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The Arf Invariants of Brieskorn–Pham Singularities

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1. INTRODUCTION

The aim of this paper is to determine the Arf invariant of the quadratic form associated to a Brieskorn–Pham singularity, as described in Corollary 4 of [1]. By quadratic form we mean a triple (V, B, Q), where

V is a vectorspace of finite dimension over the field F of two elements,

B: $V \times V \to F$ is a bilinear map, and

Q: $V \to F$ is a function such that $Q(x+y) = Q(x) + Q(y) + B(x, y)$ for all $x, y \in V$.

In particular $B(x, x) = 0$ and $B(y, x) = B(x, y)$ for all $x, y \in V$.

Two such triples (V, B, Q) and (V', B', Q') are called isometric, denoted by $=$, if there exists a linear isomorphism A: $V \to V'$ such that $Q'(Ax) = Q(x)$ and so $B'(Ax, Ay) = B(x, y)$ for all $x, y \in V$.

From two such triples (V_1, B_1, Q_1) and (V_2, B_2, Q_2) a third one (V_3, B_3, Q_3) can be constructed by taking $V_3 = V_1 \times V_2$, $B_3((x_1, x_2), (y_1, y_2)) = B_1(x_1, y_1) + B_2(x_2, y_2)$, $Q_3((x_1, x_2)) = Q_1(x_1) + Q_2(x_2)$. This construction is called orthogonal sum and denoted by $\perp$.

Any quadratic form (V, B, Q) is isometric to a repeated orthogonal sum of the following four types:

(a) V_a has basis $\{e_a, f_a\}$ and $B_a(e_a, f_a) = 1$ and $Q_a(e_a) = Q_a(f_a) = 0$,

(b) V_b has basis $\{e_b, f_b\}$ and $B_b(e_b, f_b) = 1$ and $Q_b(e_b) = Q_b(f_b) = 1$,

(c) V_c has basis $\{e_c\}$ and $Q_c(e_c) = 0$,

(d) V_d has basis $\{e_d\}$ and $Q_d(e_d) = 1$.

The summands of types (c) and (d) together form the kernel of (V, B, Q): the set of $x \in V$ such that $B(x, y) = 0$ for all $y \in V$; its dimension is called the corank of (V, B, Q).

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If the kernel contains an element x such that $Q(x)=1$, i.e., if type (d) occurs as summand then we say that the quadratic form has Arf invariant undefined, denoted by ∞. Otherwise the Arf invariant is the number modulo 2 of summands of type (b).

The Arf invariant is indeed an invariant of isometry since it is 0, 1 or ∞ according as $\tau(V, B, Q)=\sum_{x\in V}(-1)^{Q(x)}$ is positive, negative, or zero. It also determines a quadratic form of given dimension and corank up to isometry since:

$$(V_d, B_d, Q_d)\perp(V_d, B_d, Q_d)=(V_d, B_d, Q_d)\perp(V_c, B_c, Q_c),$$

$$(V_b, B_b, Q_b)\perp(V_b, B_b, Q_b)=(V_a, B_a, Q_a)\perp(V_a, B_a, Q_a),$$

$$(V_b, B_b, Q_b)\perp(V_d, B_d, Q_d)=(V_a, B_a, Q_a)\perp(V_d, B_d, Q_d).$$

Given natural numbers l, $a_1, a_2,..., a_l$ the Brieskorn–Pham singularity $f\colon \mathbb{C}^l\to\mathbb{C}$ is defined by $f(z_1, z_2,..., z_l)=\sum_{k=1}^{l}(z_k)^{a_k}$. According to [1] the following quadratic form (V, B, Q) is associated to this singularity: V has a basis of elements e_I associated to sequences $I\in\mathbb{Z}^l$ such that $0<I(k)<a_k$ for $k=1, 2,..., l$;

$$B(e_I, e_J)=1 \quad \text{for } I\neq J \quad \text{if } I-J\in\{0,1\}^l \text{ or } J-I\in\{0,1\}^l,$$

$$B(e_I, e_J)=0 \quad \text{otherwise;}$$

$$Q(e_I)=1 \quad \text{for all } I.$$

The aim of this paper is to determine the Arf invariant of this form in terms of the data $a_1, a_2,..., a_l$.

To describe the result we write $a_k=\alpha_k p_k$ where p_k is odd and α_k is a power of two. We write Ω for the map $\{p\in\mathbb{Z}\mid p \text{ is odd}\}\to\mathbb{Z}/2$ such that

$$\Omega(p)=0 \quad \text{if } p\equiv -1 \text{ or } p\equiv +1 \text{ modulo } 8,$$

$$\Omega(p)=1 \quad \text{if } p\equiv -3 \text{ or } p\equiv +3 \text{ modulo } 8.$$

We write g.c.d. for greatest common divisor.

THEOREM 2. *If* $\alpha_k=1$ *for some* k *then the Arf invariant is* $\Omega(d)+\sum_{k=1}^{l}c_k\Omega(\text{g.c.d.}(d, p_k))$, *where* $d=\text{g.c.d.}\{p_k\mid\alpha_k=1\}$ *and where* $c_k=1$ *means that* $\alpha_k>1$ *and that* $\operatorname{card}\{i\leqslant l\mid\alpha_i=A\}$ *is even for* $A>\alpha_k$.

THEOREM 3. *If* $\alpha_k\geqslant 2$ *for all* k *then the Arf invariant is*

∞ *if the number of* k *for which* $\alpha_k=A$ *is even for* $A\geqslant 4$;

1 *if the number of* k *for which* $\alpha_k=A$ *is even for* $A\geqslant 8$, *and is* 1 *modulo* 4 *for* $A=4$;

0 *otherwise.*

The proofs of the above theorems depend on the abelian case of the following theorem, which has independent interest.

THEOREM 1. *Let G be a group of odd order d. We define $g_1, g_2 \in G$ to be equivalent if g_2 is conjugate to g_1^n for n equal to some power of 2. Then the number of equivalence classes is equal to $1+\Omega(d)$ modulo 2.*

I would like to thank W. Janssen for suggesting the problem to me.

2. QUADRATIC FORMS OVER GROUP ALGEBRAS

We give another description of the explicit quadratic form of Section 1 which exploits its symmetry. To this end we consider quadratic forms in the sense of [2] over a general ring R, equipped with a map $^{-}: R \to R$ such that

$$\overline{r_1+r_2}=\overline{r_1}+\overline{r_2}, \qquad \overline{r_1 r_2}=\overline{r_2}\,\overline{r_1}, \qquad \text{and} \qquad \bar{\bar{r}}=r \text{ for } r, r_1, r_2 \in R.$$

Such a quadratic form is a triple (M, b, q), where M is a left R-module, b is a map $M \times M \to R$, q is a map $M \to R$ modulo $\{r+\bar{r} \mid r \in R\}$ such that

$$\begin{aligned} b(x, y_1+y_2)&=b(x, y_1)+b(x, y_2), \\ b(x_1+x_2, y)&=b(x_1, y)+b(x_2, y), \\ b(r_1 x, r_2 y)&=r_2 b(x, y)\overline{r_1}, \\ b(y, x)&=\overline{b(x, y)}, \\ q(x+y)&=q(x)+q(y)+b(x, y), \\ q(rx)&=rq(x)\bar{r}, \\ q(x)+\overline{q(x)}&=b(x, x). \end{aligned}$$

for $x, x_1, x_2, y, y_1, y_2 \in M$ and $r, r_1, r_2 \in R$. For $R=F$ this reduces to the notion of quadratic form defined in Section 1; the notions isometry and orthogonal sum are defined as there.

Suppose that (M, b, q) is such a quadratic form over R, and that $S \subseteq R$ is a subring invariant under $^{-}$ equipped with a map $E: R \to S$ such that

$$E(r_1+r_2)=E(r_1)+E(r_2), \qquad E(s_1 r s_2)=s_1 E(r) s_2, \qquad \overline{E(r)}=E(\bar{r})$$

for $r_1, r_2, r \in R$ and $s_1, s_2 \in S$. Then we consider M as an S-module, and composing b and q with E then gives a quadratic form over S.

In particular this applies if R is the group algebra FG over F of some finite group G: the set of formal F-linear combinations of elements of G, with multiplication defined by linear extension of the product in G, and with involution $^{-}$ defined by linear extension of the inversion in G. If H is a subgroup of G then $E: FG \to FH$ can be defined by $E(g) = g$ for $g \in H$ and $E(g) = 0$ for $g \in G - H$.

This we will apply to two situations, in preparation for the proofs of the Theorems 1 and 2.

Case 1. Let G be a group of odd order d, and $H = \{1\}$. Let $\xi \in FG$ be the sum of all $g \in G$ for which $g \neq 1$; then there exists $\eta \in FG$ such that $\eta + \bar{\eta} = \xi$ and $E(\eta) = (d-1)/2$: this η has a summand $g + 1$ for each pair $\{g, g^{-1}\}$. Let M be the free FG module on one generator e and define

$$b(xe, ye) = y\xi\bar{x}, \qquad q(xe) = x\eta\bar{x} \quad \text{for } x, y \in FG.$$

The application of $E: FG \to F$ now yields a quadratic form (M, B, Q) as in Section 1, where M has a basis consisting of the $e_g = ge$ for $g \in G$,

$$B(e_g, e_h) = E(b(ge, he)) = 0 \text{ if } g = h \text{ and } 1 \text{ if } g \neq h,$$

$$Q(e_g) = E(q(ge)) = \frac{d-1}{2} \qquad \text{for all } g.$$

Since (M, B, Q) only depends on d, its Arf invariant is a function of d; in [3, Proposition 2.6] it is checked that it is in fact $\Omega(d)$.

Case 2. Let G be the product, for $k = 1$ to l, of cyclic groups with generators T_k of order a_k. Take $\eta = \prod_{k=1}^{l} (1 + T_k) \in FG$ and $\xi = \eta + \bar{\eta} = \eta(1 + \prod_{k=1}^{l} T_k^{-1})$. Let again M be the free module on one generator e and define $b(xe, ye) = y\xi\bar{x}$, $q(xe) = x\eta\bar{x}$ for $x, y \in FG$. Then M has an F-basis consisting of elements $e_I = (\prod_{k=1}^{l} t(k, I(k)))e$ associated to sequences $I \in \mathbb{Z}^l$ such that $0 \leqslant I(k) < a_k$ for $k \leqslant l$. Here $t(k, i)$ means T_k^i if $i \neq 0$ and $\sum_{j=0}^{a_k - 1} T_k^j$ if $i = 0$. The application of E yields

$B(e_I, e_J) = 0$	if I or J contains a zero,
$B(e_I, e_J)$	is as in Section 1 otherwise;
$Q(e_I) = 0$	if I contains a zero and 1 otherwise.

Therefore up to some summands of type (c) which do not affect the Arf invariant we get exactly the quadratic form of [1].

3. Quadratic Forms over Finite Fields

In the remaining sections we are going to analyse the forms over FG described at the end of Section 2 by dissecting FG into finite fields.

Therefore it is useful to list a few properties of quadratic forms over finite fields.

First, a few remarks about the invariant τ:

PROPOSITION 1. $\tau((V_1, B_1, Q_1) \perp (V_2, B_2, Q_2)) = \tau(V_1, B_1, Q_1) \tau(V_2, B_2, Q_2)$.

The proof is obvious.

PROPOSITION 2. *If* (V, B, Q) *has trivial kernel and* $\dim(V)=2n$ *then* $\tau(V, B, Q)^2 = 2^{2n}$.

Proof. From Proposition 1 checking the types (a) and (b). Or as follows:

$$\begin{aligned}\tau(V, B, Q)^2 &= \sum_{x\in V}\sum_{y\in V}(-1)^{Q(x)+Q(y)} = \sum_{x\in V}\sum_{z\in V}(-1)^{Q(x)+Q(x+z)}\\ &= \sum_{z\in V}\sum_{x\in V}(-1)^{Q(z)+B(x,z)}\\ &= \sum_{z\in V}(-1)^{Q(z)}[2^{2n} \text{ if } B(\ ,z)=0 \text{ and } 0 \text{ otherwise}].\end{aligned}$$ ▮

COROLLARY. *The cardinality of* $\{x\in V \mid Q(x)=0\}$ *is in the above case* $2^{2n-1} \pm 2^{n-1}$ *depending on the Arf invariant.*

PROPOSITION 3. *Suppose that* $V = U \oplus W$ *and that* Q *vanishes on* U *and on* W *and therefore* B *vanishes on* $U\times U$ *and* $W\times W$. *Then* $\mathrm{Arf}(V, B, Q)=0$.

Proof. $\tau(V, B, Q) = \sum_{u\in U}\sum_{w\in W}(-1)^{Q(u+w)} = \sum_{u\in U}\sum_{w\in W}(-1)^{B(u,w)}$. Here the inner sum equals the cardinality of W if $B(u,\ )=0$ and vanishes otherwise. Therefore the sum is positive. ▮

Now a few remarks about finite fields. As is well known the group of automorphisms of a finite field K of cardinality 2^v is cyclic of order v generated by $\sigma\colon x\to x^2$. So the only possibility for an involution $^-$ is the identity, and if $v=2w$ also the map $\rho=\sigma^w$.

PROPOSITION 4. *If* $\rho(x)=x\in K$ *then* $x=y+\rho(y)$ *for some* $y\in K$, *and also* $x=z\rho(z)$ *for some* $z\in K$. *If* $\rho(x)=x^{-1}\in K$ *then* $x=z^{-1}\rho(z)$ *for some* $z\in K$.

Proof. The equation $y+\rho(y)=0$ is a polynomial equation of degree 2^w so the kernel of the map $x\to x+\rho(x)$ has at most 2^w elements, and the image of the map $y\to y+\rho(y)$ has at least $2^v/2^w=2^w$ elements. But that image is contained in that kernel; so both must coincide. The other statements are proven similarly. ▮

The trace map $K \to F$ is nonzero since $\mathrm{Tr}(x) = x + \sigma(x) + \cdots + \sigma^{v-1}(x)$ is of degree 2^{v-1} in x so vanishes in at most 2^{v-1} elements. We use this map to define a quadratic form on the F-vectorspace K for $v = 2w$ by $B(x, y) = \mathrm{Tr}(\rho(x)y)$, $Q(x) = \mathrm{Tr}(\rho(x)\, x\chi)$ where $\chi \in K$ is chosen such that $\chi + \rho(\chi) = 1$.

PROPOSITION 5. *This form (K, B, Q) has Arf invariant* 1.

Proof. Choose $\omega \in K$ such that $\mathrm{Tr}(\omega) = 1$. Then for any $x \in K - \{0\}$ the element $y = \omega\rho(x)^{-1}$ satisfies $B(x, y) = 1$; thereore we are in the situation of Proposition 2. Any $x \in K$ for which $Q(x) = 0$ satisfies $x\rho(x) + \sigma(x\rho(x)) + \cdots + \sigma^{w-1}(x\rho(x)) = 0$ which is after elimination of a factor $x\rho(x)$ a polynomial equation of degree $(2^w + 1)(2^{w-1} - 1)$ in x. So there are at most $1 + (2^{w-1} - 1)(2^w + 1) = 2^{2w-1} - 2^{w-1}$ such x. Comparing this with the corollary to Proposition 2 we get the desired result. ∎

PROPOSITION 6. *Suppose (M, b, q) is a quadratic form over K with involution ρ. Consider it as a form over F using $E = \mathrm{Tr}$. Then its Arf invariant is the K-rank of b.*

Proof. Using Proposition 4 one can choose a K-basis $\{e_1, e_2, \ldots, e_n\}$ of M such that $b(e_i, e_j) = 0$ for $i \neq j$ and such that for some r:

$$q(e_i) = \chi \qquad \text{for} \quad i \leqslant r$$

and

$$q(e_i) = 0 \qquad \text{for} \quad i > r,$$

both modulo $\{z + \rho(z) \mid z \in K\}$. Then the associated quadratic form over F consists of r copies of the situation of Proposition 5 and further has summands of type (c). Therefore the Arf invariant is r modulo 2. ∎

4. DECOMPOSITION OF *FG* AND PROOF OF THEOREM 1

Let G be a group of odd order d. We refer to Section 2 of [4] as the most concise reference for the following facts about the ring $R = FG$. It is a semisimple ring with minimum condition and therefore a direct sum of two-sided ideals R_i which are simple rings. The splitting is unique up to permutation of the summands. Each R_i is a full matrix ring $\mathrm{Mat}(K_i, n_i)$ with coefficients in a division ring K_i, in our case in a finite field. In the case of interest for Theorems 2 and 3 where G and so each R_i is commutative the dimensions n_i are all 1.

If we compute for $r \in R$ the trace of the F-linear map $x \to rx$ then we get $E(r)$; therefore $E\colon R \to F$ is determined by the ring structure. In particular on R_i it is n_i times the composition of the field trace $K_i \to F$ with the matrix trace $\mathrm{Mat}(K_i, n_i) \to K_i$.

PROPOSITION 7. *The n_i are odd.*

Proof. The F-bilinear form B on FG defined by $B(r_1, r_2) = E(r_1 r_2)$ has on basis elements $g, h \in G$ the value $B(g, h) = 1$ if $g = h^{-1}$ and 0 otherwise. Therefore it has zero kernel. On the other hand the R_i are orthogonal for this B, and if any n_i were even then E would vanish on R_i and so would B on $R_i \times R_i$. ∎

As ring with involution R can be decomposed into ideals R_i invariant under the involution and parts of the form $R_i \oplus R_j$ where the involution interchanges R_i and R_j. The second "hyperbolic" type of summand is isomorphic to $R_i \times R_i^{\mathrm{op}}$ with involution $(x, y) \to (y, x)$. Here op means opposite multiplication. The isomorphism is given by $(x, y) \to x \oplus \bar{y}$. On the center K_i of an invariant summand $R_i = \mathrm{Mat}(K_i, n_i)$ the involution induces the identity or ρ. We write $\dagger$ for the composition of matrix transpose with 1 or ρ accordingly. Then the composition of the given involution $^-$ with the constructed involution $\dagger$ is a K_i-linear automorphism of $\mathrm{Mat}(K_i, n_i)$ and thus of the form $X \to U_i X U_i^{-1}$ for some invertible matrix U_i. Then $\bar{X} = U_i X^\dagger U_i^{-1}$ for all X. From $\bar{\bar{X}} = X$ it follows that one can take $U_i^\dagger = U_i$.

PROPOSITION 8. *The involution induces the identity on the center K_i of $R_i = \mathrm{Mat}(K_i, n_i)$ only in the trivial case: for $K_i = F$ and $n_i = 1$.*

Proof. We compute the F-vectorspace $\{x \in R \mid \bar{x} = x\}$ modulo $\{y + \bar{y} \mid y \in R\}$ on both sides of the identity $FG = \bigoplus_i R_i$.

On FG it is of dimension 1 generated by $1 \in G$. From summands $R_i \times R_i^{\mathrm{op}}$ or summands $\mathrm{Mat}(K_i, n_i)$ with $^- = \rho$ on K_i there is no contribution. However a summand $\mathrm{Mat}(K_i, n_i)$ with $^- = 1$ on K_i yields as contribution: the space of $n_i \times n_i$ diagonal matrices multiplied with U_i. ∎

In view of the above result it is better to decompose FG first as $R_0 \oplus R_\infty$ where R_0 is the invariant two sided ideal generated by $\sum_{g \in G} g$, so that $R_0 = F$, and where R_∞ is the invariant ideal generated by the $g + 1 \in FG$ for $g \in G$. One can decompose R_∞ further as $\bigoplus_{i=1}^{N} R_i$ as before; the exception mentioned in Proposition 8 does not occur any more.

Now we can prove the version of Theorem 1 needed for Theorem 2.

PROPOSITION 9. *The Arf invariant of the quadratic form in Case 1 of Section 2 is the number N of nontrivial summands R_i of FG.*

Proof. In R_0 the image of η is 0 and therefore $R_0 e$ is a summand of type (c) which does not contribute to the Arf invariant. If $R_i \oplus R_j$ is a hyperbolic summand of R_∞ then it contributes 2 to N and zero to the Arf invariant, according to Proposition 3 with

$$V = (R_i \oplus R_j)e, \qquad U = R_i e, \qquad W = R_j e.$$

Proposition 8 tells us that we can apply Proposition 6 to the other summands $R_i = \mathrm{Mat}(K_i, n_i)$. Since ξ maps to 1 in R_∞ the kernel of B vanishes and the rank is n_i^2. So according to Proposition 7 one gets each time a contribution 1 to the Arf invariant. ▮

Proof of Theorem 1. For each equivalence class we consider the element in FG which is the sum of all $g \in G$ in that class. Together these elements constitute a basis of the F-vectorspace of central idempotents of FG. On the other hand every summand $R_i = \mathrm{Mat}(K_i, n_i)$ has centre K_i in which the set of idempotents is $\{0, 1\} = F$. So that vectorspace has dimension $N + 1$. ▮

5. More about Counting Summands

In this section we study the product G for $k = 1$ to l of cyclic groups C_k of odd order p_k generated by X_k. We write FG for its group algebra over F. Similar to the splitting of R as $R_0 \oplus R_\infty$ in Section 4 there is a splitting of FG into the ideal generated by $1 + X_k$ and the ideal generated by $\sum_{j=1}^{p_k} (X_k)^j$; the latter is isomorphic to the group algebra of G/C_k over F. We can therefore distinguish between the summands K_i of FG contained in the first ideal, in which X_k does not map to 1, and summands contained in the second ideal, where it does. The latter summands are just the summands of $F(G/C_k)$. A similar argument works if we look at more then one X_k.

Now we count these different kinds of summands:

Proposition 10. *Let $I, J \subseteq \{1, 2, ..., l\}$ be disjoint. Then the cardinality modulo* 2 *of the collection $N(I, J)$ of summands $K_i \subseteq FG$ in which X_k becomes* 1 *for $k \in I$ and X_k becomes $\neq 1$ for $k \in J$ is*

$$1 + \sum_{k \notin I} \Omega(p_k) \quad \text{if } J = \varnothing,$$

$$\Omega(p_k) \quad \text{if } J = \{k\},$$

$$0 \quad \text{if } J \text{ is larger.}$$

Proof. There is a bijection between $N(I, \varnothing)$ and the collection of field summands that occur in the group algebra of $\prod_{k \notin I} C_k$ over F. According to Proposition 9 the number of those is $1 + \Omega(\prod_{k \notin I} p_k)$ modulo 2. The second statement follows from this by writing $N(I, \varnothing)$ as the disjoint union of $N(I \cup \{k\}, \varnothing)$ and $N(I, \{k\})$.

The third statement follows starting from the second one by induction on the cardinality of J, by writing $N(I, J)$ as the disjoint union of $N(I \cup \{k\}, J)$ and $N(I, J \cup \{k\})$. ∎

PROPOSITION 11. *For any $J \subseteq \{1, 2, ..., l\}$ let $P(J)$ be the collection of field summands $K_i \subseteq FG$ in which X_k becomes 1 for $k \notin J$, X_k becomes $\neq 1$ for $k \in J$, and $\prod_{k=1}^{l} X_k$ becomes 1. Then the cardinality modulo 2 of $P(J)$ is:*

$$1 \quad \text{if } J = \emptyset,$$

$$0 \quad \text{if } J \text{ has cardinality one},$$

$$\Omega(\text{g.c.d.}\{p_k \mid k \in J\}) \quad \text{if } J \text{ is larger.}$$

Proof. We prove the third statement since the others are obvious. We write the collection of summands in which $X_k = 1$ for $k \notin J$ and $\prod_{k=1}^{l} X_k = 1$ as the disjoint union of the $P(I)$ with $I \subseteq J$. According to the induction hypothesis the $P(I)$ with $I \neq J$ yield a contribution $1 + \sum\{\Omega(\text{g.c.d.}\{p_k \mid k \in I\}) \mid I \subset J, \text{card}(I) \geqslant 2\}$ to the number modulo two of summands.

On the other hand these summands are precisely those that occur as summands in the group algebra of $(\prod_{k \in J} C_k)$/(subgroup generated by $\prod_{k \in J} X_k$). That is a group of order $(\prod_{k \in J} p_k)/\text{l.c.m.}\{p_k \mid k \in J\}$, where l.c.m. denotes least common multiple. The l.c.m. can be rewritten as $\prod_{\emptyset \neq I \subseteq J}(\text{g.c.d.}\{p_k \mid k \in I\})^{\varepsilon(I)}$, where $\varepsilon(I)$ is $+1$ or -1 according as I has odd or even cardinality. So the number modulo 2 of summands is $1 + \sum_{k \in J} \Omega(p_k) - \sum_{\emptyset \neq I \subseteq J} \Omega(\text{g.c.d.}\{p_k \mid k \in I\})$ according to Proposition 9. ∎

PROPOSITION 12. *For any $J \subseteq \{1, 2, ..., l\}$ the number modulo 2 of summands in which*

$$X_k = 1 \quad \text{for } k \notin J, \qquad X_k \neq 1 \quad \text{for } k \in J, \qquad \text{and} \qquad \prod_{k=1}^{l} X_k \neq 1$$

is

$$0 \quad \text{if } J = \emptyset, \qquad \text{and} \qquad \Omega(\text{g.c.d.}\{p_k \mid k \in J\}) \quad \text{otherwise.}$$

Proof. Subtract the result of Proposition 11 from the result of Proposition 10 for the case that I is the complement of J. ∎

6. THE COMPUTATION OF SOME RANKS

Since we are going to base the proof of Theorem 2 on Proposition 6 we have to compute some ranks.

Let K be a field of characteristic 2. If D is a commutative K-algebra and

$x \in D$ then we write $r(D, x)$ for the K-rank of the multiplication with $x: D \to D$.

PROPOSITION 13. *Let $\beta, \gamma \in \mathbb{N}$; let $U, V \in D$ be such that $V^\gamma = 1$. Write DC for the group algebra over D of the cyclic group C of order $\beta\gamma$ generated by S, and write $W = U(1+S)(1+VS^{-\beta}) \in DC$. Then $r(DC, W) = (\beta\gamma - \beta - 1)\, r(D, U) + r(D, U \sum_{t=0}^{\gamma-1} V^t)$.*

Proof. We may assume that $\gamma \geqslant 2$. Start with the D bases $\{e_i\}$ and $\{f_i\}$ of DC defined by $e_i = f_i = S^{i-1}$ for $1 \leqslant i \leqslant \gamma\beta$. Then

$$\begin{aligned} We_i &= U(f_i + f_{i+1} + Vf_{\gamma\beta-\beta+i} + Vf_{\gamma\beta-\beta+i+1}) && \text{if} \quad 1 \leqslant i \leqslant \beta - 1,\\ We_\beta &= U(f_\beta + f_{\beta+1} + Vf_{\gamma\beta} + Vf_1),\\ We_i &= U(f_i + f_{i+1} + Vf_{i-\beta} + Vf_{i-\beta+1}) && \text{if} \quad \beta + 1 \leqslant i \leqslant \gamma\beta - 1,\\ We_{\gamma\beta} &= U(f_{\gamma\beta} + f_1 + Vf_{\gamma\beta-\beta} + Vf_{\gamma\beta-\beta+1}). \end{aligned}$$

We define new bases $\{e'_i\}$ and $\{f'_i\}$ by

$$\begin{aligned} e'_{(j-1)\beta+i} &= \sum_{t=1}^{j} V^{j-t} e_{(t-1)\beta+i} && \text{if } 1 \leqslant i \leqslant \beta,\ 1 \leqslant j \leqslant \gamma,\\ f'_{(j-1)\beta+i} &= f_{(j-1)\beta+i} + V^j f_{(\gamma-1)\beta+i} && \text{if } 1 \leqslant i \leqslant \beta,\ 1 \leqslant j \leqslant \gamma - 1,\\ f'_{(\gamma-1)\beta+i} &= f_{(\gamma-1)\beta+i} && \text{if } 1 \leqslant i \leqslant \beta; \end{aligned}$$

then

$$\begin{aligned} We'_{(j-1)\beta+i} &= U(f'_{(j-1)\beta+i} + f'_{(j-1)\beta+i+1}) && \text{if } 1 \leqslant i \leqslant \beta - 1,\ 1 \leqslant j \leqslant \gamma - 1,\\ We'_{j\beta} &= U(f'_{j\beta} + f'_{j\beta+1} + V^j f'_1) && \text{if } 1 \leqslant j \leqslant \gamma - 2,\\ We'_{(\gamma-1)\beta} &= U(f'_{(\gamma-1)\beta} + V^{\gamma-1} f'_1),\\ We'_{(\gamma-1)\beta+i} &= 0 && \text{if } 1 \leqslant i \leqslant \beta. \end{aligned}$$

We again define new bases $\{\tilde{e}_i\}$ and $\{\tilde{f}_i\}$ by

$$\begin{aligned} \tilde{e}_i &= \sum_{t=i}^{\gamma\beta} e'_t \qquad \text{if} \quad 1 \leqslant i \leqslant \gamma\beta,\\ \tilde{f}_1 &= f'_1,\\ \tilde{f}_{(j-1)\beta+i} &= f'_{(j-1)\beta+i} + \left(\sum_{t=j}^{\gamma-1} V^t\right) f'_1 \\ &\qquad \text{if} \quad 1 \leqslant j \leqslant \gamma,\ 1 \leqslant i \leqslant \beta,\ (j-1)\beta + i \neq 1; \end{aligned}$$

then

$$W\tilde{e}_1=\left(U\sum_{t=0}^{\gamma-1}V^t\right)\tilde{f}_1,$$

$$W\tilde{e}_i=U\tilde{f}_i \quad \text{if} \quad 2\leqslant i\leqslant(\gamma-1)\beta,$$

$$W\tilde{e}_i=0 \quad \text{if} \quad (\gamma-1)\beta+1\leqslant i\leqslant\gamma\beta. \quad ∎$$

PROPOSITION 14. *With the same assumptions as in Proposition* 13 *write* $Y=U(1+S)(\sum_{t=0}^{\gamma-1}(VS^{-\beta})^t)\in DC$. *Then* $r(DC, Y)=(\beta-1)\,r(D, U)+r(D, U(1+V))$.

Proof. We start again with bases $\{e_i\}$ and $\{f_i\}$ defined by $e_i=f_i=S^{i-1}$ if $1\leqslant i\leqslant\gamma\beta$ and we define new bases $\{e_i'\}$ and $\{f_i'\}$ by

$$e_i'=e_i \quad \text{if} \quad 1\leqslant i\leqslant\beta,$$

$$e_i'=e_i-Ve_{i-\beta} \quad \text{if} \quad \beta+1\leqslant i\leqslant\gamma\beta,$$

$$f_i'=\sum_{t=1}^{\gamma}V^tf_{(\gamma-t)\beta+i} \quad \text{if} \quad 1\leqslant i\leqslant\beta,$$

$$f_i'=f_i \quad \text{if} \quad \beta+1\leqslant i\leqslant\gamma\beta;$$

then

$$Ye_i'=U(f_i'+f_{i+1}') \quad \text{if} \quad 1\leqslant i\leqslant\beta-1,$$

$$Ye_\beta'=U(f_\beta'+Vf_1'),$$

$$Ye_i'=0 \quad \text{if} \quad \beta+1\leqslant i\leqslant\gamma\beta.$$

We again define new bases $\{\tilde{e}_i\}$ and $\{\tilde{f}_i\}$ by

$$\tilde{e}_i=\sum_{t=i}^{\gamma\beta}e_t' \quad \text{if} \quad 1\leqslant i\leqslant\gamma\beta,$$

$$\tilde{f}_1=f_1',$$

$$\tilde{f}_i=f_i'+Vf_1' \quad \text{if} \quad 2\leqslant i\leqslant\gamma\beta;$$

then

$$Y\tilde{e}_1=U(1+V)\tilde{f}_1,$$

$$Y\tilde{e}_i=U\tilde{f}_i \quad \text{if} \quad 2\leqslant i\leqslant\beta,$$

$$Y\tilde{e}_i=0 \quad \text{if} \quad \beta+1\leqslant i\leqslant\gamma\beta. \quad ∎$$

PROPOSITION 15. *Let D be as before and $U \in D$. Let $\alpha_k > 1$ be a power of two for $1 \leqslant k \leqslant p$. Let G be the product for $k = 1$ to p of cyclic groups of orders α_k generated by S_k. Write $Z = U \prod_{k=1}^{p} (1 + S_k) \in DG$. Then $r(DG, Z) \equiv r(D, U)$ modulo 2.*

Proof. We apply induction on p. If $p = 0$ then $DG = D$ and $Z = U$. Otherwise we write H for the product of the cyclic groups for $k = 1$ to $p - 1$; then DG is the group algebra over DH of the cyclic group of order α_p generated by S_p. The application of Proposition 14 with $\beta = \alpha_p$ and $\gamma = 1$ yields

$$r(DG, Z) = (\alpha_p - 1)\, r\left(DH, U \prod_{k=1}^{p-1} (1 + S_k) \right)$$

which is equivalent to $r(D, U)$ modulo 2 by induction hypothesis. ∎

PROPOSITION 16. *Let $\alpha_k > \theta > 1$ be powers of two for $1 \leqslant k \leqslant p$. Let G be the product for $k = 1$ to p of cyclic groups of order α_k generated by S_k. Write*

$$Z = \left(1 + \prod_{k=1}^{p} S_k^{-\theta} \right) \prod_{k=1}^{p} (1 + S_k) \in KG.$$

Then $r(KG, Z)$ is even iff the number of k for which $\alpha_k = A$ is even for all A.

Proof. We apply induction on p. If $p = 0$ then $G = \{1\}$, $Z = 0$, $r(KG, Z) = 0$ and the above mentioned number is also zero. Otherwise we may assume that α_k is a nondecreasing function of k. We write H for the product of the first $p - 1$ cyclic groups. We apply Proposition 13 with

$$D = KH, \quad \beta = \theta, \quad \gamma = \theta^{-1} \alpha_p, \quad S = S_p, \quad U = \prod_{k=1}^{p-1} (1 + S_k), \quad V = \prod_{k=1}^{p-1} S_k^{-\theta}.$$

Then the term $(\beta\gamma - \beta - 1)\, r(D, U)$ is odd according to Proposition 15.

Suppose that $p = 1$ or $\alpha_p > \alpha_{p-1}$. Then there is only one k for which $\alpha_k = \alpha_p$. On the other hand the term $r(D, U \sum_{t=0}^{\gamma-1} V^t)$ vanishes since $V^{\gamma/2} = 1$, hence $r(KG, Z)$ is odd.

Now suppose that $\alpha_p = \alpha_{p-1}$. Then we must compute $r(KH, \prod_{k=1}^{p-1} (1 + S_k) \sum_{t=0}^{\gamma-1} (\prod_{k=1}^{p-1} S_k^{-\theta})^t)$. To do that we write N for the product of the first $p - 2$ cyclic groups and apply Proposition 14 with

$$D = KN, \quad \beta = \theta, \quad \gamma = \theta^{-1} \alpha_{p-1} = \theta^{-1} \alpha_p, \quad S = S_{p-1},$$

$$U = \prod_{k=1}^{p-2} (1 + S_k), \quad V = \prod_{k=1}^{p-2} S_k^{-\theta}.$$

Then the term $(\beta-1)\,r(D, U)$ is odd according to Proposition 15. So in this case $r(KG, Z)$ is equivalent modulo 2 to the second term

$$r(D, U(1+V)) = r\left(KN, \left(1+\prod_{k=1}^{p-2} S_k^{-\theta}\right)\prod_{k=1}^{p-2}(1+S_k)\right)$$

which by induction is even iff the number of $k \leqslant p-2$ for which $\alpha_k = A$ is even for all A. The restriction $k \leqslant p-2$ is obviously immaterial. ∎

PROPOSITION 17. *Let D be as before and $U, V \in D$. Let C be the cyclic group of order θ generated by S. Write $Z = U(1+VS^{-1})$. Then $r(DC, Z) = (\theta-1)\,r(D, U) + r(D, U(1+V^\theta))$.*

Proof. We start with the D bases $\{e_i\}$ and $\{f_i\}$ of DC defined by $e_i = f_i = S^{i-1}$ for $1 \leqslant i \leqslant \theta$. Then

$$Ze_1 = U(f_1 + Vf_\theta),$$
$$Ze_i = U(f_i + Vf_{i-1}) \qquad \text{if} \quad 2 \leqslant i \leqslant \theta.$$

We define new bases $\{e_i'\}$ and $\{f_i'\}$ by

$$e_1' = e_1 + \sum_{t=1}^{\theta-1} V^t e_{\theta-t+1},$$
$$e_i' = e_i \qquad \text{if} \quad 2 \leqslant i \leqslant \theta,$$
$$f_1' = f_1,$$
$$f_i' = f_i + Vf_{i-1} \qquad \text{if} \quad 2 \leqslant i \leqslant \theta;$$

then

$$Ze_1' = U(1+V^\theta)\,f_1',$$
$$Ze_i' = Uf_i' \qquad \text{if} \quad 2 \leqslant i \leqslant \theta. \quad ∎$$

PROPOSITION 18. *Let G be the product for $k=0$ to q of cyclic groups of orders $\alpha_k > 1$, with generators S_k. Let the α_k be powers of 2. Write $Z = (1 + S_0^{-1}\prod_{k=1}^{q} S_k^{-1})\prod_{k=1}^{q}(1+S_k) \in KG$. Then $r(KG, Z)$ is odd if* card$\{k \leqslant l \mid \alpha_k = A\}$ *is even for all $A > \alpha_0$.*

Proof. Let p be such that $\alpha_k > \alpha_0$ if $1 \leqslant k \leqslant p$ and that $\alpha_k \leqslant \alpha_0$ if $p+1 \leqslant k \leqslant q$. We write H for the group generated by the S_k with $1 \leqslant k \leqslant p$. Write $S = S_0 \prod_{k=p+1}^{q} S_k$ since S order α_0 it can be used in place of S_0 to generate G together with the other S_k. We write N for the group generated by H and S. Then $Z = Y\prod_{k=p+1}^{q}(1+S_k)$ where $Y=$

$(1+S^{-1}\prod_{k=1}^{p} S_k^{-1})\prod_{k=1}^{p}(1+S_k)\in KN$. Therefore $r(KG, Z)\equiv r(KN, Y)$ modulo 2 according to Proposition 15. Now we apply Proposition 17 with

$$D=KH, \quad \theta=\alpha_0, \qquad U=\prod_{k=1}^{p}(1+S_k), \qquad \text{and} \qquad V=\prod_{k=1}^{p} S_k^{-1}.$$

Then the first term $(\theta-1)\,r(D, U)$ is odd according to Proposition 15 and the second term $r(D, U(1+V^{\theta}))$ is dealt with in Proposition 16. ∎

7. Proof of Theorem 2

We refer to case (2) of Section 2. We write G_1 for the subgroup of G of elements of odd order and G_2 for the subgroup of elements of 2-power order. Now the canonical isomorphism $G\to G/G_2\times G/G_1=G_1\times G_2$ makes it possible to view $R=FG$ as group algebra of G_2 over $A=FG_1$. The image of $T_k\in G$ in G_1 will be denoted by X_k and the image in G_2 by S_k. So S_k is of order α_k and in particular $S_k=1$ if $\alpha_k=1$. We write Λ for the set of k for which $\alpha_k=1$. So $\Lambda\neq\varnothing$ in this section.

As in Section 4, we decompose A into summands invariant under $^-$; this induces a similar decomposition of $R=AG_2$. In particular a hyperbolic summand of A gives rise to a hyperbolic summand of R and contributes zero to the Arf invariant. Therefore we restrict our attention to the summands of A which are fields K invariant under $^-$.

The quadratic form is determined by $\eta=\prod_{k=1}^{l}(1+T_k)=\prod_{k\in\Lambda}(1+X_k)\prod_{k\notin\Lambda}(1+X_kS_k)$ hence vanishes on KG_2 if $X_k=1$ in K for some $k\in\Lambda$. Therefore we restrict our attention to fields K in which $X_k\neq 1$ for every $k\in\Lambda$. In particular $\bar{X}_k=X_k^{-1}\neq X_k$ which implies that $^-=\rho$. According to Proposition 6 the contribution of KG_2 to the Arf invariant is then equal to $r(KG_2, \xi)$ where $\xi=\eta+\bar{\eta}=(1+\prod_{k=1}^{l} X_k^{-1}\prod_{k\notin\Lambda} S_k^{-1})\prod_{k\in\Lambda}(1+X_k)\prod_{k\notin\Lambda}(1+X_kS_k)$.

A factor $1+X_kS_k$ is invertible if $X_k\neq 1$ since it has the same α_k power as $1+X_k$. For this reason we write Δ for the set of $k\in\{1, 2,..., l\}-\Lambda$ for which $X_k\neq 1$ and Γ for the set of $k\in\{1, 2,..., l\}-\Lambda$ for which $X_k=1$.

We distinguish three cases in the computation of $r(KG_2, \xi)$.

Case a. $\prod_{k=1}^{l} X_k\neq 1$. Then ξ is up to invertible factors equal to $Z=\prod_{k\in\Gamma}(1+S_k)\in KG_2$.

Write H_a for the subgroup of G_2 generated by the S_k with $k\in\Gamma$; then KG_2 can be viewed as the group algebra of G_2/H_a over KH_a; in particular KG_2 is of rank $\prod_{k\in\Delta}\alpha_k$ over KH_a; thus $r(KG_2, Z)=(\prod_{k\in\Delta}\alpha_k)\,r(KH_a, Z)$.

Therefore $r(KG_2, Z)$ is even unless $\Delta=\varnothing$, in which case it is odd according to Proposition 15. A field K contributes 1 to the Arf invariant iff

$X_k \neq 1$ for $k \in \Lambda$ and $X_k = 1$ for $k \notin \Lambda$; the number modulo 2 of such summands is Ω (g.c.d.$\{p_k \mid k \in \Lambda\}$) according to Proposition 12.

Case b. $\prod_{k=1}^{l} X_k = 1$ and $\Delta \neq \emptyset$. Then ξ is up to invertible factors equal to $Z = (1 + \prod_{k \in \Gamma \cup \Delta} S_k^{-1}) \prod_{k \in \Gamma} (1 + S_k)$.

Write H_b for the subgroup of G_2 generated by H_a and $\prod_{k \in \Delta} S_k$; then KG_2 can be viewed as the group algebra of G_2/H_b over KH_b. Then $r(KG_2, Z)$ is even unless the group G_2/H_b is trivial, which would mean that Δ would consists of only one element δ. In that case $r(KG_2, Z)$ is given by Proposition 18 which says that it is odd iff card$\{k \in \Gamma \mid \alpha_k = A\}$ is even for all $A > \alpha_\delta$. Therefore for each $\delta \notin \Lambda$ a field K can contribute to the Arf invariant only if card$\{k \leqslant l \mid k \notin \Lambda,\ k \neq \delta,\ \alpha_k = A\}$ is even for all $A > \alpha_\delta$ (which means $c_\delta = 1$ in the notation of the theorem) and if $X_k \neq 1$ for $k \in \Lambda \cup \{\delta\}$, $X_k = 1$ for $k \notin \Lambda \cup \{\delta\}$. The number modulo 2 of such fields is Ω(g.c.d.$\{p_k \mid k \in \Lambda \cup \{\delta\}\}$) according to Proposition 11.

Case c. $\prod_{k=1}^{l} X_k = 1$ and $\Delta = \emptyset$.

We have to compute $r(KG_2, Z)$ where $Z = (1 + \prod_{k \in \Gamma} S_k^{-1}) \prod_{k \in \Gamma} (1 + S_k)$.

That this rank is even can be seen by inductively applying Propositions 13 and 14 with $\beta = 1$, or by noting that this Z defines a quadratic form with coefficients in F.

8. Proof of Theorem 3

We distinguish the same three kinds of summands K as in Section 7. In cases (a) and (b) the involution $^-$ must be ρ. In case (a) the rank $r(KG_2, \xi)$ is even unless $\Delta = \emptyset$ which is in contradiction with $\prod_{k=1}^{l} X_k \neq 1$. In case (b) the rank is even unless $\Delta = \{\delta\}$ which is in contradiction with $\prod_{k=1}^{l} X_k = 1$. So in these cases there is no contribution to the Arf invariant. Therefore the only contribution comes from case (c) where $X_k = 1$ for all k and $K = F$. That is exactly the whole quadratic form when $p_k = 1$ for all k.

To apply induction on l we need two propositions which play a role similar to Propositions 13 and 14 in Section 6 for $\beta = 1$.

Proposition 19. *Let D be a commutative K algebra with involution $^-$. Let $U, V \in D$ be such that $U = \bar{U} V$, $\bar{V} = V^{-1}$, $V^\theta = 1$. Write C for the cyclic group of even order $\theta \geqslant 4$ generated by S, and C' for the cyclic group of order $\theta - 1$ generated by S'. Then the quadratic from over D defined on DC using $\eta = U(1 + S)$ is the orthogonal sum of the quadratic form defined on DC' using $\eta' = U(1 + S')$ and the quadratic form on D using $\tilde{\eta} = U \sum_{t=0}^{(\theta-2)/2} V^{2t}$.*

Proof. There is a D basis $\{e_i\}$ of DC defined by $e_i = S^{i-1}$ for $1 \leqslant i \leqslant \theta$. Let $d_i = \sum_{j=0}^{i-1} V^j$ if $1 \leqslant i \leqslant \theta - 1$; then $d_{i+1} + d_i(1+V) + d_{i-1}V = 0$ if $1 \leqslant i \leqslant \theta - 1$. We define a new basis $\{e_i'\}$ by

$$e_i' = e_i \qquad \text{if} \quad 1 \leqslant i \leqslant \theta - 2,$$

$$e_{\theta-1}' = \sum_{i=1}^{\theta} e_i,$$

$$e_\theta' = \sum_{i=1}^{\theta-1} d_i e_i.$$

This is a basis because $d_{\theta-1}$ is a unit since $(d_{\theta-1})^2 = V^{-2}$. On this basis the quadratic form (DC, b, q) is described by

$$
\begin{aligned}
b(e_i', e_j') &= \bar{U} && \text{if} \quad i+1 = j \text{ and } 1 \leqslant i, j \leqslant \theta - 2,\\
b(e_i', e_j') &= U && \text{if} \quad i-1 = j \text{ and } 1 \leqslant i, j \leqslant \theta - 2,\\
b(e_i', e_j') &= U + \bar{U} && \text{if} \quad i = j \text{ and } 1 \leqslant i, j \leqslant \theta - 2,\\
b(e_i', e_j') &= 0 && \text{otherwise if} \quad 1 \leqslant i, j \leqslant \theta - 2.\\
b(e_i', e_{\theta-1}') &= 0 && \text{if} \quad 1 \leqslant i \leqslant \theta - 1,\\
b(e_i', e_\theta') &= 0 && \text{if} \quad 1 \leqslant i \leqslant \theta - 1,\\
q(e_i') &= U && \text{if} \quad 1 \leqslant i \leqslant \theta - 2,
\end{aligned}
$$

$$q(e_{\theta-1}') = 2\theta U = 0,$$

$$
\begin{aligned}
q(e_\theta') &= \left(\sum_{i=1}^{\theta-1} \bar{d}_i d_i + \sum_{i=1}^{\theta-2} \bar{d}_{i+1} d_i\right) U = \left(\bar{d}_1 d_1 + \sum_{i=1}^{\theta-2} \bar{d}_{i+1}(d_i + d_{i+1})\right) U\\
&= \left(1 + \sum_{i=1}^{\theta-2} \bar{d}_{i+1} V^i\right) U\\
&= \left(1 + \sum_{i=1}^{\theta-2} d_{i+1}\right) U = \left(1 + \sum_{t=1}^{(\theta-2)/2} (d_{2t} + d_{2t+1})\right) U\\
&= \left(1 + \sum_{t=1}^{(\theta-2)/2} V^{2t}\right) U.
\end{aligned}
$$

On the other hand DC' has a D basis $\{f_i\}$ defined by $f_i = (S')^{i-1}$ if $1 \leqslant i \leqslant \theta - 1$. We define a new basis $\{f_i'\}$ by

$$f_i' = f_i \qquad \text{if} \quad 1 \leqslant i \leqslant \theta - 2,$$

$$f_{\theta-1}' = \sum_{i=1}^{\theta-1} f_i.$$

Then b and q assume on the f_i' the same values as on the e_i' with $i \leqslant \theta - 1$. ▮

PROPOSITION 20. *Let D, U, V, C, S, C', S' be as in Proposition 19. Then the quadratic form defined on DC using $\eta = U(1+S)\sum_{t=0}^{(\theta-2)/2}(VS)^{2t}$ is the orthogonal sum of the zero form on DC' and the form on D using $\tilde{\eta} = U$.*

Proof. There is a D basis $\{e_i\}$ of DC defined by $e_i = S^{i-1}$ if $1 \leqslant i \leqslant \theta$. We define a new basis $\{e_i'\}$ by

$$e_1' = e_1,$$

$$e_i' = e_i + V^{1-i} e_1 \qquad \text{if} \quad 2 \leqslant i \leqslant \theta.$$

Then $b(e_i', e_j') = 0$ if $i \neq 1$ or $j \neq 1$ because $b(e_i', e_j') = E(\bar{e}_i' \xi e_j')$, where $\xi = \eta + \bar{\eta} = U(1+S)\sum_{t=0}^{\theta-1}(VS)^t$ and $e_j' = S^{j-1}(1 + (VS)^{1-j})$ and thus $\xi e_j' = 0$. Therefore we are left with the quadratic form on e_1; it is given by $q(e_1) = E(\eta) = U$. ▮

Remainder of the Proof of Theorem 3. We apply induction on l. We may assume that α_k is a nondecreasing function of k.

If $\alpha_l = 2$ then $\alpha_k = 2$ for all k and the quadratic form in the theorem has an F basis consisting of one element e such that $Q(e) = 1$: the element e_I where I consists of only ones (notation of case 2 in Sect. 2). Therefore the Arf invariant is undefined in this case.

Now assume that $\alpha_l > 2$. Then we write H for the subgroup of G_2 generated by the S_i for $1 \leqslant i \leqslant l-1$ and we apply Proposition 19 with

$$D = FH, \qquad \theta = \alpha_l, \qquad S = S_l, \qquad U = \prod_{k=1}^{l-1}(1+S_k), \qquad V = \prod_{k=1}^{l-1} S_k.$$

The proposition yields a first orthogonal summand of the type to which Theorem 2 applies, with data $\alpha_1, \alpha_2, \ldots, \alpha_{l-1}, \alpha_l - 1$. According to the theorem the contribution to the Arf invariant is $\Omega(\alpha_l - 1)$, and therefore is 1 or 0 depending on whether α_l is 4 or larger.

If $l = 1$ or $\alpha_l > \alpha_{l-1}$ then the second orthogonal summand vanishes because $V^{\theta/2} = 1$. If $\alpha_l \geqslant 8$ this makes the Arf invariant zero in agreement with the claim because there is only one k for which $\alpha_k = \alpha_l$. If $\alpha_l = 4$ this makes the Arf invariant one in agreement with the claim since the number of k for which $\alpha_k = A$ is zero for $A \geqslant 8$ and one for $A = 4$.

Now assume $\alpha_l = \alpha_{l-1}$. Then write N for the subgroup of H generated by the S_k for $1 \leqslant k \leqslant l-2$. We apply Proposition 20 with

$$D = FN, \qquad \theta = \alpha_l, \qquad S = S_{l-1}, \qquad U = \prod_{k=1}^{l-2}(1+S_k), \qquad V = \prod_{k=1}^{l-2} S_k.$$

According to the proposition the Arf invariant of the second summand is then equal to the Arf invariant of a quadratic form to which the induction hypothesis applies, with data α_i for $1 \leqslant i \leqslant l-2$. However, ommitting $k=l-1$ and $k=l$ changes $\operatorname{card}\{k \leqslant l \mid \alpha_k = A\}$ by an even number if $A \geqslant 4$ and by 2 if $\alpha_l = \alpha_{l-1} = A = 4$. ▮

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