Dedicated to S. P. Novikov on his 60th birthday.

THE MORSE-NOVIKOV THEORY OF CIRCLE-VALUED FUNCTIONS AND NONCOMMUTATIVE LOCALIZATION

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Abstract. We use noncommutative localization to construct a chain complex which counts the critical points of a circle-valued Morse function on a manifold, generalizing the Novikov complex. As a consequence we obtain new topological lower bounds on the minimum number of critical points of a circle-valued Morse function within a homotopy class, generalizing the Novikov inequalities.

1. Introduction.

Let us start by recalling the way in which chain complexes are used to count the critical points of a real-valued Morse function.

A Morse function $f : M \to \mathbb{R}$ on a compact (differentiable) m-dimensional manifold M with $c_i(f)$ critical points of index i determines a handlebody decomposition of M with $c_i(f)$ i-handles. The *Morse-Smale complex* is the cellular chain complex of the corresponding CW decomposition of the universal cover $\widetilde{M}$ of M, a based f.g. free $\mathbb{Z}[\pi_1(M)]$-module chain complex $C(\widetilde{M})$ with

$$\mathrm{rank}_{\mathbb{Z}[\pi_1(M)]} C_i(\widetilde{M}) \ = \ c_i(f) \ .$$

For any ring morphism $\rho : \mathbb{Z}[\pi_1(M)] \to R$ there is induced a based f.g. free R-module chain complex

$$C(M;R) \ = \ R \otimes_{\mathbb{Z}[\pi_1(M)]} C(\widetilde{M})$$

with homology R-modules $H_*(M;R) \ = \ H_*(C(M;R))$ (which in general depend on ρ as well as R). The number $c_i(f)$ of critical points of index i is bounded from below by the minimum number $\mu_i(M;R)$ of generators in degree i of a finite f.g. free R-module chain complex which is chain equivalent to $C(M;R)$

$$c_i(f) \geq \mu_i(M;R) \ .$$

For a principal ideal domain R and a ring morphism $\rho \ : \ \mathbb{Z}[\pi_1(M)] \to R$ the R-coefficient *Betti numbers* of M are defined as usual by

$$\begin{aligned} b_i(M;R) \ &= \ \mathrm{rank}_R\big(H_i(M;R)/T_i(M;R)\big) \ , \\ q_i(M;R) \ &= \ \text{minimum number of generators of } T_i(M;R) \end{aligned}$$

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with $T_i(M;R) \subseteq H_i(M;R)$ the torsion submodule, and

$$\mu_i(M;R) \;=\; b_i(M;R) + q_i(M;R) + q_{i-1}(M;R) \;.$$

The *Morse inequalities*

$$c_i(f) \geq b_i(M;R) + q_i(M;R) + q_{i-1}(M;R)$$

are thus an algebraic consequence of the existence of the Morse-Smale complex.

Now for circle-valued functions. Given a Morse function $f : M \to S^1$ let $c_i(f)$ denote the number of critical points of index i. The *Novikov complex* of [N] is a based f.g. free chain complex $C^{Nov}(M,f)$ over the principal ideal domain

$$\mathbb{Z}((z)) \;=\; \mathbb{Z}[[z]][z^{-1}] \;,$$

such that

(i) $\mathrm{rank}_{\mathbb{Z}((z))} C_i^{Nov}(M,f) = c_i(f)$,

(ii) $C^{Nov}(M,f)$ is chain equivalent to $C\big(M;\mathbb{Z}((z))\big)$, with

$$\rho : \mathbb{Z}[\pi_1(M)] \xrightarrow{f_*} \mathbb{Z}[\pi_1(S^1)] \;=\; \mathbb{Z}[z,z^{-1}] \to \mathbb{Z}((z)) \;.$$

The chain complex $C^{Nov}(M,f)$ is constructed geometrically using the gradient flow. The *Novikov inequalities*

$$c_i(f) \geq \mu_i\big(M;\mathbb{Z}((z))\big) \;=\; b_i\big(M;\mathbb{Z}((z))\big) + q_i\big(M;\mathbb{Z}((z))\big) + q_{i-1}\big(M;\mathbb{Z}((z))\big)$$

are an algebraic consequence of the existence of a chain complex $C^{Nov}(M,f)$ satisfying (i) and (ii). The $\mathbb{Z}((z))$-coefficient Betti numbers are called the *Novikov numbers* of M. The Novikov numbers depend only on the cohomology class $\xi = f^*(1) \in H^1(M)$, and so may be denoted by

$$b_i\big(M;\mathbb{Z}((z))\big) \;=\; b_i(\xi) \;,\; q_i\big(M;\mathbb{Z}((z))\big) \;=\; q_i(\xi) \;.$$

A map $f : M \to S^1$ classifies an infinite cyclic cover $\overline{M} = f^*\mathbb{R}$ of M. We shall assume that M and $\overline{M}$ are connected, so that there is defined a short exact sequence

$$0 \to \pi \to \pi_1(M) \xrightarrow{f_*} \pi_1(S^1) = \mathbb{Z} \to 0$$

with $\pi = \pi_1(\overline{M})$. Let $z \in \pi_1(M)$ be such that $f_*(z) = 1$, so that

$$\pi_1(M) \;=\; \pi \times_\alpha \mathbb{Z} \;,\; \mathbb{Z}[\pi_1(M)] \;=\; \mathbb{Z}[\pi]_\alpha[z,z^{-1}] \;.$$

with $\alpha : \pi \to \pi; g \mapsto z^{-1}gz$ the monodromy automorphism.

Let Σ denote the set of square matrices with entries in $\mathbb{Z}[\pi_1(M)]$ having the form $1 - ze$ where e is a square matrix with entries in $\mathbb{Z}[\pi]$. A ring morphism $\mathbb{Z}[\pi_1(M)] \to R$ is called Σ-*inverting* if it sends matrices in Σ to invertible matrices over the ring R. There exists *a noncommutative localization in the sense of P. M. Cohn* [C], a ring $\Sigma^{-1}\mathbb{Z}[\pi_1(M)$ together with a ring morphism $\mathbb{Z}[\pi_1(M)] \to \Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ which has the universal property that every Σ-inverting homomorphism $\mathbb{Z}[\pi_1(M)] \to R$ has a unique factorization

$$\mathbb{Z}[\pi_1(M)] \to \Sigma^{-1}\mathbb{Z}[\pi_1(M)] \to R \ .$$

In particular, the inclusion of the group ring $\mathbb{Z}[\pi_1(M)]$ in *the Novikov completion*

$$\mathbb{Z}[\widehat{\pi_1(M)}] \ = \ \mathbb{Z}[\pi][[z]]_\alpha[z^{-1}]$$

is Σ-inverting, so that there is a factorization

$$\mathbb{Z}[\pi_1(M)] \to \Sigma^{-1}\mathbb{Z}[\pi_1(M)] \to \mathbb{Z}[\widehat{\pi_1(M)}] \ .$$

Pazhitnov [P1] extended the geometric construction of the Novikov complex to a based f.g. free chain complex $C^{Nov}(M,f)$ over $\mathbb{Z}[\widehat{\pi_1(M)}]$, such that

(i) $\mathrm{rank}_{\mathbb{Z}[\widehat{\pi_1(M)}]} C_i^{Nov}(M,f) = c_i(f)$,

(ii) $C^{Nov}(M,f)$ is chain equivalent to $C(M;\mathbb{Z}[\widehat{\pi_1(M)}])$, with

$$\rho \ = \ \text{inclusion} \ : \ \mathbb{Z}[\pi_1(M)] \to \mathbb{Z}[\widehat{\pi_1(M)}] \ .$$

Moreover, Pazhitnov [P2],[P3] showed that the geometric construction of $C^{Nov}(M,f)$ can be adjusted so that the differentials are rational, in the sense that the entries of their matrices belong to $\mathrm{im}(\Sigma^{-1}\mathbb{Z}[\pi_1(M)] \to \mathbb{Z}[\widehat{\pi_1(M)}])$. If $\pi_1(M)$ is abelian and $\alpha = 1$ the localization is a subring of the completion

$$\Sigma^{-1}\mathbb{Z}[\pi_1(M)] \ = \ (1 + z\mathbb{Z}[\pi])^{-1}\mathbb{Z}[\pi][z,z^{-1}] \subset \mathbb{Z}[\widehat{\pi_1(M)}] \ = \ \mathbb{Z}[\pi][[z]][z^{-1}]$$

so that $C^{Nov}(M,f)$ is induced from a chain complex defined over $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$.

Our main result is a direct construction of such a rational lift of the Novikov complex, which is valid for arbitrary $\pi_1(M)$ (not necessary abelian) and arbitrary α :

Main Theorem. *For every Morse function $f : M \to S^1$ there exists a based f.g. free $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$-module chain complex $\widehat{C}(M,f)$ such that*

(i) $\mathrm{rank}_{\Sigma^{-1}\mathbb{Z}[\pi_1(M)]}\widehat{C}_i(M,f) = c_i(f)$,

(ii) *$\widehat{C}(M,f)$ is chain equivalent to $\Sigma^{-1}C(\widetilde{M})$.*

The Main Theorem is proved in section 3 by cutting M at the inverse image $N = f^{-1}(x) \subset M$ of a regular value $x \in S^1$ of f, to obtain a fundamental domain $(M_N; N, zN)$ for $\overline{M}$, and considering the chain homotopy theoretic properties of the handle decomposition of the cobordism, with $c_i(f)$ i-handles.

The existence of the chain complex $\widehat{C}(M,f)$ has as immediate consequence:

Corollary (Generalized Novikov inequalities). *For any Morse function $f : M \to S^1$ and any Σ-inverting ring morphism $\rho : \mathbb{Z}[\pi_1(M)] \to R$*

$$c_i(f) \geq \mu_i(M;R) \ .$$

Recall that the number $\mu_i(M;R)$ is defined as the minimum number of generators in degree i of any f.g. free R-module chain complex, which is chain homotopy equivalent to $C(M;R)$. The numbers $\mu_i(M;R)$ are homotopy invariants of M as a space over S^1 (via f); they depend on the homotopy class of f and on the ring homomorphism ρ.

As an example consider the following ring of rational functions

$$\mathcal{R} \ = \ (1+z\mathbb{Z}[z])^{-1}\mathbb{Z}[z,z^{-1}]$$

introduced in [F]. This ring is a principal ideal domain [F]. The induced homomorphism $f_* : \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$ determines a ring homomorphism

$$\rho : \mathbb{Z}[\pi_1(M)] \to \mathcal{R} \ ; \ g \mapsto z^{f_*(g)} \quad (g \in \pi_1(M)) \ .$$

It is easy to see that ρ is Σ-inverting. The corollary implies the inequalities

$$c_i(f) \geq \mu_i(M;\mathcal{R}), \quad i = 0,1,2,\dots .$$

These inequalities coincide with the classical Novikov inequalities since

$$\mu_i(M;\mathcal{R}) \ = \ \mu_i\big(M;\mathbb{Z}((z))\big) \ = \ b_i(\xi) + q_i(\xi) + q_{i-1}(\xi) \ .$$

The equivalence between this approach (using the ring $\mathcal{R}$ of rational functions) and the original approach of Novikov (which used the formal power series ring $\mathbb{Z}((z))$) was proved in [F].

2. The endomorphism localization.

This section describes the endomorphism localization of a Laurent polynomial extension, and provides the algebraic machinery required for the construction of the chain complex $\widehat{C}(M,f)$ in section 3.

Given a ring A and an automorphism $\alpha : A \to A$ let $A_\alpha[z]$, $A_\alpha[[z]]$, $A_\alpha[z,z^{-1}]$, $A_\alpha((z))$ be the α-twisted polynomial extension rings of A, with z an indeterminate over A with $az = z\alpha(a)$ ($a \in A$); for the record, $A_\alpha[z]$ is the ring of finite polynomials $\sum\limits_{j=0}^{\infty} a_j z^j$, $A_\alpha[[z]]$ is the ring of power series $\sum\limits_{j=0}^{\infty} a_j z^j$, $A_\alpha[z,z^{-1}]$ is the ring of finite Laurent polynomials $\sum\limits_{j=-\infty}^{\infty} a_j z^j$, and $A_\alpha((z)) = A_\alpha[[z]][z^{-1}]$ is the Novikov ring of power series $\sum\limits_{j=-\infty}^{\infty} a_j z^j$ with only a finite number of non-zero coefficients $a_j \in A$ for $j < 0$.

Definition 2.1 Let Σ be the set of square matrices in $A_\alpha[z,z^{-1}]$ of the form $1 - ze$ with e a square matrix in A. The *endomorphism localization* $\Sigma^{-1}A_\alpha[z,z^{-1}]$ is the noncommutative localization of $A_\alpha[z,z^{-1}]$ inverting Σ in the sense of Cohn [C]. $\square$

By construction, $\Sigma^{-1}A_\alpha[z,z^{-1}]$ is the ring obtained from $A_\alpha[z,z^{-1}]$ by adjoining generators corresponding to the entries in formal inverses $(1-ze)^{-1}$ of elements $1-ze\in\Sigma$, and the relations given by the matrix equations

$$(1-ze)^{-1}(1-ze) ~=~ (1-ze)(1-ze)^{-1} ~=~ 1~.$$

Given an A-module B let zB be the A-module with elements zx $(x\in B)$ and A acting by

$$A\times zB\to zB~;~(a,zx)\mapsto azx ~=~ z\alpha(a)x~.$$

If B is a f.g. free A-module with basis $\{b_1,b_2,\dots,b_r\}$ then zB is a f.g. free A-module with basis $\{zb_1,zb_2,\dots,zb_r\}$. For f.g. free A-modules B,B' an $A_\alpha[z,z^{-1}]$-module morphism $f:B_\alpha[z,z^{-1}]\to B'_\alpha[z,z^{-1}]$ is a finite Laurent polynomial

$$f ~=~ \sum_{j=-\infty}^{\infty} z^j f_j ~:~ B_\alpha[z,z^{-1}]\to B'_\alpha[z,z^{-1}]$$

with coefficients A-module morphisms $f_j:z^jB\to B'$. Note that in the special case $\alpha=1:A\to A$ there is defined a natural A-module isomorphism

$$B\to zB~;~x\mapsto zx~.$$

Proposition 2.2. *(i) A ring morphism $A_\alpha[z,z^{-1}]\to R$ which sends every $1-ze\in\Sigma$ to an invertible matrix in R has a unique factorization*

$$A_\alpha[z,z^{-1}]\to\Sigma^{-1}A_\alpha[z,z^{-1}]\to R~.$$

(ii) If E is a f.g. free A-module then for any A-module morphism $e:zE\to E$ there is defined an automorphism of a f.g. free $\Sigma^{-1}A_\alpha[z,z^{-1}]$-module

$$1-ze:\Sigma^{-1}E_\alpha[z,z^{-1}]\to\Sigma^{-1}E_\alpha[z,z^{-1}]~.$$

(iii) The inclusion $A_\alpha[z,z^{-1}]\to A_\alpha((z))$ factorizes through $\Sigma^{-1}A_\alpha[z,z^{-1}]$

$$A_\alpha[z,z^{-1}]\to\Sigma^{-1}A_\alpha[z,z^{-1}]\to A_\alpha((z))$$

with $A_\alpha[z,z^{-1}]\to\Sigma^{-1}A_\alpha[z,z^{-1}]$ an injection.

Proof. (i) This is the universal property of noncommutative localization.
(ii) By construction.
(iii) Every matrix of the type $1-ze$ is invertible in $A_\alpha((z))$, with

$$(1-ze)^{-1} ~=~ 1+ze+z^2e^2+\dots~.$$

so that (i) applies. $\square$

For commutative A and $\alpha=1:A\to A$

$$\Sigma^{-1}A[z,z^{-1}] ~=~ (1+zA[z])^{-1}A[z,z^{-1}]$$

is just the usual commutative localization inverting the multiplicative subset

$$1+zA[z]\subset A[z,z^{-1}]~.$$

We shall need the following

2.3. Deformation Lemma. *Let C be an A-module chain complex of the form*

$$d_C \;=\; \begin{pmatrix} d_D & a & c \\ 0 & d_F & b \\ 0 & 0 & d_{D'} \end{pmatrix} \;:\; C_i \;=\; D_i \oplus F_i \oplus D'_i \to C_{i-1} \;=\; D_{i-1} \oplus F_{i-1} \oplus D'_{i-1}$$

where $a : F_i \to D_{i-1}$, $b : D'_i \to F_{i-1}$ and $c : D'_i \to D_{i-1}$. Suppose that the morphism $c : D'_i \to D_{i-1}$ is an isomorphism for all i. The formula

$$\widehat{d}_C \;=\; d_F - bc^{-1}a \;:\; F_i \to F_{i-1}$$

defines a "deformed differential" on F_i (i.e. $(\widehat{d}_C)^2 = 0$), and the chain complex $\widehat{C}$ defined by

$$\widehat{d}_C \;:\; \widehat{C}_i \;=\; F_i \to \widehat{C}_{i-1} \;=\; F_{i-1}$$

is chain equivalent to C.

Proof. Note that $(d_C)^2 = 0$ implies

$$(d_D)^2 \;=\; 0\;,\; (d_F)^2 \;=\; 0\;,\; (d_{D'})^2 \;=\; 0\;,$$

and also

$$\begin{aligned} d_D a + a d_F \;&=\; 0\;, \\ d_F b + b d_{D'} \;&=\; 0\;, \\ d_D c + c d_{D'} + ab \;&=\; 0\;. \end{aligned} \tag{1}$$

Using (1) we obtain

$$\begin{aligned} (\widehat{d}_C)^2 \;&=\; (d_F - bc^{-1}a)\cdot(d_F - bc^{-1}a) \\ &=\; -d_F bc^{-1}a - bc^{-1}ad_F + bc^{-1}a \cdot bc^{-1}a \\ &=\; -d_F bc^{-1}a - bc^{-1}ad_F - bc^{-1}[d_D c + c d_{D'}]c^{-1}a \\ &=\; 0\;. \end{aligned}$$

Now we define two chain maps:

$$u \;=\; (\,-bc^{-1} \quad 1 \quad 0\,) : C \to \widehat{C}, \quad \text{and} \quad v \;=\; \begin{pmatrix} 0 \\ 1 \\ -c^{-1}a \end{pmatrix} : \widehat{C} \to C\;.$$

One checks that

$$ud_C = \widehat{d}_C u, \quad d_C v = v\widehat{d}_C, \quad uv = 1_{\widehat{C}}$$

and

$$vu = 1_C - d_C w - w d_C, \quad \text{where} \quad w \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c^{-1} & 0 & 0 \end{pmatrix} : C \to C.$$

Hence u and v are mutually inverse chain equivalences. $\square$

Theorem 2.4. *Let C be a finite based f.g. free $A_\alpha[z,z^{-1}]$-module chain complex of the form*

$$C \;=\; \mathcal{C}(g-zh : D_\alpha[z,z^{-1}] \to E_\alpha[z,z^{-1}])$$

where $g : D \to E$, $h : zD \to E$ are chain maps of finite based f.g. free A-module chain complexes, and $\mathcal{C}$ denotes the algebraic mapping cone. If each $g : D_i \to E_i$ is a split injection sending basis elements to basis elements, then there is defined a $\Sigma^{-1}A_\alpha[z,z^{-1}]$-module chain complex $\widehat{C}$ such that

(i) *each $\widehat{C}_i$ is a based f.g. free $\Sigma^{-1}A_\alpha[z,z^{-1}]$-module with*

$$\mathrm{rank}_{\Sigma^{-1}A_\alpha[z,z^{-1}]}\widehat{C}_i \;=\; \mathrm{rank}_A E_i - \mathrm{rank}_A D_i \ ,$$

(ii) *there is defined a chain equivalence $\Sigma^{-1}C \to \widehat{C}$.*

Proof. Let $F_i \subseteq E_i$ be the submodule generated by the basis elements not coming from D_i, so that

$$\begin{aligned}
g \;&=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;:\; D_i \to E_i \;=\; D_i \oplus F_i \ ,\\
h \;&=\; \begin{pmatrix} e \\ f \end{pmatrix} \;:\; zD_i \to E_i \;=\; D_i \oplus F_i \ ,\\
d_E \;&=\; \begin{pmatrix} d_D & a \\ 0 & d_F \end{pmatrix} \;:\; E_i \;=\; D_i \oplus F_i \to E_{i-1} \;=\; D_{i-1} \oplus F_{i-1} \ ,\\
d_C \;&=\; \begin{pmatrix} d_D & a & 1-ze \\ 0 & d_F & -zf \\ 0 & 0 & d_D \end{pmatrix} \;:\\
&\quad C_i \;=\; (D_i \oplus F_i \oplus D_{i-1})_\alpha[z,z^{-1}] \to C_{i-1} \;=\; (D_{i-1} \oplus F_{i-1} \oplus D_{i-2})_\alpha[z,z^{-1}] \ .
\end{aligned}$$

Now apply Lemma 2.3 to the induced chain complex $\Sigma^{-1}C$ over $\Sigma^{-1}A_\alpha[z,z^{-1}]$ with

$$\begin{aligned}
d_{\Sigma^{-1}C} \;&=\; \begin{pmatrix} d_D & a & 1-ze \\ 0 & d_F & -zf \\ 0 & 0 & d_D \end{pmatrix} \;:\\
\Sigma^{-1}C_i \;&=\; \Sigma^{-1}(D_i \oplus F_i \oplus D_{i-1})_\alpha[z,z^{-1}]\\
&\qquad \to \Sigma^{-1}C_{i-1} \;=\; \Sigma^{-1}(D_{i-1} \oplus F_{i-1} \oplus D_{i-2})_\alpha[z,z^{-1}]
\end{aligned}$$

where each

$$c \;=\; 1-ze \;:\; \Sigma^{-1}(D_{i-1})_\alpha[z,z^{-1}] \to \Sigma^{-1}(D_{i-1})_\alpha[z,z^{-1}]$$

is an automorphism. Explicitly, Lemma 2.3 gives a based f.g. free $\Sigma^{-1}A_\alpha[z,z^{-1}]$-module chain complex $\widehat{C}$ with

$$\widehat{d}_C \;=\; d_F + (zf)(1-ze)^{-1}a \;:\; \widehat{C}_i \;=\; \Sigma^{-1}(F_i)_\alpha[z,z^{-1}] \to \widehat{C}_{i-1} \;=\; \Sigma^{-1}(F_{i-1})_\alpha[z,z^{-1}]$$

which satisfies (i) and (ii). $\square$

3. The chain complex $\widehat{C}(M,f)$.

Given a Morse function $f : M \to S^1$ we now construct a chain complex $\widehat{C}(M,f)$ over $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ satisfying the conditions of the Main Theorem.

Choose a regular value $x \in S^1$ for $f : M \to S^1$, and cut M along the codimension 1 framed submanifold

$$N^{m-1} \;=\; f^{-1}(x) \subset M^m$$

to obtain a fundamental domain $(M_N; N, zN)$ for the infinite cyclic cover

$$\overline{M} \;=\; \bigcup_{j=-\infty}^{\infty} z^j M_N$$

of M, with a Morse function

$$f_N \;:\; (M_N; N, zN) \to ([0,1]; \{0\}, \{1\})$$

such that f_N has exactly as many critical points of index i as f

$$c_i(f_N) \;=\; c_i(f) \;.$$

Let $\widetilde{N}$, $\widetilde{M_N}$ be the covers of N, M_N obtained from the universal cover $\widetilde{M}$ of M by pullback along the inclusions $N \to \overline{M}$, $M_N \to \overline{M}$. The cobordism $(M_N; N, zN)$ has a handle decomposition

$$M_N \;=\; N \times I \cup \bigcup_{i=0}^{m} \bigcup_{c_i(f)} h^i$$

with $c_i(f)$ i-handles $h^i = D^i \times D^{m-i}$. Let $\pi_1(\overline{M}) = \pi$ with monodromy automorphism $\alpha : \pi \to \pi$, so that $\pi_1(M) = \pi \times_\alpha \mathbb{Z}$, $\mathbb{Z}[\pi_1(M)] = \mathbb{Z}[\pi]_\alpha[z, z^{-1}]$ as in the Introduction. The relative cellular chain complex $C(\widetilde{M_N}, \widetilde{N})$ is a based f.g. free $\mathbb{Z}[\pi]$-module chain complex with

$$\mathrm{rank}_{\mathbb{Z}[\pi]} C_i(\widetilde{M_N}, \widetilde{N}) \;=\; c_i(f) \;.$$

Choose an arbitrary CW structure for N, and let $c_i(N)$ be the number of i-cells. Let M_N have the CW structure with $c_i(N) + c_i(f)$ i-cells: for each i-cell $e^i \subset N$ there is defined an i-cell $e^i \times I \subset M_N$, and for each i-handle h^i there is defined an i-cell $h^i \subset M_N$. Then $M = M_N/(N = zN)$ has a CW complex structure with $c_i(f)+c_i(N)+c_{i-1}(N)$ i-cells. The universal cover $\widetilde{M}$ has cellular $\mathbb{Z}[\pi_1(M)]$-module chain complex

$$C(\widetilde{M}) \;=\; \mathcal{C}(g - zh : C(\widetilde{N})_\alpha[z, z^{-1}] \to C(\widetilde{M_N})_\alpha[z, z^{-1}])$$

where

$$g \;:\; C(\widetilde{N}) \to C(\widetilde{M_N}) \;,\; h \;:\; zC(\widetilde{N}) \to C(\widetilde{M_N})$$

are the $\mathbb{Z}[\pi]$-module chain maps induced by the inclusions

$$g \ : \ N \to M_N \ , \ h \ : \ zN \to M_N \ .$$

Since $g : N \to M_N$ is the inclusion of a subcomplex

$$g \ = \ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ : \ C_i(\widetilde{N}) \to C_i(\widetilde{M_N}) \ = \ C_i(\widetilde{N}) \oplus C_i(\widetilde{M_N}, \widetilde{N})$$

is a split injection. Now apply Theorem 2.4 to the based f.g. free $\mathbb{Z}[\pi_1(M)]$-module chain complex

$$C(\widetilde{M}) \ = \ \mathcal{C}(g - zh : C(\widetilde{N})_\alpha[z, z^{-1}] \to C(\widetilde{M_N})_\alpha[z, z^{-1}])$$

with $D = C(\widetilde{N})$, $E = C(\widetilde{M_N})$. $\square$

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