Surgery on products. I, II.

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Communicated by Prof. W. T. van Est at the meeting of June 17, 1978

The aim of this paper is to determine how obstructions for surgery, as introduced by C. T. C. Wall in his celebrated book [W1], behave with respect to the operation of taking the cartesian product with a fixed manifold P. We accomplish this, in contrast to other existing literature, for the case of arbitrary fundamental groups, and in contrast to [R] in a geometrical way.

In the following each manifold and Poincaré-complex will be understood to be equipped with an orientation twisted by the first Stiefel-Whitney class, or equivalently an honest orientation of the orientation covering.

Let a finitely presented group G and a homomorphism $w_G\colon G \to \{\pm 1\}$ be given.

DEFINITION: *An n-manifold over* (G, w_G) *is a pair consisting of an n-manifold* P *in the above sense and a homomorphism* $\alpha\colon \pi_1(P) \to G$ *such that* $w_G\alpha$ *is the first Stiefel-Whitney class* $w_1(P)$ *of* P.

There is an obvious concept of cobordism of such objects. We denote the set of equivalence classes by $\Omega_n(G, w_G)$; it is made into an abelian group by the operation of disjoint union.

Equivalently one may look at the covering $\tilde{P}$ of P induced by α together with an orientation and a free action of G on $\tilde{P}$ such that w_G measures the preservation of orientation.

For calculations the following description is the most suitable one. Given a space X with an involution τ we consider the bordism group $\Omega_n^-(X;\tau)$ of n-manifolds together with an honest orientation, an orientation reversing involution T, and an equivariant map to X. Then we take for X the double covering of the classifying space BG which is induced by w_G and for τ the covering transformation. In case w_G is trivial we so recover the classical identification of $\Omega_n(G, 1)$ with $\Omega_n(BG)$.

The following definition is found in chapter 9 of [W1].

DEFINITION: *A surgery problem over (H, w_H) is an object determined by the following data: a finite Poincaré pair $(X, \partial X)$ in the above sense, a compact m-manifold with boundary $(M, \partial M)$ in the same sense, a map $f: (M, \partial M) \to (X, \partial X)$ of pairs of degree one, inducing a homotopy equivalence $\partial M \to \partial X$, a bundle ν over X, a stable trivialisation F of $TM \oplus f^*\nu$ and a homomorphism $\beta: \pi_1(X) \to H$ such that $w_H\beta$ is $w_1(X)$.*

We will abbreviate this object to f. There is an obvious concept of bordism between such objects; the set of bordism classes is made into an abelian group by disjoint union and is denoted by $L_m(H, w_H)$.

Given objects as above one can construct a new surgery problem over $(G\times H: w_G w_H)$ determined by the following data: the finite Poincaré pair $(P\times X, P\times\partial X)$, the $m+n$ manifold with boundary $(P\times M, P\times\partial M)$, the map $id\times f: P\times M \to P\times X$, the bundle $\nu P\times\nu$ over $P\times X$, where νP denotes the stable normal bundle of P i.e. a bundle with a framing F' of $TP\oplus\nu P$, the framing $F'\times F$ of $T(P\times M)\oplus(1\times f)^*(\nu P\times\nu) = (TP\oplus \oplus\nu P)\times(TM\oplus f^*\nu)$, and the homomorphism $\alpha\times\beta: \pi_1(P\times X) = \pi_1(P)\times \times\pi_1(X) \to G\times H$.

We will abbreviate this object to $1\times f$. This construction is well defined on classes and induces a biadditive pairing

$$\Omega_n(G, w_G)\times L_m(H, w_H) \to L_{n+m}(G\times H, w_G w_H).$$

In this paper we study this map in the case that n and m are even: $n = 2q$ and $m = 2k$.

We denote by A the integral group ring $Z[G]$ equipped with the antihomomorphic involution $-$ defined by the formula $\overline{\sum n_g g} = \sum w_G(g) n_g g^{-1}$; similarly one constructs B from (H, w_H); then $A\otimes B$ is associated with $(G\times H, w_G w_H)$. We then consider pairs (V, ψ) consisting of a stably free left B module V and a nonsingular $(-1)^k$ symmetric quadratic form ψ on V in the sense of [W2]. There is an obvious notion of direct sum of such pairs. The Grothendieck group quotiented by the subgroup of standard quadratic forms is denoted by $L_{2k}(B)$.

The importance of these concepts stems from the fact that for $k>2$ a canonical isomorphism $s: L_{2k}(H, w_H) \to L_{2k}(B)$ exists. If X is connected and β is an isomorphism (which can be arranged by a bordism of f) then $s(f) = 0$ iff one can change f by surgery (i.e. a bordism that fixes X) into a homotopy equivalence. For this reason $s(f)$ is called the surgery obstruction of f.

In this paper we prove the following

THEOREM: *There exists a free left A module K and a sesquilinear form σ on K such that $\sigma \otimes \psi$ is a nonsingular quadratic form representing $s(1 \times f)$, where ψ is one representing $s(f)$.*

The form σ is nonsingular i.e. $Ad(\sigma)$ is invertible, and is almost $(-1)^q$ symmetric in the sense that $(-1)^q(Ad\,\sigma)^{-1}(Ad\,\sigma)\dagger - 1$ is nilpotent. Furthermore (K, σ) can be expressed in terms of the Miscenko/Ranicki [R] *symmetric Poincaré complex associated with P.*

This tensor product of forms must be understood in a graded sense i.e. $(\sigma \otimes \psi)(a \otimes x, b \otimes y) = (-1)^{kq}\, \sigma(a, b) \otimes \psi(x, y)$.

The paper is organized as follows: in section 1 we introduce some notations; in section 2 we perform the low-dimensional surgery on $id \times f$; in section 3 we prove a few technical lemmas which are needed for the computation of the mid-dimensional homology of the resulting surgery-problem in section 4. In section 5 we establish a relation between the data we used about P and the description of P on the chain level as in [R]; in section 6 we describe the form σ in these terms. In section 7 we show how to represent homology-classes by immersions in the right regular homotopy class and in section 8 we count the intersections, thereby finishing the proof of the main theorem. Finally in section 9 we give another description of the form σ and an example of application of the theorem.

§ 1. DEFINITIONS AND NOTATIONS

We now consider a surgery problem f over (H, w_H) as in § 0. By a bordism of f we can arrange X to be connected and β to be an isomorphism (see [W1], p. 91), hence $\tilde{X}$ is connected and simply connected. The groups $H_i(\tilde{M})$ and $H_i(\tilde{X})$ have the structure of left $B = Z[H]$ modules because of the left H actions on $\tilde{M}$ and $\tilde{X}$; $f_* : H_i(\tilde{M}) \to H_i(\tilde{X})$ is a module homomorphism and is surjective since f is of degree one. We denote the kernel by $K_i(M)$.

By doing preliminary surgery we may suppose that f is k-connected, hence that $\tilde{M}$ is connected and simply connected and that $K_i(M)$ vanishes unless $i = k$; furthermore we may assume that $K_k(M)$ is a free B module with basis $e_1, \dots, e_r$ say ([W1], p. 49).

Each $e_j \in K_k(M)$ can be represented by an immersion $g_j : S^k \times D^k(1) \to \tilde{M}$ together with a nullhomotopy h_j of $f \circ g_j(\ , 0)$. The regular homotopy class of g_j is well determined by the condition that the stable framing of $g_j(\ , 0)^* TM$ induced by the derivative of g_j, together with the restriction to S^k of the canonical framing of $h_j^* \nu$ corresponds under $g_j(\ , 0)^*$ with the given framing F of $TM \oplus f^* \nu$.

By general position we may assume that the g_j intersect regularly i.e. on an appropriate coordinate chart the intersection looks like $(\mathbf{R}^k \times 0, 0 \times \mathbf{R}^k)$ in $\mathbf{R}^{2k}$. By choosing a Riemann structure on M which is Euclidean on the above-mentioned chart and by using the exponential map to redefine g_j we may assume that the g_j are disjoint embeddings except that for certain pairs of points (p, p') and coordinate maps $\eta_p: (D^k(1), 0) \to \to (S^k, p)$ and $\eta_{p'}$ around those points and for certain $\gamma \in H$ we have:

$$g_j(\eta_p(x), y) = \gamma^{-1} g_{j'}(\eta_{p'}(y), x) \text{ for all } x, y \text{ in } D^k(1).$$

We write D_p and $D_{p'}$ for the images of η_p and $\eta_{p'}$; $D_p(R)$ denotes the η_p image of the disc of radius R.

Now we choose a C^∞ function $\varkappa$ with values in $[0, 1]$ on each copy S^k_j of the standard k-sphere such that for any intersection-pair $\{p, p'\}$ as above $\varkappa = 0$ on $D_p(\frac{1}{2})$ and $\varkappa = 1$ on $D_{p'}(\frac{1}{2})$ or vice versa, and such that $\varkappa$ vanishes outside the images of the η.

Now we define ψ by the formula $\psi(e_j, e_{j'}) = \sum_{\gamma \in H} (g_j \cdot \gamma^{-1} g_{j'})_< \gamma$, where $\cdot$ denotes the ordinary intersection-number, which counts the number of pairs (p, p') as above with multiplicity $+1$ or -1 depending on whether $\eta_{p'}$ preserves or changes orientation, and $<$ means that we count only those pairs for which $\varkappa(p) < \varkappa(p')$, hence $\varkappa(p) = 0$ and $\varkappa(p') = 1$.

If we count all pairs, we get the equivariant intersection-number $\lambda(e_j, e_{j'})$. Since η_p changes orientation by a factor $\varepsilon_p = (-1)^k w_H(\gamma)\, \varepsilon_{p'}$ if $\eta_{p'}$ does so by a factor $\varepsilon_{p'}$, we see that $\lambda(e_j, e_{j'}) - \psi(e_j, e_{j'}) = (-1)^k \overline{\psi(e_{j'}, e_j)}$: a pair $\{p, p'\}$ with $\varkappa(p) > \varkappa(p')$ such that $g_j(\eta_p(x), y) = \gamma^{-1} g_{j'}(\eta_{p'}(y), x)$ contributes $\varepsilon_{p'}\gamma$ to $\lambda(e_j, e_{j'})$; it can also be seen as a pair with $\varkappa(p') < \varkappa(p)$ such that $g_{j'}(\eta_{p'}(y), x) = \gamma g_j(\eta_p(x), y)$ and so it contributes $\varepsilon_p \gamma^{-1}$ to $\psi(e_{j'}, e_j)$ hence $\varepsilon_{p'}\gamma = (-1)^k\, \varepsilon_p \bar{\gamma}^{-1}$ to $(-1)^k\, \overline{\psi(e_{j'}, e_j)}$.

This ψ extends to a pairing $K_k(M) \times K_k(M) \to B$ which is sesquilinear, i.e. biadditive and such that $\psi(ax, by) = b\psi(x, y)\bar{a}$ for $x, y \in K_k(M)$ and $a, b \in B$. For any left B module V, the dual $V^d = \mathrm{Hom}_B(V, B)$ has the structure of a left B module such that $(af)(v) = f(v)\bar{a}$ for $a \in B$, $f \in V^d$, $v \in V$; in particular this applies to $V = K_k(M)$. Saying that ψ is sesquilinear is equivalent to saying that the map $Ad(\psi): V \to V$ defined by the formula $((Ad\psi)x)(y) = \psi(x, y)$ is a module homomorphism.

The same applies to the symmetrisation λ of ψ; since $Ad(\lambda)$ is an isomorphism one calls ψ a nonsingular quadratic form [W2]. The class of ψ in $L_{2k}(B)$ is independent of choices and defines $s(f)$ (see [W1], p. 50).

For later use we introduce the notation $\tilde{\psi}$ for the homomorphism $V \to V$ corresponding to Ad under the isomorphism $V \to V^d$ mapping the free generators e_j of V to their duals e_j^*; thus $\tilde{\psi}(x) = \sum_{j=1}^{r} \overline{\psi(x, e_j)} e_j$. An intersection-pair $\{p, p'\}$ with $\varkappa(p) < \varkappa(p')$ as above contributes $(-1)^k \varepsilon_p \gamma^{-1} e_{j'}$ to $\tilde{\psi}(e_j)$. Notice that $\lambda(\tilde{\lambda}^{-1}\tilde{\psi}x, y) = \psi(x, y)$.

Now consider P. We suppose P triangulated; we denote the i-skeleton by P_i. In any C^∞ neighbourhood of the identity one can find a diffeomorphism $\xi: P \to P$ which puts each simplex of P in transverse position with respect to the simplices of P. In particular $\xi P_i \cap P_{n-i-1} = \emptyset$, $\xi P_i \cap P_{n-i}$ is discrete.

If we choose ξ close enough to the identity, then we can find an isotopy ξ_t such that $\xi_0 = id$ and $\xi_1 = \xi$: we embed P in some Euclidean space, connect x and ξx by a straight line segment, and project down to P; if ξ was chosen close enough to id this yields diffeomorphisms.

Now consider the path in the space of C^∞ maps $P \to P$ which is defined by ξ_{2t-1} for $t \in [\frac{1}{2}, 1]$ and by ξ_{1-2t}^{-1} for $t \in [0, \frac{1}{2}]$: in any neighbourhood of it we can find a path χ with the same endpoints such that χ puts the product of $[0, 1]$ and any simplex of P in transverse position with respect to the simplices of P, and homotopic with it. If the path is sufficiently close it consists entirely of diffeomorphisms.

Furthermore one can choose regular neighbourhoods S of P_{q-2}, Q of P_{q-1} and R of P_q small enough so that

$$\chi([0, 1] \times R) \cap S = \chi([0, 1] \times Q) \cap Q = \emptyset$$
$$\xi R \cap Q = \xi Q \cap R = \emptyset$$
$$S \subset \text{int}\,(Q) \text{ and } Q \subset \text{int}\,(R).$$

We denote by $\tilde{P}_i$ the covering of P_i induced by $\tilde{P}$; idem for Q, S etc. Notice that our diffeomorphisms, being nullhomotopic, define unique diffeomorphisms of $\tilde{P}$ with similar properties.

§ 2. SURGERY BELOW THE MIDDLE DIMENSION

We define a map $\tilde{\Omega}_j : \tilde{P} \times S^k \times D^k(\frac{1}{2}) \to \tilde{P} \times \tilde{M}$ by the formula

$$\tilde{\Omega}_j(y, x, v) = (\xi_{\varkappa(x)} y, g_j(x, v)).$$

The $\tilde{\Omega}_j$ determine disjoint embeddings $\Omega_j : Q \times S^k \times D^k(\frac{1}{2}) \to P \times M$. For suppose that $\Omega_j(y, x, v) = (\theta \times \gamma)^{-1}\, \Omega_{j'}(y', x', v')$: then $\xi_{\varkappa(x)} y = \theta^{-1} \xi_{\varkappa(x')} y'$ and $g_j(x, v) = \gamma^{-1} g_{j'}(x', v')$. Unless $x = x'$, $v = v'$ and $y = \theta^{-1} y'$ the last formula implies that for some intersection-pair $\{p, p'\}$ we have $x \in D_p$, $x' \in D_{p'}$, or vice versa i.e. $\varkappa(x) = 0$, $\varkappa(x') = 1$ or vice versa; hence we get a contradiction with $\xi Q \cap Q = \emptyset$.

Hence we can use Ω to define a manifold by glueing:

$$W = P \times M \times [0, 1] \cup \bigcup_{j=1}^{r} Q \times D_j^{k+1} \times D^k(\tfrac{1}{2}).$$

Since Ω_j is homotopic to $1 \times g_j$ we can extend $1 \times f : P \times M \to P \times X$ to a map $W \to P \times X$; similarly we can extend the framing F, and $\alpha \times \beta$ extends to an isomorphism $\pi_1(W, *) \cong G \times H$ by the van Kampen theorem.

Now ∂W is the disjoint union of $\partial_- W \cong P \times M$ and $N = \partial_+ W \cong$ $\cong (P \times M - \text{im}\,\Omega) \cup \bigcup_{j=1}^{r} (Q \times D_j^{k+1} \times S^{k-1}(\frac{1}{2}) \cup Q \times D_j^{k+1} \times D^k(\frac{1}{2}))$.

THEOREM 1: *The surgery problem $N \to P \times X$ thus obtained is $(q+k)$-connected.*

PROOF: From now on we write $K_i(W)$ for the kernel of the above-mentioned map in homology: $H_i(\tilde{W}) \to H_i(\tilde{P} \times \tilde{X})$; similarly one has $K_i(P \times M)$, $K_i(N)$ etc.

We can construct a module map $H_i(\tilde{P}) \otimes K_k(M) \to K_{i+k}(P \times M)$ which maps $\{c\} \otimes e_j$ to the class represented by $c \times g_j(S^k \times 0)$; this is an isomorphism by the Kunneth theorem.

Similarly we have a map

$$H_i(\tilde{Q}) \otimes K_k(M) \to K_{i+k+1}(W, P \times M \times [0, 1]) \cong K_{i+k+1}(W, P \times M)$$

mapping $\{c\} \otimes e_j$ to the class represented by $c \times D_j^{k+1} \times 0$. This map is isomorphic by excision and the Kunneth theorem.

From the way these maps are defined it follows that the following diagram commutes

$$\begin{array}{ccc} H_i(\tilde{Q}) \otimes K_k(M) & \longrightarrow & H_i(\tilde{P}) \otimes K_k(M) \\ \downarrow & & \downarrow \\ K_{k+i+1}(W, P \times M) & \longrightarrow & K_{k+i}(P \times M) \end{array}$$

where the upper horizontal arrow is induced by the inclusion $Q \subset P$. Since $H_i(\tilde{P}, \tilde{Q}) = H_i(\tilde{P}, \tilde{P}_{q-1}) = 0$ for $i \leqslant q-1$, that map is a surjection for $i \leqslant q-1$ and an injection for $i < q-1$, hence ∂ is, as well. So $K_{k+i}(W)$ vanishes for $k+i \leqslant k+q-1$.

It is clear that the union of N and $\bigcup_{j=1}^{r} Q \times D_j^{k+1} \times D^k(\frac{1}{2})$ is a retract of W relative to N; the intersection of these two is

$$\bigcup_{j=1}^{r} \{Q \times D_j^{k+1} \times S^{k-1}(\tfrac{1}{2}) \cup \partial Q \times D_j^{k+1} \times D^k(\tfrac{1}{2})\}.$$

Accordingly, we see that by retraction, excision and the Kunneth theorem we have an isomorphism $H_i(\tilde{Q}, \partial\tilde{Q}) \otimes K_k(M) \to K_{i+k}(W, N)$ mapping $\{c\} \otimes e_j$ to the class of $c \times 0_j \times D^k$.

Since $H_i(\tilde{Q}, \partial\tilde{Q}) \cong H_c^{n-i}(\tilde{Q}) \cong H_c^{n-i}(\tilde{P}_{q-1})$ vanishes for $n-i \geqslant q$ i.e. $i \leqslant q$, it follows that $K_{i+k}(W, N) = 0$ for $i+k \leqslant q+k$.

Substituting the above results in the long exact homology sequence of the pair (W, N) we deduce that $K_i(N) = 0$ for $i \leqslant q+k-1$. Furthermore the inclusion of N in W induces an isomorphism of fundamental groups: the inclusion $\partial Q \times D^{k+1} \times S^{k-1} \subset \partial Q \times D^{k+1} \times D^k$ does, hence $Q \times D^{k+1} \times S^{k-1} \subset Q \times D^{k+1} \times S^{k-1} \cup \partial Q \times D^{k+1} \times D^k$ does by van Kampen, hence $Q \times D^{k+1} \times S^{k-1} \cup \partial Q \times D^{k+1} \times D^k \subset Q \times D^{k+1} \times D^k$ does, hence $N \subset W$ does.

For a similar reason the inclusion $P \times M \to W$ induces an isomorphism of fundamental groups. Hence the isomorphism of $\pi_1(M)$ and $\pi_1(X)$ implies one of $\pi_1(N)$ and $\pi_1(P \times X)$. Q.E.D.

§ 3. SOME MAPS AND DIAGRAMS

In this section we prove some results which are needed for the determination of $K_{q+k}(N)$ in the next section.

LEMMA 1: *$K_{i+k}(W)$ is isomorphic to $H_i(\tilde{P}, \tilde{Q}) \otimes K_k(M)$.*

PROOF: First we note the existence of a homomorphism

$$H_i(\tilde{P}, \tilde{Q}) \otimes K_k(M) \to K_{i+k}(W),$$

mapping $\{c\} \otimes e_j$ to $\tilde{\Omega}_j(c \times S^k \times 0) \cup \partial c \times D_j^{k+1} \times 0 \subset \tilde{W}$.

Then the following diagram commutes by construction:

$$\begin{array}{ccccccc} \dots H_{i+1}(\tilde{P}, \tilde{Q}) \otimes K_k(M) & \to & H_i(\tilde{Q}) \otimes K_k(M) & \to & H_i(\tilde{P}) \otimes K_k(M) & \to & H_i(\tilde{P}, \tilde{Q}) \otimes K_k(M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_{i+k+1}(W) & \to & K_{i+k+1}(W, P \times M) & \to & K_{i+k}(P \times M) & \to & K_{i+k}(W) \end{array}$$

It follows that the map is an isomorphism, since we have seen in the proof of theorem 1 that the other vertical maps are. Q.E.D.

LEMMA 2: *There is a commutative ladder*

$$\begin{array}{ccccccc} H_i(\tilde{P} - \tilde{Q}) \otimes K_k(M) & \to & H_i(\tilde{P}) \otimes K_k(M) & \to & H_i(\tilde{P}, \tilde{P} - \tilde{Q}) \otimes K_k(M) & \to & H_{i-1}(\tilde{P} - \tilde{Q}) \otimes K_k(M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_{i+k}(N) & \longrightarrow & K_{i+k}(W) & \longrightarrow & K_{i+k}(W, N) & \longrightarrow & K_{i+k-1}(N) \end{array}$$

Here the horizontal maps are the usual ones.

PROOF: We define the vertical arrows in step (a) and discuss the commutativity of the three squares in the remaining three steps.

Step (a) We define a map $\tilde{\Psi}_j: \tilde{P} \times S^k \times D^k \to \tilde{P} \times \tilde{M}$ analoguous to $\tilde{\Omega}_j$ by the formula $\tilde{\Psi}_j(y, x, v) = (\xi_{1-\varkappa(x)}y, g_j(x, v))$. If $w \in D^k$ is of length $\frac{1}{2}$ then $\Psi_j((P-Q) \times S^k \times w) \subset P \times M$ is contained in the closure of $P \times M - \operatorname{im} \Omega$, hence in N. For suppose that $\tilde{\Psi}_j(y, x, w) = (\theta \times \gamma)^{-1} \tilde{\Omega}_{j'}(y', x', v')$, then $\xi_{1-\varkappa(x)}y = \theta^{-1}\xi_{\varkappa(x')}y'$ and $g_j(x, w) = \gamma^{-1}g_j(x', v')$. The last formula implies that either $j = j'$, $x = x'$, $w = v'$, in contradiction with $v' \in \operatorname{int}(D^k(\frac{1}{2}))$, or for some intersection-pair $\{p, p'\}$ we have $x \in D_p(\frac{1}{2})$, $x' \in D_{p'}(\frac{1}{2})$ or vice versa i.e. $\varkappa(x) = 0$, $\varkappa(x') = 1$ or vice versa. But then the first formula says that $y = \theta^{-1}y'$, contradicting the fact that $y \in \tilde{P} - \tilde{Q}$ and $y' \in \tilde{Q}$. Thus Ψ determines a homomorphism $H_i(\tilde{P} - \tilde{Q}) \otimes K_k(M) \to K_{i+k}(N)$.

The second vertical arrow is the composition of the Kunneth isomorphism $H_i(\tilde{P}) \otimes K_k(M) \cong K_{i+k}(P \times M)$ and the inclusion $\tilde{P} \times \tilde{M} \to \tilde{W}$. The third vertical map is the composition of the isomorphism $H_i(\tilde{Q}, \partial\tilde{Q}) \otimes K_k(M) \to$ $\to K_{i+k}(W, N)$ discussed in the proof of theorem 1, and the tensor-product of the excision isomorphism $H_i(\tilde{Q}, \partial\tilde{Q}) \cong H_i(\tilde{P}, \tilde{P} - \tilde{Q})$ with $(-1)^k\tilde{\lambda}$.

Step (b) The commutativity of the first square is an immediate consequence of the fact that $\tilde{\Psi}_j$ is isotopic to $1 \times g_j$ in $\tilde{P} \times \tilde{M} \subset \tilde{W}$.

Step (*c*) We have to prove the commutativity of the following diagram:

$$\begin{array}{ccccc} H_i(\tilde{Q}, \partial\tilde{Q}) \otimes K_k(M) & \to & H_i(\tilde{P}, \tilde{P}-\tilde{Q}) \otimes K_k(M) & \underset{\partial}{\to} & H_{i-1}(\tilde{P}-\tilde{Q}) \otimes K_k(M) \\ \downarrow (-1)^k\tilde{\lambda} & & & & \downarrow \\ H_i(\tilde{Q}, \partial\tilde{Q}) \otimes K_k(M) & & & & \\ \downarrow & & & & \\ K_{i+k}(W, N) & & \xrightarrow{\quad\partial\quad} & & K_{i+k-1}(N) \end{array}$$

If we start with $\{c\} \otimes e_j \in H_i(\tilde{Q}, \partial\tilde{Q}) \otimes K_k(M)$ and go along the upper side we get the class of $\tilde{\Psi}_j(\partial c \times S^k \times w) = \partial\tilde{\Psi}_j(c \times S^k \times w)$ in $K_{i+k-1}(N)$.

Let Δ denote the union of the $D_p(\frac{1}{2}) \subset S^k$, where p runs through the intersection points of g_j with the $g_{j'}$. We will see that $\tilde{\Psi}_j(c \times \Delta \times w)$ is precisely the part of $\tilde{\Psi}_j(c \times S^k \times w) \subset \tilde{P} \times \tilde{M}$ which lies in im $(\tilde{\Omega})$; then $\tilde{\Psi}_j(c \times (S^k - \Delta) \times w)$ lies in $\tilde{N}$ and can thus be viewed as an homology between $\partial\tilde{\Psi}_j(c \times \Delta \times w)$ and $\partial\tilde{\Psi}_j(c \times S^k \times w)$; subsequently we rewrite $\tilde{\Psi}_j(c \times \Delta \times w)$ in terms of $\tilde{\Omega}$ and note that the result corresponds to the other composition in the above diagram.

To prove these assertions we note that $\tilde{\Psi}_j(y, x, w) = (\theta \times \gamma)^{-1}\tilde{\Omega}_{j'}(y', x', v')$ implies that $g_j(x, w) = \gamma^{-1}g_{j'}(x', v')$, hence that $x \in D_p(\frac{1}{2})$, $x' \in D_{p'}(\frac{1}{2})$ for some intersection-pair $\{p, p'\}$; in particular, $x \in \Delta$.

On the other hand such a pair with $g_j(\eta_p(a), b) = \gamma^{-1}g_{j'}(\eta_p(b), a)$ for $a, b \in D^k(\frac{1}{2})$ gives a contribution $\varepsilon_{p'}\gamma = (-1)^k w(\gamma)\varepsilon_p\gamma$ to $\lambda(e_j, e_{j'})$, hence $\varepsilon_p\gamma^{-1}e_{j'}$ to $(-1)^k\tilde{\lambda}(e_j)$. Also we can identify the part $\varepsilon_p\tilde{\Psi}_j(c \times D_p(\frac{1}{2}) \times w)$ of $\tilde{\Psi}_j(c \times S^k \times w)$ with $\varepsilon_p\gamma^{-1}\tilde{\Omega}_{j'}(c \times p' \times D^k(\frac{1}{2}))$ since $\varkappa(p') = 1 - \varkappa(p)$; this represents $\varepsilon_p\gamma^{-1}e_{j'}$ in $K_{i+k}(W, N)$.

Step (*d*) We can use the foregoing calculation to prove the commutativity of the following diagram

$$\begin{array}{ccccc} H_i(\tilde{P}) \otimes K_k(M) & \to & H_i(\tilde{P}, \tilde{P}-\tilde{Q}) \otimes K_k(M) & \leftarrow & H_i(\tilde{Q}, \partial\tilde{Q}) \otimes K_k(M) \\ \downarrow & & & & \downarrow (-1)^k\tilde{\lambda} \\ & & & & H_i(\tilde{Q}, \partial\tilde{Q}) \otimes K_k(M) \\ & & & & \downarrow \\ K_{i+k}(W) & & \longrightarrow & & K_{i+k}(W, N) \end{array}$$

An element $\{c\}$ of $H_i(\tilde{P})$ can be represented by $d_1 + d_2$ where $d_1 \subset \tilde{Q}$ and $d_2 \subset$ closure of $\tilde{P} - \tilde{Q}$. The result on $\{c\} \otimes e_j$ of going along the left and lower side to $K_{i+k}(W, N)$ is the image of $c \times S^k \times 0$ under $1 \times g_j$ or equivalently under $\tilde{\Psi}_j$. All but $\tilde{\Psi}_j(d_1 \times \Delta \times w)$ lands in $\tilde{N}$ and can be neglected; on the other hand, $\varepsilon_p\tilde{\Psi}_j(d_1 \times D_p(\frac{1}{2}) \times w)$ can be replaced by $\varepsilon_p\gamma^{-1}\tilde{\Omega}_{j'}(d_1 \times p' \times D^k(\frac{1}{2}))$ and those terms represent $(-1)^k\tilde{\lambda}(e_j)$ as we have seen above.

Q.E.D.

LEMMA 3: *There is a map* $H_i(\tilde{R}, \tilde{Q}) \otimes K_k(M) \to K_{i+k}(N)$ *such that the following diagrams commute:*

$$\begin{array}{ccc} H_i(\tilde{R},\tilde{Q})\otimes K_k(M) & \to & H_i(\tilde{P},\tilde{Q})\otimes K_k(M) \\ \downarrow & (a) & \downarrow \\ K_{i+k}(N) & \longrightarrow & K_{i+k}(W) \end{array} \quad and \quad \begin{array}{ccc} H_i(\tilde{R})\otimes K_k(M) & \to & H_i(\tilde{R},\tilde{Q})\otimes K_k(M) \\ \downarrow & (b) & \downarrow \\ H_i(\tilde{P}-\tilde{Q})\otimes K_k(M) & \to & K_{i+k}(N) \end{array}$$

where the map $H_i(R) \to H_i(P-\tilde{Q})$ *is induced by* ξ.

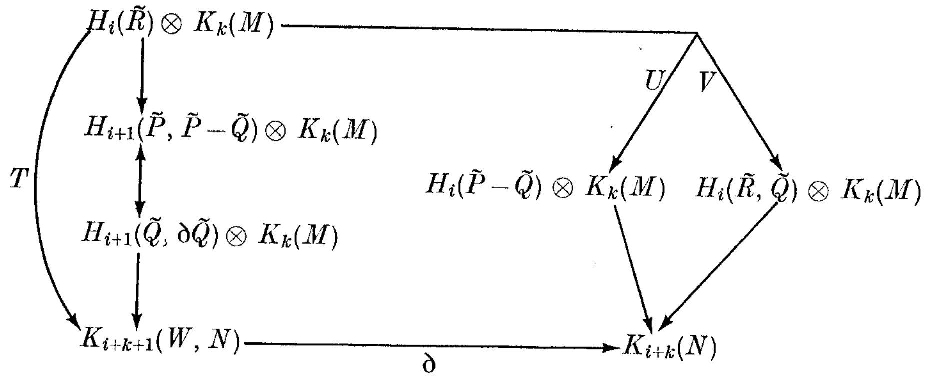
PROOF: We start by defining the homomorphism $\omega: H_i(R, \tilde{Q}) \otimes \otimes K_k(M) \to K_{i+k}(N)$ which maps $\{c\} \otimes e_j$ to the class of $\tilde{\Omega}_j(c \times S^k \times 0) \cup \cup\, \partial c \times D_j^{k+1} \times 0$ provided we choose the representing cycle c in the closure of $\tilde{R}-\tilde{Q}$; that we end up in $\tilde{N}$ follows from the fact that $\xi R \cap Q = R \cap \cap\, \xi Q = \emptyset$. By construction, then, the diagram (a) commutes. Now we consider the two compositions U and V in diagram (b). There is a homomorphism $H_i(\tilde{R}) \to H_{i+1}(\tilde{P}, \tilde{P}-\tilde{Q})$ which is induced by the map

$$([0,1]\times\tilde{R},\ \{0,1\}\times R) \to (\tilde{P}, \tilde{P}-\tilde{Q})$$

which maps (t, y) to $\xi^{-1}\xi_t\xi_t y$; furthermore $H_{i+1}(\tilde{P}, \tilde{P}-\tilde{Q}) \cong H_{i+1}(\tilde{Q}, \partial\tilde{Q})$ by excision. Together with $(-1)^k\bar{\psi}: K_k(M) \to K_k(M)$ and the identification $H_{i+1}(\tilde{Q}, \partial\tilde{Q}) \otimes K_k(M) \cong K_{i+k+1}(W, N)$, this defines a map

$$T: H_i(\tilde{R}) \otimes K_k(M) \to K_{i+k+1}(W, N).$$

We assert that $U - V = \partial T$; this is illustrated in the following diagram:



This assertion will be proved in the same style as in the foregoing. We define a map $\tilde{\Gamma}_j: [0,1] \times \tilde{P} \times S^k \times D^k(\tfrac{1}{2}) \to \tilde{P} \times \tilde{M}$ by the formula

$$\tilde{\Gamma}_j(t, y, x, v) = (\xi_{(1-\varkappa(x))t}\xi_{t+(1-t)\varkappa(x)}y,\ g_j(x, v)),$$

then

$$\tilde{\Gamma}_j(0, y, x, v) = \tilde{\Omega}_j(y, x, v) \text{ and } \tilde{\Gamma}_j(1, y, x, v) = \tilde{\Psi}_j(\xi y, x, v).$$

In this situation, Δ denotes the union of the $D_p(\tfrac{1}{2}) \subset S^k$ where p belongs to an intersection-pair $\{p, p'\}$ such that $\varkappa(p) < \varkappa(p')$ i.e. $\varkappa(p)=0$. Then $\tilde{\Gamma}_j([0,1]\times c\times(S^k-\Delta)\times w)$ is a homology in $\tilde{N}$ between

$$\partial\tilde{\Gamma}_j([0,1]\times c\times\Delta\times w)\subset \text{closure im }(\tilde{\Omega})$$

and

$$\partial\tilde{\Gamma}_j([0,1]\times c\times S^k\times w)=\tilde{\Gamma}_j(1\times c\times S^k\times w)-\tilde{\Gamma}_j(0\times c\times S^k\times w)=$$
$$=\tilde{\Psi}_j(\xi c\times S^k\times w)-\tilde{\Omega}_j(c\times S^k\times w),$$

which represents $(U-V)(c\otimes e_j)$.

To prove these claims we note that $\tilde{\Gamma}_j(t, y, x, w)=(\theta\times\gamma)^{-1}\tilde{\Omega}_{j'}(y', x', v')$ implies $g_j(x, w)=\gamma^{-1}g_{j'}(x', v')$ hence $x\in D_p(\frac{1}{2})$, $x'\in D_{p'}(\frac{1}{2})$ for some intersection-pair $\{p, p'\}$. In case $\varkappa(x)=1$, $\varkappa(x')=0$ we further get $\xi_1 y=\theta^{-1}y'$ contradicting $\xi y\in\xi\tilde{R}$, $y'\in\tilde{Q}$, $\xi\tilde{R}\cap\tilde{Q}=\emptyset$; hence $\varkappa(x)=0$ i.e. $x\in\Delta$.

On the other hand such a pair, with $g_j(\eta_p(a), b)=\gamma^{-1}g_{j'}(\eta_{p'}(b), a)$ for $a, b\in D^k(\frac{1}{2})$ gives a contribution $\varepsilon_{p'}\gamma$ to $\psi(e_j, e_{j'})$ hence $\varepsilon_p\gamma^{-1}e_{j'}$ to $(-1)^k\tilde{\psi}(e_j)$. Also we can identify $\varepsilon_p\tilde{\Gamma}_j(t\times c\times D_p(\frac{1}{2})\times w)$ with $\varepsilon_p\gamma^{-1}\tilde{\Omega}_{j'}(\xi^{-1}\xi_t\xi_t c\times 0\times \times D^k(\frac{1}{2}))$; hence $\tilde{\Gamma}_j([0,1]\times c\times\Delta\times w)$ represents $T(c\otimes e_j)$.

The map $\Xi\colon [0,1]\times[0,1]\times\tilde{P}\to\tilde{P}$ defined by the formula

$$\Xi(s,t,y)=\begin{cases}\xi^{-1}_{1-2st}\,\xi_{t(1-s)}\,\xi_{t(1-s)}y & \text{for } t\leqslant\frac{1}{2}\\ \xi^{-1}_{1-s}\,\xi_{t(1-s)}\,\xi_{t+s(t-1)}y & \text{for } t\geqslant\frac{1}{2}\end{cases}$$

satisfies

$$\Xi(0,t,y)=\xi^{-1}\xi_t\xi_t y \text{ for all } t,$$
$$\Xi(s,0,y)=\xi^{-1}y \text{ and } \Xi(s,1,y)=\xi y, \text{ for all } s.$$

Accordingly, we may replace the map $(t, y)\to\xi^{-1}\xi_t\xi_t y$ in the above statement by the map $(t, y)\to\xi_{2t-1}$ for $t\geqslant\frac{1}{2}$ and ξ^{-1}_{1-2t} for $t\leqslant\frac{1}{2}$, hence by $(t, y)\to\chi_t y$.

This has the advantage that it shows that T factorizes over $H_i(\tilde{R}, \tilde{Q})$. The above statement can now be read to say that $U=V$ provided we correct the original map $H_i(\tilde{R}, \tilde{Q})\otimes K_k(M)\to K_{i+k}(N)$ by the term given by either composition in the diagram

$$\begin{array}{ccccc}
H_i(\tilde{R},\tilde{Q})\otimes K_k(M) & & & & \\
\big\downarrow{\scriptstyle \chi\otimes(-1)^k\tilde{\psi}} & & & & \\
H_{i+1}(\tilde{P},\tilde{P}-\tilde{Q})\otimes K_k(M) & & \xrightarrow[\partial\otimes(-1)^k\tilde{\lambda}^{-1}]{} & & H_i(P-Q)\otimes K_k(M)\\
\big\uparrow{\scriptstyle \cong} & & & & \big\downarrow\\
H_{i+1}(\tilde{Q},\partial\tilde{Q})\otimes K_k(M) & \longrightarrow & K_{i+k+1}(W,N) & \longrightarrow & K_{i+k}(N)
\end{array}$$

Notice that $\tilde{\lambda}^{-1}\tilde{\psi}=(Ad\,\lambda)^{-1}(Ad\,\psi)$ is independent of the choice of base $\{e_j\}$.

This concludes the proof of the commutativity of (b); the commutativity of (a) is not disturbed by the addition of the correction term to ω.

Q.E.D.

§ 4. THE COMPUTATION OF $K_{q+k}(N)$

We define K to be the cokernel of the map

$$(-\xi, 1): H_q(\tilde{R}) \to H_q(\tilde{P}-\tilde{Q}) \oplus H_q(\tilde{R}, \tilde{Q}).$$

THEOREM 2: *$K_{q+k}(N)$ is isomorphic to $K \otimes K_k(M)$.*

PROOF: According to Lemma 3b there are maps

$$U: H_q(\tilde{P}-\tilde{Q}) \otimes K_k(M) \to K_{q+k}(N)$$

and

$$V: H_q(\tilde{R}, \tilde{Q}) \otimes K_k(M) \to K_{q+k}(N)$$

which agree on $H_q(\tilde{R}) \otimes K_k(M)$; hence there is an induced map

$$K \otimes K_k(M) \to K_{q+k}(N).$$

This will be shown to be an isomorphism.

The following diagram has exact rows

$$\begin{array}{ccccccc}
H_{q+1}(\tilde{P}, \tilde{R}) & \xrightarrow{\partial} & H_q(\tilde{R}) & \longrightarrow & H_q(\tilde{P}) & \longrightarrow & 0 \\
\downarrow 1 & & \downarrow & & \downarrow & & \\
H_{q+1}(\tilde{P}, \tilde{R}) & \xrightarrow{\partial} & H_q(\tilde{R}, \tilde{Q}) & \longrightarrow & H_q(\tilde{P}, \tilde{Q}) & \longrightarrow & 0
\end{array}$$

and the vertical maps are injective so

$$H_q(\tilde{R}, \tilde{Q})/H_q(\tilde{R}) \cong H_q(\tilde{P}, \tilde{Q})/H_q(\tilde{P}).$$

The upper row of the next diagram is exact

$$\begin{array}{ccccc}
H_{q+1}(\tilde{P}, \tilde{P}-\tilde{Q}) & \xrightarrow{\partial} & H_q(\tilde{P}-\tilde{Q}) & \longrightarrow & H_q(\tilde{P}) \\
\downarrow 1 & & \downarrow & & \downarrow \\
H_{q+1}(\tilde{P}, \tilde{P}-\tilde{Q}) & \xrightarrow{(\partial, 0)} & K = \operatorname{coker}(-\xi, 1) & \longrightarrow & H_q(\tilde{P}, \tilde{Q})
\end{array}$$

The vertical maps are injective and their cokernels are isomorphic as we have just seen; hence the lower row is exact. So we get an exact sequence

$$0 \to H_{q+1}(\tilde{P}) \to H_{q+1}(\tilde{P}, \tilde{P}-\tilde{Q}) \to K \to H_q(\tilde{P}, \tilde{Q}) \to 0$$

Now consider the diagram

$$\begin{array}{ccccccccc}
0 \to H_{q+1}(\tilde{P}) \otimes K_k(M) & \to & H_{q+1}(\tilde{P}, \tilde{P}-\tilde{Q}) \otimes K_k(M) & \to & K \otimes K_k(M) & \to & H_q(\tilde{P}, \tilde{Q}) \otimes K_k(M) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to K_{q+k+1}(W) & \to & K_{q+k+1}(W, N) & \to & K_{q+k}(N) & \to & K_{q+k}(W) & \to & 0
\end{array}$$

Of this diagram we know the following:

i) as we have just proved, the rows are exact
ii) the first and the fourth vertical maps are isomorphisms by lemma 1
iii) the second vertical map is so by the proof of theorem 1
iv) the first square commutes by lemma 2b
v) the second square commutes by lemma 2c
vi) the third square commutes by lemma 2a and lemma 3a.

The stated isomorphism now follows by application of the five lemma.

Q.E.D.

(To be continued)

MATHEMATICS

Surgery on products. II

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Communicated by Prof. W. T. van Est at the meeting of June 17, 1978

§ 5. CHAIN COMPLEXES ASSOCIATED TO P

We now prove a few lemmas which establish a relation between K and the chain map which realizes the Poincaré duality of P.

LEMMA 4: *The equivariant intersection-product* λ *defines an equivariant isomorphism* $\varrho: H_i(\tilde{P}-\tilde{P}_{n-i-1}, \tilde{P}-\tilde{P}_{n-i}) \to H_{n-i}(\tilde{P}_{n-i}, \tilde{P}_{n-i-1})^d$ *which maps* ∂ *to* $(-1)^i\partial^d$.

PROOF: This is really a matter of stating the (sign) conventions.

Definitions of cup and cap-product are such as to make the following relations hold on the chain level: $(x \cup y) \cap \alpha = x \cap (y \cap \alpha)$ for any chain α, cochains x, y; $\partial^*(x \cup y) = \partial^* x \cup y + (-1)^a x \cup \partial^* y$, where $a = \deg(x)$; $\langle x, \alpha\rangle = \varepsilon(x \cap \alpha)$, where ε denotes augmentation.

From this follows that

$$y \cap \partial\alpha = \partial(y \cap \alpha) + (-1)^a \partial^* y \cap \alpha, \text{ where } a = \deg(\alpha) - \deg(y) - 1.$$

In particular: if α is an n-cycle then $\cap\, \alpha$ is a chain map from C^* equipped with $(-1)^i\partial^*: C^{n-i} \to C^{n-i+1}$ to C equipped with ∂. The same holds when ∂^* stands for the boundary operator in the long exact sequence of a triple; and the same holds for the Cech cap.

Hence the Poincaré-Lefschetz duality map

$$\cap\, [P]: H_c^{n-i}(\tilde{P}_{n-i}, \tilde{P}_{n-i-1}) \to H_i(\tilde{P}-\tilde{P}_{n-i-1}, \tilde{P}-\tilde{P}_{n-i})$$

makes

$$(-1)^i\partial^*: H_c^{n-i}(\tilde{P}_{n-i}, \tilde{P}_{n-i-1}) \to H_c^{n-i+1}(\tilde{P}_{n-i+1}, \tilde{P}_{n-i})$$

correspond with

$$\partial: H_i(\tilde{P}-\tilde{P}_{n-i-1}, \tilde{P}-\tilde{P}_{n-i}) \to H_{i-1}(\tilde{P}-\tilde{P}_{n-i}, \tilde{P}-\tilde{P}_{n-i+1}).$$

Moreover the naturality of the cap-product, applied to the left action of $\gamma \in G$ on $\tilde{P}$ implies that $\cap\,[P]$ is equivariant if $\gamma \in G$ acts on H_i as γ_* and on H^{n-i} as $\bar{\gamma}^*$. Here $\bar{\gamma}=w(\gamma)\gamma^{-1}$; we used that $\gamma_*[P]=w(\gamma)[P]$.

For any $x \in H_c^{n-i}(\tilde{P}_{n-i}, \tilde{P}_{n-i-1})$ the map $H_{n-i}(\tilde{P}_{n-i}, \tilde{P}_{n-i-1}) \to Z[G]$ which maps α to $\sum_{\gamma\in G}\langle x, \gamma^{-1}\alpha\rangle\gamma$ is a module homomorphism. This leads to a map

$$H_c^{n-i}(\tilde{P}_{n-i}, \tilde{P}_{n-i-1}) \to H_{n-i}(\tilde{P}_{n-i}, \tilde{P}_{n-i-1})^d$$

which is an equivariant isomorphism and makes $(-1)^i\partial^*$ correspond with $(-1)^i\partial^d$. The composite with $(\cap\,[P])^{-1}$ is an equivariant isomorphism $\varrho: H_i(\tilde{P}-\tilde{P}_{n-i-1}, \tilde{P}-\tilde{P}_{n-i}) \to H_{n-i}(\tilde{P}_{n-i}, \tilde{P}_{n-i-1})^d$ which maps ∂ to $(-1)^i\partial^d$. Making use of the relation $\langle x, \alpha\rangle=(x \cap [P])\cdot\alpha$ between cap and intersection product we see that ϱ is precisely $Ad\,(\lambda)$.

DEFINITION: *From now on C_* will denote the chain complex such that $C_i=H_i(\tilde{P}_i, \tilde{P}_{i-1})$ and $\partial: C_i \to C_{i-1}$ is the boundary operator of the triple $(\tilde{P}_i, \tilde{P}_{i-1}, \tilde{P}_{i-2})$; furthermore C^* denotes the complex with $C^{n-i}=(C_{n-i})^d$ and $\delta=(-1)^i\partial^d: C^{n-i} \to C^{n-i+1}$.*

LEMMA 5: $\zeta_0=\varrho(\xi_*): (C_*, \partial) \to (C^*, \delta)$ *is a chain equivalence.*

PROOF: As we have just seen, ϱ is an isomorphism between the obvious chain complex with chain groups $H_i(\tilde{P}-\tilde{P}_{n-i-1}, \tilde{P}-\tilde{P}_{n-i})$ and C^*; e.g.

$$H_i(\tilde{P}, \tilde{P}-\tilde{P}_{n-i}) = \text{coker}\,(\partial: H_{i+1}(\tilde{P}-\tilde{P}_{n-i-2}, \tilde{P}-\tilde{P}_{n-i-1}) \to$$
$$\to H_i(\tilde{P}-\tilde{P}_{n-i-1}, \tilde{P}-\tilde{P}_{n-i}))$$

can be identified with $\text{coker}\,(\delta: C^{n-i-1} \to C^{n-i})$ using ϱ, and in fact the following diagrams correspond under ϱ:

$$\begin{array}{ccc} H_i(\tilde{P}-\tilde{P}_{n-i-1}) & \longrightarrow & H_i(\tilde{P}) \\ \downarrow & & \downarrow \\ H_i(\tilde{P}-\tilde{P}_{n-i-1}, \tilde{P}-\tilde{P}_{n-i}) & \to & H_i(\tilde{P}, \tilde{P}-\tilde{P}_{n-i}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \ker\delta & \longrightarrow & \ker\delta/\text{im}\,\delta \\ \downarrow & & \downarrow \\ C^{n-i} & \longrightarrow & \text{coker}\,\delta \end{array}$$

The diffeomorphism $\xi: \tilde{P} \to \tilde{P}$ has the property that $\xi(\tilde{P}_i) \subset \tilde{P}-\tilde{P}_{n-i-1}$, hence defines a homomorphism $H_i(\tilde{P}_i, \tilde{P}_{i-1}) \to H_i(\tilde{P}-\tilde{P}_{n-1-i}, \tilde{P}-\tilde{P}_{n-i})$ respecting ∂. Using ϱ we thus get a chain map $\zeta_0: (C_*, \partial) \to (C^*, \delta)$. The fact that ξ is homotopic to the identity map implies that it induces the identity on $H_i(\tilde{P})$ and so an application of ϱ tells us that ζ_0 induces an isomorphism between $H(C_*, \partial)$ and $H(C^*, \delta)$. Q.E.D.

LEMMA 6: *There is an exact sequence*

$$C^{q-2} \oplus C_{q+1} \to C^{q-1} \oplus C_q \to K \to 0.$$

PROOF: According to lemma 5 the canonical map

$$\frac{\operatorname{im} \delta \oplus C_q}{(-\zeta_0, 1) \operatorname{im} \partial} \to \frac{\ker \delta \oplus C_q}{(-\zeta_0, 1) \ker \partial} = \frac{H_q(\tilde{P} - \tilde{P}_{q-1}) \oplus H_q(\tilde{P}_q, \tilde{P}_{q-1})}{(-\xi_*, 1) H_q(\tilde{P}_q)} = K$$

is an isomorphism; and in particular $C^{q-1} \oplus C_q$ maps onto K. Diagram chasing now yields an exact sequence

$$0 \to \ker \partial \xrightarrow[a]{} \ker \delta \oplus C_{q+1} \xrightarrow[b]{} C^{q-1} \oplus C_q \to K \to 0$$

where $a(x) = (-\zeta_0 x, x)$ and $b(x, y) = (-x - \zeta_0 y, \partial y)$.
Lemma 5 now allows us to replace $\ker \partial$ by $\operatorname{im} \partial$ and $\ker \delta$ by $\operatorname{im} \delta$.

Q.E.D.

This invites us to make the following definition:

DEFINITION: *The chain complex* (E, ∂_E) *is given by the formulas*

$$E_i = C^{n-i-1} \oplus C_i, \quad \partial_E(x, y) = (-\delta x - \zeta_0 y, \partial y);$$

the cokernel of $\partial_E^{i+1}: E_{i+1} \to E_i$ *we denote by* K_i.

This chaincomplex is acyclic according to lemma 5; hence K_i is isomorphic to $\operatorname{im} \partial_E^i = \ker \partial_E^{i-1}$ and thus has a resolution

$$0 \to K_i \to E_{i-1} \to E_{i-2} \to \dots \to E_{-1} \to 0$$

by free modules; hence K_i is stably free. On the other hand lemma 6 identifies K with K_q.

In order to identify $\varrho(\xi^{-1})_*$ we introduce the following notation. If ω is a homomorphism between modules S and V^d, the dual ω^d is one between V^{dd} and S^d; if V is finitely generated projective we can use the isomorphism $\hat{}: V \to V^{dd}$ to identify this with a homomorphism ω^* between V and S^d. Given a map between a chain module S_a and a cochain module V^q we define $T\omega$ to be $(-1)^{aq}\omega^*$.

Now for $x \in C_a$, $y \in C_q$ we get

$$\begin{aligned}(\varrho(\xi^{-1})_* y)(x) &= \lambda((\xi^{-1})_* y, x) = \lambda(y, \xi_* x) = (-1)^{aq}\overline{\lambda(\xi_* x, y)} = \\ &= (-1)^{aq}\overline{(\varrho\xi_* x)(y)} = (-1)^{aq}\overline{(\zeta_0 x)(y)} = (-1)^{aq}\hat{y}(\zeta_0 x) \\ &= (-1)^{aq}(\zeta_0^d \hat{y})(x) = (-1)^{aq}(\zeta_0^* y)(x) = ((T\zeta_0)y)(x)\end{aligned}$$

which means that $\varrho(\xi^{-1})_* = T\zeta_0$. Hence, if we define ζ_1 by the formula $\zeta_1 c = \varrho\chi([0, 1] \times c)$ we have the identity $\delta\zeta_1 + \zeta_1\partial = \zeta_0 - T\zeta_0$.

In order to compare our ζ_s with the ϕ_s of Ranicki [R], we consider the chain complex $'C$ associated with the dual cell decomposition of P. Then an isomorphism $\omega: {'C^i} \to C_{n-i}$ exists, defined by the formula $\lambda(\omega(y), x) = y(x)$ for $y \in {'C^i}$, $x \in {'C_i}$; it is a chain map $('C^*, \delta) \to (C_*, \partial)$.

The associated map $T\omega$ satisfies $\lambda((T\omega)y, x) = y(x)$ for $y \in C^i$, $x \in C_i$ and is also a chain map.

Now if $\zeta_s: C_{n-i} \to C^{i-s}$ is defined in such a way that

$$\delta\zeta_s + (-1)^{s-1}\zeta_s\partial + (-1)^s(\zeta_{s-1} - T\zeta_{s-1}) = 0$$

and if $\phi_s: 'C^i \to 'C_{n-i+s}$ is defined by the formula $\phi_s = (-1)^{(n-s)s}(T\omega)\zeta_s\omega$, then we get the relation of [R]:

$$\partial\phi_s + (-1)^{s-1}\phi_s\delta + (-1)^{n+s-1}(\phi_{s-1} + (-1)^s T\phi_{s-1}) = 0.$$

§ 6. CONSTRUCTION OF THE SESQUILINEAR FORM ON K

The results of the last section are sufficient to prove:

THEOREM 3: *There is a sesquilinear form on K which is determined by the fact that the induced form σ on $H_{q+1}(\tilde{P} - \tilde{P}_{q-2}, \tilde{P} - \tilde{P}_{q-1}) \oplus H_q(\tilde{P}_q, \tilde{P}_{q-1})$ is given by the formula*

$$\sigma((x, y), (x', y')) = \overline{\lambda(y, \partial x')} + \lambda(\partial x + (\xi^{-1})_* y + \partial\chi_*([0, 1] \times y), y').$$

PROOF: We construct a form on K_q.

We start by identifying $E^{n-i-1} = (E_{n-i-1})^d$ with $C_i \oplus C^{n-i-1}$; under this identification $\partial^d_E: E^{n-i-1} \to E^{n-i}$ is given by the formula

$$\partial^d_E(x, y) = ((-1)^{i-1}\partial x, \partial^d y - \zeta_0^* x).$$

We want to construct a chain transformation $\Sigma: (E, \partial_E) \to (E^*, \partial^d_E)$ such that

$$\Sigma(x, y) = (\alpha_i y, \beta_i x + \gamma_i \zeta_1 y) \text{ for } x \in C^{n-i-1},\ y \in C_i;$$

here α, β, γ are signs.

Since

$$\partial^d_E\, \Sigma(x, y) = ((-1)^{n-i}\partial\alpha_i y,\ -\zeta_0^*\alpha_i y + \partial^d\beta_i x + \partial^d\gamma_i\zeta_1 y)$$

and

$$\Sigma\partial_E(x, y) = (\alpha_{i-1}\partial y,\ \beta_{i-1}(-1)^i\partial^d x - \beta_{i-1}\zeta_0 y + \gamma_{i-1}\zeta_1\partial y)$$

this works provided

$$\alpha_i = (-1)^{n-i}\alpha_{i-1},\ \beta_i = (-1)^i\beta_{i-1}$$

and

$$\beta_{i-1}\zeta_0 - \gamma_{i-1}\zeta_1\partial - \zeta_0^*\alpha_i + \partial^d\gamma_i\zeta_1 = 0.$$

But $-\zeta_0 + \zeta_1\partial + (-1)^{i(n-i)}\zeta_0^* + (-1)^{i+1}\partial^d\zeta_1 = 0$ so we must choose

$$\gamma_i = \beta_i = (-1)^{\frac{1}{2}i(i+1)}\beta_0$$

and

$$\alpha_i = (-1)^{i(n-i)+\frac{1}{2}i(i-1)}\beta_0 = (-1)^{\frac{1}{2}n(n-1)+\frac{1}{2}(n-i)(n-i-1)}\beta_0;$$

in particular $\alpha_i=\beta_i=\gamma_i$ if $n=2q$. Thus

$$\partial_E^d\,\Sigma(x,y)=\alpha_q((-1)^q\partial y,\ -\zeta_0^* y+\partial^d x+\partial^d\zeta_1 y)\in E^q \text{ for } (x,y)\in E_q.$$

Dualising the exact sequence $E_{q+1}\to E_q\to K_q\to 0$ yields an exact sequence $0\to K_q^d\to E^q\to E^{q+1}$ since K_q is finitely generated projective. The map $\partial_E^d\,\Sigma=\Sigma\,\partial_E\colon E_q\to E^q$ vanishes on im ∂_E and maps into ker ∂_E^d, hence factorizes uniquely over a map $\sigma\colon K_q\to K_q^d$. Since $\Sigma\colon E_i\to E^{n-i-1}$ is isomorphic for all i an application of the five lemma on the following diagram tells us that σ is isomorphic

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_q & \xrightarrow{\partial_E} & E_{q-1} & \xrightarrow{\partial_E} & E_{q-2} \\
 & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\Sigma} \\
0 & \longrightarrow & K_q^d & \longrightarrow & E^q & \xrightarrow{\partial_E^d} & E^{q+1}
\end{array}$$

In other words $\partial_E^d\Sigma$ defines a sesquilinear form on E_q which induces a nonsingular one on its quotient K_q, denoted by σ.

For $x,x'\in C^{q-1}$ and $y,y'\in C_q$ we get

$$\begin{aligned}
&\alpha_q\sigma((x,y),(x',y'))=\alpha_q(\sigma(x,y))(x',y')=\\
&=((-1)^q\partial y,\ -\zeta_0^* y+\partial^d x+\partial^d\zeta_1 y)(x',y')=\\
&=(-1)^q\overline{x'(\partial y)}-(\zeta_0^* y)(y')+(\partial^d x)(y')+(\partial^d\zeta_1 y)(y').
\end{aligned}$$

With the aid of ϱ this defines a form σ on

$$H_{q+1}(\tilde P-\tilde P_{q-2},\tilde P-\tilde P_{q-1})\oplus H_q(\tilde P_q,\tilde P_{q-1})$$

such that

$$\begin{aligned}
&\alpha_q\sigma((x,y),(x',y'))=(-1)^q\overline{(\varrho x')(\partial y)}-(\zeta_0^* y)(y')+(\partial^d\varrho x)(y')+\\
&+(\partial^d\zeta_1 y)(y').
\end{aligned}$$

We can rewrite the terms as follows

a) $(-1)^q\overline{(\varrho x')(\partial y)}=(-1)^q\overline{\lambda(x',\partial y)}=-\overline{\lambda(\partial x',y)}=(-1)^{q+1}\overline{\lambda(y,\partial x')}$

b) $(-\zeta_0^* y)(y')=-(\zeta_0^d\hat y)(y')=-\hat y(\zeta_0 y')=$
$=-\overline{(\zeta_0 y')(y)}=-\overline{(\varrho\xi_* y')(y)}=$
$=-\overline{\lambda(\xi_* y',y)}=(-1)^{q+1}\lambda(y,\xi_* y')$
$=(-1)^{q+1}\lambda((\xi^{-1})_* y,y')$.

c) $(\partial^d\varrho x)(y')=(\varrho x)(\partial y')=\lambda(x,\partial y')=(-1)^{q+1}\lambda(\partial x,y')$

d) $(\partial^d\zeta_1 y)(y')=(\zeta_1 y)(\partial y')=\lambda(\chi_*([0,1]\times y),\partial y')$
$=(-1)^{q+1}\lambda(\partial\chi_*([0,1]\times y),y')$.

We thus fix α_q to be $(-1)^{q+1}$. Q.E.D.

§ 7. THE REPRESENTATION OF $K_{q+k}(N)$ BY IMMERSIONS

In section 4 we have seen that $K_{q+k}(N)$ is generated by elements of the form $\tilde{\Psi}_j(\partial c \times S^k \times 0)$ where $c \in H_{q+1}(\tilde{P}-\tilde{S}, \tilde{P}-\tilde{Q})$, or $\tilde{\Omega}_j(d \times S^k \times 0)$ where $d \in H_q(\tilde{R}, \tilde{Q})$. In this section we show how to represent these classes by framed immersions of $(q+k)$-spheres in a way given by the normal data.

An element $c \in H_{q+1}(\tilde{P}-\tilde{S}, \tilde{P}-\tilde{Q})$ can be represented by a framed immersion $c: D^{q+1} \to \tilde{P}-\tilde{S}$ such that $e=\partial c$ maps S^q into $\tilde{P}-\tilde{Q}$.

LEMMA 7: *The framing of* $\tilde{\Psi}_j(e \times id \times w): S^q \times S^k \to \tilde{N}$ *got by composing* $\tilde{\Psi}: (\tilde{P}-\tilde{Q}) \times \tilde{M} \to \tilde{N}$ *with the framed immersons* $e: S^q \times D^q \to \tilde{P}-\tilde{Q}$ *and* $g_j: S^k \times D^k \to \tilde{M}$ *is in accordance with the normal data.*

PROOF: In step 1 we show that we may use $S^q \times S^k$ instead of S^{q+k} to measure intersections; in step 2 we check the framing.

Step 1: We consider the map $\tilde{\Psi}_j(e \times id \times w): S^q \times S^k \to \tilde{N}$. The image in $\tilde{P} \times \tilde{X}$ of this map factorizes over $\tilde{P} \times \tilde{M} \subset \tilde{W}$; but in $\tilde{P} \times \tilde{M}$ it is homotopic to $e \times g_j$; furthermore e is nullhomotopic in $\tilde{P}$ and fg_j is so in $\tilde{X}$; hence the afore-mentioned image in $\tilde{P} \times \tilde{X}$ is nullhomotopic. Now if $z \in S^q$ is some fixed basepoint the same is true for the restriction to $z \times S^k$. Since the map $\tilde{N} \to \tilde{P} \times \tilde{X}$ is highly-connected there exists a nullhomotopy of $\tilde{\Psi}_j(e \times id \times w): z \times S^k \to N$ whose image in $\tilde{P} \times \tilde{X}$ is the aforementioned nullhomotopy. All this means that $\tilde{\Psi}_j(e \times id \times w)$ can be extended to a map $S^q \times S^k \cup z \times D^{k+1} \to \tilde{N}$ whose image in $\tilde{P} \times \tilde{X}$ is nullhomotopic.

Let Δ be a little disc around z in S^q and consider $Y = S^q \times S^k \times [0,1] \cup \cup \Delta \times D^{k+1}$ which is a $(q+k+1)$-manifold with boundary consisting of $S^q \times S^k$ and S^{q+k}. Using a homotopy-equivalence with the situation described above we conclude that there exists a map $Y \to \tilde{N}$ whose image in $\tilde{P} \times \tilde{X}$ is nullhomotopic in such a way that this situation extends $\tilde{\Psi}_j(e \times id \times w)$ and its nullhomotopy in $\tilde{P} \times \tilde{X}$. This yields a commutative diagram of maps

$$\begin{array}{ccc} Y & \longrightarrow & \tilde{N} \\ \downarrow & & \downarrow \\ \text{cone on } Y & \rightarrow & \tilde{P} \times \tilde{X} \end{array}$$

Furthermore the tangent bundle TY has a canonical framing extending the well-known stable ones of $T(S^{q+k})$ and $T(S^q \times S^k)$. Hence, the above diagram determines a regular homotopy class of immersions $Y \to \tilde{N}$ in the way of theorem 1.1 of [W1]. By restriction to the boundary part S^{q+k} one gets a framed immersion of it, and the framing is determined by the normal data in the way of the above-mentioned theorem 1.1. We may assume that the immersed Y intersect each other and themselves regularly i.e. along arcs ending on the boundary; this implies that both boundary parts have the same intersection behaviour and in particular determine the same quadratic form. Therefore we may use the $S^q \times S^k$ boundary part to determine the quadratic form.

Step 2: The restriction of the above diagram to the $S^q \times S^k$ boundary part yields:

$$\begin{array}{ccc} S^q \times S^k & \xrightarrow{\;\Psi_j\;} & \tilde{P} \times \tilde{M} - \mathrm{im}\,\tilde{\Omega} \subset \tilde{N} \\ \downarrow & & \downarrow \\ \text{Cone on } S^q \times S^k & \longrightarrow & \tilde{P} \times \tilde{X} \end{array}$$

It is more convenient to work with the inclusion of $S^q \times S^k$ in $D^{q+1} \times D^{k+1}$ instead of the inclusion of $S^q \times S^k$ in its cone. The map on $D^{q+1} \times D^{k+1}$ is then $c \times h_j$, where h_j is the nullhomotopy of g_j in X. Furthermore, we note that everything takes place in $\tilde{P} \times \tilde{M}$, and there $\tilde{\Psi}_j(e \times id \times w)$ is isotopic to $e \times g_j(\ ,0)$.

Hence the framing is determined by the diagram

$$\begin{array}{ccc} S^q \times S^k & \xrightarrow{\;e \times g_j(\ ,0)\;} & \tilde{P} \times \tilde{M} \\ \downarrow & & \downarrow{\scriptstyle 1 \times \tilde{f}} \\ D^{q+1} \times D^{k+1} & \xrightarrow{\;c \times h_j\;} & \tilde{P} \times \tilde{X} \end{array}$$

In order to show that the framing of $(e \times g_j(\ ,0))^*T(\tilde{P} \times \tilde{M})$ determined by this diagram is equal to the product of the framing of $e^*T\tilde{P}$ orginating from $c^*T\tilde{P}$ and the framing of $g_j(\ ,0)^*T\tilde{M}$ determined by the normal data for $\tilde{f}$, we must show that the sum with the framing of

$$(e \times g_j(\ ,0))^*(1 \times \tilde{f})^*(\nu P \times \nu)$$

originating from $c^*\nu P \times h_j^*\nu$ equals $(e \times g_j(\ ,0))^*$ applied to the trivial framing of $T\tilde{P} \oplus \nu\tilde{P}$ and the framing F of $T\tilde{M} \oplus \tilde{f}^*\nu$.

Well then, this equality is the sum of the one determining the framing of $g_j(\ ,0)^*T\tilde{M}$ and the one which says that the framing of $e^*T\tilde{P} \oplus e^*\nu\tilde{P}$ originating from the bundle $c^*T\tilde{P} \oplus c^*\nu\tilde{P}$ over the disc is precisely e^* applied to the trivial framing of $T\tilde{P} \oplus \nu\tilde{P}$. Q.E.D.

An element $d \in H_q(\tilde{R}, \tilde{Q})$ can be represented by an immersion $d: D^q \to \tilde{R}$ such that ∂d maps S^{q-1} into $\tilde{Q}$.

LEMMA 8: *The framed immersion $\tilde{\Omega}_j(d \times id \times 0): D^q \times S^k \to \tilde{N}$ got by composing $\tilde{\Omega}$ with the framed immersions $d: D^q \times D^q \to \tilde{R}$ and $g_j: S^k \times D^k \to \tilde{M}$ is the restriction of a framed immersion $S^{q+k} \to \tilde{N}$ in accordance with the normal data, and has no further selfintersections.*

PROOF: The image of $d \otimes e_j$ under the original map

$$\omega\colon H_q(\tilde{R}, \tilde{Q}) \otimes K_k(M) \to K_{q+k}(N)$$

is represented by the map $v\colon S^{q+k} \to \tilde{N}$ which is the union of $\tilde{\Omega}_j(d \times id \times w)\colon D^q \times S^k \to \tilde{N}_0$ and $d \times id \times w\colon S^{q-1} \times D^{k+1} \to \tilde{Q} \times D^{k+1} \times S^{k-1} \subset \tilde{N}_1$ (where N_0 denotes $P \times M - \operatorname{im} \Omega$ and $N_1 = \overline{N - N_0}$), together with a nullhomotopy of v in $\tilde{P} \times \tilde{X}$, as is summarized in the diagram:

$$\begin{array}{ccc} D^q \times S^k \cup S^{q-1} \times D^{k+1} & \xrightarrow{\;v\;} & \tilde{N} = \tilde{N}_0 \cup \tilde{N}_1 \\ \downarrow & & \downarrow \\ D^q \times D^{k+1} & \xrightarrow{\;d \times h_j\;} & \tilde{P} \times \tilde{X} \end{array}$$

One can write down an explicit formula for a framing of v which is as stated on the first summand; to check wether it is the right one it is sufficient to check the framed immersion which one gets by composing with the collar $N \times (0, 1) \subset W$.

Secondly one notes that $W \times (-2, 2)$ is constructed from $P \times M \times \times [0, 1] \times (-2, 2)$ by attaching $Q \times D_j^{k+1} \times D^k(\frac{1}{2}) \times (-2, 2)$ according to the map $\tilde{Q} \times S^k \times D^k(\frac{1}{2}) \times (-2, 2) \to P \times M \times (-2, 2)$ which maps (y, x, v, ϱ) to $(\xi_{\varkappa(x)}y, g_j(x, v), \varrho)$; this embedding is isotopic to the one mapping (y, x, v, ϱ) to $(y, g_j(x, v), \frac{1}{10}\varrho + \varkappa(x))$. The last one is defined for all $y \in P$; thus we get an embedding $W \times (-2, 2) \subset P \times T$ where

$$T = M \times [0, 1] \times (-2, 2) \cup D^{k+1}(2) \times D^k(\tfrac{1}{2}) \times (-2, 2)$$

identifying $(\tau x, v, \varrho)$ with $(g_j(x, v), 2-\tau, \frac{1}{10}\varrho + \varkappa(x))$ for $\tau \in [1, 2]$, $x \in S^k$.

Furthermore the normal map extends to $P \times T$ so we can test things there. But the corresponding immersed sphere lies in the disc formed by d and the second term of T as described above. In fact it is immersed there in a standard way i.e. extendible to an immersion of D^{k+q+1}; that is just what the normal data prescribe. Q.E.D.

§ 8. COMPLETION OF THE PROOF

By subdividing the triangulation of P if necessary we can make certain that $K = K_q$ is free; we choose a base $f_1 \ldots f_p$ of K. Any f_i can be seen as the image of $\{a_i\} + \{b_i\}$ where $\{a_i\} \in H_{q+1}(\tilde{P} - \tilde{S}, \tilde{P} - \tilde{Q})$ and $\{b_i\} \in H_q(\tilde{R}, \tilde{Q})$. To be more precise we denote by

- ι: the map $H_{q+1}(\tilde{P} - \tilde{S}, \tilde{P} - \tilde{Q}) \otimes K_k(M) \to K_{q+k}(N)$
- ω: the uncorrected map $H_q(\tilde{R}, \tilde{Q}) \otimes K_k(M) \to K_{q+k}(N)$
- ω': the corrected map $H_q(\tilde{R}, \tilde{Q}) \otimes K_k(M) \to K_{q+k}(N)$.

Then there is a basis of $K_{q+k}(N)$ consisting of the elements corresponding to the $f_i \otimes e_j$; i.e. the elements

$$\iota(\{a_i\} \otimes e_j) + \omega'(\{b_i\} \otimes e_j) = \iota(\{a_i\} \otimes e_j + \textstyle\sum_s (\tilde{\lambda}^{-1}\tilde{\psi})_{js}\{c_i\} \otimes e_s) + \omega(\{b_i\} \otimes e_j),$$

where $\{c_i\}$ denotes $\chi_*([0, 1] \times \{b_i\})$.

We must represent these basis elements by immersions consistent with the normal data, define an ordering for the intersection pairs, and count the intersections.

First we construct the immersions. We have denoted the i-skeleton of the triangulation by P_i; similarly we denote the i-skeleton of the dual linear cell complex by ${}_iP$. Since ${}_{q+1}P \cap P_{q-2} = \emptyset$ we may assume that S was chosen such that ${}_{q+1}P \cap S = \emptyset$; similarly there must be a small regular neighbourhood Z of ${}_qP$ such that $Z \cap Q = \emptyset$. Furthermore we write Q_1 for $P - {}_qP$.

Now we consider a disc a obtained from a $(q+1)$ cell of ${}_{q+1}\tilde{P}$ by deleting a small collar along the boundary which is contained in Z. Then $a \subset \tilde{Q}_1 - \tilde{S}$ and $\partial a \subset Z \subset \tilde{P} - \tilde{Q}$ hence $(a, \partial a) \subset (\tilde{Q}_1 - \tilde{S}, \tilde{Q}_1 - \tilde{Q})$. By homotopy equivalence it follows that the corresponding element of $H_{q+1}(\tilde{Q}' - \tilde{S}, \tilde{Q}' - \tilde{Q})$ is spherical if $\tilde{Q}'$ is some regular neighbourhood of P_{q-1} containing $\tilde{Q}$; we may assume that $\tilde{Q}' \cap \xi\tilde{Q}' = \emptyset$. So we can represent this element by a smoothly embedded disc $a \subset \tilde{Q}' - \tilde{S}$ such that $\partial a \subset \tilde{Q}' - \tilde{Q}$ meets P_q transversally.

The element $\{a_i\} \in H_{q+1}(\tilde{P} - \tilde{S}, \tilde{P} - \tilde{Q})$ which is a combination of such cells can, thus, be represented by joining such embedded disc using suitable ribbons inside $\tilde{Q}' - \tilde{Q}$. This is possible since the inclusions $\tilde{Q}' - \tilde{Q} \subset \subset \tilde{Q}' - P_{q-1} \subset \tilde{Q}' \subset P$ induce isomorphisms on π_1. The same goes for the $\{c_i\}$. We have shown that $\xi a_i \cap a_{i'} = \xi a_i \cap c_{i'} = \xi c_i \cap a_{i'} = \emptyset$.

The element $\{b_i\} \in H_q(\tilde{R}, \tilde{Q})$ is a sum of q-simplices of $\tilde{P}_q$. Therefore we consider the discs obtained from these simplices by deleting collars along the boundaries contained in $\tilde{Q}$; we then join them by suitable ribbons inside $\tilde{Q} - \tilde{P}_{q-1}$. Since b_i looks like these simplices outside $\tilde{Q}$ it meets $\partial a_{i'}$, $\partial c_{i'}$ or $\xi b_{i'}$ transversally.

The foregoing discussion combined with the theory of the last section tells us how to construct the immersions. To define an ordering for the intersection pairs we can restrict our attention to the individual terms in the formula for $e_i \otimes f_j$; we define the ordering with the aid of a real valued function, like our former $\varkappa$. For the a_i and b_i term, where the immersed manifold is $S^q \times S^k$ resp. $D^q \times S^k$ we define this function by projecting to S^k and using $\varkappa$; for the c_i term we use $\varkappa - 2$.

We are going to count the intersections between

$$\iota(\{a_i\} \otimes e_j + \textstyle\sum_s (\tilde{\lambda}^{-1}\tilde{\psi})_{js}\{c_i\} \otimes e_s) + \omega(\{b_i\} \otimes e_j)$$

and

$$\iota(\{a_{i'}\} \otimes e_{j'} + \textstyle\sum_{s'} (\tilde{\lambda}^{-1}\tilde{\psi})_{j's'}\{c_{i'}\} \otimes e_{s'}) + \omega(\{b_{i'}\} \otimes e_{j'}).$$

1) Since $\xi a_i \cap \gamma^{-1} a_{i'} = \emptyset$ there is no intersection between $\tilde{\Psi}_j(\partial a_i \times S^k \times 0)$ and $(\gamma \times \theta)^{-1}\tilde{\Psi}_{j'}(\partial a_{i'} \times S^k \times 0)$ and hence no contribution from $\iota(\{a_i\} \otimes e_j)$ and $\iota(\{a_{i'}\} \otimes e_{j'})$; the same is true for the other ι terms.

2) In order to intersect $\tilde{\Psi}_j(\partial a_i \times S^k \times 0)$ with $(\theta \times \gamma)^{-1}\tilde{\Omega}_{j'}(b_{i'} \times S^k \times 0)$ we have to solve $g_j(x, 0) = \gamma^{-1}g_j(x', 0)$, $\varkappa(x) < \varkappa(x')$ and $\xi_{1-\varkappa(x)}y = \theta^{-1}\xi_{\varkappa(x')}y'$ i.e. $y = \theta^{-1}y'$; so we get a contribution $(-1)^{kq}\lambda(\partial a_i, b_{i'}) \otimes \psi(e_j, e_{j'})$ from the terms $\iota(\{a_i\} \otimes e_j)$ and $\omega(\{b_{i'}\} \otimes e_{j'})$. Similarly we get a contribution $(-1)^{kq}\lambda(b_i, \partial a_{i'}) \otimes \psi(e_j, e_{j'})$ from $\omega(\{b_i\} \otimes e_j)$ and $\iota(\{a_{i'}\} \otimes e_j)$.

3) The terms $\iota(\{c_i\} \otimes e_s)$ and $\omega(\{b_{i'}\} \otimes e_{j'})$ lead to a similar equation, but with $\varkappa(x) - 2 < \varkappa(x')$, which is universally true; this yields a contribution $(-1)^{kq}\lambda(c_i, b_{i'}) \otimes \lambda(e_s, e_{j'})$. The terms $\omega(\{b_i\} \otimes e_j)$ and $\iota(\{c_{j'}\} \otimes e_s)$ lead to the impossible formula $\varkappa(x) < \varkappa(x') - 2$ and give no contribution. Since $\lambda(\sum_s (\tilde{\lambda}^{-1}\tilde{\psi})_{js}e_s, e_{j'}) = \psi(e_j, e_{j'})$, the full contribution from the correction term is $(-1)^{kq}\lambda(c_i, b_{i'}) \otimes \psi(e_j, e_{j'})$.

4) In order to intersect $\tilde{\Omega}_j(b_i \times S^k)$ with $(\theta \times \gamma)^{-1}\tilde{\Omega}_{j'}(b_{i'} \times S^k)$ we have to solve $s_j(x, 0) = \gamma^{-1}g_{j'}(x', 0)$, $\varkappa(x) < \varkappa(x')$ and $\xi_{\varkappa(x)}y = \theta^{-1}\xi_{\varkappa(x')}y$, i.e. $y = \xi y'$; so we get a contribution $(-1)^{kq}\lambda(\xi^{-1}b_i, b_{i'}) \otimes \psi(e_j, e_{j'})$ from $\omega(b_i \otimes e_j)$ and $\omega(b_{i'} \otimes e_{j'})$.

Comparing the above result with that at the end of § 6 we see that the total intersection is $\sigma((a_i, b_i), (a_{i'}, b_{i'})) \otimes \psi(e_j, e_{j'})$ up to a factor $(-1)^{kq}$. This concludes the proof of the theorem.

§ 9. SOME REMARKS AND AN EXAMPLE

LEMMA 9: *The form σ is almost $(-1)^q$ symmetric in the sense that it satisfies $\sigma^* = (-1)^q\sigma(1+r)$, where r is nilpotent.*

PROOF: From the defining formula

$$\Sigma(x, y) = (\alpha_i y, \beta_i x + \beta_i \zeta_1 y) \text{ for } x \in C^{n-i-1},\ y \in C_i$$

it follows that

$$\begin{aligned}\Sigma^*(x, y) &= (\beta_{n-i-1}y, \alpha_{n-i-1}x + \beta_{n-i-1}\zeta_1^* y) \\ &= (-1)^{\frac{1}{2}n(n-1)}(\alpha_i y, \beta_i x + \beta_i(T\zeta_1)y),\end{aligned}$$

where

$$(T\zeta_1)y = (-1)^{\frac{1}{2}i(n-i-1)}\zeta_1^* y.$$

Thus $\Sigma^{-1}\Sigma^* = (-1)^{\frac{1}{2}n(n-1)}(1+r)$ where $r(x, y) = ((T\zeta_1)y - \zeta_1 y, 0)$, hence $r^2 = 0$. For $n = 2q$ this implies that σ is indeed almost $(-1)^q$ symmetric.

Q.E.D.

Let us look at the algebraic properties of such an almost $(-1)^q$ symmetric form σ.

THEOREM 4: *Tensor product with an almost $(-1)^q$ symmetric form induces a well-defined map $L_{2k}(B) \to L_{2k+2q}(A \otimes B)$.*

PROOF: If ψ is a nonsingular quadratic form with symmetrisation

$\lambda = \psi + (-1)^k \psi^*$ then $\sigma \otimes \psi$ has symmetrisation

$$\sigma \otimes \psi + (-1)^{k+q} \sigma^* \otimes \psi^* = \sigma \otimes (\psi + (-1)^k \psi^*) +$$
$$+ (-1)^k ((-1)^q \sigma^* - \sigma) \otimes \psi^* = \sigma \otimes \lambda + (-1)^k \sigma r \otimes \psi^* =$$
$$= (\sigma \otimes \lambda)(1 + (-1)^k r \otimes \lambda^{-1} \psi^*),$$

which is invertible, hence $\sigma \otimes \psi$ is nonsingular.

In the following computation we restrict ourselves to the case $r^2 = 0$; then $(\sigma + \sigma r) r = \sigma r$ and

$$1 - r = (1+r)^{-1} = ((-1)^q \sigma^{-1} \sigma^*)^{-1} = \sigma^{-1}((-1)^q \sigma (\sigma^*)^{-1}) \sigma =$$
$$= \sigma^{-1}(1 + r^*) \sigma = 1 + \sigma^{-1} r^* \sigma,$$

hence $r^* \sigma r = -\sigma r^2 = 0$.

For any ϕ, $\sigma \otimes \psi$ is isomorphic to

$$(1 - r \otimes \lambda^{-1}\phi)^* (\sigma \otimes \psi)(1 - r \otimes \lambda^{-1} \phi) =$$
$$= \sigma \otimes \psi - \sigma r \otimes \psi \lambda^{-1} \phi - r^* \sigma \otimes (\lambda^{-1} \phi)^* \psi$$

which is equivalent to

$$\sigma \otimes \psi - \sigma r \otimes \psi \lambda^{-1} \phi - (-1)^{k+q} \sigma^* r \otimes \psi^* \lambda^{-1} \phi =$$
$$= \sigma \otimes \psi - \sigma r \otimes \psi \lambda^{-1} \phi - (-1)^k \sigma r \otimes \psi^* \lambda^{-1} \phi =$$
$$= \sigma \otimes \psi - \sigma r \otimes \phi = \sigma \otimes \psi + \sigma \otimes \phi - (-1)^q \sigma^* \otimes \phi,$$

which is equivalent to $\sigma \otimes (\psi + \phi - (-1)^k \phi^*)$.

Furthermore $(1 \otimes \phi)^* (\sigma \otimes \psi)(1 \otimes \phi) = \sigma \otimes \phi^* \psi \phi$, and $\sigma \otimes$ preserves isotropic subspaces. Q.E.D.

Now we discuss another interpretation of (K, σ). The map $H_q(\tilde{R}, \tilde{Q}) \to H_q(\tilde{P} - \xi^{-1}\tilde{Q}, \tilde{Q})$ induced by inclusion and the map $H_q(\tilde{P} - \tilde{Q}) \to H_q(\tilde{P} - \xi^{-1}\tilde{Q}, \tilde{Q})$ induced by ξ^{-1} agree on $H_q(\tilde{R})$ hence yield a map $K \to H_q(\tilde{P} - \xi^{-1}\tilde{Q}, \tilde{Q})$. The following diagram commutes and has exact rows.

$$\begin{array}{ccccccccc}
0 \to H_{q+1}(\tilde{P}) & \to & H_{q+1}(\tilde{P}, \tilde{P} - \tilde{Q}) & \longrightarrow & K & \longrightarrow & H_q(\tilde{P}, \tilde{Q}) \to 0 \\
\downarrow {\scriptstyle 1 \text{ or } \xi^{-1}} & & \downarrow {\scriptstyle \xi^{-1}} & & \downarrow & & \downarrow {\scriptstyle 1} \\
0 \to H_{q+1}(\tilde{P}) & \to & H_{q+1}(\tilde{P}, \tilde{P} - \xi^{-1}\tilde{Q}) & \to & H_q(\tilde{P} - \xi^{-1}\tilde{Q}, \tilde{Q}) & \to & H_q(\tilde{P}, \tilde{Q}) \to 0
\end{array}$$

An application of the five lemma tells us that $K \cong H_q(\tilde{P} - \xi^{-1}\tilde{Q}, \tilde{Q})$.

On the one hand there exists an intersection-pairing between

$$H_q(\tilde{P} - \xi^{-1}\tilde{Q}, \tilde{Q}) = H_q(\tilde{P} - \xi^{-1}\tilde{Q} - \tilde{Q}, \partial \tilde{Q})$$

and

$$H_q(\tilde{P} - \tilde{Q}, \xi^{-1}\tilde{Q}) = H_q(\tilde{P} - \xi^{-1}\tilde{Q} - \tilde{Q}, \partial \xi^{-1}\tilde{Q}).$$

On the other hand there exists a map

$$H_q(\tilde{P} - \xi^{-1}\tilde{Q}, \tilde{Q}) \to H_q(\tilde{P} - \tilde{Q}, \xi^{-1}\tilde{Q})$$

which maps the chain c to the chain $\xi c - \chi([0, 1] \times \partial c)$, since $\chi([0, 1] \times \tilde{Q}) \subset \subset \tilde{P} - \tilde{Q}$. Together this defines a form on $H_q(\tilde{P} - \xi^{-1}\tilde{Q}, \tilde{Q})$. If we pull this form back to $H_{q+1}(\tilde{P} - \tilde{S}, \tilde{P} - \tilde{Q}) \oplus H_q(\tilde{R}, \tilde{Q})$ then we get precisely the formula at the end of § 6. Hence the above construction describes σ.

We conclude this paper with an application of the foregoing theory.

EXAMPLE: We take $H = \{1, t\}$ and $w_H(t) = -1$.
The map $L_{2k}(B) \to L_{2k+2q}(B[H])$ induced by multiplication with real projective $2q$ space is precisely tensoring with t^q.

PROOF: This P has a cellular triangulation with one cell in each dimension $\leqslant 2q$; hence C_i is a free $Z[H]$ module of rank one for $0 \leqslant i \leqslant 2q$; thus E_i is free of rank two if $0 \leqslant i \leqslant 2q-1$ resp. one if $i = -1$ or $2q$. So we deduce inductively from the existence of an exact sequence $0 \to K_{i+1} \to \to E_i \to K_i \to 0$ that K_i is stably free of rank one for $1 \leqslant i \leqslant 2q$, hence free; $K_0 = E_{-1}$ is rank one since $K_{-1} \subset E_{-2} = 0$. But any almost $(-1)^q$ symmetric form σ on a free module on one generator e is of the form $\sigma(e, e) = \pm 1$ resp. $\pm t$ if q is even resp. odd, where t is the generator of H. Both choices of the sign are equivalent since $\sigma(te, te) = t\sigma(e, e)\bar{t} = -\sigma(e, e)$. Q.E.D.

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