and proceeds along  $\xi_2$  in the positive direction and then around  $\gamma'$  until first returning to  $p$  (along  $\xi_1$ ). Choose parametrizations such that each of  $\delta, \varepsilon$  is regular, and  $\gamma'$  is the composition of  $\delta$  with  $\varepsilon$ . If  $\text{tw}(\gamma) = (W)$  then some cyclic permutation of  $W$  is of the form  $uv$  where  $(u) = \text{tw}(\delta)$  and  $(v) = \text{tw}(\varepsilon^{-1})$ , and  $\text{tw}(\gamma') = (uv^{-1})$ .

Let  $Q$  be an open disc neighbourhood of  $q$  (the common intersection point of the curves of  $\Sigma^*$ ) not meeting  $\gamma'$  and not containing the base-point  $x$  of  $M$ , and let  $N = M - Q$ . The curves  $\gamma, \delta, \varepsilon$  and  $\gamma'$  can be thought of as curves in  $N$ . Any word  $w$  in the letters  $a_i^{\pm 1}, b_j^{\pm 1}, s_k^{\pm 1}$  represents an element of  $\pi_1(N, x)$  as well as an element of  $\pi_1(M, x)$ : we continue to denote the element of  $\pi_1(M, x)$  by  $[w]$ , and we denote the element of  $\pi_1(N, x)$  by  $[w]'$ .

In Lemmas (2.1)–(2.7) which follow we shall understand *nullhomotopic* to mean nullhomotopic in  $N$ , and by a *null word* or *null subword* we shall mean a word (subword) which represents  $1 \in \pi_1(N, x)$ . Clearly a word which represents  $1 \in \pi_1(N, x)$  also represents  $1 \in \pi_1(M, x)$ .

**(2.1) Lemma.** *If  $u \neq v$  then  $\delta, \varepsilon$  and  $\gamma'$  are not nullhomotopic and are direct (as curves on  $N$ ).*

*Proof.* If  $\delta$  is nullhomotopic then  $\text{tw}(\gamma)$  contains a null subword, which is impossible since  $\gamma$  is an elementary representative curve. If  $\delta$  is not direct on  $N$  then by [7, Lemma (2.4)] it contains a nullhomotopic loop  $\lambda$ . Since  $\gamma$  is simple the arcs of  $\mathcal{C}$  are mutually disjoint so the vertex of  $\lambda$  lies in  $K$  where

$$K = (\xi_1 \cup \xi_2) \cap (\mathcal{C} - (\kappa_1 \cup \kappa_2)).$$

But  $\xi_1, \xi_2$  are arcs meeting only once so  $\lambda \not\subset D^0$  and hence  $\text{tw}(\gamma)$  again contains a null subword, which is impossible. Therefore  $\delta$  is direct. Similar arguments apply to  $\varepsilon$ .

If  $\gamma'$  is nullhomotopic then  $[uv^{-1}]' = 1 \in \pi_1(N, x)$  and since  $uv$  is cyclically reduced this implies  $u = v$ . If  $\gamma'$  is not direct it contains a nullhomotopic loop  $\mu$  which cannot lie in  $D^0$  and by the above cannot be contained in  $\delta$  or in  $\varepsilon$ . The vertex  $k$  of  $\mu$  must lie in  $K$ , and this then implies that  $\mu$  contains the whole of  $\delta$  or the whole of  $\varepsilon$ . In the first case  $uv^{-1}$  is of the form  $uzv$  where  $z, v$  are subwords and either  $uz$  is null and  $v$  is non-empty, or  $vu$  is null and  $z$  is non-empty. Therefore either  $vu = y^{-1}uu$  or  $uv = uuz^{-1}$  where the arc  $kp_1p_2$  or  $p_1p_2k$  of  $\gamma$  gives rise to the subword  $uu$ . It is easy to verify that this is impossible since  $\gamma$  is simple and  $k$  lies between the arcs  $\kappa_1$  and  $\kappa_2$  in  $D$ . Thus  $\mu$  cannot contain the whole of  $\delta$ .

Similarly, it cannot contain the whole of  $\varepsilon$ . This is a contradiction, so  $\gamma'$  is direct.

We now invoke the theory of *winding numbers* developed in [7]. The following is a brief summary of part of the theory that we shall need here, with the non-orientable case included for completeness. We refer to [7, §§ 1, 2 and 5] for precise definitions and further explanations.

The *winding number*  $\omega_X(\gamma)$  of an oriented regular closed curve  $\gamma$  with respect to a (continuous) non-vanishing vector field  $X$  on a (smooth) surface  $N$  ( $\partial N \neq \emptyset$ ) is defined to be the number of times the tangent vector to  $\gamma$  rotates relative to the  $X$ -vector on going once round  $\gamma$ , reduced mod 2 if  $N$  is non-orientable<sup>1</sup>. If  $N$  is orientable the sign depends on a choice of orientation of  $N$ . If two curves are direct and homotopic (but not nullhomotopic) they have the same winding number ([7, Theorem (5.3)]). Homotopy classes (base-point free) correspond to conjugacy classes of elements of  $\pi_1(N, x)$ , and so we can define the *winding number*  $\omega_X(C)$  of the conjugacy class  $C \subset \pi_1(N, x)$  as an element of  $\mathbb{Z}$  ( $N$  orientable) or  $\mathbb{Z}_2$  ( $N$  non-orientable) by

$$\omega_X(C) = \omega_X(\gamma)$$

where  $\gamma$  represents  $C$  and is direct. Finally, for  $c \neq 1 \in \pi_1(N, x)$  let

$$\omega_X(c) = \omega_X(C)$$

where  $C$  = conjugacy class of  $c$ , and let  $\omega_X(1) = 1$ .

Now let  $X$  be any continuous non-vanishing vector field on the surface  $N = M - Q$  which we are considering. Such a vector field exists since  $\partial N \neq \emptyset$ .

If  $w$  is a word representing an element  $[w]'$  of  $\pi_1(N, x)$  we shall write  $\omega_X(w)$  instead of  $\omega_X([w]')$ .

**(2.2) Lemma.**  $\omega_X(\gamma') = \omega_X(\delta) + \omega_X(\varepsilon)$ .

*Proof.* Immediate, since  $\gamma'$  is the composition  $\delta \circ \varepsilon$ .

**(2.3) Lemma.** If  $u \neq v$  then  $\omega_X(uv^{-1}) = \omega_X(u) + \omega_X(v^{-1})$  where  $u, v$  are as in Lemma (2.1).

*Proof.* Immediate from Lemmas (2.1) and (2.2) and the definition of  $\omega_X(u)$  etc.

This is a key result in the construction of the algorithm. It completes our study of the simple case.

<sup>1</sup> In the non-orientable case with  $\gamma$  orientation-reversing it is necessary to choose a starting-point  $x_0$  on  $\gamma$  at which the tangent to  $\gamma$  is in the same direction as  $X(x_0)$ , and to transport an orientation at  $x_0$  around  $\gamma$ : see [7, § 5].

*The Non-Simple Case*

The following lemma applies whether  $M$  is orientable or not.

**(2.4) Lemma.** *For every conjugacy class  $C$  containing elements of the form  $[W]$  where  $W$  is completely reduced and not of the form  $VV$  for any subword  $V$  there is a regular elementary representative curve  $\gamma$  with  $\text{tw}(\gamma) = W$  and with the following property. If  $\xi$  and  $\eta$  are any two paths in  $\gamma$  with the same beginning points and same end points, and  $\xi \cap \eta$  consists of a finite number of points, then the closed curve consisting of the composition  $\xi \circ \eta^{-1}$  is not nullhomotopic in  $N$ .*

*Proof.* Let  $\gamma_1$  be any regular elementary representative curve for  $C$  with  $\text{tw}(\gamma) = W$ , and suppose  $\xi$  and  $\eta$  are two paths of  $\gamma_1$  described as above, but with  $\xi \circ \eta^{-1}$  nullhomotopic (in  $N$ ). Let  $\xi'$  be a path covering  $\xi$  in the universal cover  $M'$  of  $M$  and let  $\eta'$  be a lift of  $\eta$  to  $M'$  starting at the starting point of  $\xi'$ . Each of  $\xi'$  and  $\eta'$  is an arc since  $\gamma$  is direct (Lemma (1.10)). Since  $\xi \circ \eta^{-1}$  is nullhomotopic it follows by the homotopy lifting property that  $\eta'$  ends at the end point of  $\xi'$ . Suppose without loss of generality that the number of points  $\xi' \cap p^{-1}(P_1)$  is greater than or equal to the number of points of  $\eta' \cap p^{-1}(P_1)$  where  $P_1$  is the set of double points of  $\gamma_1$  and  $p: M' \rightarrow M$  is the covering map; if this is not so, rename  $\xi$  and  $\eta$ . Let  $\xi''$  and  $\eta''$  denote arcs in  $p^{-1}(\gamma_1)$  obtained by extending  $\xi'$  and  $\eta'$  slightly, i.e. to include no more points of  $p^{-1}(P_1)$  or points of  $p^{-1}(\gamma_1 \cap \Sigma^*)$ .

Let the number of points of  $\eta' \cap p^{-1}(P_1)$  (including the end-points of  $\eta'$ ) be  $n (\geq 2)$ , the number of those belonging to  $\xi' \cap \eta'$  being  $k (\geq 2)$ . There is a homotopy of  $\xi''$  (keeping its end-points fixed) to a path  $h\xi''$  near  $\eta''$  which meets  $\eta''$  exactly once or not at all and which meets  $p^{-1}(\gamma_1)$  a total of  $n - k + 1$  or  $n - k$  times respectively (not counting the end-points of  $h\xi''$ ). Suppose  $h\xi''$  to be chosen sufficiently close to  $\eta''$  so that  $h\xi'' \cap p^{-1}(\Sigma^*)$  has the same number of points as  $\eta'' \cap p^{-1}(\Sigma^*)$ . Since the end-points of  $\xi'$  project to distinct points of  $M$  (otherwise  $\xi$  and  $\eta$  would be closed curves and  $\gamma = \xi \circ \eta$  would be homotopic in  $N$  to  $2\xi$ , contradicting the hypothesis on the conjugacy class  $C$ ) the homotopy of  $\xi''$  to  $h\xi''$  can be chosen to induce a homotopy of  $\gamma_1$  on  $M$ , keeping  $\eta$  fixed, to a curve  $\gamma_2$  whose number of self-intersections is less than that of  $\gamma_1$ . Since  $\xi \circ \eta^{-1}$  is nullhomotopic on  $N$  and  $\pi_1(N, x)$  is the free group on the generators of  $\pi_1(M, x)$  it follows that the contribution to  $\text{tw}(\gamma_1)$  from  $\xi$  is exactly the same as the contribution from  $\eta$ , which is the same as the contribution from  $ph\xi''$ , and so  $\text{tw}(\gamma_1) = \text{tw}(\gamma_2)$ . Without altering  $\text{tw}(\gamma_2)$  or increasing the number of self-intersections we may now replace  $\gamma_2$  by a regular curve which is an elementary representative curve. By repeating this whole process a finite number of times we must eventually obtain a regular curve  $\gamma_m$  which is an elementary representative curve for  $C$  satisfying

the condition of the lemma, since a curve with no self-intersections automatically satisfies the condition. The lemma is thus proved.

An elementary representative curve satisfying the condition of Lemma (2.4) will be called *minimal*.

Suppose now that  $c \in \pi_1(M, x)$  does not contain simple curves. Suppose the conjugacy class  $C$  of  $c$  is as in Lemma (2.4), and let  $\gamma$  be a minimal regular elementary representative curve for  $C$  lying in  $N$ . The arcs of  $\mathcal{C}$  cannot all be disjoint. Assume the construction of  $\mathcal{C}$  and the parametrization of  $\gamma$  to be such that whenever two arcs  $\kappa_1, \kappa_2 \in \mathcal{C}$  meet, the tangents to  $\kappa_1$  and  $\kappa_2$  at  $r = \kappa_1 \cap \kappa_2$  coincide.

Let  $\kappa_1, \kappa_2$  be two arcs of  $\mathcal{C}$  which meet; let  $r = \kappa_1 \cap \kappa_2$ . Let  $\delta_r$  be the curve which begins at  $r$  and proceeds along  $\kappa_1$  and then continues around  $\gamma$  until first returning to  $r$ , and let  $\varepsilon_r$  be the curve which begins at  $r$  and proceeds along  $\kappa_1^{-1}$  and then continues around  $\gamma^{-1}$  until first returning to  $r$ , each parametrized so that  $\gamma$  is the composition  $\delta_r \circ \varepsilon_r^{-1}$ . Now the composition  $\delta_r \circ \varepsilon_r$  has two cusps at  $r$ ; let  $\gamma'_r$  be a regular curve in  $N$ , which is the same as  $\delta_r \circ \varepsilon_r$  outside a closed disc  $B$  where  $r \in B^0 \subset B \subset D^0$  and  $(B \cap \mathcal{C}) \subset \kappa_1 \cup \kappa_2$ , obtained by rounding off the cusps of  $\delta_r \circ \varepsilon_r$  at  $r$  in such a way that  $B \cap \gamma'_r$  is the disjoint union of two arcs. If  $\text{tw}(\gamma) = (W)$  then  $(W) = (uv)$  where  $(u) = \text{tw}(\delta_r)$ ,  $(v) = \text{tw}(\varepsilon_r^{-1})$ , and  $\text{tw}(\gamma'_r) = (uv^{-1})$ .

In order to save space in the proofs which follow we now revert to our assumption that  $M$  is *orientable*. The results still hold, however, in the non-orientable case.

Clearly  $\delta_r, \varepsilon_r$  are not nullhomotopic, since  $\gamma$  is direct.

**(2.5) Lemma.** *The curves  $\delta_r$  and  $\varepsilon_r$  are direct on  $N$ .*

*Proof.* Suppose  $\delta_r$  contains a nullhomotopic loop  $\lambda$  with vertex  $s$ . Since  $\gamma$  is direct,  $\lambda$  must contain  $r$ . Thus  $\lambda = \xi \circ \eta^{-1}$  where  $\xi$  and  $\eta$  are both paths in  $\gamma$  from  $s$  to  $r$ . This contradicts the minimality of  $\gamma$  and hence  $\delta_r$  contains no nullhomotopic loop and is therefore direct. Similarly,  $\varepsilon_r$  is direct.

Now let  $X$  be any continuous non-vanishing vector field on  $N$  as before.

**(2.6) Lemma.** *If  $\gamma'_r$  is also direct on  $N$  then*

$$\omega_X(uv^{-1}) = \omega_X(u) + \omega_X(v^{-1}) \pm 1.$$

*Proof.* By choosing  $B$  sufficiently small the contributions to  $\omega_X(\gamma'_r)$  from  $\delta_r$  and  $\varepsilon_r$  outside  $B$  may be made arbitrarily close to  $\omega_X(\delta_r)$ ,  $\omega_X(\varepsilon_r)$  respectively, and the contribution from  $\gamma'_r \cap B$  may be made arbitrarily close to  $\pm 1$  since the contributions from rounding off the cusps are both arbitrarily close to  $+\frac{1}{2}$  or to  $-\frac{1}{2}$ . See Fig. 2. Hence  $\omega_X(\gamma'_r)$  may be made arbitrarily close to  $\omega_X(\delta_r) + \omega_X(\varepsilon_r) \pm 1$ , and since winding numbers are

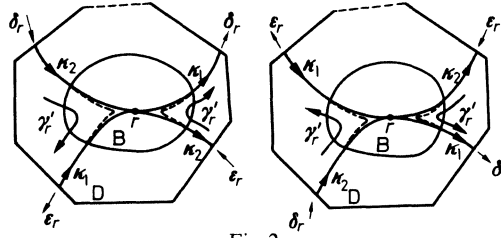


Fig. 2

integer-valued we have

$$\omega_X(\gamma'_r) = \omega_X(\delta_r) + \omega_X(\epsilon_r) \pm 1.$$

The result then follows from Lemma (2.5) and the definitions of  $\omega_X(u)$  etc.

*Note.* In the non-orientable case it is necessary to pay close attention to starting-points and to transportation of orientations around curves.

**(2.7) Lemma.** *Every minimal elementary representative curve  $\gamma$  for  $C$  either contains at least one double point  $r_*$  for which  $\gamma'_{r_*}$  is direct on  $N$ , or has total word  $(W)$  with  $W = V^m$  where  $V$  is a subword and  $m \geq 2$ .*

*Proof.* For  $\kappa \in \mathcal{C}$  let  $e(\kappa)$  denote the end-point (as opposed to the beginning-point) of  $\kappa$ . If  $\kappa_i \cap \kappa_j \neq \emptyset$  and  $r = \kappa_i \cap \kappa_j$  let  $D(r)$  denote the “triangle” (a topological 2-disc) in  $D$  bounded by the arcs  $re(\kappa_i)$ ,  $e(\kappa_i)e(\kappa_j)$ ,  $e(\kappa_j)r$  in  $\kappa_i$ ,  $\partial D$  and  $\kappa_j$ , respectively, and not containing the rest of  $\kappa_i \cup \kappa_j$ . For  $\kappa_1, \kappa_2 \in \mathcal{C}$  write  $\kappa_1 \sim \kappa_2$  if, after identifying segments of  $\partial D$  to give  $\Sigma^*$ , the points  $e(\kappa_1)$  and  $e(\kappa_2)$  lie on the same curve  $\chi$  of  $\Sigma^*$  and  $\kappa_1$  and  $\kappa_2$  meet  $\chi$  from the same side. Let  $\tau(\kappa)$  denote the successor to  $\kappa$ , i.e. the arc of  $\mathcal{C}$  which follows  $\kappa$  in the sequence of the arcs which describes  $\gamma$ ; define inductively  $\tau^0(\kappa) = \kappa$ ,  $\tau^n(\kappa) = \tau(\tau^{n-1}(\kappa))$ ,  $n = 1, 2, \dots$ . Given any double point  $r$  of  $\gamma$  we have  $r = \kappa_i \cap \kappa_j$  for some  $\kappa_i, \kappa_j \in \mathcal{C}$ ; let  $n(r) \leq \infty$  be the least upper bound of all integers  $n$  such that  $\tau^s(\kappa_i) \sim \tau^s(\kappa_j)$  for all  $s$  with  $0 \leq s \leq n$ . If  $\kappa_i \sim \kappa_j$  put  $n(r) = -1$ . Note that if  $n(r) = \infty$  then  $\tau^s(\kappa_i) \sim \tau^s(\kappa_j)$  for all integers  $s$ , and it is easy to see that in that case some non-trivial cyclic permutation of  $W$  is the same as  $W$  (where  $(W) = \text{tw}(\gamma)$ ) and hence that  $W$  is of the form  $V^m$  where  $V$  is a subword and  $m \geq 2$ . We thus assume  $n(r) < \infty$  for the remainder of the proof.

Let  $n_0$  be the minimum of  $n(r)$  taken over all double points  $r$ , and let  $r_0$  be a double point for which  $n(r_0) = n_0$ . Number the arcs of  $\mathcal{C}$  so that  $r_0 = \kappa_1 \cap \kappa_2$ , and construct the curve  $\gamma'_{r_0}$  as above. If  $\gamma'_{r_0}$  is direct we take  $r_* = r_0$ . If  $\gamma'_{r_0}$  is not direct it contains a nullhomotopic loop  $\lambda$  with vertex  $r_1$ , say, and since  $\text{tw}(\gamma)$  is completely reduced  $\lambda$  must contain at least one of the two arcs of  $\gamma'_{r_0} \cap B_0$  (where  $B_0$  denotes the small disc containing  $r_0$

inside which the cusps of  $\delta_{r_0} \circ \varepsilon_{r_0}$  are rounded off to give  $\gamma'_{r_0}$  otherwise  $\gamma$  would contain  $\lambda$ . In fact, since  $\gamma$  is minimal,  $\lambda$  must contain both arcs of  $\gamma'_{r_0} \cap B_0$ , since otherwise two arcs  $\xi, \eta$  going from  $r_1$  to  $r_0$  (or  $r_0$  to  $r_1$ ) in  $\gamma$  would be such that  $\xi \circ \eta^{-1}$  is nullhomotopic. Hence  $\lambda$  contains the whole of  $\delta_{r_0}$  or the whole of  $\varepsilon_{r_0}$ , and by renumbering  $\kappa_1$  and  $\kappa_2$  if necessary we may assume  $\lambda$  contains  $\delta_{r_0}$ . This means  $u$  must be of the form  $pq$  and  $v = phq$  where  $p, q, h$  are (possibly empty) subwords. The complementary loop to  $\lambda$  in  $\gamma$  contributes the subword  $h$ . We have  $r_1 \in \tau^l(\kappa_2)$  where  $l = \text{length of } p$  (taken to be zero if  $p$  is empty).

Let  $\kappa_3 = \tau^l(\kappa_2)$ , and let  $\kappa_4$  be the other arc of  $\mathcal{C}$  containing  $r_1$  (i.e.  $r_1 \in \kappa_3 \cap \kappa_4$ ).

First suppose  $n_0 \geq 0$ , so  $\kappa_i \sim \kappa_j$  for all  $\kappa_i, \kappa_j$  with  $\kappa_i \cap \kappa_j \neq \emptyset$ . Corresponding to the fact that  $u$  and  $v$  both begin with the subword  $p$  we have

$$\tau^s(\kappa_1) \sim \tau^s(\kappa_2), \quad s = 0, 1, \dots, l-1$$

and then since  $v = phq$  and  $u$  also ends with  $q$  we have

$$\tau^s(\kappa_4) \sim \tau^{l+s}(\kappa_1), \quad s = 0, 1, \dots, l'-1$$

where  $l' = \text{length of } q$ . However,  $\tau^{l'}(\kappa_4) = \kappa_1$  and  $\tau^{l+l'}(\kappa_1) = \kappa_2$  and so in fact

$$\tau^s(\kappa_4) \sim \tau^{l+s}(\kappa_1), \quad s = 0, 1, \dots, l' + n_0.$$

By definition of  $n(r_1)$  we have

$$\tau^s(\kappa_3) \sim \tau^s(\kappa_4), \quad s = 0, 1, \dots, n(r_1)$$

and hence

$$\tau^{l+s}(\kappa_1) \sim \tau^{l+s}(\kappa_2), \quad s = 0, 1, \dots, \min(n(r_1), l' + n_0)$$

because  $\kappa_3 = \tau^l(\kappa_2)$ . Therefore by definition of  $n_0 = n(r_0)$  we have

$$l + \min(n(r_1), l' + n_0) \leq n_0.$$

The choice of  $r_0$  implies  $n_0 \leq n(r_1)$  and so

$$l = 0$$

and

$$n(r_1) = n_0 = n(r_0)$$

as  $l'$  cannot be zero if  $l = 0$ . Thus

$$\kappa_3 = \kappa_2$$

and  $r_1$  lies between  $r_0$  and  $e(\kappa_2)$  on  $\kappa_2$ . Also the sub-arc  $r_1 e(\kappa_4)$  of  $\kappa_4$  is disjoint from the sub-arc  $r_0 e(\kappa_1)$  of  $\kappa_1$ , since if  $r' \in r_1 e(\kappa_4) \cap r_0 e(\kappa_1)$  there would exist two arcs  $\xi, \eta$  from  $r'$  to  $r_0$  in  $\gamma$  with  $\xi \circ \eta^{-1}$  nullhomotopic, thus contradicting the minimality of  $\gamma$ . Hence the “triangle”  $D(r_1)$  is strictly contained in  $D(r_0)$ .

If  $n_0 = -1$  then we arrive at the same results as these, although the proof has to be worded slightly differently.

Now either  $\gamma'_{r_1}$  is direct (in which case we take  $r_* = r_1$ ) or by replacing  $r_0$  by  $r_1$  in the above (remember  $n(r_1) = n_0$ ) we obtain a double point  $r_2$  with  $n(r_2) = n_0$  and  $D(r_2)$  strictly contained in  $D(r_1)$ . The point  $r_2$  lies on  $\kappa_2 = \kappa_3$  or on  $\kappa_4$  according to which of  $\delta_{r_1}$  or  $\varepsilon_{r_1}$  is contained in the nullhomotopic loop of  $\gamma'_{r_1}$ . If  $\gamma'_r$  is not direct for any double point  $r$  we thus obtain by induction a sequence of double points  $r_0, r_1, r_2, \dots$  with  $n(r_i) = n_0$  and  $D(r_{i+1})$  strictly contained in  $D(r_i)$ ,  $i = 0, 1, 2, \dots$ . The sequence must have repeats since the number of double points is finite. However, it is impossible for the sequence to repeat if  $D(r_{r+1}) \subset D(r_i)$  strictly. This contradiction shows that there must exist some double point  $r_*$  for which  $\gamma'_{r_*}$  is direct, which proves the lemma.

We now have all the ingredients that are needed for the construction of the algorithm, except in the case when  $\text{tw}(\gamma)$  is of the form  $(V^m)$  ( $m \geq 2$ ). However, this case will easily be disposed of using the following well-known result. For a proof see e.g. [17] or [19], or [10, Theorem 4.2].

**(2.8) Lemma.** *Let  $c = b^m \in \pi_1(M, x)$  where  $b$  is non-trivial and  $m \geq 2$ . Then  $c$  contains simple closed curves if and only if  $m = 2$  and  $b$  contains simple closed orientation-reversing curves.*

### § 3. The Algorithm

We have been assuming above that the number of generators of  $\pi_1(M, x)$  is at least 5. In order to give a comprehensive solution to the problem of finding whether an element of  $\pi_1(M, x)$  contains simple closed curves we shall first consider the exceptional cases (i.e. where  $\pi_1(M, x)$  has at most four generators) and then give the algorithm for the general case (including  $M$  non-orientable). In cases (i)–(vi) we shall by abuse of notation not distinguish between an element of  $\pi_1(M, x)$  and a word which represents it.

*Case (i):  $\pi_1(M, x) = \{s_1, \dots, s_r | s_1 \dots s_r = 1\}$  where  $1 \leq r \leq 4$ . ( $M = \text{disc}$  with  $r - 1$  holes.)*

When  $r = 1$  there is no problem. Otherwise, we observe that  $\pi_1(M, x)$  can be presented as the free group generated by  $s_1, \dots, s_{r-1}$ . The whole of §§ 1 and 2 continue to be valid with *cyclic reduction* (see Definition (1.4)) in the free group playing the role of *complete reduction* in the original presentation. For example, the analogues of Lemmas (1.8) and (1.10) continue to hold in this context. We can therefore apply to this case the algorithm as it is formulated for the general case (vii) below, with  $s_r$  replaced by  $s_{r-1}^{-1} \dots s_1^{-1}$  and “completely reduced” then replaced by “cyclically reduced”. Of course, these remarks apply equally well when  $r > 4$ .

Case (ii):  $\pi_1(M, x) = \{u_1 | u_1^2 = 1\}$ . ( $M = \text{projective plane.}$ )

Clearly both homotopy classes contain simple closed curves.

Case (iii):  $\pi_1(M, x) = \{u_1, s_1 | u_1^2 s_1 = 1\}$ . ( $M = \text{Möbius strip.}$ )

Here  $\pi_1(M, x)$  can be considered as the infinite cyclic group generated by  $u_1$ , and the only non-trivial elements represented by simple closed curves are  $u_1^{\pm 1}$  and  $u_1^{\pm 2}$ .

Case (iv):  $\pi_1(M, x) = \{u_1, s_1, s_2 | u_1^2 s_1 s_2 = 1\}$ . ( $M = \text{Möbius strip with (open) disc removed.}$ )

Bearing in mind Case (iii) it is easy to show that the only non-trivial elements represented by simple closed curves are those conjugate to  $u_1$ ,  $u_1^2$ ,  $u_1 s_1$ ,  $(u_1 s_1)^2$ ,  $s_1$ ,  $u_1^2 s_1$ , or their inverses. In order to check whether or not a given word represents an element conjugate to one of these one replaces  $s_2$  by  $s_1^{-1} u_1^{-2}$  and then checks for the conjugacy in the free group generated by  $s_1$  and  $u_1$ .

Case (v):  $\pi_1(M, x) = \{a_1, b_1 | a_1 b_1 a_1^{-1} b_1^{-1} = 1\}$ . ( $M = \text{torus.}$ )

It is well known that the element  $a_1^p b_1^q$  contains simple curves if and only if  $(p, q) = (0, 0)$ ,  $(0, \pm 1)$ ,  $(\pm 1, 0)$  or  $p$  and  $q$  are coprime. This may easily be proved by observing that any non-trivial element  $c$  contains simple closed curves if and only if there is an automorphism of  $\pi_1(M, x)$  taking  $c$  to  $a_1$ . A different proof can be found in [17].

Case (vi):  $\pi_1(M, x) = \{u_1, u_2 | u_1^2 u_2^2 = 1\}$ . ( $M = \text{Klein bottle.}$ )

It is easy to deduce from the results about isotopy classes of simple curves on  $M$  in [12] that the only simple curves are those whose homotopy classes are conjugate to  $1$ ,  $u_1^{\pm 1}$ ,  $u_2^{\pm 1}$  or  $u_1^{\pm 2}$ . A full list of these conjugate elements may be computed easily by writing  $u_1 = a$ ,  $u_1 u_2 = b$  so that  $\pi_1(M, x) = \{a, b | aba^{-1}b = 1\}$ , and it consists of the elements  $1$ ,  $(u_1 u_2)^{\pm 1}$ ,  $u_1^{\pm 2}$ , and  $(u_1(u_1 u_2)^n)^{\pm 1}$  for any  $n = 0, \pm 1, \pm 2, \dots$ .

Case (vii): *The general case.*

Given an element  $c \in \pi_1(M, x)$  first express  $c$  as a word  $w$  in the generators  $a_i, b_j, s_k$  (or  $u_i, s_k$ ) of  $\pi_1(M, x)$  and their inverses, and then reduce  $w$  completely (Definition (1.4)). Denote the resulting completely reduced word by  $W$ . If  $W$  is empty (i.e.  $c = 1$ ) or consists of only one letter, then  $[W]$ , and hence  $c$ , contains simple closed curves. Therefore suppose  $W = x_1 x_2 \dots x_p$  where  $p \geq 2$ . There are  $\frac{1}{2}p(p-1)$  ways of expressing the cyclic word  $(W)$  as  $(uv)$  where  $u, v$  are non-empty subwords: call each such expression a *division* of  $W$ .

Let  $X$  be any continuous non-vanishing vector field on  $N = M - Q$ , where  $Q$  is an open neighbourhood of the base-point of a dual dissection of  $M$  as in § 2.

If  $c$  contains simple curves it follows from the results in § 2, culminating in Lemma (2.3), that

$$\omega_X(uv^{-1}) = \omega_X(u) + \omega_X(v^{-1})$$

for every division  $(W) = (uv)$  with  $u \neq v$ .



If  $c$  does not contain simple curves, then by Lemmas (2.6), (2.7), and (2.8) either  $W = V^m (m \geq 2)$  or there exists a division  $(W) = (uv)$  with

$$\omega_X(uv^{-1}) \neq \omega_X(u) + \omega_X(v^{-1}).$$

Therefore the final result may be stated as follows:

*The element  $c$  contains simple curves if and only if a completely reduced word  $W$  obtained from  $c$  as in § 1 satisfies one of the conditions:*

- (i)  $W$  is empty (i.e.  $c = 1$ ) or contains only one letter,
- (ii)  $W = V^2$  where  $[V]$  contains an orientation-reversing simple closed curve,
- (iii)  $W$  contains at least 2 letters,  $W \neq V^m$  for any  $V$  and any  $m \geq 2$ , and for every division  $(W) = (uv)$  the equation

$$\omega_X(uv^{-1}) = \omega_X(u) + \omega_X(v^{-1})$$

is satisfied, where  $X$  is any continuous non-vanishing vector field on  $N = M - Q$ .

*Note.* In case (ii) the algorithm must be applied in turn to  $[V]$  (which is completely reduced since  $(V^2)$  is completely reduced), but case (ii) will not arise a second time since if  $V = Z^2$  then  $[V]$  can contain only orientation-preserving curves.

The algorithm is clearly not complete, however, until we are able effectively to compute whether or not the equation

$$\omega_X(uv^{-1}) = \omega_X(u) + \omega_X(v^{-1})$$

is satisfied for each division  $(W) = (uv)$ . To do this we will invoke the Formulae 1 or 2 of [7, § 7]: see below.

It is to be expected that if the equation is satisfied for one choice of vector field  $X$  it will be satisfied for all vector fields: whether  $c$  contains simple curves or not does not depend on  $X$ . Now  $\omega_X(u)$  means  $\omega_X([u]')$ , but we cannot necessarily apply Formulae 1 (1') or 2 from [7] immediately, since we are using the generators of  $\pi_1(M, x)$  to generate the free group  $\pi_1(N, x)$ . However, to apply the machinery of [7] to  $N$  we need only introduce another generator  $s_{r+1}$  corresponding to the boundary of  $Q$ . If  $\bar{u}$  denotes a *cyclically reduced* word obtained from  $u$  (here  $\bar{u}$  is not necessarily completely reduced) then  $\bar{u}$  contains no subword which is null in  $\pi_1(N, x)$ . Hence  $\omega_X([u]') = \omega_X([\bar{u}]')$  since  $[u]'$  is conjugate to  $[\bar{u}]'$ , and then calculating  $\omega_X([\bar{u}]')$  by Formulae 1' or 2 of [7] we have

$$\omega_X(u) - \varphi_X(\bar{u}) = T(\bar{u}) \quad (M \text{ orientable})$$

or

$$\omega_X(u) - \varphi'_X(\bar{u}) = T(\bar{u}) \pmod{2} \quad (M \text{ non-orientable}),$$

where  $\varphi_X(\varphi'_X)$  is the unique homomorphism  $\pi_1(N, x) \rightarrow \mathbb{Z}(\mathbb{Z}_2)$  coinciding with the function  $\omega_X$  on the chosen generators of  $\pi_1(M, x)$  (here we

identify a word in the generators with the element of  $\pi_1(N, x)$  which it represents), and where  $T(\bar{u})$  is defined as follows:

$$T(\bar{u}) = P(\bar{u}) - (\bar{A}(\bar{u}) + B(\bar{u}) + \bar{S}(\bar{u}))$$

where  $\bar{u} = y_1 y_2 \dots y_q$  ( $q > 1$ ) and (with  $y_{q+1}$  denoting  $y_1$ )

$P(\bar{u})$  = number of  $y'_m$ s ( $1 \leq m \leq q$ ) for which  $y_m^{-1}$  comes *before*  $y_{m+1}$  in the ordering

$$a_1, b_1^{-1}, a_1^{-1}, b_1, a_2, b_2^{-1}, a_2^{-1}, b_2, \dots, a_g^{-1}, b_g, s_1, s_1^{-1}, s_2, \dots, s_r^{-1}$$

for  $M$  orientable, or the ordering

$$u_1, u_1^{-1}, u_2, u_2^{-1}, \dots, u_n, u_n^{-1}, s_1, s_1^{-1}, s_2, \dots, s_r^{-1}$$

for  $M$  non-orientable,

$\bar{A}(\bar{u})$  = number of  $y'_m$ s which are  $a_i^{-1}$  for some  $i$ ,

$B(\bar{u})$  = number of  $y'_m$ s which are  $b_j$  for some  $j$ , and

$\bar{S}(\bar{u})$  = number of  $y'_m$ s which are  $s_k^{-1}$  for some  $k$ .

If  $q = 1$  we define  $T(\bar{u}) = 0$  if  $\bar{u} \neq u_i^{-1}$  ( $1 \leq i \leq n$ ), or  $T(\bar{u}) = 1 \in \mathbb{Z}_2$  if  $\bar{u} = u_i^{-1}$ , since  $\omega_X(c) + \omega_X(c^{-1}) = 0$  or  $1$  according to whether  $c$  contains orientation-preserving or -reversing curves (see Proposition (2.2) of [7]).

There are similar expressions involving  $\omega_X(v^{-1})$  and  $\omega_X(uv^{-1})$ .

Now  $\varphi_X(\bar{u}) = \varphi_X(u)$  etc., and

$$\varphi_X(uv^{-1}) = \varphi_X(u) + \varphi_X(v^{-1})$$

since  $\varphi_X$  is a homomorphism. The same applies to  $\varphi'_X$ . Hence the formula

$$\omega_X(uv^{-1}) = \omega_X(u) + \omega_X(v^{-1})$$

reduces to

$$T(\overline{uv^{-1}}) = T(\bar{u}) + T(\overline{v^{-1}})$$

in the orientable case, or

$$T(\overline{uv^{-1}}) \equiv T(\bar{u}) + T(\overline{v^{-1}}) \pmod{2}$$

in the non-orientable case, and is patently independent of the vector field  $X$ . Therefore we see that condition (iii) in the algorithm may be replaced by:

(iii)'  $W$  contains at least 2 letters,  $W \neq V^m$  for any  $V$  and for any  $m \geq 2$ , and

$$T(\overline{uv^{-1}}) \equiv T(u) + T(\overline{v^{-1}}) \pmod{2}$$

for every division  $(W) = (uv)$ .

A few examples illustrating the working of the algorithm are given in [6].

#### § 4. Application of Winding Numbers to Homeomorphisms of Surfaces

By [9], [11, 13], and [12, 14], [5] it is known that any homeomorphism of a closed surface  $M$  may be expressed as a product of

- (i) homeomorphisms isotopic to the identity;
- (ii) “twists” (defined in [11]) about members of a finite family of simple closed curves as shown in Fig. 3 ( $M$  orientable) or in [5, Fig. 2] ( $M$  non-orientable);

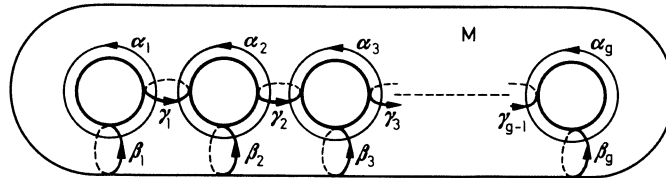


Fig. 3

and if  $M$  is orientable

- (iii) a fixed orientation-reversing homeomorphism;

or if  $M$  is non-orientable

- (iii)' a  $Y$ -homeomorphism, as defined in [12].

A homeomorphism  $h$  keeping  $x$  fixed induces an automorphism  $h_* : \pi_1(M, x) \rightarrow \pi_1(M, x)$ , and if  $x$  is not fixed then  $h_*$  can be defined only up to inner automorphism. However, if  $C$  is a conjugacy class of elements in  $\pi_1(M, x)$  then  $h_*C$  is a well-defined conjugacy class. If  $\omega_X(k_*C)$  is known<sup>2</sup> (for some vector field  $X$ ) whenever  $k$  is a homeomorphism isotopic to the identity, to a twist or to a  $Y$ -homeomorphism, then a knowledge of  $\omega_X(h_*C)$  for a general  $h$  leads to information about the composition of  $h$  in terms of twists, etc. In particular, taking  $h = \text{identity}$  may throw some light on the problem of finding a set of relations for the generators of the *homeotopy group* or *mapping class group*  $\Lambda_M$  which is the group of homeomorphisms of  $M$  factored by those isotopic to the identity. (See e.g. [1, 2] for some recent results relating to this problem.)

The following three lemmas concerning the effects of homeomorphisms on winding numbers are given without proof. The first is trivial (since  $k_*C = C$ ), the second is elementary, and the third is a little more complicated technically but essentially a matter of straightforward verification. All non-orientable surfaces will be supposed to have even genus so that winding numbers of conjugacy classes are defined, and vector fields will be assumed to be *continuous and non-vanishing* on  $M$  minus a point.

<sup>2</sup> We recall from [7, § 6] that for a closed surface  $M$  winding numbers of conjugacy classes are defined as elements of  $\mathbb{Z}_{2g-2}$  ( $M$  orientable) or  $\mathbb{Z}_2$  ( $M$  non-orientable, and of even genus), and vector fields are taken to be defined on  $M$  minus one point.

**(4.1) Lemma.** *If  $k$  is isotopic to the identity then  $\omega_X(k_*C) = \omega_X(C)$  for all  $C$ .*

**(4.2) Lemma.** *Let  $B, C$  be conjugacy classes in  $\pi_1(M, x)$ , where  $B$  is represented by a simple closed curve  $\beta$ , and let  $h_\beta$  be a twist about  $\beta$ . Denote by  $B', C'$  the homology classes corresponding to  $B, C$  with coefficients in  $\mathbb{Z}$  if  $M$  is orientable (and oriented) or in  $\mathbb{Z}_2$  if  $M$  is non-orientable, and let  $n \in (\mathbb{Z} \text{ or } \mathbb{Z}_2)$  be the intersection-number  $(B' \cdot C')$ . Then*

$$\omega_X(h_{\beta*}C) = \omega_X(C) + n\omega_X(B)$$

as an element of  $\mathbb{Z}_{2g-2}$  or  $\mathbb{Z}_2$  as appropriate.

**(4.3) Lemma.** *Let  $y$  be a  $Y$ -homeomorphism defined using a space  $Y \subset M$  (see [12]) and let  $\mu_1, \mu_2$  be two disjoint simple closed orientation-reversing curves on  $Y$  whose  $\mathbb{Z}_2$ -homology classes generate  $H_1(Y; \mathbb{Z}_2)$ . Then*

$$\omega_X(y_*C) = \omega_X(C) + n_1 + n_2$$

where  $n_i \in \mathbb{Z}_2$  ( $i = 1, 2$ ) is the intersection-number of the homology class  $C'$  with the homology class of  $\mu_i$ .

The following result has a proof similar to that of Lemma (4.2) but is more elementary. These Lemmas (4.2) and (4.4) are mutually independent.

**(4.4) Lemma.** *With the notation of Lemma (4.2),  $(h_{\beta*}C)' = C' + nB'$ .*

We now apply the foregoing Lemmas to the word problem in  $A_M$ .

Let  $M$  be a closed orientable surface of genus  $g$ . By [13], the group  $G_M$  of homeomorphisms of  $M$  is generated by an orientation-reversing homeomorphism, homeomorphisms isotopic to the identity, and twists about curves  $\alpha_i$  ( $1 \leq i \leq g$ ),  $\beta_i$  ( $1 \leq i \leq g$ ) and  $\gamma_i$  ( $1 \leq i \leq g-1$ ) as in Fig. 3. Let  $K$  denote the union of all these curves, and let  $N$  be a regular neighbourhood of  $K$  in  $M$ , i.e. a smooth closed neighbourhood of  $K$  which contracts to  $K$ . All twists may be assumed to take place in  $N$ . If  $h \in G_M$  let  $\{h\} \subset A_M$  denote the isotopy class of  $h$ .

Observe that  $H_1(M; \mathbb{Z})$  is the free abelian group generated by the homology classes of the  $\alpha_i$  and  $\beta_i$ , and  $H_1(N; \mathbb{Z})$  is similarly generated by the homology classes of the  $\alpha_i$ ,  $\beta_i$  and  $\gamma_i$ .

**(4.5) Lemma.** *Let  $h \in G_M$  and let  $k$  be a homeomorphism of  $N$  given by  $k = h_1 \dots h_r$  where  $\{h\} = \{h_1\} \dots \{h_r\}$ , and each  $h_i$  is a "generating" twist about a curve in  $K$ . Choose one of the  $\alpha_i$ , say  $\alpha_j$ , and suppose*

$$(k\alpha_j)'' = \sum_{i=1}^g \lambda_i \alpha_i'' + \sum_{i=1}^g \mu_i \beta_i'' + \sum_{i=1}^{g-1} \nu_i \gamma_i''$$

where here " denotes homology class in  $N$ , and the curves of  $K$  are oriented as shown in Fig. 3. Then

(i) if the homeomorphism  $h$  of  $M$  induces the identity on  $H_1(M; \mathbb{Z})$  we have

$$\sum_{i=1}^g \mu_i = 0, \quad \lambda_j = 1, \quad \lambda_i = 0 \ (i \neq j)$$

and (ii) if  $h$  is isotopic to the identity the above equations hold and also

$$\sum_{i=1}^{g-1} (g-i) \mu_i \equiv 0 \pmod{2g-2}.$$

*Proof.* On  $M$  we have  $\beta_i - \beta_{i+1} \sim -\gamma_i$  ( $1 \leq i \leq g-1$ ) and so in case (i) since  $h\alpha_j \sim \alpha_j$  on  $M$  it follows by equating coefficients of homology classes in  $M$  that  $\lambda_j = 1$ ,  $\lambda_i = 0$  ( $i \neq j$ ) and

$$\begin{aligned} 0 &= \mu_1 - v_1 \\ 0 &= \mu_2 + v_1 - v_2 \\ &\vdots \\ 0 &= \mu_{g-1} + v_{g-2} - v_{g-1} \\ 0 &= \mu_g + v_{g-1}. \end{aligned} \tag{*}$$

Hence by addition  $\mu_1 + \dots + \mu_g = 0$ .

By Lemmas (4.2) and (4.4) it follows that for any vector field  $X$  on  $M$  minus one point we have

$$\omega_X(k\alpha_j) = \sum_{i=1}^g \lambda_i \omega_X(\alpha_i) + \sum_{i=1}^g \mu_i \omega_X(\beta_i) + \sum_{i=1}^{g-1} v_i \omega_X(\gamma_i)$$

where we may assume the  $\alpha_i$  etc. to be regular and  $k$  to be smooth. Since  $\beta_i$ ,  $\gamma_i$  and  $\beta_{i+1}^{-1}$  are the boundary components of a surface of genus zero in  $M$  it follows by [7, Lemma (5.7) and § 6] that

$$\omega_X(\beta_i) + \omega_X(\gamma_i) - \omega_X(\beta_{i+1}) \equiv 1 \pmod{2g-2}.$$

Now the  $\omega_X(\alpha_i)$  and  $\omega_X(\beta_i)$  can be given arbitrary values  $\pmod{2g-2}$  by a suitable choice of  $X$  (see [7]; in particular Theorem (5.6) and the end of § 6). In case (ii) since  $h\alpha_j \simeq \alpha_j$  we have

$$\omega_X(k\alpha_j) \equiv \omega_X(\alpha_j) \pmod{2g-2}$$

(see Theorem (5.3) of [7]). Therefore by equating coefficients of the  $\omega_X(\alpha_i)$  and  $\omega_X(\beta_i)$  the Eqs. (\*) of case (i) arise again  $\pmod{2g-2}$  and we obtain also

$$\sum_{i=1}^{g-1} v_i \equiv 0 \pmod{2g-2}.$$

Combining this with (\*) gives the result.

*Note.* In view of the note at the end of § 5 in [7], the hypothesis (i) of Lemma (4.5) suffices to prove the entire result modulo 2.

These equations may be used to give information about  $\Lambda_M$  in some particular cases as follows.

**(4.6) Corollary.** *If  $g = 2$  it is impossible to express the isotopy class of the twist  $h_{\beta_j}$  ( $j = 1$  or  $2$ ) in terms of the other generators of  $\Lambda_M$ .*

(In fact generators and relations for  $\Lambda_M$  are known when  $g = 2$ ; see [2].)

*Proof.* Suppose  $\{h_{\beta_j}\} = \{h\}$  where  $j$  is either 1 or 2 and  $\{h\} = \{h_1\} \dots \{h_r\}$  where none of the  $h_i$  are  $h_{\beta_j}^{\pm 1}$ . Let  $k = h_1 \dots h_r$  and apply the lemma to  $k^{-1}h_{\beta_j}$  and  $\alpha_j$ . The Eqs. (i) and (ii) are

$$\begin{aligned}\lambda_j &= 1, \quad \lambda_i = 0 \ (i \neq j) \\ \mu_1 + \mu_2 &= 0 \\ \mu_1 &\equiv 0 \pmod{2}\end{aligned}$$

where  $\mu_i$  is the coefficient of  $\beta_i''$  in  $(k^{-1}h_{\beta_j}\alpha_j)''$ . But  $\mu_j = \pm 1$  (depending on the orientation of  $M$ ) since  $\alpha_j$  crosses  $\beta_j$  once, and this is a contradiction.

**(4.7) Corollary.** *If  $g = 3$  it is impossible to express  $\{h_{\beta_2}\}$  in terms of the other generators, and if  $\{h_{\beta_j}\} = \{h\}$  where  $h_{\beta_j}^{\pm 1}$  does not appear in  $h$  ( $j = 1, 3$ ) then  $h_{\beta_{4-j}}$  appears in  $h$ .*

(The first (and probably both) of these facts can also be easily deduced from known results about generators and relations in  $\Lambda_M$ ; see [2].)

*Proof.* Applying the lemma to  $k^{-1}h_{\beta_j}$  and  $\alpha_j$  as above gives the equations  $\mu_1 + \mu_2 + \mu_3 = 0$  and  $2\mu_1 + \mu_2 \equiv 0 \pmod{4}$ . These cannot hold when  $\mu_2 = \pm 1$  or when  $\mu_1 + \mu_3 = \pm 1$ .

As  $g$  increases, the information about  $\Lambda_M$  obtained in this way becomes more complicated and correspondingly less useful. Nevertheless, these examples illustrate at least the possibility of using winding numbers to obtain algebraic results. The possibility would become more real if the structure of  $\omega_X : \pi_1(M, x) \rightarrow \mathbb{Z}_{2g-2}$  were better understood.

Some similar results may be derived when  $M$  is non-orientable; if  $n$  is even, using Lemma (4.3). When  $n$  is odd an isotopy will not in general preserve winding numbers, and the above methods cannot be used.

If  $\gamma$  is a simple closed curve homologous to zero ( $\mathbb{Z}$ ) on  $M$ , which we take to be orientable, then (cf. Lemma (4.2)) a twist about  $\gamma$  preserves the homology class and the winding number with respect to any  $X$  of any other curve on  $M$ . Since homeomorphisms exist which preserve homology but not winding numbers (see example below) it follows that

homology-preserving homeomorphisms are not in general expressible as products of twists about curves which are homologous to zero. However, the following conjecture may be true. If so it would provide a useful link between the geometric and algebraic analyses of homeomorphisms.

**Conjecture.** *If  $M$  is orientable ( $\partial M \neq \emptyset$ ) with a given continuous non-vanishing vector field  $X$  on  $M$ , and  $h$  is a diffeomorphism of  $M$  such that*

(i)  $h_* : H_1(M; \mathbb{Z}) \rightarrow H_1(M; \mathbb{Z})$  *is the identity and*

(ii)  $\omega_X(h\gamma) = \omega_X(\gamma)$  *for all regular curves  $\gamma$  on  $M$ ,*

*then  $h$  is a product of twists about curves which are homologous to zero relative to  $\partial M$ .*

Condition (ii) could be replaced by:

(ii)'  $\omega_X(h_* C) = \omega_X(C)$  *for all conjugacy classes  $C$  in  $\pi_1(M, x)$ .*

If  $\partial M = \emptyset$  (where  $X$  is defined on  $M$  minus one point) then the conjecture with (ii)' is the more natural but is easily seen to be false, since  $\omega_X(C)$  is an element only of  $\mathbb{Z}_{2g-2}$ . For example, consider two simple closed regular curves  $\gamma_1, \gamma_2$ , forming the oriented boundary of a surface  $N$  of genus 2 contained in a closed orientable surface  $M$  of genus 3. Then  $\omega_X(\gamma_1) + \omega_X(\gamma_2) \equiv 0 \pmod{4}$  by [7, Lemma (5.7)]. Suppose  $\gamma_1 \sim 0$ , and choose  $X$  such that  $\omega_X(\gamma_1) = 3, \omega_X(\gamma_2) = 1$  (the singularity of  $X$  may be chosen not to lie on  $N$ ). Let  $h = h_{\gamma_1}^2 h_{\gamma_2}^{-2}$ . Then  $h$  preserves homology and preserves  $\omega_X(C)$  ( $\in \mathbb{Z}_4$ ) for all  $C$ , but if  $\gamma$  meets  $\gamma_1$  and  $\gamma_2$  once each then  $\omega_X(h\gamma) = \omega_X(\gamma) \pm 4$ , so  $h$  is not a product of twists about curves homologous to zero.

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