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HOLONOMY DISPLACEMENTS IN THE HOPF BUNDLES OVER $\mathbb{C}H^n$ AND THE COMPLEX HEISENBERG GROUPS

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Abstract. For the "Hopf bundle" $S^1 \to S^{2n,1} \to \mathbb{C}H^n$, horizontal lifts of simple closed curves are studied. Let γ be a piecewise smooth, simple closed curve on a complete totally geodesic surface S in the base space. Then the holonomy displacement along γ is given by

$$V(\gamma) = e^{\lambda\, A(\gamma) i},$$

where $A(\gamma)$ is the area of the region on the surface S surrounded by γ; $\lambda = 1/2$ or 0 depending on whether S is a complex submanifold or not.

We also carry out a similar investigation for the complex Heisenberg group $\mathbb{R} \to \mathcal{H}^{2n+1} \to \mathbb{C}^n$.

1. Introduction

Consider the Hopf fibration $S^1 \to S^3 \to S^2$. Let γ be a simple closed curve on S^2. Pick a point in S^3 over $\gamma(0)$, and take the unique horizontal lift $\widetilde{\gamma}$ of γ. Since $\gamma(1) = \gamma(0)$, $\widetilde{\gamma}(1)$ lies in the same fiber as $\widetilde{\gamma}(0)$ does. We are interested in understanding the difference between $\widetilde{\gamma}(0)$ and $\widetilde{\gamma}(1)$. The following equality was already known [2]:

$$V(\gamma) = e^{\frac{1}{2} A(\gamma) i},$$

where $V(\gamma)$ is the holonomy displacement along γ, and $A(\gamma)$ is the area of the region surrounded by γ.

In this paper, we shall generalize this fact to (higher dimensional) pseudo-spheres and the complex Heisenberg group. First we look at the fibration of the pseudo-sphere $S^{2n,1}$

$$S^1 \to S^{2n,1} \to \mathbb{C}H^n$$

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a principal S^1-bundle over the complex hyperbolic space $\mathbb{C}H^n$. Let S be a complete totally geodesic surface in the base space $\mathbb{C}H^n$, and ξ_S be the pullback bundle over S. Let γ be a piecewise smooth, simple closed curve on S parametrized by $0 \le t \le 1$, and $\widetilde{\gamma}$ its horizontal lift. The pullback over the curve γ is called a *Hopf torus* (so $\widetilde{\gamma}$ is a curve on the Hopf torus). Then

$$\widetilde{\gamma}(1) = e^{\frac{1}{2}A(\gamma)i} \cdot \widetilde{\gamma}(0) \quad \text{or} \quad \widetilde{\gamma}(0),$$

depending on whether S is a complex submanifold or not, where $A(\gamma)$ is the area of the region on the surface S surrounded by γ (See Theorem 3.3).

We also carry out a similar investigation for the complex Heisenberg group. Let $1 \to \mathbb{R} \to \mathcal{H}^{2n+1} \to \mathbb{C}^n \to 1$ be the central short exact sequence of the complex Heisenberg group. Let S be a complete totally geodesic plane in $\mathbb{C}^n$, and ξ_S be the pullback bundle over S. Let γ be a piecewise smooth, simple closed curve on S. Then

$$V(\gamma) = e(\xi_S) \cdot A(\gamma),$$

where $A(\gamma)$ is the area of the region on the surface S surrounded by γ, and the number $e(\xi_S)$ is determined by the equality $[\mathbf{v}, \mathbf{w}] = e(\xi_S)e_{2n+1}$ for an orthonormal basis $\{\mathbf{v}, \mathbf{w}\}$ for the tangent space of S (See Theorem 4.1).

2. Preliminaries

The proof of the statement in the introduction (in the case of the Hopf fibration $S^1 \to S^3 \to S^2$) uses the Gauss-Bonnet theorem. For $S^1 \to S^{2,1} \to \mathbb{C}H^1$, such is not available because the base space is not compact. Therefore, we cannot apply the arguments in [2] directly, and need to develop a new method of proof. It turns out that $S^{2,1}$ is the building blocks for higher dimensional cases.

Let $F \to E \xrightarrow{p} B$ be a principal F-bundle ($F = \mathbb{R}^1$ or S^1) of Riemannian manifolds, with B a 2-dimensional complete manifold and p a Riemannian submersion. For a simple closed cure $\gamma(t), 0 \le t \le 1$ on B, the *holonomy displacement* $V(\gamma)$ along γ is defined as follows: Let $\widetilde{\gamma}(t)$ be the horizontal lift of γ. Then

$$\widetilde{\gamma}(1) = V(\gamma) \cdot \widetilde{\gamma}(0)$$

for some $V(\gamma) \in F$. We shall establish a technical lemma which will be used later.

Lemma 2.1. *Suppose $V(\gamma) = \lambda(\gamma)A(\gamma)\ (F = \mathbb{R}^1)$, or $e^{\lambda(\gamma)A(\gamma)i}\ (F = S^1)$ for a constant $\lambda(\gamma)$, where $A(\gamma)$ is the area of the region on B surrounded by a piecewise smooth simple closed curve γ. If $\lambda(\gamma)$ is constant for all γ's which are the boundaries of rectangular regions, then it is constant for every piecewise smooth simple closed curve γ.*

Proof. Let us assume that $F = \mathbb{R}$. The case of $F = S^1$ will be similar. Let γ_0 be a curve on B. Since the region surrounded by γ_0 is compact, we may assume that this region is contained completely in one local patch. Let

$$\varphi : \mathbb{R}^2 \to U \subset B$$

be a local chart, and $p^{-1}(U) \approx U \times F$. For notational simplicity, we shall identify $\mathbb{R}^2$ with U (and suppress φ). Let $\Omega(U, \gamma_0(0))$ and $\Omega(\mathbb{R}, 0)$ be the space of paths emanating from $\gamma_0(0)$ and $0 \in \mathbb{R}$, respectively. For each $\gamma \in \Omega(U, \gamma_0(0))$, let ω_γ be the unique curve in $\mathbb{R}$ so that $\eta(t) = \gamma(t) \cdot \omega_\gamma(t)$ is the horizontal lift of γ. This defines a map

$$\mathfrak{Z} : \Omega(U) \longrightarrow \Omega(\mathbb{R})$$

by $\mathfrak{Z}(\gamma)(t) = \omega_\gamma(t)$. We use the sup metrics ρ on both $\Omega(U)$ and $\Omega(\mathbb{R})$. That is,

$$\rho(\gamma_1, \gamma_2) = \sup_{t \in [0,1]} d(\gamma_1(t), \gamma_2(t)),$$

where d is the distance function on U. A similar definition holds for $\Omega(\mathbb{R})$. We wish to show that $\mathfrak{Z}$ is continuous at γ_0. Let $\epsilon > 0$ be given. By the continuity of the connection, for each $t \in [0, 1]$, there is an open neighborhood W of $\gamma(t)$ such that any piecewise smooth curve in W has a horizontal lift which lies in $W \times (-\epsilon/2, \epsilon/2)$. Since $\gamma_0(I)$ is compact, we can find $\delta > 0$ such that if $\rho(\gamma_0, \gamma) < \delta$, then $\rho(\omega_{\gamma_0}, \omega_\gamma) < \epsilon$. This proves that $\mathfrak{Z}$ is continuous.

Any piecewise smooth simple closed curve can be approximated by a sequence of piecewise linear curves which are sums of boundaries of rectangular regions. Since $\lambda(\gamma)$ is constant for rectangular regions, the same is true for any piecewise smooth simple closed curve. □

Next, we need to know all complete totally geodesic submanifolds of the base space of the principal bundle $S^1 \to S^{2n,1} \to \mathbb{C}H^n$. Since $S^{2n,1}$ is a symmetric space, the following gives a complete answer.

Proposition 2.2 ([1, XI Theorem 4.3])**.** *Let* (G, H, σ) *be a symmetric space and* $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ *the canonical decomposition. Then there is a natural one-to-one correspondence between the set of linear subspaces* $\mathfrak{m}'$ *of* $\mathfrak{m}$ *such that* $[[\mathfrak{m}', \mathfrak{m}'], \mathfrak{m}'] \subset \mathfrak{m}'$ *and the set of complete totally geodesic submanifolds* M' *through the origin* 0 *of the affine symmetric space* $M = G/H$*, the correspondence being given by* $\mathfrak{m}' = T_0(M')$*.*

3. The bundle $S^1 \to S^{2n,1} \to \mathbb{C}H^n$

We shall study the bundle

$$U(1) \longrightarrow U(1,n)/U(n) \xrightarrow{p} U(1,n)/(U(1) \times U(n)).$$

Note that $U(1,n)/U(n) \cong S^{2n,1}$, and $U(1,n)/(U(1) \times U(n)) \cong \mathbb{C}H^n$, where $S^{2n,1} = H^{1,2n} = \{(z_0, \ldots, z_n) \in \mathbb{C}^n : \ -|z_0|^2 + \sum_{i=1}^n |z_i|^2 = -1\}$. For more information on $S^{2n,1}$, see [3]. We first consider the case when $n = 1$. Rather than using $U(1) \to U(1,1)/U(1) \to U(1,1)/(U(1) \times U(1))$, we shall use

$$U(1) \to SU(1,1) \to SU(1,1)/U(1).$$

Here $SU(1,1) = \{A \in \mathrm{GL}(2,\mathbb{C}) \ : \ AJA^* = J \text{ and } \det(A) = 1\}$ where $J = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

From now on, we shall use the convention of $\mathfrak{gl}(n,\mathbb{C}) \subset \mathfrak{gl}(2n,\mathbb{R})$ by

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \longrightarrow \begin{bmatrix} x_{11} & -y_{11} & x_{12} & -y_{12} \\ y_{11} & x_{11} & y_{12} & x_{12} \\ x_{21} & -y_{21} & x_{22} & -y_{22} \\ y_{21} & x_{21} & y_{22} & x_{22} \end{bmatrix}.$$

The group $SU(1,1)$ has the following natural representation into $\mathrm{GL}(4,\mathbb{R})$:

$$w = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \\ -w_2 & w_1 & -w_4 & w_3 \\ w_3 & -w_4 & w_1 & -w_2 \\ w_4 & w_3 & w_2 & w_1 \end{bmatrix}$$

with the condition $w_1^2 + w_2^2 - w_3^2 - w_4^2 = 1$. In fact, the map

$$w_1 + w_2 i + w_3 j + w_4 k \longmapsto w$$

is a monomorphism from the unit quaternions into $\mathrm{GL}(4,\mathbb{R})$. Therefore,

$$SU(1,1) \cong S^{2,1}.$$

The circle group

$$S^1 = \left\{ \begin{bmatrix} e^{iz} & 0 \\ 0 & e^{-iz} \end{bmatrix} \ : \ 0 \le z \le 2\pi \right\}$$

is a subgroup of $SU(1,1)$, and acts on $SU(1,1)$ as right translations, freely with quotient $\mathbb{C}H^1$, the complex hyperbolic line, giving rise to the fibration

$$S^1 \longrightarrow SU(1,1) \longrightarrow \mathbb{C}H^1.$$

In order to understand the projection map better, let $\widetilde{w}$ be the "i-conjugate" of w (replace w_2 by $-w_2$). That is,

$$\widetilde{w} = \begin{bmatrix} w_1 & -w_2 & w_3 & w_4 \\ w_2 & w_1 & -w_4 & w_3 \\ w_3 & -w_4 & w_1 & w_2 \\ w_4 & w_3 & -w_2 & w_1 \end{bmatrix}.$$

Then,

$$w\widetilde{w} = \begin{bmatrix} w_1^2+w_2^2+w_3^2+w_4^2 & 0 & 2(w_1w_3-w_2w_4) & 2(w_2w_3+w_1w_4) \\ 0 & w_1^2+w_2^2+w_3^2+w_4^2 & -2(w_2w_3+w_1w_4) & 2(w_1w_3-w_2w_4) \\ 2(w_1w_3-w_2w_4) & -2(w_2w_3+w_1w_4) & w_1^2+w_2^2+w_3^2+w_4^2 & 0 \\ 2(w_2w_3+w_1w_4) & 2(w_1w_3-w_2w_4) & 0 & w_1^2+w_2^2+w_3^2+w_4^2 \end{bmatrix}$$

and

$$(w_1^2 + w_2^2 + w_3^2 + w_4^2)^2 - (2w_1w_3 - 2w_2w_4)^2 - (2w_2w_3 + 2w_1w_4)^2 = 1.$$

Clearly, $\mathbb{C}H^1$ can be identified with the following

$$\mathbb{C}H^1 = \left\{ \begin{bmatrix} x & 0 & y & z \\ 0 & x & -z & y \\ y & -z & x & 0 \\ z & y & 0 & x \end{bmatrix} : x^2 - y^2 - z^2 = 1,\ x > 0 \right\}.$$

Therefore, the map

$$p : SU(1,1) \longrightarrow \mathbb{C}H^1$$

defined by $p(w) = w\widetilde{w}$ has the following properties:

$$\begin{aligned} p(wv) &= wp(v)\widetilde{w} \quad \text{for all } w, v \in SU(1,1), \\ p(wv) &= p(w) \quad \text{if and only if} \quad v \in S^1. \end{aligned}$$

This shows that the map p is, indeed, the orbit map of the principal bundle $S^1 \to SU(1,1) \to \mathbb{C}H^1$. The Lie group $SU(1,1)$ will have a left-invariant Riemannian metric given by the following orthonormal basis on the Lie algebra $\mathfrak{su}(1,1)$

$$e_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Notice that e_1 and e_2 correspond to $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ i \end{smallmatrix}\right]$ in $\mathfrak{gl}(2, \mathbb{C})$ and $[e_1, e_2] = -2e_3$. Consider the subset of $SU(1,1)$:

$$\begin{aligned} T &= \left\{ \begin{bmatrix} \cosh x & (\sinh x)e^{-iy} \\ (\sinh x)e^{iy} & \cosh x \end{bmatrix} : x \geq 0,\ 0 \leq y \leq 2\pi \right\} \\ &= \left\{ \begin{bmatrix} \cosh x & 0 & (\sinh x)(\cos y) & (\sinh x)(\sin y) \\ 0 & \cosh x & -(\sinh x)(\sin y) & (\sinh x)(\cos y) \\ (\sinh x)(\cos y) & -(\sinh x)(\sin y) & \cosh x & 0 \\ (\sinh x)(\sin y) & (\sinh x)(\cos y) & 0 & \cosh x \end{bmatrix} \right\} \end{aligned}$$

which is the exponential image of

$$\mathfrak{m} = \left\{ \begin{bmatrix} 0 & \bar{\xi}^t \\ \xi & 0 \end{bmatrix} : \xi \in \mathbb{C} \right\}.$$

Of course, $SU(1,1)$ is topologically a product $S^1 \times \mathbb{C}H^1$. The map p restricted to T is just the squaring map; that is,

$$p(w) = w^2, \quad w \in T.$$

Theorem 3.1. *Let $S^1 \to SU(1,1) \to \mathbb{C}H^1$ be the natural fibration. Let γ be a piecewise smooth, simple closed curve on $\mathbb{C}H^1$. Then the holonomy displacement along γ is given by*

$$V(\gamma) = e^{\frac{1}{2}A(\gamma)i} \in S^1,$$

where $A(\gamma)$ is the area of the region on $\mathbb{C}H^1$ enclosed by γ.

Proof. Let $\gamma(t)$ be a closed loop on $\mathbb{C}H^1$ with $\gamma(0)=p(I_4)$. Therefore,

$$\gamma(t)=\begin{bmatrix} \cosh 2x(t) & 0 & \sinh 2x(t)\cos y(t) & \sinh 2x(t)\sin y(t) \\ 0 & \cosh 2x(t) & -\sinh 2x(t)\sin y(t) & \sinh 2x(t)\cos y(t) \\ \sinh 2x(t)\cos y(t) & -\sinh 2x(t)\sin y(t) & \cosh 2x(t) & 0 \\ \sinh 2x(t)\sin y(t) & \sinh 2x(t)\cos y(t) & 0 & \cosh 2x(t) \end{bmatrix}.$$

Let

$$\widetilde{\gamma}(t)=\begin{bmatrix} \cosh x(t) & 0 & \sinh x(t)\cos y(t) & \sinh x(t)\sin y(t) \\ 0 & \cosh x(t) & -\sinh x(t)\sin y(t) & \sinh x(t)\cos y(t) \\ \sinh x(t)\cos y(t) & -\sinh x(t)\sin y(t) & \cosh x(t) & 0 \\ \sinh x(t)\sin y(t) & \sinh x(t)\cos y(t) & 0 & \cosh x(t) \end{bmatrix}$$

with $x(t)\geq 0$ so that $p(\widetilde{\gamma}(t))=\gamma(t)$ ($\widetilde{\gamma}$ is a lift of γ), and let

$$\omega(t)=\begin{bmatrix} \cos z(t) & -\sin z(t) & 0 & 0 \\ \sin z(t) & \cos z(t) & 0 & 0 \\ 0 & 0 & \cos z(t) & \sin z(t) \\ 0 & 0 & -\sin z(t) & \cos z(t) \end{bmatrix}.$$

Put $\eta(t)=\widetilde{\gamma}(t)\cdot\omega(t)$. We still have $p(\eta(t))=\gamma(t)$, and η is another lift of γ. We wish η to be the horizontal lift of γ. That is, we want $\eta'(t)$ to be orthogonal to the fiber at $\eta(t)$. The condition is that $\langle \eta'(t),(\ell_{\eta(t)})_*(e_3)\rangle=0$, or equivalently, $\langle (\ell_{\eta(t)^{-1}})_*\eta'(t),e_3\rangle=0$. That is,

$$\eta(t)^{-1}\cdot\eta'(t)=\alpha_1e_1+\alpha_2e_2$$

for some $\alpha_1,\alpha_2\in\mathbb{R}$. From this, we get the following equation:

$$z'(t)=\sinh^2 x(t)y'(t). \tag{3–1}$$

By virtue of Lemma 2.1, it will be enough to prove the statement for a particular type of curves as follows: Suppose we are given a rectangular region in the xy-plane

$$p\leq x\leq p+a,\quad q\leq y\leq q+b.$$

Consider the image R of this rectangle in $\mathbb{C}H^1$ by the map

$$(x,y)\mapsto \mathbf{r}(x,y)=(\cosh 2x,(\sinh 2x)(\cos y),(\sinh 2x)(\sin y)),$$

$\mathbb{C}H^1$ with the $(+--)$ metric. The area of R can be calculated as follows:

$$\mathbf{r}_x\times\mathbf{r}_y=((2\cosh 2x)(\sinh 2x),-(2\sinh^2 2x)(\cos y),-(2\sinh^2 2x)(\sin y)).$$

Now

$$\begin{aligned} ||\mathbf{r}_x\times\mathbf{r}_y|| &= 2|\sinh 2x|,\quad (+--)\text{-norm} \\ &= 2\sinh 2x \quad (\text{because } x\geq 0). \end{aligned}$$

Thus, the area is

$$\int_q^{q+b}\int_p^{p+a} 2\sinh 2x\; dxdy=\Big[\big[2\sinh^2 x\big]_p^{p+a}\Big]_q^{q+b}=2b(\sinh^2(p+a)-\sinh^2(p)).$$

On the other hand, the change of $z(t)$ along the boundary of this region can be calculated using condition (3–1). Label the four vertices by $A(p,q)$,

$B(p+a,q)$, $C(p+a,q+b)$, and $D(p,q+b)$. AB can be parametrized by $x(t)=p+at$, $y(t)=q$, $t\in[0,1]$ so that $y'(t)=0$. For BC, $x(t)=p+a$, $y(t)=q+bt$, $t\in[0,1]$. Then

$$z(1)-z(0)=\int_0^1 z'(t)dt=\int \sinh^2(p+a)b\;dt=b\cdot\sinh^2(p+a).$$

Similarly, $z(t)$ does not change along CD, but on DA, $x(t)=p$, $y(t)=q+b-bt$, $t\in[0,1]$. So

$$z(1)-z(0)=\int_0^1 z'(t)dt=\int \sinh^2(p)(-b)dt=-b\cdot\sinh^2(p).$$

Thus the total vertical change of z-values, $z(1)-z(0)$, along the perimeter of this rectangle is $b\cdot(\sinh^2(p+a)-\sinh^2(p))$ which is $1/2$ times the area. □

Now we turn to the general case

$$S^1\longrightarrow S^{2n,1}\xrightarrow{p}\mathbb{C}H^n.$$

We are viewing $S^{2n,1}\cong U(1,n)/U(n)$, and $\mathbb{C}H^n\cong U(1,n)/(U(1)\times U(n))$. The Lie algebra of $U(1,n)$ is $\mathfrak{u}(1,n)$, and has the following canonical decomposition: $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$, where

$$\mathfrak{h}=\mathfrak{u}(1)+\mathfrak{u}(n)=\left\{\begin{bmatrix}\lambda & 0\\ 0 & B\end{bmatrix}\;:\;\lambda+\bar\lambda=0,\ B\in\mathfrak{u}(n)\right\}$$

and

$$\mathfrak{m}=\left\{\begin{bmatrix}0 & \bar\xi^t\\ \xi & 0\end{bmatrix}\;:\;\xi\in\mathbb{C}^n\right\}.$$

Lemma 3.2. *A 2-dimensional subspace $\mathfrak{m}'$ of $\mathfrak{m}\subset\mathfrak{u}(1,n)$ gives rise to a complete totally geodesic submanifold of $\mathbb{C}H^n$ if and only if either*

(1) *$\mathfrak{m}'$ is J-invariant (i.e., has a complex structure), or*
(2) *$\mathfrak{m}'$ has tangent vectors $\mathbf{v}$ and $\mathbf{w}$ such that $\overline{\mathbf{v}}\mathbf{w}-\mathbf{v}\overline{\mathbf{w}}=0$.*

Furthermore, for each of these cases, the pullback of the bundle $S^1\to S^{2n,1}\to\mathbb{C}H^n$ by the inclusion is isomorphic to the standard bundle $S^1\to SU(1,1)\to\mathbb{C}H^1$ for (1), or the product bundle $S^1\times\mathbb{C}H^1$ for (2), respectively.

Proof. With the notation m as above, let $\mathbf{v}$ and $\mathbf{w}$ be elements of $\mathfrak{m}$ whose ξ's are given by

$$\begin{bmatrix}x_1+iy_1\\ x_2+iy_2\\ \cdots\\ x_n+iy_n\end{bmatrix}\quad\text{for }\mathbf{v}\text{ and}\quad\begin{bmatrix}a_1+ib_1\\ a_2+ib_2\\ \cdots\\ a_n+ib_n\end{bmatrix}\quad\text{for }\mathbf{w}.$$

Then by Proposition 2.2, $\mathfrak{m}'$ is a totally geodesic sub-manifold if and only if $[[\mathfrak{m}',\mathfrak{m}'],\mathfrak{m}']\subset\mathfrak{m}'$. Some calculations show the following equality

$$[[\mathbf{v},\mathbf{w}],\mathbf{v}]=\sum_{k=1}^n(x_ka_k+y_kb_k)\mathbf{v}-\sum_{k=1}^n(x_k^2+y_k^2)\mathbf{w}-3\sum_{k=1}^n(x_kb_k-y_ka_k)(i\mathbf{v})$$

holds. Therefore, $[[\mathbf{v},\mathbf{w}],\mathbf{v}] = p\mathbf{v} + q\mathbf{w}$ for some real p and q if and only if $i\mathbf{v} = p\mathbf{v} + q\mathbf{w}$ has solution for some real p and q.

Suppose $\mathfrak{m}'$ has a complex structure. Then we can take $\mathbf{v}$ and $\mathbf{w}$ in $\mathfrak{m}'$ so that $i\mathbf{v} = \mathbf{w}$ (so $a_k = -y_k$ and $b_k = x_k$ for all $k = 1, 2, \ldots, n$). Thus, $[[\mathbf{v},\mathbf{w}],\mathbf{v}] = p\mathbf{v}+q\mathbf{w}$ has a solution for p and q. Suppose $\overline{\mathbf{v}}\mathbf{w}-\mathbf{v}\overline{\mathbf{w}} = 2\mathrm{Im}(\overline{\mathbf{v}}\mathbf{w}) = 2\sum_{k=1}^{n}(x_k b_k - y_k a_k) = 0$. Then clearly $[[\mathbf{v},\mathbf{w}],\mathbf{v}] = p\mathbf{v}+q\mathbf{w}$ has a solution for p and q.

Conversely, suppose $[[\mathbf{v},\mathbf{w}],\mathbf{v}] = p\mathbf{v} + q\mathbf{w}$ has a solution for p and q. Then $i\mathbf{v} = p\mathbf{v} + q\mathbf{w}$ must have a real solution for p and q. Suppose $\sum_{k=1}^{n}(x_k b_k - y_k a_k) \neq 0$. Then, at least one of the summands is non-zero, say $x_1 b_1 - y_1 a_1 \neq 0$. This means that we can find a new basis for $\mathrm{span}\{\mathbf{v},\mathbf{w}\}$ with

$$x_1 = 1,\ y_1 = 0; \quad a_1 = 0,\ b_1 = 1.$$

Then the equation $p\mathbf{v} + q\mathbf{w} = i\mathbf{v}$ is quickly reduced to $p = 0$ and $q = 1$ (from $k = 1$), and hence we obtain $x_k = b_k$, $y_k = -a_k$ for all $k = 2, \ldots, n$. This shows $\mathbf{w} = i\mathbf{v}$, and the space spanned by $\mathbf{v}$ and $\mathbf{w}$ has a complex structure.

For the second part of the statement, it is enough to observe that

$$[\mathbf{v},\mathbf{w}] = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix},$$

where $\lambda = \overline{\mathbf{v}}\mathbf{w} - \mathbf{v}\overline{\mathbf{w}}$. If $\lambda = 0$, then the distribution $\mathfrak{m}'$ is integrable, and the bundle is trivial. □

By combining Theorem 3.1 and Lemma 3.2, we have now:

Theorem 3.3. *Let $S^1 \to S^{2n,1} \to \mathbb{C}H^n$ be the natural fibration. Let S be a complete totally geodesic 2-dimensional surface in $\mathbb{C}H^n$, and ξ_S be the pullback bundle over S. Let γ be a piecewise smooth, simple closed curve on S. Then the holonomy displacement along γ is given by*

$$V(\gamma) = e^{\frac{1}{2}A(\gamma)i} \text{ or } e^{0i} \ \in S^1,$$

where $A(\gamma)$ is the area of the region on the surface S surrounded by γ, depending on whether S is a complex submanifold or not.

Since the length of η is half of the length of γ, we have:

Corollary 3.4. *Suppose γ is a piecewise smooth, simple closed curve parametrized by arc length. Then the Hopf torus in $S^{2n,1}$ over γ is isometric to the torus generated by the lattice $\{(2\pi, 0), (A(\gamma)/2, L(\gamma)/2)\}$ in $\mathbb{R}^2$, where $L(\gamma)$ is the length of γ.*

4. The complex Heisenberg group $\mathcal{H}^{2n+1}$

We consider $\mathcal{H}^{2n+1}$. This is $\mathbb{R} \times \mathbb{C}^n$ with group operation given by

$$(s, \mathbf{z})(t, \mathbf{z}') = (s + t + 2\ \mathrm{Im}\{\overline{\mathbf{z}}\mathbf{z}'\},\ \mathbf{z} + \mathbf{z}'),$$

where $\mathrm{Im}\{\overline{\mathbf{z}}\mathbf{z}'\}$ is the imaginary part of the complex number $\overline{z}_1 z_1' + \overline{z}_2 z_2' + \cdots + \overline{z}_n z_n'$ for $\mathbf{z} = (z_1, z_2, \ldots, z_n)$, $\mathbf{z}' = (z_1', z_2', \ldots, z_n') \in \mathbb{C}^n$. This is a 2-step nilpotent Lie group with center $\mathcal{Z}(\mathcal{H}^{2n+1}) = \mathbb{R}$. In the case of $n = 1$, $\mathcal{H}^3$ is isomorphic to the ordinary 3-dimensional Heisenberg group by

$$(4z - 2xy, x + iy) \longleftrightarrow \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

For the sake of computations, we use the following affine representation of $\mathcal{H}^{2n+1}$ into $\mathrm{Aff}(2n+1) \subset \mathrm{GL}(2n+2)$:

$$\left(s, \begin{bmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ \cdots \\ \cdots \\ x_n + iy_n \end{bmatrix} \right) \longrightarrow \begin{bmatrix} 1 & -2y_1 & 2x_1 & \cdots & -2y_n & 2x_n & s & \\ 0 & 1 & 0 & \cdots & & 0 & 0 & x_1 \\ 0 & 0 & 1 & \cdots & 0 & 0 & y_1 & \\ . & . & . & \cdots & . & . & . & \\ . & . & . & \cdots & . & . & . & \\ 0 & 0 & 0 & \cdots & 1 & 0 & x_n & \\ 0 & 0 & 0 & \cdots & 0 & 1 & y_n & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \end{bmatrix}.$$

The Lie algebra has a following orthonormal basis

$$e_1 = \begin{bmatrix} 0 & 0 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ . & . & . & \cdots & . & . & . \\ . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ . & . & . & \cdots & . & . & . \\ . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$\vdots \qquad\qquad \vdots$$

$$e_{2n} = \begin{bmatrix} 0 & 0 & 0 & \cdots & -2 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ . & . & . & \cdots & . & . & . \\ . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \quad e_{2n+1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ . & . & . & \cdots & . & . & . \\ . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

which defines a left-invariant Riemannian metric on $\mathcal{H}^{2n+1}$. The short exact sequence of groups

$$1 \longrightarrow \mathbb{R} \longrightarrow \mathcal{H}^{2n+1} \xrightarrow{p} \mathbb{C}^n \longrightarrow 1$$

is a fiber bundle, which is topologically trivial. The left invariant metric naturally induces a connection on this principal $\mathbb{R}$-bundle. There is a unique metric on $\mathbb{C}^n$ (standard Euclidean metric) which makes the projection map p a Riemannian submersion.

Theorem 4.1. *Let $1 \to \mathbb{R} \to \mathcal{H}^{2n+1} \xrightarrow{p} \mathbb{C}^n \to 1$ be the central short exact sequence of the complex Heisenberg group. Let S be a complete totally geodesic plane in $\mathbb{C}^n$, and ξ_S be the pullback bundle over S. Let γ be a piecewise smooth, simple closed curve on S. Then*

$$V(\gamma) = e(\xi_S) \cdot A(\gamma),$$

where $A(\gamma)$ is the area of the region on the surface S surrounded by γ, and the number $e(\xi_S)$ is determined by the equality $[\mathbf{v}, \mathbf{w}] = e(\xi_S)e_{2n+1}$ for an orthonormal basis $\{\mathbf{v}, \mathbf{w}\}$ for the tangent space of S.

Proof. Every complete totally geodesic submanifold of $\mathbb{C}^n$ is an $\mathbb{R}$-linear subspace of $\mathbb{C}^n$. Therefore $S = \mathrm{span}\{\mathbf{v}, \mathbf{w}\}$ for some orthonormal basis $\mathbf{v}, \mathbf{w}$ where $\mathbf{v} = \sum_{j=1}^n (a_j + ib_j)$ and $\mathbf{w} = \sum_{j=1}^n (c_j + id_j) \in \mathbb{C}^n$. Then γ is of the form

$$\gamma(t) = x(t)\mathbf{v} + y(t)\mathbf{w} \in S \subset \mathbb{C}^n,$$

where $x(t)$ and $y(t)$ are scalars. We want to find a curve $z(t)$ in $\mathbb{R}$ so that $\eta(t) = (z(t), \gamma(t))$ is orthogonal to the fiber for every t. In other words,

$$\langle \eta'(t), (\ell_{\eta(t)})_* e_{2n+1} \rangle = 0,$$

where ℓ is the left translation. This is equivalent to $\langle (\ell_{\eta(t)}^{-1})_* \eta'(t), e_{2n+1} \rangle = 0$. Note that $\eta(t)^{-1} = (-z(t) + 2\,\mathrm{Im}\{\overline{\gamma(t)}\gamma(t)\}, -\gamma(t))$. Using the affine representation, $(\ell_{\eta(t)^{-1}})_* \eta'(t)$ is

$$\begin{bmatrix} 0 & -2(x'(t)b_1 + y'(t)d_1) & \cdots & z'(t) - 2(x(t)y'(t) - x'(t)y(t))\,\mathrm{Im}\{\overline{\mathbf{v}}\mathbf{w}\} \\ 0 & 0 & \cdots & x'(t)a_1 + y'(t)c_1 \\ 0 & 0 & \cdots & x'(t)b_1 + y'(t)d_1 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & x'(t)a_n + y'(t)c_n \\ 0 & 0 & \cdots & x'(t)b_n + y'(t)d_n \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $\mathrm{Im}\{\overline{\mathbf{v}}\mathbf{w}\} = \sum_{j=1}^n (a_j d_j - c_j b_j)$. Note that $\mathrm{Im}\{\overline{\mathbf{v}}\mathbf{w}\} = \mathrm{Im}\{\overline{\mathbf{v}}'\mathbf{w}'\}$ for any orthonormal basis $\{\mathbf{v}', \mathbf{w}'\}$. The equation $\langle (\ell_{\eta(t)^{-1}})_* \eta'(t), e_{2n+1} \rangle = 0$ gives rise to

$$z'(t) - 2(x(t)y'(t) - x'(t)y(t))\,\mathrm{Im}\{\overline{\mathbf{v}}\mathbf{w}\} = 0. \tag{4–1}$$

Suppose we are given a rectangular region on xy-plane

$$p \le x \le p + a, \quad q \le y \le q + b.$$

Consider the image R of this rectangle in $S \subset \mathbb{C}^n$ by the map

$$(x, y) \mapsto x\mathbf{v} + y\mathbf{w}.$$

Then R is a rectangle with vertices $p\mathbf{v} + q\mathbf{w}$, $(p+a)\mathbf{v} + q\mathbf{w}$, $(p+a)\mathbf{v} + (q+b)\mathbf{w}$, $p\mathbf{v} + (q+b)\mathbf{w}$. Let $\gamma(t)$ be the piecewise linear boundary curve. It can be represented by $((p + 4at)\mathbf{v}, q\mathbf{w})$ for $0 \le t \le 1/4$, $((p+a)\mathbf{v}, (q + b(4t-1))\mathbf{w})$ for

$1/4 \le t \le 1/2$, $((p+a(3-4t))\mathbf{v}, (q+b)\mathbf{w})$ for $1/2 \le t \le 3/4$, $(p\mathbf{v}, (q+b(4-4t))\mathbf{w})$ for $3/4 \le t \le 1$.

Then, from the equation (4–1),

$$z(1) - z(0) = 2\int_0^1 (x(t)y'(t) - x'(t)y(t))\ \mathrm{Im}\{\overline{\mathbf{v}}\mathbf{w}\}dt = 4ab\ \mathrm{Im}\{\overline{\mathbf{v}}\mathbf{w}\}.$$

On the other hand,

$$[\mathbf{v}, \mathbf{w}] = 4\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & \mathrm{Im}\{\overline{\mathbf{v}}\mathbf{w}\} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ . & . & . & \cdots & . & . & . \\ . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

This means that $[\mathbf{v}, \mathbf{w}] = 4\,\mathrm{Im}\{\overline{\mathbf{v}}\mathbf{w}\}e_{2n+1} = e(\xi_S)\ e_{2n+1}$ so that $V(\gamma) = e(\xi_S) \cdot A(\gamma)$ with $e(\xi_S) = 4\,\mathrm{Im}\{\overline{\mathbf{v}}\mathbf{w}\}$.

Having shown the statement for rectangular regions, now we apply Lemma 2.1 to conclude that the same formula holds for any piecewise smooth, simple closed curve. □

Corollary 4.2. *Suppose γ is a piecewise smooth, simple closed curve parametrized by arc-length. Then the Hopf cylinder in $\mathcal{H}^{2n+1}$ over γ is isometric to the cylinder generated by the translation $(x, y) \mapsto (x + e(\xi_S)\,A(\gamma),\ y + L(\gamma)\,)$ on $\mathbb{R}^2$, where $L(\gamma)$ is the length of γ.*

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