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AN ALGEBRAIC CLASSIFICATION OF SOME EVEN-DIMENSIONAL KNOTS

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§0. INTRODUCTION

AN n-KNOT k is a smooth submanifold Σ^n of S^{n+2}, where Σ^n is homeomorphic to the n-sphere S^n. When $n = 2q-1$ or $2q$, the knot is called *simple* if its complement has the homotopy $(q-1)$-type of S^1: this is the most that can be asked if k is not to be trivial (except perhaps when $n = 2$). The simple $(2q-1)$-knots, $q \geq 2$, have been classified by J. Levine[7] in terms of their Seifert matrices modulo S-equivalence.

A simple $2q$-knot is called *odd* if the q^{th} homotopy group of its complement has no 2-torsion. This paper provides a classification of odd simple $2q$-knots, $q \geq 3$, in terms of an algebraic gadget called a $(-1)^q$-form, modulo an equivalence relation called T-equivalence.

I should like to thank Andrew Ranicki for many helpful conversations: the notation used here is modelled on his work.

§1. ϵ-FORMS

Let ϵ denote $\pm$. Let $P \cong Z^{2n}$ and $P^* = \text{Hom}_Z(P, Z)$. A *Seifert map* is a homomorphism $\theta: P \to P^*$ such that $\theta + \epsilon\theta^*: P \to P^*$ is an isomorphism, where θ^* is the dual of θ and P^{**} is identified with P. Define $\theta(a,b) = \theta(b)(a)$ and if $F \subseteq P$ let the *annihilator* of F be $F^\perp = \{x \in P: \theta(F, x) = 0\}$. A subgroup F of P is *self-annihilating* if $F = F^\perp$. Note that this implies that F is a direct summand of rank n.

An ϵ-*form* is a quadruple (θ, F, G, ϕ) where θ is a Seifert map with domain P, F and G are self-annihilating subgroups of P, and there is an exact sequence of Abelian groups

$$0 \to F + G + 2P \hookrightarrow P \xrightarrow{i} \Pi \xrightarrow{h} F \cap G \to 0$$

with a bilinear pairing $\phi: \Pi \times \Pi \to Z_2$ such that for $a \in P$, $b \in \Pi$,

$$\phi(ia, b) \equiv \theta(a, hb) \qquad \text{mod } 2$$
$$\phi(b, ia) \equiv \theta(hb, a).$$

It is easy to see that $F \cap G$ is a direct summand of P, of rank r, say.

An isomorphism between two ϵ-forms (θ, F, G, ϕ) and (θ', F', G', ϕ') is a pair of maps (f, g) satisfying

$$f: P \xrightarrow{\cong} P', \qquad g: \Pi \xrightarrow{\cong} \Pi', \qquad fF = F', \qquad fG = G',$$

$$\begin{array}{ccccccccc} 0 \longrightarrow & F + G + 2P & \longrightarrow & P & \longrightarrow & \Pi & \longrightarrow & F \cap G & \longrightarrow 0 \\ & \downarrow f & & \downarrow f & & \downarrow g & & \downarrow f & \\ 0 \longrightarrow & F' + G' + 2P' & \longrightarrow & P' & \longrightarrow & \Pi' & \longrightarrow & F' \cap G' & \longrightarrow 0 \end{array}$$

commutes and

$$\theta'(fa, fb) = \theta(a, b) \quad \forall a, b \in P, \qquad \phi'(ga, gb) = \phi(a, b) \quad \forall a, b \in \Pi.$$

An ϵ-form (θ, F, G, ϕ) is called *odd* if the torsion subgroup of $P/(F+G)$ has odd order.

At this point we prove two technical lemmas about ϵ-forms which will be needed later.

LEMMA 1.1. *If* (θ, F, G, ϕ) *is an* ϵ-*form, then there is a subgroup* T *of* Π *such that* $h|_T: T \to F \cap G$ *is an isomorphism and* $\phi|_{T\times T}$ *is symmetric.*

Proof. Let $b_1, \ldots, b_r \in \Pi$ be such that $hb_1, \ldots, hb_r$ is a basis of $F \cap G$. Let $a_1, \ldots, a_r \in P$

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be such that $(\theta+\epsilon\theta^*)(a_j,x)=1$ if $x=hb_j$, 0 otherwise; this is possible as $\theta+\epsilon\theta^*$ is an isomorphism. Then

$$\begin{aligned}\phi(b_1+ia_k,b_l)-\phi(b_l,b_1+ia_k)\\ &=\phi(b_1,b_l)-\phi(b_l,b_1)+\phi(ia_k,b_l)-\phi(b_l,ia_k)\\ &\equiv\phi(b_1,b_l)-\phi(b_l,b_1)+\theta(a_k,hb_l)-\theta(hb_l,a_k)\\ &\equiv\phi(b_1,b_l)-\phi(b_l,b_1)+\delta_{kl}.\end{aligned}$$

Let L be the subset of $2,\dots,r$ for which $\phi(b_1,b_k)\neq\phi(b_k,b_1)$, and define $b'_1=b_1+\sum_{k\in L} ia_k$. Then $\phi(b'_1,b_k)=\phi(b_k,b'_1)$ for $2\le k\le r$ and $hb'_1=hb_1$. Iterate this process to obtain b'_2, b'_3, etc. □

We call T a *symmetric* subgroup of Π.

LEMMA 1.2. *Let (θ,F,G,ϕ) be an odd ϵ-form, and R, T symmetric subgroups. If $b_1,\dots,b_r$ is a basis of R and $b'_1,\dots,b'_r$ a basis of T such that $hb_j=hb'_j$ for $1\le j\le r$, then there exists $a_1,\dots,a_r\in P$ and $\lambda_{jk}\in Z$ $(1\le j,k\le r)$ with the following properties.*

(i) $\theta(a_j,x)+\epsilon\theta(x,a_j)=1$ if $x=hb_j$ and 0 otherwise.

(ii) $b'_k=b_k+\sum_{s=1}^{r}\lambda_{ks}ia_s$, $\forall k$.

(iii) $\lambda_{kl}\equiv\lambda_{lk}$ mod 2, $\forall k,l$.

Proof. The existence of $a_1,\dots,a_r\in P$ with property (i) follows because $\theta+\epsilon\theta^*$ is an isomorphism. Because the ϵ-form is odd, $ia_1,\dots,ia_r$ is a basis of Imi, and so b'_k-b_k can be expressed as $\sum_{s=1}^{r}\lambda_{ks}ia_s$ for some λ_{ks}.

$$\phi(b'_k,b'_l)=\phi(b_k,b_l)+\Sigma_s\lambda_{ls}\phi(b_k,ia_s)+\Sigma_s\lambda_{ks}\phi(ia_s,b_l).$$

Since ϕ is symmetric on R and T, we obtain

$$\Sigma_s\lambda_{ls}[\phi(b_k,ia_s)-\phi(ia_s,b_k)]+\Sigma_s\lambda_{ks}[\phi(ia_s,b_l)-\phi(b_l,ia_s)]=0$$

and so

$$-\Sigma_s\delta_{ks}\lambda_{ls}+\Sigma_s\delta_{ls}\lambda_{ks}\equiv 0 \mod 2$$
$$\lambda_{kl}\equiv\lambda_{lk}.$$ □

§2. *T*-EQUIVALENCES

If M,N,P,Q are free Abelian groups of finite rank, elements of $\mathrm{Hom}_Z(M\oplus N,P\oplus Q)$ can be displayed as matrices

$$f=\begin{pmatrix}\alpha\beta\\ \gamma\delta\end{pmatrix}: M\oplus N\to P\oplus Q,\qquad f\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\alpha(x)+\beta(y)\\ \gamma(x)+\delta(y)\end{pmatrix},$$

where $\alpha\in\mathrm{Hom}_Z(M,P)$, $\beta\in\mathrm{Hom}_Z(N,P)$, $\gamma\in\mathrm{Hom}_Z(M,Q)$, $\delta\in\mathrm{Hom}_Z(N,Q)$.

Moreover, $\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}$ will be denoted by $\alpha\oplus\delta$.

Throughout this section, $U\cong V\cong R\cong S\cong Z$, and the lower case letters will denote a generator: thus $U=\langle u\rangle$. We define the following moves on an ϵ-form (θ,F,G,ϕ).

T0. $(\theta,F,G,\phi)\to(\hat\theta,\hat F,\hat G,\hat\phi)$, $\hat P=P\oplus R\oplus S$,

$$\hat\theta=\begin{pmatrix}\theta&0&0\\0&0&1\\0&0&0\end{pmatrix}\quad\text{or}\quad\begin{pmatrix}\theta&0&0\\0&0&0\\0&1&0\end{pmatrix}$$

$\hat F=F\oplus R$, $\hat G=G\oplus S$, $\hat\Pi=\Pi$, $\hat\phi=\phi$.

T1. $\theta\mapsto\theta+\psi-\epsilon\psi^*$ where $\psi: P\to P^*$ has rank one and $\psi F=0$, $\psi^*G=0$.

T2. $(\theta,F,G,\phi)\to(\hat\theta,\hat F,\hat G,\hat\phi)$,

$\hat P=P\oplus R\oplus S$, $\hat F=F\oplus S$, $\hat G=G\oplus S$,

$$\hat\theta=\begin{pmatrix}\theta&\alpha&0\\0&0&1\\0&0&0\end{pmatrix}\quad\text{or}\quad\begin{pmatrix}\theta&0&0\\ \beta&0&0\\0&1&0\end{pmatrix}$$

$\hat\Pi=\Pi\oplus(R/2R)\oplus S$ where $0\to F+G+S+2R+2P\to P\oplus R\oplus S\xrightarrow{i\oplus(2)\oplus 0}\Pi\oplus(R/2R)\oplus S\xrightarrow{h\oplus 0\oplus 1}(F\cap G)\oplus S\to 0.$

and (2): $R\to R/2R$ is the quotient map.

$$\hat{\phi}|\Pi \times \Pi = \phi$$
$$\hat{\phi}|(\Pi \oplus S) \times S = 0 = \hat{\phi}|S \times (\Pi \oplus S).$$

Note that $\hat{\phi}$ is determined elsewhere by $\hat{\theta}$.

T3. $(\theta, F, G, \phi) \mapsto (\theta, \hat{F}, \hat{C}, \hat{\phi})$

$$Q \cong Z^{2n-2},\ P = Q \oplus R \oplus S \oplus U \oplus V$$

$$\theta = \left(\begin{array}{c|cc} \psi & 0 & \alpha \\ \hline 0 & 0 & 1 \\ -\epsilon\alpha^* & 0 & 0 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|cc} \psi & 0 & \alpha \\ \hline 0 & 0 & 0 \\ -\epsilon\alpha^* & 1 & 0 \end{array}\right)$$

where $\psi: Q \oplus R \oplus S \to (Q \oplus R \oplus S)^*$.

$$F = A \oplus \langle s - mv \rangle \oplus V, \qquad A \subseteq Q,$$
$$G = B \oplus S \oplus \langle u - mr + pv \rangle, \qquad B \subseteq Q,$$
$$\hat{F} = A \oplus \langle s - mv \rangle \oplus U,$$
$$\hat{G} = G.$$

It follows from these assumptions that $\hat{\phi}$ induces $\bar{\phi}$ on $\bar{\Pi} = \hat{\Pi}/\hat{\imath}(V + R)$, and we have an exact sequence $0 \to A + B + 2Q \to Q \xrightarrow{\bar{\imath}} \bar{\Pi} \xrightarrow{\bar{h}} A \cap B \to 0$. Consider the exact sequence

$$0 \to A + B + S + 2R + 2Q \to Q \oplus R \oplus S \xrightarrow{\bar{\imath} \oplus (2) \oplus 0} \bar{\Pi} \oplus (R/2R) \oplus S$$
$$\xrightarrow{\bar{h} \oplus 0 \oplus 1} (A \cap B) \oplus S \to 0.$$

Because $iv = 0$ and $iu = mr$, this sequence determines $\Pi = \bar{\Pi} \oplus (R/2R) \oplus S$, and ϕ is given by $\phi|\bar{\Pi} \times \bar{\Pi} = \bar{\phi}$, ϕ symmetric on $(\bar{\Pi} \oplus S) \times S \cup S \times (\bar{\Pi} \oplus S)$.

The moves T0–3 generate an equivalence relation on the set of ϵ-forms which will be called *T-equivalence*.

§3. STATEMENT OF RESULTS

It will be shown in the sequel that any simple $2q$-knot, $q \geq 3$, gives rise to a $(-1)^q$-form, via a Seifert surface.

THEOREM 3.1. *Let (θ, F, G, ϕ) be a $(-1)^q$-form. If $q \geq 3$, there is a simple $2q$-knot giving rise to (θ, F, G, ϕ).*

THEOREM 3.2. *Let k be a simple $2q$-knot, $q \geq 3$. Then any two $(-1)^q$-forms arising from k are T-equivalent.*

THEOREM 3.3. *Let k, $\bar{k}$ be two odd simple $2q$-knots, $q \geq 3$, giving rise to $(-1)^q$-forms (θ, F, G, ϕ) and $(\bar{\theta}, \bar{F}, \bar{G}, \bar{\phi})$ respectively. If these forms are T-equivalent, then k and $\bar{k}$ are isotopic.*

Remark. In §13, Theorem 3.1 is refined to Theorem 13.1; this completes the algebraic classification of odd simple knots in terms of ϵ-forms and T-equivalence.

§4. CROSS-SECTIONS OF A KNOT

Let k be a $2q$-knot, (S^{2q+2}, Σ^{2q}), and let S^{2q+1} denote the equatorial sphere of S^{2q+2}. Suppose that Σ^{2q} meets S^{2q+1} transversely in an equatorial sphere S^{2q-1} of Σ^{2q} so that Σ^{2q} is the union of two smooth $2q$-balls along their common boundary S^{2q-1}. Let k' be the knot (S^{2q+1}, S^{2q-1}), and denote the two null-cobordisms of k' by b_+, b_-.

If Y is a smooth proper submanifold of a manifold X, then the *complement* of Y in X is the closed complement of a tubular neighbourhood N of Y in X, where $N \cap \partial X$ is a tubular neighbourhood of ∂Y in ∂X. Let K denote the complement of Σ^{2q} in S^{2q+2}; we shall abbreviate this to "K is the complement of k". Similarly, let K' be the complement of k' and K^ϵ the complement of b_ϵ, $\epsilon = \pm$. We shall always take K^ϵ to be the restriction of K to the appropriate hemisphere B_ϵ^{2q+2} of S^{2q+2}, and $K' = K \cap S^{2q+1}$.

In these circumstances, k' is a *cross-section of* k if (K^ϵ, K') is q-connected for $\epsilon = \pm$.

It is known[5] that any knot k is spanned by a Seifert surface V; so that $\Sigma^{2q} = \partial V$ where V is a smooth submanifold of S^{2q+2}. If V meets S^{2q+1} transversely, then the intersection is a Seifert surface V' of k'. V' is a *cross-section* of V if (V^ϵ, V') is q-connected for $\epsilon = \pm$, where $V^\epsilon = V \cap B_\epsilon^{2q+2}$.

PROPOSITION 4.1. *Let k be a $2q$-knot, $q \geq 3$, spanned by a Seifert surface V. Then k has a cross-section k' spanned by a cross-section V' of V.*

Proof. Regard Σ^{2q} as the union of two $2q$-balls, $B_-^{2q} \cup_{S^{2q-1}} B_+^{2q}$. Let $N_- \cup N_+$ be a tubular neighbourhood of Σ^{2q}, with $(N_\epsilon, B_\epsilon^{2q})$ an unknotted ball pair such that $V_\epsilon = V \cap N_\epsilon$ is a tubular neighbourhood of B_ϵ^{2q} rel S^{2q-1} in V. Since V has a tubular neighbourhood rel ∂V of the form $V \times B^1$, it is clear that a handle decomposition of V based on V_- gives rise to a handle decomposition of a tubular neighbourhood N of V, by handle $\to$ handle $\times B^1$. Note that N_- and N_+ are exceptions to this: indeed N_+ is not a handle, but a $(2q+2)$-ball added by a face.

Choosing a handle decomposition of $M = \overline{S^{2q+2} - N}$, we obtain a handle decomposition of S^{2q+2} based on N_- (in which N_+ appears as a $(2q+2)$-handle).

We may add these handles in order of increasing index, in the usual way. Regard the handles as being added to $N_- - B_-^{2q}$, and let L be the manifold obtained when all the q-handles have been added. Since K is a homology circle, L has the homology of $S^1 \times B^{2q+1}$ with some q-handles added. For each $(q+1)$-handle of V, add a trivial pair of $(q+1)$, $(q+2)$-handles to M and move the new $(q+1)$-handle of M over the $(q+1)$-handle obtained from V. After perhaps moving some of the $(q+1)$-handles of M over each other, $L \cup$ (suitable $(q+1)$-handles of M) is a homology circle. Call this manifold L_1. Then $L_1 \cup B_-^{2q}$ is a homotopy ball, and hence a ball, B_-^{2q+2} say. Let $B_+^{2q+2} = \overline{S^{2q+2} - B_-^{2q+2}}$. The common boundary S^{2q+1} contains a knot k' which is the required cross-section of k, spanned by V'; for K^ϵ has a handle decomposition based on K' containing only handles of index at least $q+1$, and similarly for V^ϵ, V'. □

Remark. By taking handle decompositions of V^+, V^- based on V', we can see that every cross-section arises in the manner described above.

Recall that a knot k is *r-simple* if K has the homotopy r-type of a circle.

LEMMA 4.2. *If k is r-simple, then so is every cross-section, and if V is r-connected so is every cross-section, for $r < q$. Conversely, if k' is r-simple so is k and if V' is r-connected so is V.*

The proof is easy.

§5. THE ϵ-FORM OF A KNOT

Let k be a simple $2q$-knot, $q \geq 3$; then by Lemma 4.2 any cross-section k' is also simple. By a result of Levine[5], k has a Seifert surface V which is $(q-1)$-connected, so that V has a cross-section V' which is also $(q-1)$-connected. V has a tubular neighbourhood, mod ∂V, of the form $V \times B^1$, where $B^1 = [-1, 1]$ and $+1$ corresponds to the positive normal direction. We may assume that $V' \times B^1 \subset S^{2q+1}$.

V' has homology only in dimension q, so that $H_q(V')$ is free of rank $2n$, say. Setting $P = H_q(V') \cong Z^{2n}$, the map $H_q(V') \to H_q(V' \times 1)$ together with Alexander duality provides a map $\theta: P \to P^*$. Alternatively we may define $\theta: P \times P \to Z$ by $\theta(a, b) = L(z_a, z_b \times 1)$ where z_a, z_b are cycles representing a, b, and L denotes linking in S^{2q+1}. $\theta + (-1)^q\theta^*$ defines the intersection pairing on V', and so θ is a Seifert map.

Define $F = \ker(H_q(V') \to H_q(V^-))$ and $G = \ker(H_q(V') \to H_q(V^+))$; work of Levine[6] shows that F and G are self-annihilating subgroups of P.

Let $\Pi = \pi_{q+1}(V)$; we define a homotopy linking $\phi: \Pi \times \Pi \to Z_2$. If $a, b \in \Pi$, they may be represented by embedded spheres z_a, z_b, each of which is unknotted in S^{2q+2}. The spheres z_a, $z_b \times 1$ are disjoint and the complement of z_a in S^{2q+2} has the homotopy type of S^q. Thus $z_b \times 1$ defines an element of $\pi_{q+1}(S^q) \cong Z_2$, denoted by $\phi(a, b)$. Clearly ϕ is bilinear.

From the Mayer–Vietoris sequence, $0 \to H_{q+1}(V) \xrightarrow{\partial} H_q(V') \to H_q(V^-) \oplus H_q(V^+) \to$, we see that $H_{q+1}(V) \xrightarrow[\cong]{\partial} F \cap G$. Define h to be the composite $\pi_{q+1}(V) \xrightarrow{H} H_{q+1}(V) \xrightarrow{\partial} F \cap G$ where H is the Hurewicz map. From the diagram

$$\begin{array}{ccccc} \pi_{q+1}(V^-) & \xrightarrow{i_*} & \pi_{q+1}(V) & \xrightarrow{j_*} & \pi_{q+1}(V, V^-) \longrightarrow \\ & & \downarrow H & & \downarrow H \\ 0 & \longrightarrow & H_{q+1}(V) & \longrightarrow & H_{q+1}(V, V^-) \longrightarrow \end{array}$$

we see that $\ker h = \ker j_* = \operatorname{Im} i_*$. Up to homotopy type, V^- is the wedge of n q-spheres, so that $\pi_q(V^-) \cong H_q(V^-)$ may be identified with P/F and (by the work of Hilton[9]) $\pi_{q+1}(V^-)$ with $P/(F+2P)$. With these identifications, $\ker i_* = (G+F+2P)/(F+2P)$, and so we have an exact sequence

$$0 \to F+G+2P \to P \xrightarrow{i} \Pi \xrightarrow{h} F \cap G \to 0$$

where ia is just the composite of a map representing $a \in P = \Pi_q(V')$ with the non-zero element of $\pi_{q+1}(S^q)$.

Let $a \in P$, $b \in \Pi$. The homological linking of z_a and $z_{Hb} \times 1$ in S^{2q+2} is $\theta(a, hb)$, so $\phi(ia, b) \equiv \theta(a, hb) \bmod 2$.

Let $h_1^\epsilon, \ldots, h_n^\epsilon$ be the handles of $V^\epsilon(\epsilon = \pm)$ based on $V' \times B^1$ in the cross-section. Let $p: V' \times B^1 \to V' \times 1$ be projection. Suppose that p (the attaching sphere of h_i^-) coincides with that of h_i^+ for $1 \le i \le r$. Then these attaching spheres represent a basis of $F \cap G$, and the union of the cores of h_i^-, h_i^+ $(1 \le i \le r)$ with the collars of their attaching spheres in $V' \times B^1$ represent a basis of a symmetric subgroup T of Π. In these circumstances we say that T is *well-represented.*

§6. SYMMETRIC SUBGROUPS

In this section we show that a symmetric subgroup can be well-represented when the ϵ-form is odd.

LEMMA 6.1. *Let* $M \cong \#_{i=1}^{n} (S^q \times S^{q+1})_i\text{-int } B^{2q+1}$, $q \ge 3$, *and let*

$a_i \in \pi_q(M)$ *be represented by* $(S^q \times 0)_i$,
$b_i \in \pi_{q+1}(M)$ *be represented by* $(0 \times S^{q+1})_i$,
$c_i \in \pi_{q+1}(M)$ *be represented by* $(\xi \times 0)_i$,

where ξ *is the non-zero element of* $\pi_{q+1}(S^q) \cong Z_2$. *Suppose that* $d_i \in \pi_{q+1}(M)$ *are such that* $hd_i = hb_i$, $1 \le i \le n$, *where h is the Hurewicz map. Then* d_i, $1 \le i \le n$, *are represented by a set of disjoint embedded spheres if and only if*

$$d_i = b_i + \sum_{j=1}^{n} \lambda_{ij} c_j \quad \text{with} \quad \lambda_{ij} + (-1)^q \lambda_{ji} \equiv 0 \pmod 2.$$

Proof. First we prove necessity. Clearly d_i must have the form $b_i + \sum_{j=1}^{n} \lambda_{ij} c_j$ if $hb_i = hd_i$. Let $d_i' = b_i + \sum_{j \ne 1} \lambda_{ij} c_j = d_i - \lambda_{ii} c_i$. The d_i are a set of disjoint embedded spheres, and we can arrange for d_i to meet a_i transversely in a single point, and miss all the other a_j. Thus the d_i' can be represented by a set of disjoint embedded spheres, such that d_i' meets a_i transversely in a single point and misses all the other a_j. Now b_i and c_j have trivial normal bundles, and so therefore has d_i'. Thus we can write $M \cong \#_{i=1}^{n} (S^q \times S^{q+1})_i\text{-int } B^{2q+1}$, as above, but with d_i' represented by $(0 \times S^{q+1})_i$.

If $[\,,\,]$ denotes the Whitehead product, then $\sum_{i=1}^{n} [a_i, b_i] = \iota_* \eta$, where η is a generator of $\pi_{2q}(\partial M) \cong Z$ and ι_* is the map $\pi_{2q}(\partial M) \to \pi_{2q}(M)$ induced by inclusion. The same equation holds with d_i' in place of b_i, so we have

$$\sum_{i=1}^{n} \left[a_i, \sum_{j \ne i} \lambda_{ij} c_j \right] = 0$$

from which it follows that $\lambda_{ij} + (-1)^q \lambda_{ji} = 0$, $i \ne j$. This is equivalent to the equation above.

To prove sufficiency, consider first replacing b_i by $b_i + c_j$, $j \ne i$. We may represent $b_i + c_j$ by an embedded sphere with trivial normal bundle, meeting a_i transversely in a single point. Therefore a tubular neighbourhood of the wedge of the spheres representing a_i and $b_i + c_j$ has the form $S^q \times S^{q+1}\text{-int } B^{2q+1}$, so we may split this off as part of a connected sum. Thus $M \cong (S^q \times S^{q+1}) \# N\text{-int } B^{2q+1}$, where $S^q \times 0$ represents a_i, $0 \times S^{q+1}$ represents $b_i + c_j$. Now $\pi_i(M) \cong \pi_i(S^q \times S^{q+1}) \oplus \pi_i(N)$ for $i \le q+1$: this follows from the formula for the homotopy groups of a wedge of two spaces and the relative Hurewicz theorem[2]. Thus in N, hb_i is a spherical class, and the necessity condition shows that it must be represented by $b_i + c_i$.

To replace b_i by $b_i + c_i$ is easy: we only need appeal to standard embedding theorems. □

LEMMA 6.2. *Assume that (θ, F, G, ϕ) is an odd ϵ-form, and let T be a symmetric subgroup of Π. Then T is well-represented.*

Proof. By moving the handles of V over each other, we may change base in F and G. In this way we can obtain some symmetric subgroup R of Π which is well-represented. Let $b_1, \dots, b_r$ be the basis of R determined by the handle decomposition, and let $b'_1, \dots, b'_r$ be a basis of T such that $hb'_1 = hb_i$ for $1 \le i \le r$. Then Lemma 1.2 and 6.1 complete the proof. □

§7. WHEN A KNOT IS DETERMINED BY ITS ϵ-FORM

PROPOSITION 7.1. *Provided that $q \ge 3$ and $H_q(V)$ has no 2-torsion, a simple $2q$-knot k with Seifert surface V is determined up to isotopy by its $(-1)^q$-form.*

Proof. S^{2q+1} has a tubular neighbourhood of the form $S^{2q+1} \times B^1$; as V meets S^{2q+1} transversely, we may assume that $V \cap (S^{2q+1} \times B^1) = V' \times B^1$. Arrange that $V^\epsilon(\epsilon = \pm)$ has a handle decomposition on $V' \times B^1$ as described at the end of §5, and let $\alpha_i^\epsilon \in P = H_q(V')$ be the element determined by the attaching sphere of h_i^ϵ. Thus $\alpha_i^+ = \alpha_i^-$ for $1 \le i \le r$. Recall that $\theta + (-1)^q\theta^*$ is the intersection pairing on V', which we shall denote by $\alpha.\beta$. Since $F = F^\perp$, there is a basis $\alpha_1^-, \dots, \alpha_n^-, \gamma_1, \dots, \gamma_n$ of P such that $\alpha_i^- \cdot \gamma_j = \delta_{ij}$ and $\gamma_i \cdot \gamma_j = 0$ for all i, j.

Let $\hat{k}$ be another such knot, and distinguish the machinery associated with $\hat{k}$ by $\hat{}$. Let $(f, q)\colon (\theta, F, G, \phi) \to (\hat{\theta}, \hat{F}, \hat{G}, \hat{\phi})$ be an isomorphism between the $(-1)^q$-forms of the two knots, and let $\hat{\alpha}_i^\epsilon = f\alpha_i^\epsilon$, $\hat{\gamma}_i = f\gamma_i$, $1 \le i \le n$.

Recall that $S^{2q+2} = B_-^{2q+2} \cup (S^{2q+1} \times B^1) \cup B_+^{2q+2}$, and let D_i denote the core of h_i^-. Allowing the boundaries to move within ∂B_-^{2q+2}, we may isotop D_i onto $\hat{D}_i$, $1 \le i \le n$. Now we resort to an argument of Levine[7]. Let v_i be the positive unit normal field to h_i^- on D_i. By the tubular neighbourhood theorem, we may assume that h_i^- is the orthogonal complement of v_i in a normal disc bundle neighbourhood N_i of $D_i = \hat{D}_i$ in B^{2q+2}. Therefore, if we can homotop v_i to $\hat{v}_i$, we obtain an isotopy of h_i^- to $\hat{h}_i^-$ within N_i. Since we are willing to allow movement on the boundary, v_i is homotopic to $\hat{v}_i$, and we obtain the desired isotopy.

Each basis element $\alpha_1^-, \dots, \alpha_n^-, \gamma_1, \dots, \gamma_n$ of $H_q(V')$ may be represented by a handle of V', and from the argument of Levine[7] we see that as

$$\hat{\theta}(f\alpha, f\beta) = \theta(\alpha, \beta) \quad \forall \alpha, \beta \in P,$$

V' may be isotoped onto $\hat{V}'$ without disturbing the h_i^-, and so we may isotop b_- to coincide with $\hat{b}_-$.

Let C_i denote the core of h_i^+. If $1 \le i \le r$, $D_i \cup (\partial D_i \times B^1) \cup C_i$ is an embedded $(q+1)$-sphere representing an element $b_i \in \Pi$, and $b_1, \dots, b_r$ is a basis of a symmetric subgroup T. Put $\hat{b}_i = gb_i$, $1 \le i \le r$; by Lemma 6.2, $\hat{T} = g(T)$ is well-represented and we may arrange that $\hat{D}_i \cup (\partial\hat{D}_i \times B^1) \cup \hat{C}_i$ represents $\hat{b}_i$. Isotop C_1 onto $\hat{C}_1$ keeping the boundary fixed. The obstruction to isotoping C_2 onto $\hat{C}_2$ keeping the boundary fixed and without disturbing C_1 may be identified with $\phi(b_1, b_2) - \hat{\phi}(\hat{b}_1, \hat{b}_2) = 0$. Continuing in this way, we isotop C_i onto $\hat{C}_i$, $1 \le i \le r$. To isotop h_i^+ onto $\hat{h}_i^+$, $1 \le i \le r$, we adopt the same method as above; the obstruction may be identified with $\phi(b_i, b_i) - \hat{\phi}(\hat{b}_i, \hat{b}_i)) = 0$.

By a change of basis, we can arrange that $\alpha_i^+ = \sum_{j=1}^{n} a_{ij}\alpha_j^- + d_i\gamma_i$, $r < i \le n$, where the d_i are the torsion numbers of $H_q(V)$. The same tactics can now be tried on C_i, $r < i \le n$, and then h_i^+, but with this difference: the obstruction at each stage may be identified with an isotopy in S^{2q+1} of the handle of V' corresponding to γ_i, using the fact that d_i is odd. The isotopy brings the handle back to its original position. Thus the obstruction can be removed by allowing the handles of V' corresponding to the γ_i to move, and this is allowable because it does not affect b_- adversely.

□

§8. CHANGE OF CROSS-SECTIONS

We begin to investigate the extent to which k determines (θ, F, G, ϕ), where k is a simple $2q$-knot. Given a $(q-1)$-connected V spanning k, to what extent can the cross-section be changed? We can pass from one handle decomposition of V to any other by moving handles over one another and by adding or deleting cancelling pairs of index $r, r+1$: these are the well-known moves employed in the proof of, say, the h-cobordism theorem. The only move that affects the ϵ-form is that of adding (or deleting) a cancelling pair of index $q, q+1$, and this gives rise to the move T0 of §2 (or its inverse).

It may happen that in the construction of a cross-section there are two handles h_+^{q+1}, h_-^{q+1} of $S^{2q+2}-N$ (where N is a tubular neighbourhood of V), with $h_\epsilon^{q+1} \subset B_\epsilon^{2q+2}$; and that $\overline{(B_-^{2q+2}-h_-^{q+1})} \cup h_+^{q+1}$ forms another cross-section. In other words, when deciding which $(q+1)$-handles of $S^{2q+2}-N$ to add in order to cancel the q-handles homologically, we can use h_+^{q+1} in place of h_-^{q+1}.

Suppose that for a fixed V there are given two cross-sections C_1 and C_2; then these correspond to two handle decompositions G_1, G_2 of $S^{2q+2}-N$. As above, it is possible to pass from G_1 to G_2 via a sequence of handle moves. Those moves involving only handles below (above) the middle dimension do not affect the cross-section, so we may ignore them. Moreover, if any handle pairs of index $(q, q+1)$ need to be added, then they may be introduced at the beginning and so be assumed to form part of G_1; dually, any cancelling pairs of this index may be left to form part of G_2.

Let $h_1^{q+1}, \dots, h_m^{q+1}$ be the $(q+1)$-handles of G_1 which are contained in B_{1-}^{2q+2}, where the subscript 1 corresponds to C_1; these are the $(q+1)$-handles of G_1 which are used to cancel homologically in the construction of C_1. Add trivial handle pairs $\hat{h}_i^{q+1}, \hat{h}_i^{q+2}, 1 \le i \le m$, to G_1, and move $\hat{h}_i^{q+1}$ over h_i^{q+1} for each i. Now replace h_i^{q+1} by $\hat{h}_i^{q+1}$ to obtain a new cross-section in which all the original $(q+1)$-handles of G_1 are contained in B_+^{2q+2}. All the handle moves which involve moving the $(q+1)$-handles over one another may now be performed without disturbing the new cross-section. After a change of basis in the $\hat{h}_1^{q+1}, \dots, \hat{h}_m^{q+1}$, if necessary, we may replace each $\hat{h}_i^{q+1}$ by the appropriate handle in B_+^{2q+2} to obtain the cross-section C_2.

Thus we only need to consider the effect on (θ, F, G, ϕ) of replacing one handle h_-^{q+1} by another, h_+^{q+1}, as described above.

LEMMA 8.1. *This procedure induces a move* T1 *on* (θ, F, G, ϕ).

Proof. The effect of replacing h_-^{q+1} by h_+^{q+1} is to perform two surgeries of index $q+1$ on S^{2q+1} to obtain another equatorial sphere: thus we have a compact manifold $M \subset S^{2q+2}\times I$ with $M \cap (S^{2q+2}\times t) \cong S^{2q+1}$, $t = 0, 1$. M has two critical levels with respect to the height function induced by $S^{2q+2}\times I \to I$, each of index $q+1$, and $M \cap (V \times I) = V' \times I$ where $V \times I \subset S^{2q+2}\times I$ by (inclusion × identity).

M splits $S^{2q+2}\times I$ into two components, L^+ and L^-, and the Mayer–Vietoris sequence yields, $0 \to H_{q+1}(M) \xrightarrow{J^+\oplus J^-} H_{q+1}(L^+)\oplus H_{q+1}(L^-) \to 0$, where $H_{q+1}(L^+) \cong Z \cong H_{q+1}(L^-)$. Thus we may take as a basis for $H_{q+1}(M)$ elements ξ and η, being generators of $\ker J^-$ and $\ker J^+$, with $\xi\cdot\eta = 1 = (-1)^{q+1}\eta\cdot\xi$ and $\xi\cdot\xi = \eta\cdot\eta = 0$.

Consider $V'\times I \subset M$; suppose that $\alpha_1^0, \dots, \alpha_{2n}^0$ is a basis for $H_q(V'\times 0)$. Let $\tilde{\alpha}_i = \alpha_i^0 \times I \subset V' \times I$, so that $\partial\tilde{\alpha}_i = \alpha_i^1 - \alpha_i^0$ and $\alpha_1^1, \dots, \alpha_{2n}^1$ is a basis for $H_q(V'\times 1)$.

If $\tilde{\alpha}_i$ is regarded as a cycle of $H_{q+1}(M, \partial M)$, then $\tilde{\alpha}_i \sim \iota(a_i\xi - b_i\eta)$ where $\iota\colon H_{q+1}(M) \to H_{q+1}(M, \partial M)$ is the obvious isomorphism. It follows that $\tilde{\alpha}_i - \iota(a_i\xi - b_i\eta)$ is a chain with boundary $\alpha_i^1 - \alpha_i^0$ representing 0 in $H_{q+1}(M, \partial M)$. If $\tilde{\underline{\alpha}}_i$ denotes $\tilde{\alpha}_i$ pushed off $V'\times I$ in the positive direction, then

$$\begin{aligned}\theta_1(\alpha_i^1, \alpha_j^1) - \theta_0(\alpha_i^0, \alpha_j^0) &= [\tilde{\alpha}_i - \iota(a_i\xi - b_i\eta)]\cdot[\tilde{\underline{\alpha}}_j - \iota(a_j\xi - b_j\eta)] \\ &= -(a_i\xi - b_i\eta)\cdot(a_j\xi - b_j\eta) \\ &= a_ib_j + (-1)^{q+1}a_jb_i.\end{aligned}$$

Thus if the matrix of θ_t with respect to $\alpha_1^t, \dots, \alpha_{2n}^t$ is A_t, $t = 0, 1$, and a, b denote the column vectors with entries a_i, b_i, we have $A_1 - A_0 = ab' + (-1)^{q+1}ba'$.

Suppose that $\alpha_1^0, \dots, \alpha_n^0$ is a basis of F, and let D_i be the $(q+1)$-chain in V^- with boundary α_i^0. Then $D_i \times I \subset V^-\times I$ is a chain with boundary $\tilde{\alpha}_i$ mod $V^-\times\partial I$; thus $\iota^{-1}\tilde{\alpha}_i \in \ker J^-$ and so $b_i = 0$, $1 \le i \le n$.

With respect to the basis $\alpha_1^0, \dots, \alpha_{2n}^0$ and its dual, ab' represents a map $\psi\colon P \to P^*$. It is easy to check that ψ has rank one and $\psi(F) = 0$; moreover any such map is represented by a matrix of the form ab' with $b_i = 0$, $1 \le i \le n$. Dually, it can be checked that $\psi^*(G) = 0$. □

§9. SURGERY ON V

Let k be a simple $2q$-knot, $q \geq 3$. By results of Levine[5], there exists a $(q-1)$-connected Seifert surface of k. If V_0, V_1 are two such surfaces, Levine has shown[7] that there exists a cobordism between them, $V \subset S^{2q+2} \times I$, where $\partial V = V_0 \cup (\partial V_0 \times I) \cup V_1$ and $\partial V_0 \times I \subset S^{2q+2} \times I$ by (inclusion $\times$ identity). Applying the results of [5] we may arrange for V to be $(q-1)$-connected.

LEMMA 9.1. *Let* $f\colon V \to I$ *be the restriction to* V *of the projection* $S^{2q+2} \times I \to I$. *We may isotop* V, rel $V_0 \cup V_1$, *so that* f *has critical levels of index* q, $q+1$, *and* $q+2$ *only, which appear in order of increasing index.*

Proof. Let X be $S^{2q+2} \times I$ split open along V, so that ∂X contains two copies U^+, U^- of V. X_0 denotes $S^{2q+2} \times 0$ split open along V_0, $U_0^\epsilon = U^\epsilon \cap X_0$, etc. Let $i^\epsilon\colon H_q(U^\epsilon, U_0^\epsilon) \to H_q(X, X_0)$, $\epsilon = \pm$, be induced by inclusion. We can use V to construct the universal (infinite cyclic) cover $\tilde{Y}$ of $Y = S^{2q+2} \times I - S^{2q} \times I$, as in [8]. Since $H_q(\tilde{Y}, \tilde{Y}_0) \cong \pi_q(\tilde{Y}, \tilde{Y}_0) \cong \pi_q(Y, Y_0) = 0$, the argument of [8] using integer coefficients shows that at least one of i^+, i^- is singular. If say $\alpha \in \ker i^+$, then since by the Hurewicz theorem $\pi_q(U^+, U_0^+) \cong H_q(U^+, U_0^+)$, α may be represented by a singular disc. Applying results of Haefliger[1], α may be represented by a q-ball properly embedded in (U^+, U_0^+). By Hurewicz's theorem, $H_q(X, X_0) \cong \pi_q(X, X_0)$, so α is null-homotopic in (X, X_0). Repeated application of Haefliger's results[1] shows that there is a $(q+1)$-ball B^{q+1} properly embedded in $(X, \partial X)$ with the following properties. $\partial B^{q+1} = B_+^q \cup B_-^q$, $B_+^q \cap B_-^q = S^{q-1}$, B_+^q is properly embedded in (U^+, U_0^+) so as to represent α, and B_-^q is properly embedded in (X_0, U_0^+).

By considering a tubular neighbourhood of B^{q+1} we may isotop V so that $\alpha \in H_q(V, V_0)$ is represented by the core of the handle corresponding to a critical level of f: see [3] for details in the *PL* case. Continuing in this way, we reduce to the case $H_q(V, V_0) = 0$, and dually we may arrange for $H_q(V, V_1) = 0$. A similar argument now works for $H_{q+1}(V, V_0)$; the details are omitted. □

Suppose now that V has a single critical level, of index q. If V_0' is a cross-section of V_0, then (V_0, V_0') is a q-connected pair; thus in the notation used above S^{q-1} may be homotoped (and therefore isotoped) to lie within V_0'. We should like to isotop B_-^q into $S^{2q+1} \times 0$, keeping S^{q-1} fixed. Let X_0' denote $S^{2q+1} \times 0$ split along V_0'; then (X_0, X_0') is q-connected (recall the construction of a cross-section), and as S^{q-1} is null-homotopic in V_0' this is enough to show that B_-^q can be homotoped (and therefore isotoped) into $S^{2q+1} \times 0$ keeping S^{q-1} fixed.

We are thus able to obtain a cross-section V_1' of V_1 from the cross-section V_0' of V_0; the cobordism $V' = V \cap (S^{2q+1} \times I)$ between V_0' and V_1' has a single critical level, of index q.

Now suppose that V has a single critical level, of index $q+1$. By the remarks above, V_1 may be obtained from V_0 by a surgery embedded in S^{2q+2}. Thus if $V_0 \times B^1$ is a tubular neighbourhood of V_0 in S^{2q+2}, there is an embedding of a $(2q+2)$-ball, $i\colon B^{q+1} \times B^{q+1} \to S^{2q+2}$, meeting $V_0 \times B^1$ in $i(\partial B^{q+1} \times B^{q+1}) \subset V_0 \times \eta$ where $\eta = 1$ or -1, and such that $V_1 = \overline{V_0 - (i(\partial B^{q+1} \times \frac{1}{2}B^{q+1}) \times 0)} \cup (i(\partial B^{q+1} \times \partial \frac{1}{2}B^{q+1}) \times [0, \eta]) \cup i(B^{q+1} \times \partial \frac{1}{2}B^{q+1})$.

We can choose a handle decomposition of V_0, involving handles only in dimensions 0, q, $q+1$, such that $i(\partial B^{q+1} \times B^{q+1}) \times 0 \subset V_0 \times 0$ is $h^0 \cup h^q$ and Imi is the $(q+1)$-handle of the complement used to cancel h^q in the construction of a cross-section V_0' of V_0. Then $V_0' = V_1'$ a cross-section of V_1, but the roles of $B^{q+1} \times 0$ and $0 \times \frac{1}{2}B^{q+1}$ are interchanged. Thus if $\alpha \in H_q(V_0')$ is represented by $i(\partial B^{q+1} \times x) \times 0$ for suitable $x \in \partial B^{q+1}$, and β by $i(y \times \partial B^{q+1}) \times 0$ for suitable $y \in \partial B^{q+1}$, then $\alpha \in \ker(H_q(V_1') \to H_q(V_1^-))$ and $\beta \in \ker(H_q(V_0') \to H_q(V_0^-))$. Indeed, α is represented by the boundary of a $(q+1)$-ball $i(B^{q+1} \times x) \cup (i(\partial B^{q+1} \times x) \times [0, \eta])$ embedded in S^{2q+1} and meeting V_0 only in $i(\partial B^{q+1} \times x) \times 0$.

§10. T2 AND T3

LEMMA 10.1. *In the situation of the previous section, let* V *and* V' *each have a single critical level, of index* q. *Then the* $(-1)^q$*-form of* V_1 *is obtained by a* T2*-move on the* $(-1)^q$*-form of* V_0.

Proof. By work of Levine[7], θ has the form shown. A generator of S is represented by the belt sphere of the surgery performed on V_0', and this is the equator of the belt sphere of the surgery performed on V_0. This sphere is the boundary of the cocore of the surgery, and so $\hat{\phi}$ has the form shown. □

LEMMA 10.2. In the situation of the previous section, let V have a single critical level, of index $q+1$. Then there are cross-sections of V_0 and V_1 such that the corresponding $(-1)^q$-forms are related by a move T3 or its inverse.

Proof. By the proof of [5; Lemma 5] we may assume that the q^{th} Betti number of V_0 exceeds that of V_1. Let $m\gamma \in H_q(V_0)$ be the element on which the surgery is performed, γ being primitive and $m \in Z^+$. Choose a handle decomposition $h^0 \cup h_1{}^q \cup \cdots \cup h_l{}^{q+1}$ of V_0 so that a spine of $h^0 \cup h_1{}^q$ represents γ; we may also arrange that the attaching sphere of $h_1{}^{q+1}$ coincides with the belt sphere of $h_1{}^q$, and that no other $h_i{}^{q+1}$ meets $h_1{}^q$. Introduce a trivial pair h^q, h^{q+1}, and move h^q $-m$ times over $h_1{}^q$. Then the spine of $h^0 \cup h^q$ represents $m\gamma$, and so we are now in the position described at the end of §9. There is a cross-section V_0' of V_0 with the following properties. $H_q(V_0') = Q \oplus \langle r\rangle \oplus \langle s\rangle \oplus \langle u\rangle \oplus \langle v\rangle$, where Q is supported by $\partial(h^0 \cup h_2{}^q \cup \cdots \cup h_l{}^q)$, r is homologous to a spine of $h^0 \cup h_1{}^q$, s is represented by the attaching sphere of $h_1{}^{q+1}$, u is homologous to a spine of $h^0 \cup h^q$ and v is represented by the belt sphere of h^q. Thus u corresponds to α in §9, and v corresponds to β. The belt sphere of $h_1{}^q$ represents $s-mv$, and the attaching sphere of h^{q+1} represents $u-mr+pv$ for some $p \in Z$.

If (θ, F, G, ϕ) is the $(-1)^q$-form arising from V_0', and $(\theta, \hat{F}, \hat{G}, \hat{\phi})$ corresponds to V_1', then θ has the form shown in T3 since u is spanned by a $(q+1)$-ball embedded in S^{2q+1}. It follows from the remarks above that $F, G, \hat{F}, \hat{G}$ have the form shown in T3. It is easy to check that θ vanishes on $(V+R)\times(\hat{F}\cap\hat{G})$, so that $\hat{\phi}$ vanishes on $\hat{\imath}(V+R)\times\hat{\Pi}$, and similarly $\hat{\phi}$ vanishes on $\hat{\Pi}\times\hat{\imath}(V+R)$; thus $\hat{\phi}$ induces $\bar{\phi}$ on $\bar{\Pi} = \hat{\Pi}/\hat{\imath}(V+R)$. Since $iv = 0$ and $iu = mr$, ϕ and Π are determined by the second exact sequence of T3 and conditions on ϕ.

Thus $(\theta, \hat{F}, \hat{G}, \hat{\phi})$ is obtained from (θ, F, G, ϕ) by a move T3. □

§11. REALISING THE MOVES GEOMETRICALLY

LEMMA 11.1. *The move* T1 *may be effected geometrically.*

Proof. Let $\alpha_1, \ldots, \alpha_n$ be a basis of F, and extend this to a basis $\alpha_1, \ldots, \alpha_{2n}$ of P. With respect to this basis and its dual, ψ is represented by a matrix ab', where $b_i = 0$ for $1 \le i \le n$ (cf the proof of Lemma 8.1). Embed S^q in $S^{2q+1} - V'$ so that its linking with the basis $\alpha_1, \ldots, \alpha_{2n}$ is described by the vector $-b$. Since $b_i = 0$ for $1 \le i \le n$, we may extend S^q to a proper embedding of B^{q+1} in $B_-{}^{2q+2}$ which does not meet V^-. Use this B^{q+1} to perform a surgery on S^{2q+1}, obtaining a manifold $T \subset S^{2q+2}$.

Let $B_+{}^{q+1}$ be the cocore of the surgery, oriented so that $B^{q+1}\cdot B_+{}^{q+1} = +1$. Thus if T_+ is the closed complement of T which contains $B_+{}^{2q+2}$, $B_+{}^{q+1}$ is properly embedded in T_+. Embed another $\underline{S}^q$ in $T \cap S^{2q+1}$ with the following properties:

(i) $\underline{S}^q$ is homologous to $\partial B_+{}^{q+1}$ in T,

(ii) the linking in S^{2q+1} of $\underline{S}^q$ with $\alpha_1, \ldots, \alpha_{2n}$ is described by the vector $(-1)^{q+1}a$.

The orientation of $\underline{S}^q$ is determined by that of $B_+{}^{q+1}$.

By duality, the condition $\psi^*G = 0$ ensures that there is a $(q+1)$-ball $\underline{B}^{q+1}$ properly embedded in T_+, with boundary $\underline{S}^q$, which does not meet V^+. Use this $\underline{B}^{q+1}$ to perform a surgery on T, obtaining a new equatorial sphere $\underline{S}^{2q+1}$.

S^q is spanned by a $(q+1)$-ball in $\underline{S}^{2q+1}$, and the union of this with B^{q+1} is a sphere S^{q+1} which after a trivial isotopy represents a basis element ξ of $H_{q+1}(M)$ where $M \subset S^{2q+2}\times I$ is the cobordism defined by the two surgeries we have performed. If $\tilde{\alpha}_i = \alpha_i \times I \subset V' \times I$, then the intersection in M of ξ and $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{2n}$ is described by the vector $-b$.

Similarly, the sphere $\underline{S}^q$ gives rise to a sphere $\underline{S}^{q+1}$ representing a basis element η of $H_{q+1}(M)$, and the intersection in M of η and $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{2n}$ is described by the vector $(-1)^{q+1}a$.

By construction, $\xi\cdot\xi = \eta\cdot\eta = 0$, $\xi\cdot\eta = 1$. Since $\xi\cdot\tilde{\alpha}_i = -b_i$ and $\eta\cdot\tilde{\alpha}_i = (-1)^{q+1}a_i$ we have $\tilde{\alpha}_i = \iota(a_i\xi - b_i\eta)$ for $1 \le i \le 2n$, where $\iota\colon H_{q+1}(M) \to H_{q+1}(M, \partial M)$ is the usual isomorphism.

The result now follows from the proof of Lemma 8.1. □

LEMMA 11.2. *The move* T2 *can be realised geometrically.*

Proof. The work of Levine[7] shows how to realise the enlargement of θ by a surgery on V', this surgery being embedded in S^{2q+1}. A tubular neighbourhood of S^{2q+1} in S^{2q+2} is of the form $S^{2q+1}\times B^1$, and V meets this in $V'\times B^1$ embedded product wise. Thickening the surgery in the obvious way produces the desired result. □

LEMMA 11.3. *The move* T3 *and its inverse can be realised geometrically.*

Proof. To obtain the move T3, note that by work of Levine[7] we could embed a $(q+1)$-ball B^{q+1} in S^{2q+1} so that $B^{q+1} \cap V' = \partial B^{q+1}$ which represents $u \in H_q(V')$. Recalling that S^{2q+1} has a tubular neighbourhood $S^{2q+1} \times B^1$, move B^{q+1} into B_-^{2q+2}, and use it to perform surgery on V: the result is to induce the move T3 as desired.

To obtain the inverse of T3, note that since $F = F^\perp$ and $\theta(u, v)$ or $\theta(v, u) = 0$, there is a $(q+1)$-ball B^{q+1} embedded in B_-^{2q+2} and meeting V^- in ∂B^{q+1}; moreover we may arrange that $\partial B^{q+1} \subset V' \times -1 \subset S^{2q+1} \times B^1$. Using B^{q+1} to perform a surgery on V, we obtain a move $(\mathrm{T3})^{-1}$; the form of ϕ on $(\bar{\Pi} \oplus S) \times S$ is determined by the homotopy linking of B^{q+1} mod ∂B^{q+1} with balls in V^- whose boundaries represent a basis of $\hat{F}$.

§12. PROOFS OF THE MAIN THEOREMS

Proof of Theorem 3.1. This is clear from the work of Kervaire[4] and Levine[6].

Proof of Theorem 3.2. This is a consequence of Lemmas 8.1, 9.1, 10.1, 10.2.

PROPOSITION 12.1. *Let $k, \bar{k}$ be two simple $2q$-knots, $q \geq 3$, giving rise to the same $(-1)^q$-forms (θ, F, G, ϕ). If k is odd, k is isotopic to $\bar{k}$.*

Proof. Let V, $\bar{V}$ be Seifert surfaces of $k, \bar{k}$ giving rise to the form (θ, F, G, ϕ). By the work of Levine[5], we may perform surgery on V to obtain V_1 with $H_q(V_1)$ 2-torsion-free, and by the previous sections this involves algebraic moves $(\mathrm{T0})^{\pm 1}$, T1, $(\mathrm{T3})^{\pm 1}$, only. These algebraic moves may be realised geometrically on $\bar{V}$ to obtain $\bar{V}_1$. By Proposition 7.1, V_1 and $\bar{V}_1$ are isotopic. □

LEMMA 12.2. *The move* $(\mathrm{T2})^{-1}$ *may be realised geometrically on an odd knot k.*

Proof. Let V be a Seifert surface giving rise to (θ, F, G, ϕ). By Theorem 3.1, there is a knot $\bar{k}$ with surface $\bar{V}$ giving rise to $(\mathrm{T2})^{-1}$ (θ, F, G, ϕ). Realise T2 on $\bar{V}$ to obtain $\bar{\bar{V}}$. By Proposition 12.1, k is isotopic to $\bar{\bar{k}}$. □

Proof of Theorem 3.3. All the algebraic moves can be realised geometrically. □

§13. CONCLUSION

We have not quite classified the odd simple $2q$-knots: Theorem 3.1 needs to be strengthened slightly.

Let V be a $(q-1)$-connected Seifert surface of a $2q$-knot, $q \geq 3$. Let V' be a cross-section of V and (θ, F, G, ϕ) its associated $(-1)^q$-form. We define $F^0 \subseteq P^*$ by $F^0 = \{f \in P^*: f(x) = 0, \forall x \in F\}$.

If $y \in F$, then $\theta y \in F^0$; similarly if $y \in G$ then $\theta y \in G^0$. Thus θ induces a map $\boldsymbol{\theta}: P/(F+G) \to P^*/(F^0 + G^0)$.

Recalling that by Alexander duality $P^* = H_q(S^{2q+1} - V')$, it is easy to see that $F^0 = \ker(H_q(S^{2q+1} - V') \to H_q(B_-^{2q+2} - V^-))$ with a similar statement for G^0. Thus in a natural way $P/(F+G) \cong H_q(V)$ and $P^*/(F^0 + G^0) \cong H_q(S^{2q+2} - V)$. As θ corresponds to translating q-cycles off V' in the positive normal direction, so does $\boldsymbol{\theta}$ with V. Similar remarks are true for θ^*, $\boldsymbol{\theta}^*$.

Following Kervaire [4; II.4], we call V and (θ, F, G, ϕ) *minimal* if $\boldsymbol{\theta}$ and $\boldsymbol{\theta}^*$ are injections. Kervaire has shown that in these circumstances $T_q(V) \cong T_q(\tilde{K})$, where $T_q(X)$ denotes the Z-torsion subgroup of $H_q(X)$ and $\tilde{K}$ is the universal cover of K.

Thus we have the following:

THEOREM 13.1. *If $q \geq 3$ and (θ, F, G, ϕ) is T-equivalent to an odd minimal $(-1)^q$-form, then there is an odd simple $2q$-knot giving rise to (θ, F, G, ϕ).*

Remark. The $Z[t]$-module $H_q(\tilde{K})$ is presented by $t\boldsymbol{\theta} - \boldsymbol{\theta}^*$; see Kervaire[4] for details. From this can be derived the usual Alexander invariants.

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