Transformation Groups, Vol. 11, No. 3, 2006, pp. 387–426 

# ORBITS OF TRIPLES IN THE SHILOV BOUNDARY OF A BOUNDED SYMMETRIC DOMAIN

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**Abstract.** Let $\mathcal{D}$ be a bounded symmetric domain of tube-type, $S$ its Shilov boundary, and $G$ the neutral component of its group of biholomorphic transforms. We classify the orbits of $G$ in the set $S \times S \times S$.

## Introduction

Let $\mathcal{D}$ be a bounded symmetric domain, realized as a circular domain in a (finite-dimensional ) complex vector space $V$. Let $G := \mathrm{Aut}(\mathcal{D})_0$ be the identity component of its group of biholomorphic transforms of $\mathcal{D}$ and let $S$ be the Shilov boundary of $\mathcal{D}$. The action of any element of $G$ extends to a neighbourhood of $\overline{\mathcal{D}}$, and hence $G$ acts on $S$. It is well known that this action is transitive. The main result of the present paper is a classification of the $G$-orbits in the set $S \times S \times S$ of triples in $S$, when $\mathcal{D}$ is of *tube-type*.

The action of $G$ on $S \times S$ can be easily studied as an application of Bruhat theory, and the description of the orbits is the same, whether $\mathcal{D}$ is of tube-type or not. But for triples, there is a drastic difference between tube-type domains and nontube-type domains. In the first case, there is a finite number of orbits in $S \times S \times S$, whereas there are an infinite number of orbits for a nontube-type domain.

Let $r$ be the rank of $\mathcal{D}$. The notion of *$r$-polydisk* (and its corresponding Shilov boundary called *$r$-torus*) plays an important role in the analysis of the orbits. On one hand, they are the "complexifications" of the *maximal flats* of $\mathcal{D}$ (in the sense of the geometry of Riemannian symmetric spaces). On the other hand, an $r$-polydisk in the usual sense is a set of the form

$$\Delta^r = \left\{ \sum_{j=1}^{r} \zeta_j x_j \mid |\zeta_j| < 1, 1 \leqslant j \leqslant r \right\} ,$$

DOI: 10.1007/s00031-005-1117-2.

*The first author acknowledges partial support from the EU (TMR Network Harmonic Analysis and Related Problems).

Received April 10, 2005. Accepted September 30, 2005.
*MSC*: 32M15, 53D12, 22F30.
*Keywords*: bounded symmetric domain, tube-type domain, Shilov boundary, face, Maslov index, flag manifold, Jordan triples.

Published online August 15, 2006.

where the $x_j$ are linearly independent elements in $V$. The space $V$ has a natural structure of a positive Hermitian Jordan triple system and, in particular, it has a natural (Banach) norm, called the *spectral norm*, for which the domain $\mathcal{D}$ is realized as the open unit ball. One of the results we prove is that such a polydisk, constructed on vectors $x_j$ of norm 1 lies in $\mathcal{D}$ if and only if the $(x_j)_{1\leqslant j\leqslant r}$ form a *Jordan frame* for $V$.

Fix an $r$-torus $T \subseteq S$ arising as the Shilov boundary of an $r$-polydisk associated to a Jordan frame. The main step towards the classification of the orbits of $G$ in $S\times S\times S$ is the result that any triple in $S$ can be sent by an element of $G$ to a triple in $T$. This requires that $\mathcal{D}$ is of tube-type, and this property really distinguishes tube-type domains from nontube-type domains. Once this result is obtained, the classification becomes easy, because the problem is reduced to the case of a polydisk, and further, using the product structure, to the case of the unit disk in $\mathbb{C}$, where the situation is easy to analyze. The generalized Maslov index (see [CO01], [Cl04b]) comes in as a subtle invariant for triples.

A special case of this theorem was known before. If $\mathcal{D}$ is the Siegel domain (the unit ball in the space of complex symmetric matrices $\mathrm{Sym}_r(\mathbb{C})$), then the group $G$ is the projective symplectic group $\mathrm{PSp}_{2r}(\mathbb{R}) := \mathrm{Sp}_{2r}(\mathbb{R})/\{\pm\mathbf{1}\}$, and the Shilov boundary of $\mathcal{D}$ can be identified with the Lagrangian manifold (the set of Lagrangian subspaces of $\mathbb{R}^{2r}$). Then the orbits of triples of Lagrangians have been described (see [KS90, p. 492]), using linear symplectic algebra techniques. Related results can be found in [FMS04], and in particular their Proposition 4.3 (which they deduce from [KS90]) is, for this specific example, equivalent to our Theorem 3.1. The main point of [FMS04] is a description of the orbits of the action of the maximal compact subgroup group $\mathrm{U}_n(\mathbb{C})$ of $\mathrm{Sp}_{2n}(\mathbb{R})$ on triples of Lagrangians, but this is a different problem.

As explained in the appendix, the bounded symmetric domains of tube-type can be described in terms of Euclidean Jordan algebras. More precisely, the irreducible ones are in one-to-one correspondence with simple Euclidean Jordan algebras. From the table in [FK94, p. 213] (see also [Be00]) it is easy to give the following table, where for each simple Euclidean Jordan algebra $E$, we list the group $L$ of linear transforms of $E$ preserving the cone $\Omega$, the group $G$ of holomorphic diffeomorphisms of the bounded symmetric domain $\mathcal{D}$, and the Shilov boundary $S$ as compact Riemannian symmetric space. There are four infinite series and one exceptional case. From the point of view of flag manifold (see below), $S$ is realized as $G/P$, where the (maximal) parabolic subgroup $P$ is the semidirect product of $L$ (Levi component) and $E$ (unipotent radical).

Table 1.

| $E$ | $\mathrm{Sym}_n(\mathbb{R})$ | $\mathrm{Herm}_n(\mathbb{C})$ | $\mathrm{Herm}_n(\mathbb{H})$ | $\mathbb{R}^{1,n-1}$ | $\mathrm{Herm}_3(\mathbb{O})$ |
|---|---|---|---|---|---|
| $L$ | $\mathrm{GL}_n(\mathbb{R})$ | $\mathrm{GL}_n(\mathbb{C})$ | $\mathrm{GL}_n(\mathbb{H})$ | $\mathrm{SO}_0(1,n-1)\times\mathbb{R}^*$ | $\mathbb{E}_{6(-26)}\times\mathbb{R}^*$ |
| $G$ | $\mathrm{PSp}_{2n}(\mathbb{R})$ | $\mathrm{PU}_{n,n}(\mathbb{C})$ | $\mathrm{PSO}^*(4n)$ | $\mathrm{SO}_{2,n}(\mathbb{R})_0$ | $\mathbb{E}_{7(-25)}$ |
| $S$ | $\mathrm{U}_n(\mathbb{C})/\mathrm{O}_n(\mathbb{R})$ | $\mathrm{U}_n(\mathbb{C})$ | $\mathrm{U}_{2n}(\mathbb{C})/\mathrm{SU}(n,\mathbb{H})$ | $\mathbb{S}^1\times\mathbb{S}^{n-1}/\mathbb{Z}_2$ | $\mathrm{U}(1)\mathbb{E}_6/\mathbb{F}_4$ |

The Shilov boundary $S$ of a bounded domain is in particular a generalized flag manifold of $G$, i.e., of the form $G/P$, where $P$ is a parabolic subgroup of $G$. A nice description of $P$ is obtained after performing a Cayley transform. The domain $\mathcal{D}$ is transformed to an unbounded domain $\mathcal{D}^C$ which is a Siegel domain of type II and the group $P$ is the group of all affine transformations preserving $\mathcal{D}^C$ (see Section 1 for details). The group

$P$ has some specific properties: it is a maximal parabolic subgroup of $G$, conjugate to its opposite. Moreover, one can show that the domain $\mathcal{D}$ is of tube-type if and only if the unipotent radical $U$ of $P$ is abelian. A natural question arises to which extent results similar to the ones obtained in this paper could be valid for other generalized flag manifolds. The natural background for this problem is the following. If $P_1, \dots, P_k$ are parabolic subgroups of a connected semisimple group $G'$, then the product manifold

$$M := G'/P_1 \times \cdots \times G'/P_k$$

is called a multiple flag manifold of finite type if the diagonal action of $G'$ on $M$ has only finitely many orbits. For $k = 1$ we always have only one orbit, and for $k = 2$ the finiteness of the set of orbits follows from the Bruhat decomposition of $G'$. For $G' = \mathrm{GL}_n(\mathbb{K})$ or $G' = \mathrm{Sp}_{2n}(\mathbb{K})$ and $\mathbb{K}$ an algebraically closed field of characteristic zero, it has been shown in [MWZ99], [MWZ00] that finite type implies $k \leqslant 3$, and for $k = 3$ the triples of parabolics leading to multiple flag manifolds of finite type are described and the $G'$-orbits in these manifolds classified. The main technique to achieve these classifications was the representation theory of quivers. In [Li94], Littelmann considers general simple algebraic groups over $\mathbb{K}$ and describes all multiple flag manifolds of finite type for $k = 3$ under the assumption that $P_1$ is a Borel subgroup and $P_2$, $P_3$ are maximal parabolics. Actually, Littelmann considers the condition that $B = P_1$ has a dense orbit in $G'/P_2 \times G'/P_3$, but the results in [Br86], [Vi86] (see also [Po86, p. 314]) show that this implies the finiteness of the number of $B$-orbits and hence the finiteness of the number of $G'$-orbits in $G'/B \times G'/P_2 \times G'/P_3$. From Littelmann's classification one can easily read off that for a maximal parabolic $P$ in $G'$ the triple product $(G'/P)^3$ is of finite type if and only if the unipotent radical $U$ of $P$ is abelian and in two exceptional situations. If $U$ is abelian, then $P$ is the maximal parabolic defined by a 3-grading of $\mathfrak{g}' = \mathbf{L}(G')$, so that $G'/P$ is the conformal completion of a Jordan triple (see [BN05] for a discussion of such completions in an abstract setting). This case was also studied in [RRS92]. The first exceptional case, where $U$ is not abelian, corresponds to $G' = \mathrm{Sp}_{2n}(\mathbb{K})$, where $G'/P = \mathbb{P}_{2n-1}(\mathbb{K})$ is the projective space of $\mathbb{K}^{2n}$, $U$ is the $(2n-1)$-dimensional Heisenberg group, and the Levi complement is $\mathrm{Sp}_{2n-2}(\mathbb{K}) \times \mathbb{K}^\times$. In the other exceptional case, $G' = \mathrm{SO}_{2n}(\mathbb{K})$ and $G'/P$ is the highest weight orbit in the $2^n$-dimensional spin representation of the covering group $\widetilde{G}' = \mathrm{Spin}_{2n}(\mathbb{K})$ of $G'$. Here $U \cong \Lambda^2(\mathbb{K}^n) \oplus \mathbb{K}^n$ is also a 2-step nilpotent group and the Levi complement acts like $\mathrm{GL}_n(\mathbb{K})$ on this group. It seems that the positive finiteness results have a good chance to carry over to the split forms of groups over more general fields and in particular to $\mathbb{K} = \mathbb{R}$, but for real groups not much seems to be known about multiple flag manifolds of finite type.

If $M = (G'/P)^3$ is a multiple flag manifold of finite type, $P$ is conjugate to its opposite, and $P = U \rtimes L$ is a Levi decomposition of $P$, then $L$ is the simultaneous stabilizer of a pair in $(G'/P)^2$ with an open orbit, and this implies that the conjugation action of $L$ on $U$ has only finitely many orbits. A closely related but different problem is the question when the conjugation action of $P$ on $U$ has finitely many orbits. According to a result of Richardson, every parabolic $P$ has a dense orbit in its unipotent radical $U$, but this does not imply finiteness. For more specific results on this question we refer to [RRS92], [PR97], and [HR99].

It is perhaps worthwhile to stress that the proofs we give are one more occurrence of the interaction between complex analysis of bounded symmetric domains and the geometry of convex sets in the normed space $V$. The notions of extremal points or faces of a convex set do play an important role in our study.

The contents of the paper is as follows. In Section 1 we first recall several facts on bounded symmetric domains. Our main sources are Loos' lecture notes [Lo77] and Satake's book [Sa80]. For results concerning Euclidean Jordan algebras we use [FK94]. The main result of Section 1 is a classification of the $G$-orbits in the set of quasi-invertible (= transversal) pairs in $\overline{\mathcal{D}}$ (Theorem 1.7). For this classification, there would be no gain in assuming that $\mathcal{D}$ is of tube-type, so that the theorem is proved in full generality. However, for the analysis of $G$-orbits in $S \times S \times S$ (assuming $\mathcal{D}$ to be of tube-type), we only need the classification result for transversal pairs $(x, y)$, where $x \in S$ and $y \in \overline{\mathcal{D}}$. For this case we give a more direct shorter proof (see Lemma I.20), but we think that the general case might also be useful in other situations.

The main tool for the classification of $G$-orbits in $S \times S \times S$ is the characterization of the transversality relation on $\overline{\mathcal{D}}$ in terms of faces of the compact convex set $\overline{\mathcal{D}}$: Two elements $x, y \in \overline{\mathcal{D}}$ are transversal if and only if they are not contained in a proper face of $\overline{\mathcal{D}}$ (Theorem 2.6). This characterization is also valid for nontube-type domains. A key concept for the classification is the notion of the rank of a face $F$ of $\overline{\mathcal{D}}$. For an irreducible domain $\mathcal{D}$ of rank $r$ it takes values in the set $\{0, 1, \ldots, r\}$ and classifies the $G$-orbits in the set of faces of $\mathcal{D}$. It is normalized in such a way that the rank of $\overline{\mathcal{D}}$ as a face is zero and that the extreme points, i.e., the elements in the Shilov boundary, are faces of rank $r$. If $\mathrm{Face}(x_1, \ldots, x_n)$ denotes the face generated by the subset $\{x_1, \ldots, x_n\}$ of $\overline{\mathcal{D}}$, then the function

$$\overline{\mathcal{D}}^n \longrightarrow \{0, 1, \ldots, r\}, \quad (x_1, \ldots, x_n) \mapsto \operatorname{rank} \mathrm{Face}(x_1, \ldots, x_n),$$

is an invariant for the $G$-action on $\overline{\mathcal{D}}^n$.

In these terms, two elements $x, y \in \overline{\mathcal{D}}$ are transversal if and only if $\operatorname{rank} \mathrm{Face}(x, y) = 0$. In Section 3 we use this fact to show that for a domain $\mathcal{D}$ of tube-type every triple in $S$ is conjugate to a triple in the Shilov boundary $T$ of a maximal polydisk $\Delta^r$ defined by a Jordan frame. This reduces the classification of $G$-orbits in $S \times S \times S$ to the description of intersections of these orbits with $T^3$. This is fully achieved in Section 5 by assigning a 5-tuple of integer invariants to each orbit and by showing that triples with the same invariant lie in the same orbit. The first four components of this 5-tuple are

$$(\operatorname{rank} \mathrm{Face}(x_1, x_2, x_3), \operatorname{rank} \mathrm{Face}(x_1, x_2), \operatorname{rank} \mathrm{Face}(x_2, x_3), \operatorname{rank} \mathrm{Face}(x_1, x_3)).$$

The fifth component is defined as the Maslov index $\iota(x_1, x_2, x_3)$ which is discussed in some detail in Section 4. Note that if $(x_1, x_2, x_3)$ is transversal in the sense that all pairs $(x_1, x_2)$, $(x_2, x_3)$, $(x_3, x_1)$ are transversal, then the first four components of the invariant vanish, which implies that the $G$-orbits in the set of transversal triples are classified by the Maslov index.

We conclude the paper (Section 6) with a brief discussion of how the classification of the $G$-orbits in $S \times S$ can be interpreted in terms of the Bruhat decomposition of $G$. Note that, although $S$ is always a generalized flag manifold of the real group $G$, the unipotent radical of the corresponding parabolic is abelian if and only if the domain $\mathcal{D}$ is of tube-type. If this is the case, then [Li94] and [RRS92] imply that the complexification $G_{\mathbb{C}}$

acts with finitely many orbits on $(G_{\mathbb{C}}/P_{\mathbb{C}})^3$. For each $G_{\mathbb{C}}$-orbit $M \subseteq (G_{\mathbb{C}}/P_{\mathbb{C}})^3$ meeting the totally real submanifold $(G/P)^3$, the intersection $M \cap (G/P)^3$ is totally real in $M$, hence a real form of $M$, and [BS64, Cor. 6.4] implies that $G$ has only finitely many orbits in $M \cap (G/P)^3$. Alternatively, one can argue with Whitney's theorem [Wh57] that the set of real points of a complex variety has only finitely many connected components which coincide with the $G$-orbits in our case. In view of this argument, it's not the finiteness of the $G$-orbits but their classification and the relation to the Maslov index that is the main point of the present paper.

In [RRS92, Theorem 1.2(b)] one also finds a classification of the $G_{\mathbb{C}}$-orbits in $(G_{\mathbb{C}}/P_{\mathbb{C}})^2$ which turns out to be the same as in the real case (see Theorem 6.1).

A final Appendix gives a short presentation of the relation between positive Hermitian Jordan triple systems and bounded symmetric domains on the one hand, and between Euclidean Jordan algebras and tube-type domains on the other. This Appendix is designed for readers not familiar with the language of Jordan algebra and/or the Jordan triple system.

We thank L. Kramer and H. Rubenthaler for comments and references concerning multiple flag manifolds of finite type. We also thank several anonymous editors of this journal for numerous remarks and for pointing out reference [RRS92].

## 1. Classification of orbits of transversal pairs in the boundary

Let $\mathcal{D}$ be an irreducible circular bounded symmetric domain, so that $\mathcal{D}$ is the open unit ball for a norm on a complex vector space $V$ ([Lo77, Theorem 4.1]). In this section we describe the $G$-orbits in the set of quasi-invertible pairs of elements in the closure of $\mathcal{D}$ (see Theorem 1.7 below). Here we do not have to assume that $\mathcal{D}$ is of tube-type.

*The associated Jordan triple.* On $V$ we consider the Hermitian Jordan triple product $\{\cdot\,,\cdot\,,\cdot\}\colon V^3 \to V$ that is uniquely determined by the property that, for each $v \in V$, the vector field given by the function

$$\xi_v\colon V \longrightarrow V, \quad z \mapsto v - \{z, v, z\},$$

generates a one-parameter group of automorphisms of $\mathcal{D}$ ([Lo77, Lemma 4.3]). Note that, for each $v \in V$, the map $(z, w) \mapsto \{z, v, w\}$ is symmetric and complex bilinear, and that, for each $a, b \in V$, the map $z \mapsto \{a, z, b\}$ is antilinear. For $x, y \in V$ we define $Q(x)$ and $x\Box y \in \mathrm{End}(V)$ by

$$Q(x) \cdot y := \{x, y, x\} \quad \text{and} \quad x\Box y \cdot z := \{x, y, z\}.$$

The Jordan triple structure on $V$ used by Loos is $\{x, y, z\}' = 2\{x, y, z\}$, so that his quadratic representation is given by $Q'(x, y) = 2\{x, y, z\}$, but since Loos defines $Q'(x)$ as $\frac{1}{2}Q'(x, x)$, we obtain the same operators $Q(x) = Q'(x)$.

*Tripotents and Peirce decomposition.* An element $e \in V$ is called a *tripotent* if $e = \{e, e, e\}$. For a tripotent $e \in V$ let $V_j := V_j(e)$ denote the $j$-eigenspace of the operator $2e\Box e$. Then we obtain the corresponding *Peirce decomposition of* $V$:

$$V = V_0 \oplus V_1 \oplus V_2$$

([Lo77, Theorem 3.13]). Since $e\Box e$ is a Jordan triple derivation, we have the Peirce rules

$$\{V_i, V_j, V_k\} \subseteq V_{i-j+k}, \tag{1.1}$$

which imply in particular that each space $V_j$ is a Jordan subtriple. In addition, we have

$$V_0 \Box V_2 = V_2 \Box V_0 = \{0\}. \tag{1.2}$$

The Jordan triple $V$ also carries a Jordan algebra structure, denoted $V^{(e)}$, given by

$$ab := L(a) \cdot b := \{a, e, b\}.$$

Then $e$ is an idempotent in $V^{(e)}$ because $ee = \{e, e, e\} = e$, and the Peirce decomposition of $V$ with respect to the tripotent $e$ coincides with the Peirce decomposition of the Jordan algebra $V^{(e)}$ with respect to the idempotent $e$.

The multiplication operators in $V^{(e)}$ are given by $L(a) = a\Box e$, so that $L(e)|_{V_2} = \mathrm{id}_{V_2}$ implies that $(V_2, e)$ is a unital Jordan subalgebra of $V^{(e)}$. For the quadratic representation in $V^{(e)}$ we have

$$P(e) = 2L(e)^2 - L(e^2) = 2L(e)^2 - L(e) = (2L(e) - \mathbf{1})L(e),$$

so that $P(e) = Q(e)^2$ vanishes on $V_0 \oplus V_1$ and restricts to the identity on $V_2$. It follows, in particular, that $(V_2, e, Q(e))$ is an involutive Jordan algebra (see [Lo77, Theorem 3.13]).

*Orbits in* $\overline{\mathcal{D}}$. Two tripotents $e, f \in V$ are said to be *orthogonal* if $f \in V_0(e)$. In view of the Peirce rules (1.2), this implies $\{f, f, e\} = \{e, f, f\} = (e\Box f) \cdot f = 0$, so that we also have $e \in V_0(f)$, i.e., orthogonality is a symmetric relation. If this is the case, then $e + f$ is also a tripotent because the relations $e\Box f = f\Box e = 0$ lead to

$$\{e + f, e + f, e + f\} = \{e, e, e + f\} + \{f, f, e + f\} = \{e, e, e\} + \{f, f, f\} = e + f.$$

We call a nonzero tripotent $e$ *primitive* if it cannot be written as a sum of two nonzero orthogonal tripotents and $e$ is said to be *maximal* if there is no nonzero tripotent orthogonal to $e$. A maximal tuple $(c_1, \ldots, c_r)$ of mutually orthogonal primitive tripotents is called a *Jordan frame in* $V$ and $r = \operatorname{rank} \mathcal{D}$ is called the *rank of* $\mathcal{D}$. We fix a Jordan frame $(c_1, \ldots, c_r)$. For $k = 0, 1, \ldots, r$ we then obtain tripotents

$$e_k := c_1 + \cdots + c_k,$$

where it is understood that $e_0 = 0$.

We recall that each bounded symmetric domain $\mathcal{D}$ can be decomposed in a unique fashion as a direct product of indecomposable, also called *irreducible*, bounded symmetric domains:

$$\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_m. \tag{1.3}$$

Then the connected group $G := \operatorname{Aut}(\mathcal{D})_0$ satisfies

$$G \cong G_1 \times \cdots \times G_m, \qquad \text{where} \qquad G_j := \operatorname{Aut}(\mathcal{D}_j)_0. \tag{1.4}$$

If $\mathcal{D}$ is irreducible, then $G$ has exactly $r+1$ orbits in the closure $\overline{\mathcal{D}}$ of $\mathcal{D}$ in $V$ and $e_0, \ldots, e_r$ form a set of representatives (see [Sa80, Theorem III.8.7]). For $k=0$ we have $G \cdot e_0 = \mathcal{D}$ and for $k=r$ we obtain the *Shilov boundary* $G \cdot e_r = S$ ([Sa80, Theorem III.8.14]). We define the *rank of* $x \in \overline{\mathcal{D}}$ by

$$\operatorname{rank} x = k \quad \text{for} \quad x \in G \cdot e_k$$

and observe that the rank function is $G$-invariant and classifies the $G$-orbits in $\overline{\mathcal{D}}$.

If $\mathcal{D}$ is not irreducible, then (1.3/4) imply that the orbit of $x = (x_1, \ldots, x_m) \in \overline{\mathcal{D}} = \prod_{j=1}^m \overline{\mathcal{D}_j}$ is determined by the $m$-tuple

$$(\operatorname{rank} x_1, \ldots, \operatorname{rank} x_m) \in \mathbb{N}_0^m.$$

Here $(0, \ldots, 0)$ corresponds to elements in $\mathcal{D}$ and $(\operatorname{rk} \mathcal{D}_1, \ldots, \operatorname{rk} \mathcal{D}_m)$ to elements in the product set $S = S_1 \times \cdots \times S_m$.

*Spectral decomposition and spectral norm.* Let $K$ be the stabilizer of $0 \in \mathcal{D}$ in $G$. Then $K$ acts as a group of automorphisms on the Jordan triple $V$ and each element $z \in V$ is conjugate under $K$ to an element in $\operatorname{span}_{\mathbb{R}}\{c_1, \ldots, c_r\}$. For $k \cdot z = \sum_{j=1}^r \lambda_j c_j$ the number

$$|z| := \max\{|\lambda_1|, \ldots, |\lambda_r|\}$$

is called the *spectral norm of* $z$. Then the elements $\widetilde{c}_j := k^{-1} \cdot c_j$ are orthogonal tripotents with

$$z = \sum_{j=1} \lambda_j \widetilde{c}_j,$$

which is the spectral decomposition of $z$. The spectral norm $|\cdot|$ is indeed a norm on $V$ with

$$\mathcal{D} = \{z \in V \mid |z| < 1\}. \tag{1.5}$$

The following theorem relates the holomorphic arc-components in $\partial\mathcal{D}$ to the tripotents in $V$.

**Theorem 1.1.** ([Lo77, Theorem 6.3]) *For each holomorphic arc-component $A$ of $\partial\mathcal{D}$ there exists a tripotent $e$ in $A$ such that*

$$A = A_e := e + \mathcal{D}_e, \quad \textit{where} \quad \mathcal{D}_e := \mathcal{D} \cap V_0(e),$$

*is a bounded symmetric domain in $V_0(e)$. The map $e \mapsto A_e$ yields a bijection from the set of nonzero tripotents of $V$ onto the set of holomorphic arc-components of $\partial\mathcal{D}$. The Shilov boundary $S$ coincides with the set of maximal tripotents.*

*An element $x \in \overline{\mathcal{D}}$ is contained in $A_e$ if and only if*

$$e = \lim_{n \to \infty} Q(x)^n \cdot x. \tag{1.6}$$

*Conformal completion of $V$.* Let $G_{\mathbb{C}}$ denote the universal complexification of the connected real Lie group $G$, and $\tau$ the antiholomorphic involution of $G_{\mathbb{C}}$ for which $G$ is the identity component of the fixed point group $G_{\mathbb{C}}^\tau$. Then the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$

has a faithful realization by polynomial vector fields of degree $\leqslant 2$ on $V$, which leads to a 3-grading

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-,$$

where $V \cong \mathfrak{g}_+$ is the space of constant vector fields, $\mathfrak{g}_0$ consists of linear vector fields, and $\mathfrak{g}_-$ is the set of quadratic vector fields corresponding to the maps $z \mapsto Q(z)\cdot v = \{z, v, z\}$ for $v \in V$. By construction of the triple product, the vector fields $\xi_v$ correspond to elements of the real Lie algebra $\mathfrak{g} = \mathbf{L}(G)$, which implies that $\tau$ maps the constant vector field $v$ to the quadratic vector field $z \mapsto -\{z, v, z\}$. Hence $\tau$ reverses the grading of $\mathfrak{g}_{\mathbb{C}}$, i.e., $\tau(\mathfrak{g}_j) = \mathfrak{g}_{-j}$ for $j \in \{+, -, 0\}$. The Jordan triple structure on $V \cong \mathfrak{g}_+$ then satisfies

$$\{x, y, z\} = \frac{1}{2}[[x, \tau \cdot y], z]. \tag{1.7}$$

The subgroups

$$G^{\pm} := \exp \mathfrak{g}_{\pm} \quad \text{and} \quad G^0 := \{g \in G_{\mathbb{C}} \mid (\forall j)\ \operatorname{Ad}(g)\mathfrak{g}_j = \mathfrak{g}_j\}$$

satisfy

$$G^{\pm} \cap G^0 = \{\mathbf{1}\} \quad \text{and} \quad (G^{\pm} \rtimes G^0) \cap G^{\mp} = \{\mathbf{1}\}.$$

Therefore $P^{\pm} := G^{\pm}G^0 \cong G^{\pm} \rtimes G^0$ are subgroups of $G_{\mathbb{C}}$, and we obtain an embedding

$$V \hookrightarrow X := G_{\mathbb{C}}/P^-, \quad v \mapsto \exp v \cdot P^-,$$

called the *conformal completion of* $V$. The elements of $G^+$ act on $V \subseteq X$ by translations

$$t_v : x \mapsto x + v \tag{1.8}$$

because $\exp v \exp x P^- = \exp(v + x)P^-$. We further have $\tau(G^{\pm}) = G^{\mp}$ and $\tau(G^0) = G^0$.

For $w \in V$, we write $\widetilde{t}_w$ for the map $X \to X$ induced by the element $\exp(-\tau(w)) = (\tau(\exp w))^{-1}$. For $v \in V$ the condition $\widetilde{t}_w \cdot v \in V$, where $V$ is considered as a subset of $X$, is then equivalent to the invertibility of

$$\mathbf{1} + \operatorname{ad} v \operatorname{ad}(-\tau \cdot w) + \frac{1}{4}(\operatorname{ad} v)^2(\operatorname{ad} \tau \cdot w)^2 = \mathbf{1} - \operatorname{ad} v \operatorname{ad}(\tau \cdot w) + \frac{1}{4}(\operatorname{ad} v)^2 \circ \tau \circ (\operatorname{ad} \tau)^2 \circ \tau \tag{1.9}$$

([BN05, Cor. 1.10]). In view of (1.7), this is precisely the Bergman operator

$$B(v, w) = \mathbf{1} - 2v\Box w + Q(v)Q(w).$$

We further have in $V$ the relation

$$\widetilde{t}_w \cdot v = B(v, w)^{-1} \cdot (v - Q(v) \cdot w). \tag{1.10}$$

**Remark 1.1.** (*Quasi-invertibility and transversality*) A pair $(x, y) \in V$ is called *quasi-invertible* if $B(x, y) \in \operatorname{End}(V)$ is invertible. We write $x \top y$ if $(x, y)$ is quasi-invertible and say that $x$ is *transversal* to $y$. We write $x^{\top} := \{y \in V \mid x \top y\}$ for the set of all elements in $V$ transversal to $x$.

In the Jordan algebra $V^{(y)}$ with the product $ab := \{a, y, b\}$ we have $L(a) = a\Box y$ and $P(a) = Q(a)Q(y)$ ([NO04, App. A]), so that

$$B(x, y) = \operatorname{id}_V - 2L(x) + P(x),$$

and in the unital Jordan algebra $V^{(y)} \times \mathbb{R}$ with the identity element $\mathbf{1} := (0, 1)$ we have

$$\mathbf{1} - 2L(x) + P(x) = P(\mathbf{1}, \mathbf{1}) - 2P(\mathbf{1}, x) + P(x, x) = P(\mathbf{1} - x),$$

i.e., the quasi-invertibility of $(x, y)$ is equivalent to the quasi-invertibility of $x$ in the Jordan algebra $V^{(y)}$.

**Remark 1.2.** (*The $\mathfrak{sl}_2$-triple associated to a tripotent*) Let $e \in V$ be a tripotent, $f := \tau(e)$, $h := [e, f]$ and $\mathfrak{g}_e := \mathrm{span}_{\mathbb{R}}\{h, e, f\}$. Then

$$[h, e] = 2\{e, e, e\} = 2e \quad \text{and} \quad [h, f] = \tau[\tau h, e] = -\tau[h, e] = -2\tau e = -2f,$$

so that $\mathfrak{g}_e \cong \mathfrak{sl}_2(\mathbb{R})$ is a three-dimensional subalgebra of $\mathfrak{g}$ with $\mathfrak{g}_e^\tau = \mathbb{R}(e + f)$.

(a) The operator $\mathrm{ad}_V\, h = 2e\Box e$ is diagonalizable with possible eigenvalues $0, 1, 2$. The corresponding eigenspace decomposition $V = V_0 \oplus V_1 \oplus V_2$ is the Peirce decomposition of the Jordan algebra $V^{(e)}$ with multiplication $ab := \{a, e, b\}$ with respect to the idempotent $e$, i.e., $2L(e) \cdot v_j = jv_j$ for $j = 0, 1, 2$.

(b) We observe that $P(e) = 2L(e)^2 - L(e^2) = (2L(e) - \mathbf{1})L(e)$. For $\lambda \in \mathbb{R}$ we therefore have for

$$\begin{aligned} B(e, (1-\lambda)e) &= B((1-\lambda)e, e) = \mathbf{1} - (1-\lambda)2e\Box e + (1-\lambda)^2 Q(e)^2 \\ &= \mathbf{1} - (1-\lambda)2L(e) + (1-\lambda)^2 P(e) = \mathbf{1} - (1-\lambda)2L(e) + (1-\lambda)^2(2L(e) - \mathbf{1})L(e) \end{aligned}$$

the relation

$$B(e, (1-\lambda)e)v_j = \lambda^j v_j, \quad j = 0, 1, 2.$$

(c) From $Q(e) = Q(Q(e)e) = Q(e)^3$ we conclude that the antilinear map $Q(e)$ is diagonalizable over $\mathbb{R}$ with eigenvalues in $\{1, 0, -1\}$, so that $Q(e)^2 = P(e) = (2L(e) - \mathbf{1})L(e)$ implies that

$$\ker Q(e) = \ker P(e) = V_0 \oplus V_1. \tag{1.11}$$

From $V_0 \Box V_2 = V_2 \Box V_0 = \{0\}$ we obtain, for $x, y \in V_0$,

$$\begin{aligned} B(e + x, e + y) \cdot e &= e - 2(e + x)\Box(e + y) \cdot e + Q(e + x)Q(e + y)e \\ &= e - 2e - 2x\Box y \cdot e + Q(e + x)(Q(e) \cdot e + Q(y) \cdot e + 2\{e, e, y\}) \\ &= -e - 2(e\Box y) \cdot x + Q(e + x) \cdot e \\ &= -e + (Q(e) \cdot e + Q(x) \cdot e + \{e, e, x\}) = 0. \end{aligned}$$

**Theorem 1.2.** ([Lo77, Theorem 8.11]) *Let $e \in V$ be a tripotent and $V^{(e)}$ the corresponding Jordan algebra with product $ab = \{a, e, b\}$. Identifying $e \in V$ with an element of $\mathfrak{g}_+$, the partial Cayley transform corresponding to $e$ is defined by $C_e := \exp\left(\frac{\pi}{4}(e - \tau \cdot e)\right) \in G_{\mathbb{C}}$, and in Jordan theoretic terms it is given as a partially defined map on $V$ by*

$$C_e = t_e \cdot B(e, (1 - \sqrt{2})e) \cdot \widetilde{t}_e.$$

*In particular,*

$$C_e^{-1}(V) \cap V = \{v \in V \mid B(e, v) \in \mathrm{GL}(V)\} = e^\top.$$

In [Lo77] Loos writes $B(e, -e)^{\frac{1}{2}}$ instead of $B(e, (1-\sqrt{2})e)$, which makes sense because

$$B(e, (1 - \sqrt{2})e)^2 = B(e, (1 - 2)e) = B(e, -e)$$

is diagonalizable and the eigenvalues $1, \sqrt{2}$ and $2$ of $B(e, (1 - \sqrt{2})e)$ are positive (Remark 1.2).

**Remark 1.3.** The preceding theorem implies in particular that the condition for an element $x \in V$ to lie in the domain of the Cayley transform is precisely the transversality condition $e \top x$. If $x_2$ is the Peirce component of $x$ in $V_2$, then [Lo77, Prop. 10.3] says that $e \top x$ is equivalent to the invertibility of $e - x_2$ in the unital Jordan algebra $(V_2, e)$.

**Definition 1.4.** A Hermitian scalar product $\langle \cdot\,, \cdot \rangle$ on $V$ is said to be *associative* if for $x, y, z, w \in V$ we have

$$\langle \{x, y, z\}, w\rangle = \langle x, \{y, z, w\}\rangle,$$

which is equivalent to

$$(z \Box y)^* = y \Box z \quad \text{for} \quad y, z \in V.$$

According to [Lo77, Cor. 3.16], a scalar product with this property is given by

$$\langle x, y\rangle := \operatorname{tr}(x \Box y),$$

and for $0 \neq x \in V$ the operator $x \Box x$ is nonzero and positive semidefinite. In this sense $(V, \{\cdot\,, \cdot\,, \cdot\})$ is a *positive Hermitian Jordan triple.* □

**Lemma 1.3.** *Let $e \in V$ be a tripotent, $V_j := V_j(e)$ its Peirce spaces, and $z \in V_0$ with $|z| \leqslant 1$. Further let $f := \lim_{n\to\infty} Q(z)^n \cdot z$ denote the unique tripotent contained in the holomorphic arc-component of $z$. Then $\phi(z) := Q(z + e)\,|_{V_1}\colon V_1 \to V_1$ is an antilinear operator which is symmetric with respect to the real scalar product $(z, w) := \operatorname{Re}\operatorname{tr}(z \Box w)$, and for $z \in V_1$ we have $\phi(z)v = 2\{z, v, e\}$.*

*If $|z| < 1$, then $\phi(z) + \mathbf{1}$ is injective ($\mathbf{1}$ stands for $\mathrm{id}_{V_1}$), and for $|z| = 1$ its kernel is*

$$\operatorname{Fix}(-Q(e + f)) \cap V_1(f) \cap V_1(e).$$

*Proof.* For $v \in V_1$ we have

$$\phi(z)v = \{z + e, v, z + e\} = Q(z)v + Q(e)v + 2Q(z, e)v,$$

and $Q(e)v \in V_{4-1} = V_3 = \{0\}$ as well as $Q(z)v \in V_{0-1} = V_{-1} = \{0\}$ by the Peirce relations (1.1), so that $\phi(z)v = 2\{z, v, e\}$.

According to [Lo77, Lemma 6.7], the operator $\phi(z)$ on $V_1$ is symmetric with respect to the real scalar product $(\cdot\,, \cdot)$ on $V_1$, hence diagonalizable over $\mathbb{R}$ with real eigenvalues.

Let $v \in V_1$ be an eigenvector for $\phi(z)$ corresponding to the eigenvalue $\lambda \in \mathbb{R}$, i.e., $Q(z + e) \cdot v = \lambda v$. Inductively, we get

$$Q(Q(z + e)^n \cdot (z + e)) \cdot v = \lambda^{2n+1} \cdot v$$

for all $n \in \mathbb{N}_0$ from

$$\begin{aligned} Q(Q(z + e)^n \cdot (z + e)) \cdot v &= Q(Q(z + e)Q(z + e)^{n-1} \cdot (z + e)) \cdot v \\ &= Q(z + e)Q(Q(z + e)^{n-1} \cdot (z + e))Q(z + e) \cdot v \\ &= Q(z + e)Q(Q(z + e)^{n-1} \cdot (z + e)) \cdot \lambda v \\ &= \lambda Q(z + e) \cdot (\lambda^{2n-1} \cdot v) = \lambda^{2n+1} v. \end{aligned}$$

Since the inclusion $V_0 \hookrightarrow V$ is isometric with respect to the spectral norm ([Lo77, Theorem 3.17]), we have

$$e + z \in e + \overline{\mathcal{D}_e} = \overline{A_e} \subseteq \overline{\mathcal{D}},$$

and the limit $f = \lim_{n\to\infty} Q(z)^n \cdot z$ is a tripotent in $V_0(e)$ (Theorem 1.1).

As a consequence of the Peirce relations (1.2), we obtain

$$Q(e+z).(e+z) = Q(e)e + Q(z)z = e + Q(z)z,$$

and, inductively,

$$Q(e+z)^n \cdot (e+z) = e + Q(z)^n \cdot z \longrightarrow e + f.$$

Therefore

$$\lim_{n\to\infty} \lambda^{2n+1} v = \lim_{n\to\infty} Q(Q(z+e)^n \cdot (z+e)) \cdot v = Q(e+f) \cdot v,$$

and the existence of the limit implies that $|\lambda| \leqslant 1$. If $|\lambda| < 1$, then $Q(e+f) \cdot v = 0$ and, otherwise, $Q(e+f) \cdot v = \lambda v$. It follows in particular that each eigenvector for $Q(e+z)$ on $V_1$ is also an eigenvector of $Q(e+f)$.

Suppose that $|\lambda| = 1$. As a consequence of the Peirce rules, the sum $e+f$ is a Jordan tripotent (1.3), and from $Q(e+f).v = \lambda v$ and $\ker Q(e+f) = V_0(e+f) \oplus V_1(e+f)$ (Remark 1.2), we derive $v \in V_2(e+f)$, so that $(e+f)\Box(e+f) = e\Box e + f\Box f$ implies that $v \in V_1(f)$.

On the other hand, $Q(e+f)$ is an antilinear involution of $V_2(e+f) \supseteq V_1(e) \cap V_1(f)$. We conclude that

$$\ker(\phi(z) + \mathbf{1}) = \ker(\phi(f) + \mathbf{1}) = \mathrm{Fix}(-Q(e+f)) \cap V_1(f) \cap V_1(e). \quad \Box$$

To classify the $G$-orbits of transversal pairs in $\overline{\mathcal{D}}$, we need a more explicit description of the image

$$\mathcal{D}^C := C_e(\mathcal{D})$$

of $\mathcal{D}$ under the partial Cayley transform $C_e$ in terms of the Peirce decomposition of $V$. To this end, we introduce the following notation.

**Definition 1.5.** Let $e \in V$ be a tripotent.

(1) $(V_2, e, Q(e))$ is a unital involutive Jordan algebra. We write $v^* := Q(e)v$ for the involution on $V_2$ and observe that $V_2 = E \oplus iE$ for $E := \{v \in V \mid v^* = v\}$. In this sense,

$$\mathrm{Re}\, v = \tfrac{1}{2}(v + v^*) = \tfrac{1}{2}(v + Q(e)v)$$

is the component of $v$ in the real form $E$ of $V_2$. The real Jordan algebra $E$ is Euclidean and we write $E_+ := \{a^2 \mid a \in E\}$ for its closed positive cone. For $v, w \in E$ we write $v > w$ for $v - w \in \mathrm{int}(E_+)$ and $v \geqslant w$ for $v - w \in E_+$.

(2) For $z \in V_0$ we define the antilinear map

$$\phi(z)\colon V_1 \longrightarrow V_1, \quad v \mapsto 2\{e, v, z\} = Q(e+z) \cdot v.$$

(Due to the different normalization, the factor 2 is not present in [Lo77].)

(3) We also define a Hermitian map

$$F\colon V_1 \times V_1 \longrightarrow V_2, \quad (z, w) \mapsto \{z, w, e\},$$

with

$$F(z,w)^* = F(w,z) \quad \text{and} \quad F(z,z) > 0 \quad \text{for} \quad 0 \neq z \in V_1.$$

For $u \in V_0$ with $|u| < 1$ we further define a real bilinear map

$$F_u(z,w) = F(z, (\mathbf{1} + \phi(u))^{-1} \cdot w),$$

where we recall from Lemma 1.3 that $\mathbf{1} + \phi(u)$ is invertible.

In the following proposition the missing factor $\frac{1}{2}$ in front of $F$, compared to [Lo77], is due to our different normalization of the triple product.

**Proposition 1.4.** ([Lo77, Theorem 10.8]) *We have*

$$\mathcal{D}^C = C_e(\mathcal{D}) = \{v = v_2 + v_1 + v_0 \in V_2 \oplus V_1 \oplus V_0 \mid |v_0| < 1, \mathrm{Re}(v_2 - F_{v_0}(v_1, v_1)) > 0\}.$$

To determine the closure of $\mathcal{D}^C$, we need the following lemma, because there might be elements $x_0 \in \partial\mathcal{D} \cap V_0$ for which the operator $\phi(x_0) + \mathbf{1}$ is not invertible.

**Lemma 1.5.** *Let $F$ be a finite-dimensional Euclidean vector space, $(A_n)_{n\in\mathbb{N}}$ a sequence of positive definite operators on $F$ converging to $A$, and $(v_n)_{n\in\mathbb{N}}$ a sequence of elements of $F$ converging to $v$. If the sequence $A_n^{-1/2} v_n$ is bounded, then $v \in \mathrm{im}(A)$.*

*Proof.* Since $A$ is symmetric, we have $\mathrm{im}(A) = \ker(A)^\perp$. Let $w \in \ker(A)$. We have to show that $\langle v, w\rangle = 0$. Since the sequence $A_n^{-1/2} v_n$ is bounded, it contains a convergent subsequence, and we may thus assume that it converges to some $u \in F$. Then we get

$$\begin{aligned}\langle v, w\rangle &= \lim_{n\to\infty} \langle v_n, w\rangle = \lim_{n\to\infty} \langle A_n^{1/2} A_n^{-1/2} v_n, w\rangle \\ &= \lim_{n\to\infty} \langle A_n^{-1/2} v_n, A_n^{1/2} w\rangle = \langle u, A^{1/2} w\rangle = \langle u, 0\rangle = 0.\end{aligned}$$

This completes the proof. □

**Lemma 1.6.** *For each element $v = v_2 + v_1 + v_0 \in \overline{\mathcal{D}^C}$ we have $v_1 \in \mathrm{im}(\mathbf{1} + \phi(v_0))$.*

*Proof.* Let $(v^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}^C$ converging to $v$ and write $v_j^n$, $j = 0, 1, 2$, for its Peirce components.

We pick a linear functional $f \in E^*$ in the interior of the dual cone of $E_+$, so that $f(x) > 0$ holds for $0 \neq x \in E_+$, and observe that this implies that

$$(v, w) := f(\mathrm{Re}\, F(v, w))$$

defines a real scalar product on $V_1$. The argument in [Lo77, p. 10.6] shows that for each $z \in V_0$ the operator $\phi(z)$ is symmetric with respect to this scalar product. According to Lemma 1.3, all its eigenvalues $\lambda$ satisfy $|\lambda| \leqslant 1$ and even $|\lambda| < 1$ for $|z| < 1$, so that $\mathbf{1} + \phi(z)$ is a positive semidefinite symmetric operator which is positive definite for $|z| < 1$.

From $v^n \in \mathcal{D}^C$ we get

$$|v_0^n| < 1 \quad \text{and} \quad \mathrm{Re}\, F_{v_0^n}(v_1^n, v_1^n) \leqslant \mathrm{Re}\, v_2^n,$$

which implies that

$$\begin{aligned} f(v_2^n) &\geqslant f(\operatorname{Re} F_{v_0^n}(v_1^n, v_1^n)) = f(\operatorname{Re} F(v_1^n, (\mathbf{1}+\phi(v_0^n))^{-1}v_1^n)) \\ &= (v_1^n, (\mathbf{1}+\phi(v_0^n))^{-1}v_1^n) = ((\mathbf{1}+\phi(v_0^n))^{-1/2}v_1^n, (\mathbf{1}+\phi(v_0^n))^{-1/2}v_1^n). \end{aligned}$$

Therefore the sequence $(\mathbf{1}+\phi(v_0^n))^{-1/2}v_1^n$ in $V_1$ is bounded, and Lemma 1.5 implies that

$$v_1 = \lim_{n\to\infty} v_1^n \in \operatorname{im}(\mathbf{1}+\phi(v_0)). \quad \square$$

From the preceding lemma one easily derives an explicit description of the closure of $\mathcal{D}^C$ because the operator $(\mathbf{1}+\phi(v_0))^{-1}$ is well defined on $\operatorname{im}(\mathbf{1}+\phi(v_0))$. This leads to

$$\overline{\mathcal{D}^C} = \big\{v \in V \mid |v_0| \leqslant 1, v_1 \in \operatorname{im}(\phi(v_0)+\mathbf{1}), \operatorname{Re}\big(v_2 - F(v_1, (\mathbf{1}+\phi(x_0))^{-1}v_1)\big) \geqslant 0\big\}.$$

Since we do not need this description in the following, we leave the details of its verification to the reader.

**Theorem 1.7.** (Orbits of transversal pairs) *Let $\mathcal{D}$ be an irreducible bounded symmetric domain, not necessarily of tube-type. If $(x,y) \in \overline{\mathcal{D}}$ is a transversal pair with* $\operatorname{rk} x = k$, *then there exists a $g \in G$ with $g.(x,y) = (e_k, z)$ with $e_k = c_1 + \cdots + c_k$ and*

$$z = -(c_{j+1} + \cdots + c_k) + \sum_{l=k+1}^{r} \lambda_l c_l, \quad -1 \leqslant \lambda_{k+1} \leqslant \cdots \leqslant \lambda_r \leqslant 1.$$

*Proof.* Since $\mathcal{D}$ is irreducible, $G$ acts transitively on the set of elements of rank $k$, so that we may without loss of generality assume that $x = e := e_k$. We then have to show that each $G_e$-orbit in $e^\top \cap \overline{\mathcal{D}}$ contains an element of the form

$$-(c_{j+1} + \cdots + c_k) + \sum_{l=k+1}^{r} \lambda_l c_l, \quad -1 \leqslant \lambda_{k+1} \leqslant \cdots \leqslant \lambda_r \leqslant 1.$$

We recall the notation from Definition 1.5. For $y > 0$ in $E$ we then find, with Remark 1.2,

$$B(e-y, e) = \operatorname{id}_V - 2L(e-y) + P(e-y) = P(e-(e-y)) = P(y). \tag{1.12}$$

Let $Q := G_{A_e}$ denote the stabilizer of the holomorphic arc-component $A_e$ of $e$ in $\partial\mathcal{D}$ (which is a maximal parabolic subgroup of $G$). Then the group $Q^C := C_e \circ Q \circ C_e^{-1}$ acts naturally on $\mathcal{D}^C = C_e(\mathcal{D})$ and we also put

$$Q_e^C := C_e \circ G_e \circ C_e^{-1} \subseteq Q^C,$$

where $G_e$ is the stabilizer of $e$ in $G$.

From [Lo77, Lemma 10.7] we now obtain

$$\begin{aligned} Q^C = \{ & t_b \circ t_{v+F(v,v)} \exp(2e\Box v) P(y) \exp(\xi_w) \cdot k \mid \\ & b \in iE, v \in V_1, 0 < y \in E, w \in V_0, k \in K_e\}, \end{aligned}$$

where $K_e := \{g \in G \mid g\cdot 0 = 0, g\cdot e = e\} \subseteq \mathrm{Aut}(V)_e$ is the set of all automorphisms of the Jordan triple $V$ fixing $e$, and $P(y)$ is the quadratic representation of the Jordan algebra $V^{(e)}$ (Remark 1.1). From the proof of [Lo77, Theorem 9.15] and the description of the Lie algebra $\mathbf{L}(Q^C)$ in [Lo77, Prop. 10.6] it follows that, for $b \in iE, v \in V_1, 0 < y \in E$ and $k \in K_e$, we have

$$t_b \circ t_{v+F(v,v)} \exp(2e\Box v)P(y)k \in Q_e^C.$$

Moreover, the explicit calculations in the proof of [Lo77, Theorem 10.8] further imply that the map

$$V_0 \longrightarrow A_e = e + (\mathcal{D} \cap V_0), \quad w \mapsto \exp(\xi_w) \cdot e,$$

is bijective and that the Cayley transform fixes each $\xi_w$. This implies that

$$Q_e^C = \{t_b \circ t_{v+F(v,v)} \exp(2e\Box v)P(y) \cdot k \mid b \in iE, v \in V_1, 0 < y \in E, k \in K_e\}.$$

We observe that for $v \in V_1$ the Peirce rules imply that $e\Box v$ is a nilpotent operator on $V$ mapping $V_j \to V_{j+1}$. For $x = x_2 + x_1 + x_0 \in \overline{\mathcal{D}^C}$ the $V_1$-component of

$$t_{v+F(v,v)} \exp(2e\Box v) \cdot x$$

is given by

$$x_1 + v + \phi(x_0) \cdot v,$$

and since $-x_1 \in \mathrm{im}(\mathbf{1} + \phi(x_0))$ by Lemma 1.6, there is a unique $v \in \mathrm{im}(\mathbf{1} + \phi(x_0))$ with

$$t_{v+F(v,v)} \exp(2e\Box v).x \in V_2 \oplus V_0.$$

From that we conclude that each $Q_e^C$-orbit in $V$ through an element $y = y_2+y_1+y_0 \in \overline{\mathcal{D}^C}$ contains an element of the form

$$x_2 + x_0 \quad \text{with} \quad |x_0| \leqslant 1 \quad \text{and} \quad \mathrm{Re}\, x_2 \geqslant 0.$$

Applying elements of the form $t_v$, $v \in iE$, we may further assume that $x_2 \in E$, so that we have an element in $E_+ \times \mathcal{D}_e$. From the explicit description of $Q_e^C$ we derive that the intersection of the orbit of $x_2 + x_0 \in E + V_0$ with $E + V_0$ contains the orbit of $x_2 + x_0$ under the group $Q'' := P(E_+)K_e$.

The orbits of $Q''$ on the set $E_+ \times \overline{\mathcal{D}}_e$ are products of orbits of the automorphism group $G(E_+)$ of the symmetric cone $E_+$ in $E$, and orbits of the identity component of the group $K_e$ on $\mathcal{D}_e$. Since the action of the group $K_e$ preserves the Peirce decomposition, it acts on $\mathcal{D}_e \subseteq V_0$ as a subgroup of $\mathrm{Aut}(V_0)$. The identity component of the latter group is obtained by exponentiating elements of the Lie subalgebra $V_0 + \tau(V_0) + [V_0, \tau(V_0)] \subseteq \mathfrak{g}_{\mathbb{C}}$ (here we use that $\mathcal{D}_e = \mathcal{D} \cap V_0$ is an irreducible bounded symmetric domain; see Theorem 1.1), and all the elements of this subalgebra commute with the element $e \in V_2$ by the Peirce rules (1.2). Hence the image of $K_e$ in $\mathrm{Aut}(V_0)$ contains the identity component of $\mathrm{Aut}(V_0)$.

For $e = e_k = c_1 + \cdots + c_k$, the orbits of $G(E_+)_0$, which coincide with the orbits of the full group $G(E_+)$, are represented by the elements

$$e_0 = 0,\ e_1 = c_1, \ldots, e_j = c_1 + \cdots + c_j, \ldots, e_k = e$$

([FK94, Prop. IV.3.2]). Since $(c_{k+1}, \dots, c_r)$ is a Jordan frame in $V_0$, each orbit of $\mathrm{Aut}(V_0)_0$ in $V_0$ contains an element of the form

$$\sum_{l=k+1}^{r} \lambda_l c_l, \quad \lambda_{k+1} \leqslant \cdots \leqslant \lambda_r$$

(see [FK94, Prop. X.3.2]).

Next we transfer this information back to the bounded picture, i.e., to $G_e$-orbits in $\overline{\mathcal{D}}$. According to [Lo77, Prop. 10.3], we have

$$C_e(x_2 + x_0) = C_e(x_2) + x_0 = (e + x_2)(e - x_2)^{-1} + x_0 \quad \text{for } x_2 \in V_2, x_0 \in V_0. \quad (1.13)$$

For $e_j = c_1 + \cdots + c_j$, $j \leqslant k$, the element $e + e_j$ is invertible in $V_2$, and we obtain for $\widetilde{e}_j := (e_j - e)(e_j + e)^{-1} = -C_e(-e_j) = C_e^{-1}(e_j)$ that $C_e(\widetilde{e}_j) = e_j$. An explicit calculation in the associative Jordan algebra generated by $c_1, \dots, c_k$ quickly shows that

$$\widetilde{e}_j = -(e - e_j) = -e + e_j = -c_{j+1} - \cdots - c_k.$$

This completes the proof. □

For the special case $k = r$, i.e., $e \in S$, we have $V_0 = \{0\}$, so that $\mathcal{D}^C$ is the Siegel domain

$$\mathcal{D}^C = \{v = v_2 + v_1 \in V_2 \oplus V_1 = V \mid \mathrm{Re}(v_2 - F(v_1, v_1)) > 0\}$$

of type II. In this case the orbits of $Q''_e$ are represented by elements of the form $-e + e_j$, $j = 0, \dots, r$, so that we obtain only finitely many orbits. Observe that $\mathrm{rk}(-e+e_j) = r-j$, so that, even if $Q''$ is not connected, it cannot have less orbits in $e^\top$ than its identity component.

There would be no substantial gain in the proof of Theorem 1.7 by assuming that $\mathcal{D}$ is of tube-type. However, in the sequel we will need only a special case of the theorem, for which an easy direct proof (independent of the proof of Theorem 1.7) can be offered.

**Lemma 1.8.** *Suppose that $\mathcal{D}$ is irreducible and of tube-type, let $x \in S$ and $z \in \overline{\mathcal{D}}$, and assume that $x \top z$. There exists $g \in G$ and an integer $k$, $0 \leqslant k \leqslant r$, such that*

$$g(x) = e_r \quad \textit{and} \quad g(z) = -\sum_{j=k+1}^{r} c_j = e_k - e_r.$$

*(If $k = r$, use the convention that $\sum_{j=r+1}^{r} c_j = 0$.)*

*Proof.* As $G$ is transitive on $S$, there is no restriction in assuming that $x = e := e_r$. Now the transversality condition is equivalent to $z$ belonging to the domain $V^\times + e$ of the Cayley transform $C(z) := C_e(z) := (e + z)(e - z)^{-1}$ (see (1.13)). Set $\zeta = C(z)$ (Theorem 1.2). Then $\zeta \in E_+ + iE$. The point $e$ is sent by the Cayley transform "to infinity", in such a way that the stabilizer of $e$ in $G$ corresponds via conjugation by the Cayley transform to a subgroup of the affine group of $E^{\mathbb{C}}$, denoted by $Q_e^C$, namely, the semidirect product of the translations by an element of $iE$ and the group $G(E_+)$ (after complexification to $E^{\mathbb{C}}$ of its action on $E$). By using a translation, we see that in the

$Q_e^C$-orbit of $\zeta$, there is an element of the form $\eta \in E_+$. Since $\mathcal{D}$ is irreducible, the $G(E_+)$-orbits in $E_+$ are known to be exactly the $r+1$ orbits of the elements $e_k = \sum_{j=1}^k c_j$, with $k = 0, 1, \ldots, r$ (see [FK94, Prop. IV.3.2]). But now the inverse Cayley transform of the element $\sum_{j=1}^{k} c_j$ is the element $e_k - e = -\sum_{j=k+1}^{r} c_j$. Hence the result. □

## 2. Transversality and faces

In this section we keep the notation from Section 1. In particular, $\mathcal{D}$ is a circular irreducible bounded symmetric domain of rank $r$ in $V$. The main result of this section is that transversality of two elements $x, y \in \overline{\mathcal{D}}$ is equivalent to the geometric property that $x$ and $y$ do not lie in a proper face of the compact convex set $\overline{\mathcal{D}}$ (Theorem 2.6).

**Definition 2.1.** (a) We call a nonempty convex subset $F$ of a convex set $C$ a *face* if for $0 < t < 1$ and $c, d \in C$ the relation $tc + (1-t)d \in F$ implies $c, d \in F$. We write $\mathcal{F}(C)$ for the set of nonempty faces of $C$. A face $F$ is called *exposed* if there exists a linear functional $f \colon V \to \mathbb{R}$ with

$$F = f^{-1}(\max f(C)).$$

An *extreme point* $e \in C$ is a point for which $\{e\}$ is a face, i.e., $tc + (1-t)d = e$ for $c, d \in C$ and $0 < t < 1$ implies $c = d = e$. We write $\mathrm{Ext}(C)$ for the set of extreme points of $C$.

The set of all faces of $C$ has a natural order structure given by set inclusion whose maximal element is $C$ itself. All extreme points of $C$ are minimal elements of this set, but $C$ need not have any extreme points.

Obviously, the intersection of any family of faces is a face. We thus define for a subset $M \subseteq C$ the *face generated by* $M$ by

$$\mathrm{Face}(M) := \bigcap \{F \subseteq C \mid F \in \mathcal{F}(C), M \subseteq F\}.$$

(b) For a convex set $C$ in the vector space $V$ we write

$$\mathrm{algint}(C) := \{x \in C \mid (\forall v \in C - C)(\exists \varepsilon > 0)\ x + [0, \varepsilon]v \subseteq C\}$$

for its *algebraic interior*. If $V$ is finite-dimensional , then $\mathrm{algint}(C)$ is the interior of $C$ in the affine subspace it generates.

**Remark 2.2.** (a) Suppose that $C$ is a convex subset of a finite-dimensional vector space having nonempty interior. Then all proper faces of $C$ are contained in the boundary $\partial C$ and, conversely, the Hahn–Banach Separation Theorem implies that each boundary point is contained in a proper exposed face.

(b) For any nonempty convex subset of a finite-dimensional real vector space the algebraic interior is nonempty. Hence, if $x$ belongs to the algebraic interior of a face $F$, then $F$ is generated by $\{x\}$.

(c) Since every face $E$ of a face $F$ of $C$ is also a face of $C$, faces of exposed faces of $C$ are faces of $C$. On the other hand, every proper face is contained in an exposed face (see (a)), so that we obtain inductively that, for each face $F$, there exists a sequence of faces

$$F_0 = F \subseteq F_1 \subseteq \cdots \subseteq F_n = C$$

for which $F_i$ is an exposed face of $F_{i+1}$ for $i = 0, \ldots, n-1$.

**Proposition 2.1.** *The proper faces of the convex set $\overline{\mathcal{D}}$ are the closures of the holomorphic arc-components in $\partial\mathcal{D}$ and the Shilov boundary is the set of extreme points of $\overline{\mathcal{D}}$.*

*In particular, the group $G$ acts on the set $\mathcal{F}(\overline{\mathcal{D}})$ of faces of $\overline{\mathcal{D}}$.*

*Proof.* For the fact that $S$ is the set of extreme points of $\overline{\mathcal{D}}$ we refer to [Lo77, Theorem 6.5].

Next we use [Sa80, Lemma III.8.11, Theorem III.8.13] to see that the proper exposed faces $F$ of $\overline{\mathcal{D}}$ are the closures of the holomorphic arc-components in $\partial\mathcal{D}$. Since the action of the group $G$ on $\overline{\mathcal{D}}$ permutes the holomorphic arc-components in $\partial\mathcal{D}$, it also permutes the exposed faces of $\overline{\mathcal{D}}$.

We now claim that each face of $\overline{\mathcal{D}}$ is exposed. Since every face $F$ of $\overline{\mathcal{D}}$ is generated by a suitable element $x \in F$ (Remark 2.2), it suffices to show that the face generated by any element $x \in \partial\mathcal{D}$ is exposed. Let $A_x$ be the holomorphic arc-component of $\partial\mathcal{D}$ containing $x$. Then $\overline{A_x}$ is an exposed face of $\overline{\mathcal{D}}$ with $\operatorname{algint}(\overline{A_x}) = A_x$ (Theorem 1.1). Therefore the face generated by $x$ coincides with $\overline{A_x}$, showing that every face of $\overline{\mathcal{D}}$ is exposed. □

**Remark 2.3.** From the preceding proposition we know that the map $F \mapsto \operatorname{algint}(F)$ is a $G$-equivariant bijection between the set $\mathcal{F}(\overline{\mathcal{D}})$ of faces of $\overline{\mathcal{D}}$ and the set of holomorphic arc-components in $\overline{\mathcal{D}}$.

If $\mathcal{D}$ is irreducible, we define the *rank of a face* by $\operatorname{rk} F := k$ if $\operatorname{algint}(F)$ consists of elements of rank $k$. Since two holomorphic arc-components are conjugate under $G$ if and only if their elements have the same rank (see Theorem 1.1), the rank function

$$\operatorname{rk}\colon \mathcal{F}(\overline{\mathcal{D}}) \longrightarrow \{0, \ldots, r\}$$

classifies the $G$-orbits in $\mathcal{F}(\overline{\mathcal{D}})$. The stabilizer of a proper face (resp., a holomorphic arc-component in $\partial\mathcal{D}$) is a maximal parabolic subgroup of $G$ ([Sa80, Cor. III.8.6]).

If $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_m$ is a direct product of the irreducible domains $\mathcal{D}_j$, then each face $F$ of $\overline{\mathcal{D}}$ is a product $F_1 \times \cdots \times F_m$ of faces $F_j \in \mathcal{F}(\overline{\mathcal{D}}_j)$, so that the $G$-orbits in

$$\mathcal{F}(\mathcal{D}) \cong \mathcal{F}(\mathcal{D}_1) \times \cdots \times \mathcal{F}(\mathcal{D}_m)$$

are classified by the $m$-tuple $(\operatorname{rk} F_1, \ldots, \operatorname{rk} F_m)$.

In the following we shall prove that for two elements $x, y \in \overline{\mathcal{D}}$ transversality is equivalent to the *geometric transversality relation* $\operatorname{Face}(x, y) = \overline{\mathcal{D}}$. We start with the easy implication.

**Proposition 2.2.** *If $x, y \in \overline{\mathcal{D}}$ are transversal, then they are not contained in a proper face, i.e.,* $\operatorname{Face}(x, y) = \overline{\mathcal{D}}$.

*Proof.* If $x$ and $y$ are not geometrically transversal, then $F := \operatorname{Face}(x, y)$ is a proper face of $\overline{\mathcal{D}}$, hence of the form

$$F = F_e = e + (\overline{\mathcal{D}} \cap V_0(e)) = (e + V_0(e)) \cap \overline{\mathcal{D}}$$

for some tripotent $e \in V$ (Theorem 1.1, Proposition 2.1, and [Sa80, Lemma III.8.10] for the second equality). Then $x, y \in F$ implies that $x, y \in e + V_0(e)$, so that Remark 1.2 leads to $B(x, y).e = 0$. Thus $x$ and $y$ are not transveral. This proves the assertion. □

**Example 2.4.** We consider the $r$-dimensional polydisk

$$\mathcal{D} := \Delta^r := \{z \in \mathbb{C}^r \mid \max_j |z_j| < 1\} \subseteq V = \mathbb{C}^r.$$

Let $(c_1, \ldots, c_r)$ denote the canonical basis of $\mathbb{C}^r$. The corresponding Jordan triple structure is given by

$$\{x, y, z\} = (x_1\overline{y_1}z_1, \ldots, x_r\overline{y_r}z_r).$$

An element $z \in \mathbb{C}^r$ is a tripotent if $|z_j|^2 z_j = z_j$ holds for each $j$, which means that either $z_j = 0$ or $|z_j| = 1$. We have

$$\operatorname{rk} z = |\{j \mid |z_j| = 1\}|,$$

and the tripotents of maximal rank form the $n$-dimensional torus $S = \mathbb{T}^n$, the Shilov boundary of $\Delta^r$.

Since the faces of $\overline{\mathcal{D}} = \overline{\Delta^r}$ are Cartesian products of faces of the closed unit disk

$$\overline{\Delta} = \{z \in \mathbb{C} \mid |z| \leqslant 1\},$$

each face $F \in \mathcal{F}(\overline{\Delta^r})$ is a product $F_1 \times \cdots \times F_r$ of closed unit disks and points in the boundary of $\Delta$. For a subset $M \subseteq \overline{\Delta^r}$, it follows that the face generated by $M$ is given by

$$\operatorname{Face}(M) = F_1 \times \cdots \times F_r, \quad F_j = \begin{cases} \{s\} & \text{if } m_j = s \in \partial\Delta \text{ for all } m \in M, \\ \Delta & \text{otherwise.} \end{cases}$$

It follows, in particular, that $x, y \in \overline{\mathcal{D}}$ are contained in a proper face if and only if $x_j = y_j \in \partial\Delta$ holds for some $j$.

For $k \leqslant r$ we consider the tripotent $e_k := c_1 + \ldots + c_k$. Then

$$V_2 = \mathbb{C}^k \times \{0\}^{r-k} \quad \text{and} \quad V_0 = \{0\}^k \times \mathbb{C}^{r-k}.$$

An element $x \in \overline{\Delta^r}$ is transversal to $e_k$ if and only if $e_k - (x_1, \ldots, x_k, 0, \ldots, 0)$ is invertible in the unital Jordan algebra $(V_2, e_k)$, which means that the first $k$ components of $x$ are different from 1 (Remark 1.3). That this is not the case means that one component $x_j$, $j \leqslant k$, equals 1, and therefore $\operatorname{Face}(e_k, x) \neq \overline{\mathcal{D}}$. If, conversely, $\operatorname{Face}(e_k, x) \neq \overline{\mathcal{D}}$, then $e_k, x$ are contained in a proper face of $\overline{\Delta^r}$ which implies that $x_j = 1$ for some $j \leqslant k$.

**Proposition 2.3.** *Let $e \in V$ be a tripotent, $V = \sum_{j=0}^{2} V_j$ the corresponding Peirce decomposition, and $p_j\colon V \to V_j$ the projection along the other Peirce components. Then each $V_j$ is a positive Hermitian Jordan triple and we have*

$$\mathcal{D}_j = V_j \cap \mathcal{D} = p_j(\mathcal{D}).$$

*In particular, each map $p_j\colon V \to V_j$ is a contraction with respect to the spectral norms determined by the domains $\mathcal{D}$ and $\mathcal{D}_j$.*

*Proof.* Let $\langle\cdot,\cdot\rangle$ be an associative Hermitian scalar product on $V$ (Definition 1.4). Then the Peirce decomposition is orthogonal with respect to $\langle\cdot,\cdot\rangle$, so that it provides an orthogonal decomposition of $V$ into three Jordan subtriples ([Lo77, Theorem 3.13]).

Clearly the restriction of the scalar product to each $V_j$ provides an associative scalar product on $V_j$ and for each $v \in V_j$ the operator $v\Box v$ is positive semidefinite on $V$, which implies, in particular, that its restriction to $V_j$ is positive semidefinite. Hence each $V_j$ is a positive Hermitian Jordan triple.

According to [Lo77, Theorem 3.17], the inclusion maps $V_j \hookrightarrow V$ are isometric with respect to the spectral norm, which means that

$$\mathcal{D}_j = V_j \cap \mathcal{D} = \{z \in V_j \mid |z| < 1\}$$

holds for the corresponding bounded symmetric domains.

To see that the projections $p_j$ are contractive with respect to the spectral norm, let $v \in V$ and $v_j = p_j(v)$ its component in $V_j$. For each unit vector $w \in V_j$ the orthogonality of the Peirce decomposition implies that

$$\langle v\Box v \cdot w, w\rangle = \sum_{k,l=0}^{2} \langle v_k \Box v_l \cdot w, w\rangle = \sum_{k=0}^{2} \langle v_k \Box v_k \cdot w, w\rangle \geqslant \langle v_j \Box v_j \cdot w, w\rangle,$$

which leads for the spectral norm $|v_j|$ to

$$\begin{aligned} |v_j|^2 = \|v_j\Box v_j\|_{V_j} &= \sup\{\langle v_j\Box v_j.w, w\rangle \mid w \in V_j, \langle w,w\rangle = 1\} \\ &\leqslant \sup\{\langle v\Box v.w, w\rangle \mid w \in V_j, \langle w,w\rangle = 1\} \\ &\leqslant \sup\{\langle v\Box v.w, w\rangle \mid w \in V, \langle w,w\rangle = 1\} = |v|^2. \end{aligned}$$

Since the inclusion $V_j \hookrightarrow V$ is isometric, $p_j$ is a contraction with respect to the spectral norm and, therefore, $\mathcal{D}_j \subseteq p_j(\mathcal{D}) \subseteq \mathcal{D}_j$ proves equality. □

**Corollary 2.4.** *If $F$ is a proper face of $\overline{\mathcal{D}}_j$, then $p_j^{-1}(F)$ is a proper face of $\overline{\mathcal{D}}$.*

**Definition 2.5.** Suppose that $e \in V$ is a tripotent with $V_2(e) = V$, so that $Q(e)$ is an antilinear involution on $V$ turning $(V, e, Q(e))$ into an involutive unital Jordan algebra. As in Section 1, we endow $V$ with the spectral norm $|z|$ whose open unit ball is $\mathcal{D}$.

A *state of the unital involutive Jordan algebra $V$* is a linear functional $f\colon V \to \mathbb{C}$ with

$$1 = f(e) = \|f\| := \sup|f(\mathcal{D})|.$$

**Remark 2.6.** If $f$ is a state on $V$ and $y \in \overline{\mathcal{D}}$ with $f(y) = 1$, then $e$ and $y$ lie in the proper face $\{z \in \overline{\mathcal{D}} \mid \operatorname{Re} f(z) = 1\}$.

**Proposition 2.5.** *If $y \in \overline{\mathcal{D}}$ and $e - y$ is not invertible in the unital Jordan algebra $(V, e)$, there exists a state $f$ of $V$ with $f(y) = 1$.*

*Proof.* We endow $V$ with the associative scalar product $\langle z, w\rangle := \operatorname{tr}(z\Box w)$ (see Definition 1.4).

By assumption, $e - y$ is not invertible, which implies that the left multiplication $L(e-y) = (e-y)\Box e$ is not invertible. Pick $v \in \ker L(e-y)$ with $\langle v, v\rangle = 1$. We consider the linear functional

$$f\colon V \longrightarrow \mathbb{C}, \quad f(z) := \langle L(z)\cdot v, v\rangle,$$

satisfying $f(e) = \langle v, v\rangle = 1$ and

$$f(y) = \langle L(y) \cdot v, v\rangle = \langle L(e) \cdot v, v\rangle = f(e) = 1.$$

It remains to show that $f$ is a state. Let $E := \{z \in V \mid z^* = Q(e)z = z\}$ denote the Euclidean Jordan algebra with $V \cong E \otimes_{\mathbb{R}} \mathbb{C}$ and unit element $e$. We write $E_+$ for the closed positive cone in $E$. This is the set of all those elements $z$ for which there exists a system $c_1, \ldots, c_k$ of orthogonal idempotents with $e = c_1 + \cdots + c_k$ and nonnegative real numbers $\lambda_j$ with

$$z = \sum_{j=1}^{k} \lambda_j c_j.$$

For such elements $z \in E_+$ we then have

$$f(z) = \sum_{j=1}^{k} \lambda_j \langle L(c_j) \cdot v, v\rangle = \sum_{j=1}^{k} \lambda_j \langle c_j \Box c_j \cdot v, v\rangle \geqslant 0$$

because $L(c_j) = c_j \Box e = c_j \Box c_j$ follows from $c_j \Box (e - c_j) = 0$ (1.2) and the operators $c_j \Box c_j$ are positive semidefinite on $V$ ([Lo77, Cor. 3.16]). We conclude that $f(E) \subseteq \mathbb{R}$, so that $f(z^*) = \overline{f(z)}$ for all $z \in V$.

From $Q(e)^{-1} = Q(e)$ we derive $Q(Q(e).z) = Q(e)Q(z)Q(e) = Q(e)Q(z)Q(e)^{-1}$, so that $Q(e)\colon z \mapsto z^*$ is a Jordan triple automorphism of $V$, hence an isometry for the spectral norm $|\cdot|$ on $V$. This implies that $Q(e)\mathcal{D} = \mathcal{D}$ and, therefore, that for $z = x + iy \in \mathcal{D}$, $x, y \in E$, we have

$$|x| = \tfrac{1}{2}|z + z^*| \leqslant \tfrac{1}{2}(|z| + |z^*|) = |z|.$$

For the map $\mathrm{Re}\colon V \to E, z \mapsto \frac{1}{2}(z + z^*)$ this means that $\mathcal{D}_E := \mathcal{D} \cap E = \mathrm{Re}(\mathcal{D})$.

For the functional $f$ we thus obtain

$$\|f\| = \sup |f(\mathcal{D})| = \sup \mathrm{Re}\, f(\mathcal{D}) = \sup f(\mathrm{Re}\,\mathcal{D}) = \sup f(\mathcal{D}_E).$$

In view of the Spectral Theorem for Euclidean Jordan algebras ([FK94]), we have

$$\mathcal{D}_E = (e - E_+) \cap (-e + E_+) \subseteq e - E_+,$$

so that $f(z) \geqslant 0$ for $z \in E_+$ leads to $\|f\| = \sup f(\mathcal{D}_E) = f(e) = 1$. This means that $f$ is a state. $\Box$

**Theorem 2.6.** *Two elements $x, y \in \overline{\mathcal{D}}$ are transversal if and only if they are not contained in a proper face, i.e.,*

$$x \top y \iff \mathrm{Face}(x, y) = \overline{\mathcal{D}}.$$

*Proof.* In view of Proposition 2.1, geometric transversality is also invariant under the action of the group $G$. On the other hand, transversality is invariant under $G$ ([CO01]), so that it suffices to assume that $x = e$ is a Jordan tripotent. In view of Proposition 2.2, it suffices to show that if $e$ is not transversal to $y \in \overline{\mathcal{D}}$, then both $e$ and $y$ lie in a proper face of $\overline{\mathcal{D}}$.

For $e = 0$ we have $\mathrm{Face}(x, e) = \overline{\mathcal{D}}$ because $e \in \mathcal{D} = \mathrm{algint}(\overline{\mathcal{D}})$ and also $e \top x$ for all $x \in \overline{\mathcal{D}}$ because $B(x, e) = \mathrm{id}_V$.

We may therefore assume that $e \neq 0$. We have to show that if $e$ and $y$ are not transversal, then they are contained in a proper face of $\overline{\mathcal{D}}$. That $y$ is not transversal to $e$ is equivalent to the element $e - y_2$ being not invertible in the unital Jordan algebra $V_2(e)$ (Remark 1.3). In view of Proposition 2.5, combined with Remark 2.6, $e$ and $y_2$ are contained in a proper face $F$ of the convex set $\overline{\mathcal{D}}_2$. Hence $e$ and $y$ are contained in the proper face $p_2^{-1}(F)$ of $\overline{\mathcal{D}}$ (Corollary 2.4). □

**Example 2.7.** Let $p, q \in \mathbb{N}$, $r := \min(p, q)$, and $\|\cdot\|$ denote the Euclidean norm on $\mathbb{C}^p$ (resp., $\mathbb{C}^q$). On the matrix space $V := \mathrm{M}_{p,q}(\mathbb{C}) \cong \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^p)$ we write $|X|$ for the corresponding operator norm. Then

$$\mathcal{D} := \{X \in \mathrm{M}_{p,q}(\mathbb{C}) \mid |X| < 1\}$$

is a bounded symmetric domain. The pseudo-unitary group $\mathrm{U}_{p,q}(\mathbb{C})$ acts transitively on $\mathcal{D}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z := (az + b)(cz + d)^{-1},$$

the effectivity kernel of this action is $\mathbb{T}\mathbf{1}$, so that $G = \mathrm{Aut}(\mathcal{D})_0 \cong \mathrm{PU}_{p,q}(\mathbb{C})$. The 3-grading of $\mathfrak{g}_{\mathbb{C}}$ is induced by the 3-grading of $\mathfrak{gl}_{p+q}(\mathbb{C})$ given by

$$\mathfrak{gl}_{p+q}(\mathbb{C})_+ = \begin{pmatrix} 0 & \mathrm{M}_{p,q}(\mathbb{C}) \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{gl}_{p+q}(\mathbb{C})_0 = \begin{pmatrix} \mathfrak{gl}_p(\mathbb{C}) & 0 \\ 0 & \mathfrak{gl}_q(\mathbb{C}) \end{pmatrix},$$

and

$$\mathfrak{gl}_{p+q}(\mathbb{C})_- = \begin{pmatrix} 0 & 0 \\ \mathrm{M}_{q,p}(\mathbb{C}) & 0 \end{pmatrix}.$$

We further have

$$\mathfrak{u}_{p,q}(\mathbb{C}) = \Big\{ \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} \mid a^* = -a, d^* = -d \Big\}.$$

The vector field associated to the one-parameter group given by $\exp\left(t \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ is given by $z \mapsto b - az - zd - zcz$, so that the Jordan triple structure on $V = \mathrm{M}_{p,q}(\mathbb{C})$ satisfies $Q(z)(w) = zw^*z$, which leads to

$$\{a, b, c\} = \tfrac{1}{2}(ab^*c + cb^*a).$$

In particular, the Bergman operator satisfies

$$\begin{aligned} B(v, w)z &= z - 2v\Box w.z + Q(v)Q(w)z = z - (vw^*z + zw^*v) + v(wz^*w)^*v \\ &= (\mathbf{1} - vw^*)z(\mathbf{1} - w^*v). \end{aligned}$$

From that it follows that $v \top w$ is equivalent to the invertibility of $\mathbf{1} - w^* v$ in the algebra $\mathrm{M}_q(\mathbb{C})$.

An element $e \in \mathrm{M}_{p,q}(\mathbb{C})$ is a tripotent if and only if $ee^*e = e$, which implies that $ee^*$ and $e^*e$ are orthogonal projections, and that $e$ defines a partial isometry $\mathbb{C}^q \to \mathbb{C}^p$. If $K := \ker(e)$ and $R := \mathrm{im}(e)$, then the face $F_e$ of $\overline{\mathcal{D}}$ consists of all matrices $z \in \overline{\mathcal{D}}$ with $z \cdot v = e \cdot v$ for $v \in \ker(e)^\perp$. For $k = \mathrm{rank}(e)$ and an orthonormal basis $v_1, \dots, v_k$ of $\ker(e)^\perp$ and $w_i := e \cdot v_i$, we have

$$F_e = \{z \in \overline{\mathcal{D}} \mid (\forall i)\ \langle z v_i, w_i \rangle = 1\}.$$

From this description of the faces of $\overline{\mathcal{D}}$ it follows that an element $z \in \overline{\mathcal{D}}$ is contained in a proper face if and only if its restriction to some one-dimensional subspace of $\mathbb{C}^q$ is isometric, i.e., if and only if $|z| = 1$. Two elements $z, w$ generate a proper face if and only if there exists a unit vector $v \in \mathbb{C}^q$ for which $z \cdot v = w \cdot v$ is a unit vector in $\mathbb{C}^p$.

A Jordan frame is given by the matrices $c_j := E_{jj}$, $j = 1, \dots, r$, with a single nonzero entry 1 in position $(j, j)$. The rank of $\mathcal{D}$ is $r$ and $e_r := c_1 + \cdots + c_r$ is a maximal tripotent with

$$S = G.e_r = \begin{cases} \{z \in \mathrm{M}_{p,q}(\mathbb{C}) \mid z^*z = \mathbf{1}\} & \text{if } q \leqslant p, \\ \{z \in \mathrm{M}_{p,q}(\mathbb{C}) \mid zz^* = \mathbf{1}\} & \text{if } p \leqslant q. \end{cases}$$

For $q \leqslant p$ this is the set of isometries $\mathbb{C}^q \hookrightarrow \mathbb{C}^p$ and for $p \leqslant q$ this is the set of all adjoints of isometries $\mathbb{C}^p \to \mathbb{C}^q$.

Let $e_k := c_1 + \cdots + c_k$ be the canonical tripotent of rank $k$. Writing an element $z \in \mathrm{M}_{p,q}(\mathbb{C})$ as a block matrix

$$z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$$

with $z_{11} \in \mathrm{M}_k(\mathbb{C}), z_{12} \in \mathrm{M}_{k,q-k}(\mathbb{C}), z_{21} \in \mathrm{M}_{p-k,k}(\mathbb{C}), z_{22} \in \mathrm{M}_{p-k,q-k}(\mathbb{C})$, we have

$$\begin{aligned} 2\{e, e, z\} = ee^*z + ze^*e &= \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} + \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2z_{11} & z_{12} \\ z_{21} & 0 \end{pmatrix}. \end{aligned}$$

This shows that

$$V_2(e_k) \cong \mathrm{M}_k(\mathbb{C}), \quad V_1(e_k) \cong \mathrm{M}_{k,q-k}(\mathbb{C}) \oplus \mathrm{M}_{p-k,k}(\mathbb{C}), \quad \text{and} \quad V_0(e_k) \cong \mathrm{M}_{p-k,q-k}(\mathbb{C}),$$

and therefore

$$F_e = \Big\{ \begin{pmatrix} \mathbf{1} & 0 \\ 0 & z \end{pmatrix} \mid z \in \mathrm{M}_{p-k,q-k}(\mathbb{C}), |z| \leqslant 1 \Big\}.$$

For $k = r$ we see, in particular, that $V_0(e_r) = 0$.

## 3. Orbits of triples in the Shilov boundary

In this section we obtain the key result for our classification of triples in $S$ in the tube-type case. We show that if $(c_1, \dots, c_r)$ is a Jordan frame in $E$, then each $G$-orbit in $S \times S \times S$ meets the Shilov boundary $T \cong \mathbb{T}^r$ of the corresponding polydisk. We further show that the polydisks arising in this result can also be characterized directly as the intersections of $\mathcal{D}$ with $r$-dimensional subspaces of $V$, or, equivalently, as isometric images of polydisks under affine maps $\mathbb{C}^r \to V$, mapping $\Delta^r$ isometrically into $\mathcal{D}$. In particular, we show that any such affine map is linear.

### 3.1. The conjugacy theorem

**Theorem 3.1.** *Suppose that $\mathcal{D} \subseteq V$ is of tube-type, $(c_1, \dots, c_r)$ is a Jordan frame in $V$, and*

$$T := S \cap \operatorname{span}\{c_1, \dots, c_r\} = \Big\{ \sum_{j=1}^{r} \lambda_j c_j \mid (\forall j)\ |\lambda_j| = 1 \Big\}$$

*is the corresponding $r$-torus in $S$. Then, for each triple $(e, f, g) \in S$, there exists a $g \in G$ with $g \cdot e, g \cdot f, g \cdot h \in T$.*

*Proof.* Since Jordan frames and $G$ decompose according to the decomposition of $\mathcal{D}$ into products of irreducible domains, it suffices to prove the assertion for irreducible domains. We prove the assertion by induction on the rank $r$ of $\mathcal{D}$. Observe that the algebraic interior of any face $F$ of $\mathcal{D}$ is a bounded symmetric space of tube-type. In fact, let $E$ be a Euclidean Jordan algebra which has $V$ as its complexification. Let $F$ be a face of rank $k$. Then $F$ contains a tripotent $c$ of rank $k$ and there exists a Jordan frame $(c_1, \dots, c_r)$ in $E$ such that $c = \sum_{j=1}^{k} \lambda_j c_j$ with $|\lambda_j| = 1$ for $1 \leqslant j \leqslant k$. Then $V_0(c)$ is the complexification of the Euclidean Jordan algebra $E_0(c) = E_0(c_1 + \cdots + c_k)$. For $z \in V_0(c)$, the spectral norm relative to $V_0(c)$ coincides with the spectral norm in $V$, and so $V_0(c) \cap \mathcal{D} = \mathcal{D}_0$ is the bounded symmetric domain of tube-type associated to the Euclidean Jordan algebra $E_0(c)$. As

$$\operatorname{algint}(F) = \operatorname{algint}(F_c) = c + (\mathcal{D} \cap V_0(c)) = c + \mathcal{D}_0,$$

we see that $\operatorname{algint}(F)$ is a bounded symmetric domain of tube-type.

*Case* 1: If $\operatorname{Face}(e, f, h)$ is proper, then its algebraic interior is a bounded symmetric domain of tube-type $\mathcal{D}'$ of smaller rank and $(e, f, h)$ are contained in its Shilov boundary. In fact, according to Theorem 1.1 and Proposition 2.1, for each face $F$ of $\overline{\mathcal{D}}$ corresponding to the holomorphic arc-component $A = \operatorname{algint}(F)$, the Shilov boundary of $A$ is given by

$$S_A = \operatorname{Ext}(\overline{A}) = \operatorname{Ext}(F) = \operatorname{Ext}(\overline{\mathcal{D}}) \cap F = S \cap F.$$

Since every element of $\operatorname{Aut}(\mathcal{D}')_0$ is the restriction of an element of $\operatorname{Aut}(\mathcal{D})$ [Sa80, Lemma III.8.1], in this case the result follows from the induction hypothesis if $r > 1$. If $r = 1$, then each proper face of $\overline{\mathcal{D}}$ is an extreme point, so that the assumption that $e, f, h$ lie in a proper face implies $e = f = h$. In this case we further have $c_1 \in S$, so that the assertion follows from the transitivity of the action of $G$ on $S$.

*Case* 2: We assume that some pair $(e, f)$, $(f, h)$ or $(e, h)$ is transversal. We may without loss of generality assume that $(e, f)$ is transversal. Then $\operatorname{Face}(e, f, h) \supseteq F(e, f) = \overline{\mathcal{D}}$ by Theorem 2.6, and $G.(e, f)$ contains $(e, -e)$ because $\operatorname{rk} f = \operatorname{rk} e = r$ (Lemma 1.8). Therefore the orbit of $(e, f, h)$ contains an element of the form $(e, -e, h)$. Now the assertion follows from the Spectral Theorem for unitary elements in $V$ (see [FK94, Prop. X.2.3]) and (A.4) in the Appendix.

*Case* 3: $\operatorname{Face}(e, f, h) = \overline{\mathcal{D}}$, but neither $(e, f)$, nor $(f, h)$ or $(e, h)$ is transversal. Since $G$ acts transitively on $S$, we may without loss of generality assume that $e = e_r = c_1 + \cdots + c_r$. Consider the proper face $F := \operatorname{Face}(f, h)$ of $\overline{\mathcal{D}}$. Then we have

$$\overline{\mathcal{D}} = \operatorname{Face}(e, f, h) = \operatorname{Face}(\{e\} \cup F),$$

and for any $x \in \operatorname{algint}(F)$ we obtain

$$\overline{\mathcal{D}} = \operatorname{Face}(\{e\} \cup F) = \operatorname{Face}(e, x),$$

which means that $e$ and $x$ are transversal (Theorem 2.6).

Now we need the classification of $G$-orbits in the set of transversal pairs, which shows that the pair $(e, x)$ is conjugate to an element of the form $(e, -e + e_j)$ (Lemma 1.8). The face

$$\begin{aligned} F' &= \operatorname{Face}(-e + e_j) = -\operatorname{Face}(e - e_j) = -(e - e_j) + (V_0(e - e_j) \cap \mathcal{D}) \\ &= (e_j - e) + (V_2(e_j) \cap \mathcal{D}) \end{aligned}$$

is a bounded symmetric domain of tube-type of rank $j$, and $(e, f, h)$ is conjugate to a triple of the form $(e, f', h')$ where $f', h'$ are two elements in the Shilov boundary of $F'$, where they are transversal because they generate $F'$ as a face (Theorem 2.6). Next we observe that the Peirce rules imply that by exponentiating elements of the centralizer of $e - e_j$ in $\mathfrak{g}$ we generate the identity component $G^0$ of the group $\operatorname{Aut}(\mathcal{D} \cap V_0(e - e_j))$ and its elements $g$ act on $e_j - e + z$ by

$$g \cdot (e_j - e + z) = (e_j - e) + g \cdot z$$

because they commute with the translation $t_{e_j - e}$. Now we conclude the proof by applying the special case of transversal elements which has already been taken care of, to see that the $G^0$-orbit of $(e, f', h')$ intersects $T$. $\square$

**Remark 3.1.** If $\mathcal{D}$ is not of tube-type, then the Cayley transform $C = C_e$ leads to a realization of $\mathcal{D}$ as a Siegel domain $\mathcal{D}^C$ of type II, and since $C_e(-e) = 0$, the stabilizer $G_{e,-e}$ of $\pm e$ in $G$ corresponds to the stabilizer $Q^C_{e,-e} := C_e(G_{e,-e})$ of 0 in the affine group $Q^C_e$, and the identity component of this group is $G(E_+)_0 K_e$ (see the proof of Theorem 1.7). The Shilov boundary of $\mathcal{D}^C$ is the set

$$\{(v_2, v_1) \in V = V_2 \oplus V_1 \mid \operatorname{Re} v_2 = F(v_1, v_1)\},$$

and from this description it is clear that no element $v_2 + v_1$ with $v_1 \neq 0$ is conjugate under $Q^C_{e,-e}$ to an element in $\operatorname{span}_{\mathbb{R}}\{c_1, \ldots, c_r\} \subseteq V_2$. Therefore the condition that $\mathcal{D}$ is of tube-type is necessary for the conclusion of Theorem 3.1.

**Example 3.2.** The simplest example of a bounded symmetric domain not of tube-type is the matrix ball $\mathcal{D} \subseteq \mathbb{C}^n$ for $n > 1$. Its rank is $r = 1$ and in this case $G \cong \mathrm{PSU}_{n,1}(\mathbb{C})$ (see Example 2.7).

To $z \in \mathcal{D}$ we assign the one-dimensional subspace $L_z := \mathbb{C}\begin{pmatrix} z \\ 1 \end{pmatrix} \in \mathbb{C}^{n+1}$. Endowing $\mathbb{C}^{n+1}$ with the indefinite Hermitian form $h$ given by

$$h(z, w) := z_1\overline{w_1} + \cdots + z_n\overline{w_n} - z_{n+1}\overline{w_{n+1}},$$

we see that $\mathcal{D}$ corresponds to the set of lines on which $h$ is negative definite, and its Shilov boundary, the sphere $S \cong \mathbb{S}^{2n-1}$, corresponds to the set of isotropic lines. In this picture the action of $SU_{n,1}(\mathbb{C})$ on $\mathcal{D}$ comes from the natural action of this group on the one-dimensional subspaces of $\mathbb{C}^{n+1}$.

Fixing a unit vector $e \in S$, the pair $(e, -e)$ corresponds to two different isotropic lines $L_e$ and $L_{-e}$ in $\mathbb{C}^{n+1}$, and the stabilizer of this pair in $U_{n,1}(\mathbb{C})$ fixes the nondegenerate subspace $L_e + L_{-e}$, and also its orthogonal complement of dimension $n-1$. We conclude that $U_{n,1}(\mathbb{C})_{e,-e} \cong \mathbb{R}^\times \times U_{n-1}(\mathbb{C})$, and that no line $L_z \not\subseteq L_e + L_{-e}$ can be moved by $U_{n,1}(\mathbb{C})$ into the plane $L_e + L_{-e}$. On the other hand, the set of isotropic lines in the plane $L_e + L_{-e}$ corresponds to the circle in $S$ obtained by intersecting $S$ with the boundary of a one-dimensional disk $\Delta \subseteq \mathcal{D}$ of size 1 which, in particular is a polydisk of maximal rank. This shows quite directly that there are triples in $S$ that cannot be moved into the one-dimensional space $\mathbb{C}e$, so that Theorem 3.1 does not hold.

That Theorem 3.1 fails in this context, can be expressed quantitatively by the observation that

$$F(\mathbb{C}v_1, \mathbb{C}v_2, \mathbb{C}v_3) := \frac{h(v_1, v_2)h(v_2, v_3)h(v_3, v_1)}{h(v_2, v_1)h(v_3, v_2)h(v_1, v_3)}$$

is a well defined function on the set of triples of pairwise different isotropic lines in $\mathbb{C}^{n+1}$ which is invariant under the pseudo-unitary group $U_{n,1}(\mathbb{C})$. The function $F$ is related to the *Cartan invariant* (for a presentation and a generalization of this invariant we refer to [Cl05]).

**Example 3.3.** The matrix ball $\mathcal{D} \subseteq M_n(\mathbb{C})$ is a symmetric domain of tube-type with Shilov boundary $S = U_n(\mathbb{C})$, the unitary group. The maximal polydisks in $\mathcal{D}$ are obtained by intersecting $\mathcal{D}$ with the set of all matrices that are diagonal with respect to some fixed orthonormal basis of $\mathbb{C}^n$ with respect to the standard scalar product. A particular Jordan frame consists of the matrix units $c_j := E_{jj}$, $j = 1, \dots, n$, whose span is the set of diagonal matrices. Therefore Theorem 3.1 states that each triple $(s_1, s_2, s_3)$ of unitary matrices can be diagonalized by an element $g \in U_{n,n}(\mathbb{C})$, acting on $U_n(\mathbb{C})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = (az + b)(cz + d)^{-1}.$$

The compact subgroup $U_n(\mathbb{C}) \times U_n(\mathbb{C})$ acts linearly by $(a, d).z = azd^{-1}$, and under this group each pair $(s_1, s_2)$ is conjugate to a pair of the form $(\mathbf{1}, s_2')$, where the stabilizer of $\mathbf{1}$ is the diagonal subgroup, acting on the second component by $(a, a^{-1}) \cdot s_2 = as_2a^{-1}$, so that $s_2'$ can be diagonalized by conjugating with a suitable element $a \in U_n(\mathbb{C})$. This means that the diagonalizability of pairs reduces to classical linear algebra, but diagonalizability of triples requires the nonlinear action of $U_{n,n}(\mathbb{C})$ and Theorem 3.1.

A classification of the conjugation orbits of $U_n(\mathbb{C})$ in $U_n(\mathbb{C})^2$ is given in [FMS04], but since $U_n(\mathbb{C})$ is much smaller than $U_{n,n}(\mathbb{C})$, this classification leads to infinitely many orbits.

### 3.2. Polydisk in bounded symmetric domains

Let $\mathcal{D} \subseteq V$ be a bounded symmetric domain of rank $r$ and $\Delta^r \subseteq \mathbb{C}^r$ the $r$-dimensional unit polydisk. We endow $\mathbb{C}^r$ with the metric defined by the sup-norm

$$|z| := \max\{|z_1|, \dots, |z_r|\}$$

and $V$ by the metric defined by the spectral norm, also denotes $|z|$.

**Theorem 3.2.** *Any affine isometric map $f\colon \mathbb{C}^r \to V$ mapping $\overline{\Delta^r}$ into $\overline{\mathcal{D}}$ is linear and preserves the rank, i.e., for each $x \in \overline{\Delta^r}$ we have*

$$\operatorname{rk} f(x) = \operatorname{rk} x.$$

*Moreover, it is a morphism of Jordan triples and $f(e_1, \ldots, e_r)$ is a Jordan frame.*

*Proof.* Let $x_0 := f(0)$. Then $\ell(x) := f(x) - x_0$ defines an isometric linear map $\ell\colon \overline{\Delta}^r \to V$. Since $\ell$ is linear and isometric, it maps the open unit ball $\Delta^r$ in $\mathbb{C}^r$ into the open unit ball $\mathcal{D}$ of $(V, |\cdot|)$, so that it also maps $\overline{\Delta^r}$ isometrically into $\overline{\mathcal{D}}$.

Let $f_1, \ldots, f_r$ denote the images of the canonical basis in $\mathbb{C}^r$ under $\ell$. Then the coordinate projections

$$\chi_j\colon L := \operatorname{span}\{f_1, \ldots, f_r\} = \operatorname{im}(\ell) \longrightarrow \mathbb{C}, \quad \sum_j \lambda_j f_j \mapsto \lambda_j,$$

are linear maps with $\|\chi_j\| = 1$ because $\ell\colon \mathbb{C}^r \to L$ is an isometric inclusion. Using the Hahn–Banach theorem, we find extensions $\chi_j\colon V \to \mathbb{C}$ with the same norm. Then the map

$$\chi := (\chi_1, \ldots, \chi_r)\colon V \longrightarrow \mathbb{C}^r$$

satisfies $\|\chi\| = 1$ and $\chi \circ \ell = \mathrm{id}$. It follows in particular that $\chi(\mathcal{D}) \subseteq \Delta^r$.

Since $\chi$ maps $\overline{\mathcal{D}}$ into $\overline{\Delta^r}$, we have an order-preserving map

$$\chi^*\colon \mathcal{F}(\overline{\Delta^r}) \longrightarrow \mathcal{F}(\overline{\mathcal{D}}), \quad F \mapsto \chi^{-1}(F),$$

and the corresponding map

$$\ell^*\colon \mathcal{F}(\overline{\mathcal{D}}) \longrightarrow \mathcal{F}(\overline{\Delta^r}), \quad F \mapsto \ell^{-1}(F),$$

satisfies

$$\ell^* \circ \chi^* = (\chi \circ \ell)^* = \mathrm{id}\,.$$

We conclude that $\chi^*$ is an order-preserving injection. This entails, in particular, that for each strictly increasing chain

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_r$$

of faces of $\overline{\Delta^r}$, the images under $\chi^*$ form a strictly increasing chain of faces of $\overline{\mathcal{D}}$. Since $r$ is the rank of $\mathcal{D}$, the maximal chains in $\mathcal{F}(\overline{\mathcal{D}})$ are of length $r$, which implies that $\chi^*$ preserves the rank of faces. Since the rank of an element $x \in \overline{\mathcal{D}}$ coincides with the rank of the face it generates, we further see that for $z \in \overline{\Delta^r}$ we have

$$\operatorname{rk} \ell(z) = \operatorname{rk} \operatorname{Face}(\ell(z)) = \operatorname{rk} \ell^*(\operatorname{Face}(z)) = \operatorname{rk}(\operatorname{Face}(z)) = \operatorname{rk} z.$$

Therefore $\ell$ preserves the rank.

Moreover, $\ell$ maps the Shilov boundary $\mathbb{T}^r$, consisting of the elements of maximal rank, into the Shilov boundary $S$ of $\mathcal{D}$. The relation

$$f(\overline{\Delta^r}) = x_0 + \ell(\overline{\Delta^r}) \subseteq \overline{\mathcal{D}}$$

implies

$$-x_0 + \ell(\overline{\Delta^r}) = -(x_0 + \ell(\overline{\Delta^r})) \subseteq \overline{\mathcal{D}},$$

so that for each $z \in \mathbb{T}^r$ we have

$$\ell(z) = \tfrac{1}{2}((\ell(z) + x_0) + (\ell(z) - x_0)) \in S,$$

so that $S = \operatorname{Ext}(\overline{\mathcal{D}})$ implies $x_0 = 0$, and hence $f = \ell$ is linear.

For $i \in \{1, \ldots, r\}$ we consider the corresponding face

$$F := \{z \in \overline{\Delta^r} \mid z_i = 1\} \in \mathcal{F}(\overline{\Delta^r}).$$

Then $F$ is the closure of an $(r-1)$-dimensional affine polydisk, and $f|_F : F \to \overline{\mathcal{D}}$ is an affine isometry into a face $F_c \in \mathcal{F}(\overline{\mathcal{D}})$, where $c$ is a primitive tripotent (Theorem 1.1, Proposition 2.1). Applying the first part of the proof with $\mathcal{D}$ replaced by $\operatorname{algint}(F')$ to the corresponding map

$$\overline{\Delta}^{r-1} \longrightarrow F_c - c, \quad z \mapsto f(z_1, \ldots, z_{i-1}, 1, z_i, \ldots, z_r) - c,$$

we see that this map is linear, hence maps 0 to 0, which leads to $f(e_i) = c$. For $i \neq j$ the element $e_i + e_j \in \overline{\Delta^r}$ is contained in the face generated by $e_i$, which implies that $f(e_i + e_j) = f(e_i) + f(e_j)$ is contained in the face generated by $f(e_i)$. From Theorem 1.1 we now derive

$$f(e_j) = f(e_i + e_j) - f(e_i) \in V_0(f(e_i)),$$

so that the primitive tripotents $f(e_i)$, $i = 1, \ldots, r$, are mutually orthogonal. Hence the linear map $f: \mathbb{C}^r \to V$ is a morphism of Lie triples systems. □

**Corollary 3.3.** *Suppose that $\mathcal{D}_1 \subseteq V_1$ and $\mathcal{D}_2 \subseteq V_2$ are circular bounded symmetric domains of the same rank. Then any affine isometric map $f: V_1 \to V_2$ mapping $\overline{\mathcal{D}}_1$ into $\overline{\mathcal{D}}_2$ is linear and rank-preserving.*

*Proof.* Let $r := \operatorname{rk} \mathcal{D}_1 = \operatorname{rk} \mathcal{D}_2$ and fix a polycylinder $\mathcal{D}_0 := \Delta^r \subseteq \mathcal{D}_1$ defined by a Jordan frame $(c_1, \ldots, c_r)$. For $V_0 := \operatorname{span}\{c_1, \ldots, c_r\}$ we then obtain by restriction an isometric map $f_0: V_0 \to V_2$ mapping $\overline{\mathcal{D}}_0 \to \overline{\mathcal{D}}_2$. In view of Theorem 3.2, this map is linear, which implies $f(0) = f_0(0) = 0$, and thus $f$ is linear.

Moreover, $f_0$ is rank-preserving by Theorem 3.2, which implies that $f$ is also rank-preserving. □

**Corollary 3.4.** *If $r = \operatorname{rank} \mathcal{D}$, then any isometric linear embedding $f: \Delta^r \hookrightarrow \mathcal{D}$ is equivariant in the sense that there exists a subgroup $G_1 \subseteq \operatorname{Aut}(\mathcal{D}_0)$ and a surjective homomorphism $G_1 \to \operatorname{Aut}(\Delta^r)_0 \cong \operatorname{PSU}_{1,1}(\mathbb{C})^r$ such that $f$ is equivariant with respect to the action of $G_1$ on $\Delta^r$ and $\mathcal{D}$.*

*Proof.* If $(e_1, \ldots, e_r)$ is the canonical basis in $\mathbb{C}^r$, then $(c_1, \ldots, c_r) := (f(e_1), \ldots, f(e_r))$ is a Jordan frame, so that

$$\mathfrak{g}_1 := \sum_{j=1}^{r} \mathfrak{g}_{c_j} \subseteq \mathfrak{g}$$

is isomorphic to $\mathfrak{su}_{1,1}(\mathbb{C})^r \cong \mathfrak{sl}_2(\mathbb{R})^r$ (see Remark 1.2), the Lie algebra of the group $\operatorname{Aut}(\Delta^r)_0 \cong \operatorname{PSU}_{1,1}(\mathbb{C})$. We may now put $G_1 := \langle \exp \mathfrak{g}_1 \rangle \subseteq G$, and the assertion follows. □

## 4. The Maslov index

To define the integers classifying the $G$-orbits in $S \times S \times S$, we need in particular the Maslov index, a certain $G$-invariant function $\iota\colon S \times S \times S \to \mathbb{Z}$. In this section we explain how the Maslov index can be defined for bounded symmetric domains of tube-type which are not necessarily irreducible, hence extending the definition given in [CO01], [CO03], [Cl04b]. Using Theorem 3.1, we further derive a list of properties of the Maslov index and show that it can be characterized in an axiomatic fashion by these properties. Actually, this was our original motivation to prove Theorem 3.1.

Let us first consider the case of the unit disk $\Delta$. Then the group $G$ is $\mathrm{PSU}_{1,1}(\mathbb{C})$ acting by homographies on $\Delta$, and its Shilov boundary is the unit circle $\mathbb{T}$. The *Maslov index*

$$\iota = \iota_{\mathbb{T}} : \mathbb{T} \times \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{Z}$$

is defined by

- $\iota(x, y, z) = 0$ if two of the elements of the triplet coincide.
- $\iota(x, y, z) = \pm 1$ if $(x, y, z)$ is conjugate under $G$ to $(1, -1, \mp i)$.

If $\Delta^r$ denotes the $r$-polydisk, then the identity component of the group $\mathrm{Aut}(\Delta^r)$ is $G = \mathrm{PSU}_{1,1}(\mathbb{C})^r$ and the Shilov boundary of $\Delta^r$ is $\mathbb{T}^r$. The Maslov index $\iota = \iota_{\mathbb{T}^r} : \mathbb{T}^r \longrightarrow \mathbb{R}$ is defined by

$$\iota((x_1, \dots, x_r), (y_1, \dots, y_r), (z_1, \dots, z_r)) := \iota(x_1, y_1, z_1) + \cdots + \iota(x_r, y_r, z_r).$$

Now consider an irreducible bounded symmetric domain $\mathcal{D}$ of tube-type with Shilov boundary $S$. The Maslov index $\iota = \iota_S : S \times S \times S \longrightarrow \mathbb{Z}$ is defined in [CO01], [CO03], [Cl04b]. As the definition is involved, we won't repeat it here, but it has the following property, which, in the light of Theorem 3.1 and because of the invariance of this index under $G$, is characteristic: For any Jordan frame $(c_1, \dots, c_r)$, let

$$T = \Big\{ \sum_{j=1}^{r} t_j c_j \mid |t_j| = 1, 1 \leqslant j \leqslant r \Big\}$$

be the $r$-torus which is the Shilov boundary of the associated $r$-polydisk. Then, for any three points $x, y, z$ in $T$, one has

$$\iota_S(x, y, z) = \iota_T(x, y, z). \tag{4.1}$$

Last, we extend now the definition of the Maslov index to any bounded symmetric domain $\mathcal{D}$ in the following way. Assume that $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_m$ is the decomposition of $\mathcal{D}$ as a product of irreducible domains. Then the identity component of the group of biholomorphic automorphisms of $\mathcal{D}$ is the product

$$G = \mathrm{Aut}(\mathcal{D}_1)_0 \times \cdots \times \mathrm{Aut}(\mathcal{D}_m)_0,$$

and the Shilov boundary $S$ of $\mathcal{D}$ is the product $S = S_1 \times \cdots \times S_m$ of the corresponding Shilov boundaries. Then the Maslov index $\iota = \iota_S$ is defined by

$$\iota(x, y, z) := \iota_{S_1}(x_1, y_1, z_1) + \cdots + \iota_{S_r}(x_l, y_l, z_l).$$

**Theorem 4.1.** *The Maslov index has the following properties:*

(M1) *It is invariant under the group $G$.*

(M2) *It is an alternating function with respect to any permutation of the three arguments.*

(M3) *It satisfies the cocycle property $\iota(x, y, z) = \iota(x, y, w) - \iota(x, z, w) + \iota(y, z, w)$.*

(M4) *It is additive in the sense that if $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$, so that $S = S_1 \times S_2$, then*

$$\iota_S(x, y, z) = \iota_S((x_1, x_2), (y_1, y_2), (z_1, z_2)) = \iota_{S_1}(x_1, y_1, z_1) + \iota_{S_2}(x_2, y_2, z_2).$$

(M5) *If $\Phi : \mathcal{D}_1 \longrightarrow \mathcal{D}_2$ is an equivariant holomorphic embedding of bounded symmetric domains of tube-type of equal rank, then $\iota_{S_2} \circ \Phi = \iota_{S_1}$.*

(M6) *It is normalized by $\iota_{\mathbb{T}}(1, -1, -i) = 1$ for the Shilov boundary $\mathbb{T}$ of the unit disk $\Delta$.*

*Proof.* Properties (M1)–(M3) are known for irreducible domains ([CO01], [Cl04b]), and the extension of these properties to products of irreducible domains is obvious. Property (M4) obviously holds by the way we have defined the Maslov index.

For property (M5), let $r$ be the common rank of the two domains. We may assume that $\mathcal{D}_1$ and $\mathcal{D}_2$ are given in a circular realization as unit balls in spaces $V_1$ (resp., $V_2$). Then $\phi(0) \in \mathcal{D}_2$, and there is some $g_2 \in G_2 := \mathrm{Aut}(\mathcal{D}_2)_0$ with $g_2.\phi(0) = 0$. Then $\psi(z) := g_2.\phi(z)$ defines an equivariant embedding $\mathcal{D}_1 \to \mathcal{D}_2$ which is linear because $\psi(0) = 0$.

Let $(x, y, z) \in S_1$ and pick $g_1 \in G_1 := \mathrm{Aut}(\mathcal{D}_1)_0$ such that $g_1.(x, y, z)$ is contained in the span of a Jordan frame $(c_1, \ldots, c_r)$ (Theorem 3.1), hence in the Shilov boundary $T_1$ of the corresponding polydisk $\Delta^r$ in $\mathcal{D}_1$. From the equivariance of $\phi$ we derive the existence of some $\widetilde{g}_1 \in G_2$ with $\phi \circ g_1 = \widetilde{g}_1 \circ \phi$. Then $\psi(\Delta^r)$ is a maximal polydisk in $\mathcal{D}_2$ with Shilov boundary $T_2 := \psi(T_1)$, so that (4.2) implies that

$$\begin{aligned}
\iota_{S_1}(x, y, z) &= \iota_{S_1}(g_1 \cdot x, g_1 \cdot y, g_1 \cdot z) = \iota_{T_1}(g_1 \cdot x, g_1 \cdot y, g_1 \cdot z) \\
&= \iota_{T_2}(\psi(g_1 \cdot x), \psi(g_1 \cdot y), \psi(g_1 \cdot z)) = \iota_{S_2}(\psi(g_1 \cdot x), \psi(g_1 \cdot y), \psi(g_1 \cdot z)) \\
&= \iota_{S_2}(g_2\phi(g_1 \cdot x), g_2\phi(g_1 \cdot y), g_2\phi(g_1 \cdot z)) = \iota_{S_2}(\phi(g_1 \cdot x), \phi(g_1 \cdot y), \phi(g_1 \cdot z)) \\
&= \iota_{S_2}(\widetilde{g}_1\phi(x), \widetilde{g}_1\phi(y), \widetilde{g}_1\phi(z)) = \iota_{S_2}(\phi(x), \phi(y), \phi(z)).
\end{aligned}$$

Property (M6) is a consequence of the definition. □

**Remark 4.1.** Note that (M2) and (M3) mean that $\iota_S$ is a $\mathbb{Z}$-valued Alexander–Spanier 2-cocycle on $S$.

Before we turn to the general case in the following section, we recall the classification of triples in the circle, the Shilov boundary of the unit disk.

**Example 4.2.** We consider the case $\Delta := \{z \in \mathbb{C} \mid |z| < 1\}$. Then $G = \mathrm{PSU}_{1,1}(\mathbb{C})$ acts by

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \cdot z = (az + b)(cz + d)^{-1}.$$

The Shilov boundary is $S = \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Identifying $S$ with the projective line $\mathbb{P}_1(\mathbb{R})$ and $G$ with $\mathrm{PSL}_2(\mathbb{R})$, we immediately see that there are exactly two $G$-orbits in $S \times S$, represented by

$$(1, 1) \quad \text{and} \quad (1, -1),$$

i.e., the diagonal in $S \times S$ and the set $(S \times S)_\top$ of transversal pairs. Since the action of $G$ on $S$ preserves the orientation of a triple, it follows that we have six orbits in $S \times S \times S$, represented by

$$(1,1,1), \quad (1,1,-1), \quad (1,-1,1), \quad (1,-1,-1), \quad (1,-1,-i), \quad \text{and} \quad (1,-1,i).$$

**Remark 4.3.** As a function assigning to any triple in the Shilov boundary of any bounded symmetric domain $\mathcal{D}$ an integer, the Maslov index is uniquely determined by the properties (M1), (M2) and (M4)–(M6).

In view of Example 4.2, the Maslov index for $\mathcal{D} = \Delta$ is uniquely determined by (M1), (M2) and (M6). By (M4) it is also determined for polydisks.

If $\mathcal{D}$ is any bounded symmetric domain of rank $r$ and $(s_1, s_2, s_3) \in S \times S \times S$, then Theorem 3.1 implies that it can be conjugate by some $g \in G$ to a triple in the Shilov boundary $T \cong \mathbb{T}^r$ of a maximal polydisk, so that Corollary 3.4, (M1) and (M5) lead to

$$\iota_S(s_1, s_2, s_3) = \iota_S(g \cdot s_1, g \cdot s_2, g \cdot s_3) = \iota_T(g \cdot s_1, g \cdot s_2, g \cdot s_3).$$

We conclude that $\iota_S$ is determined uniquely by (M1), (M2), together with (M3)–(M6).

### 4.1. A classical case: The Lagrangian manifold

Let $E$ be a real vector space of dimension $2r$ and let $\omega$ be a symplectic form on $E$. The symplectic group $\mathrm{Sp}(E, \omega)$ is the group of linear automorphisms which preserve $\omega$. A *Lagrangian* is a maximal totally isotropic subspace of $E$, hence of dimension $r$. The set $\Lambda_r$ of all Lagrangians is a compact submanifold of the Grassmannian $\mathrm{Gr}_r(E)$ of $r$-dimensional subspaces of $E$. Then the group $G := \mathrm{PSp}(E, \omega) := \mathrm{Sp}(E, \omega)/\{\pm \mathbf{1}\}$ acts transitively and effectively on $\Lambda_r$. Choosing a symplectic basis in $E$, we may identify $E$ with $\mathbb{R}^r \times \mathbb{R}^r$, the symplectic form being the standard one, namely,

$$\omega((\xi, \eta), (\xi', \eta')) = \xi^\top \eta' - \eta^\top \xi'. \tag{4.2}$$

Let us consider the complex vector space $V = \mathrm{Sym}_r(\mathbb{C})$ of complex $r \times r$ symmetric matrices, and let $\mathcal{D}$ be the unit ball with respect to the operator norm. The space $V$ is an involutive unital Jordan algebra with real form $\mathrm{Sym}_r(\mathbb{R})$, involution $z^* = \overline{z}$ and Jordan product $x * y := \frac{1}{2}(xy + yx)$. The spectral norm on $V$ coincides with the operator norm, and the unit ball is then a bounded symmetric domain. To make connection with symplectic geometry, observe that the graph of a symmetric matrix is a complex isotropic subspace in $\mathbb{C}^r \times \mathbb{C}^r$ for the symplectic structure (4.2). Let, moreover, $h$ be the Hermitian form on $\mathbb{C}^r \times \mathbb{C}^r$ given by

$$h((\xi, \eta), (\xi', \eta')) = \xi^\top \overline{\xi'} - \eta^\top \overline{\eta'} = (\xi')^* \xi - (\eta')^* \eta.$$

The Hermitian form $h$ has signature $(r, r)$. Now to any $x \in V$, associate its graph

$$\ell_x = \{(\xi, x.\xi) \mid \xi \in \mathbb{C}^r\}.$$

The condition that $x$ is in the unit ball is equivalent to the fact that $\mathbf{1} - xx^*$ is positive definite, which in turn implies that the restriction of $h$ to $\ell_x$ is positive definite.

Conversely, any (complex) Lagrangian in $\mathbb{C}^r \times \mathbb{C}^r$ on which the restriction of $h$ is positive definite is the graph of some complex symmetric matrix in the unit ball. The Shilov boundary of $\mathcal{D}$ is the manifold of unitary symmetric matrices, and the corresponding graphs are the (complex) Lagrangians on which the restriction of the form $h$ is identically 0. Let $C$ be the map from $\mathbb{R}^r \times \mathbb{R}^r$ to $\mathbb{C}^r \times \mathbb{C}^r$ given by

$$C(\xi, \eta) = \Big( \frac{\xi + i\eta}{\sqrt{2}}, \frac{\xi - i\eta}{\sqrt{2}} \Big) .$$

Then an elementary computation shows that the complexification of the image under $C$ of a (real) Lagrangian is a (complex) Lagrangian on which the restriction of $h$ is identically 0, and vice versa. This gives a one-to-one correspondence between $\Lambda_r$ and $S$. Moreover, the natural action of $G$ on $\Lambda_r$ is transferred to an action on $S$ and realizes an isomorphism of the real symplectic group and the group $\mathrm{Sp}_{2r}(\mathbb{C}) \cap \mathrm{U}_{r,r}(\mathbb{C})$, which generalizes the isomorphism of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SU}_{1,1}(\mathbb{C})$.

The matrices $E_{11}, \dots, E_{rr}$ form a Jordan frame in $\mathrm{Sym}_r(\mathbb{C})$. The corresponding $r$-torus is

$$T := \Big\{ \begin{pmatrix} e^{i\theta_1} & 0 & \dots & 0 \\ 0 & e^{i\theta_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{i\theta_r} \end{pmatrix} \mid \theta_j \in \mathbb{R}, 1 \leqslant j \leqslant r \Big\} .$$

The graph of an element of $T$ is the $r$-space generated by

$$(e_1, e^{i\theta_1} e_1), \dots, (e_r, e^{i\theta_r} e_r),$$

or, equivalently, by

$$(e^{-i\frac{\theta_1}{2}} e_1, e^{i\frac{\theta_1}{2}} e_1), \dots, (e^{-i\frac{\theta_r}{2}} e_r, e^{i\frac{\theta_r}{2}} e_r).$$

Observe that $(e^{-i\frac{\theta_j}{2}} e_j, e^{i\frac{\theta_j}{2}} e_j) = C(\cos\frac{\theta_j}{2} e_j, \sin\frac{\theta_j}{2} e_j)$ to get that the corresponding Lagrangian $\ell(\theta_1, \theta_2, \dots, \theta_r)$ in $\Lambda_r$ is generated by

$$\Big( \cos\frac{\theta_1}{2} e_1, -\sin\frac{\theta_1}{2} e_1 \Big), \dots, \Big( \cos\frac{\theta_r}{2} e_r, -\sin\frac{\theta_r}{2} e_r \Big) .$$

In this case, one can then reformulate Theorem 3.1 as follows.

**Theorem 4.2.** *Let $\ell_1, \ell_2, \ell_3$ be three arbitrary Lagrangians in a symplectic vector space $E$ of dimension $2r$. Then there exists a symplectic basis $e_1, \dots, e_r, f_1, \dots, f_r$ such that each of the three Lagrangians is generated by*

$$\cos\theta_1 e_1 + \sin\theta_1 f_1, \dots, \cos\theta_r e_r + \sin\theta_r f_r$$

*for appropriate choices of the $(\theta_j)_{1 \leqslant j \leqslant r}$.*

The classification result (Theorem 5.2 below) for the case $S = \Lambda_r$ can also be found in [KS90, p. 492].

## 5. The classification of triples

In this section we complete the classification of $G$-orbits in the set $S \times S \times S$ of triples in $S$ by first assigning to each triple an increasing 5-tuple of integers $N = (n_1, n_2, n_3, n_4, n_5) \in \{0, \dots, r\}^5$ depending only on its orbit. Then we exhibit for each such 5-tuple a standard triple with this invariant and, finally we show that two different standard triples belong to different orbits.

**Definition 5.1.** To any triple $(x_1, x_2, x_3)$ in $S \times S \times S$, we may associate five integers:

(1) the ranks of the three faces (see Remark 2.3):

$$n_{12} = \operatorname{rank}\operatorname{Face}(x_1, x_2), \quad n_{2,3} = \operatorname{rank}\operatorname{Face}(x_2, x_3), \quad n_{3,1} = \operatorname{rank}\operatorname{Face}(x_3, x_1);$$

(2) the rank of the face generated by the triple

$$n_{1,2,3} = \operatorname{rank}\operatorname{Face}(x_1, x_2, x_3);$$

(3) the Maslov index $\iota(x_1, x_2, x_3)$.

Clearly the action of $G$ preserves these integers.

When $x_1, x_2, x_3$ are contained in the boundary of a polydisk (see Section III), then these integral invariants are easy to compute (see Example 2.4).

**Lemma 5.1.** *Let $e = \sum_{j+1}^r c_j$ be a Peirce decomposition of the unit and, for $\kappa = 1, 2, 3$, let*

$$x_\kappa = \sum_{j=1}^r \xi_j^{(\kappa)} c_j, \qquad \textit{where} \qquad |\xi_j^{(\kappa)}| = 1 \quad \textit{for all} \quad j \in \{1, \dots, r\}.$$

*Then*

$$n_{\kappa,\kappa'} = |\{j \mid \xi_j^{(\kappa)} = \xi_j^{(\kappa')}\}|, \quad n_{1,2,3} = |\{j \mid \xi_j^{(1)} = \xi_j^{(2)} = \xi_j^{(3)}\}|,$$

*and*

$$\iota(x_1, x_2, x_3) = \sum_{j=1}^r \iota(\xi_j^{(1)}, \xi_j^{(2)}, \xi_j^{(3)}).$$

**Definition 5.2.** We now describe the *standard triples* associated to a (fixed) Jordan frame $(c_1, \dots, c_r)$. Let $N = (n_1, n_2, n_3, n_4, n_5)$ be a 5-tuple of integers such that

$$0 \leqslant n_1 \leqslant n_2 \leqslant n_3 \leqslant n_4 \leqslant n_5 \leqslant r \ .$$

Then the *standard triple of type* $N$ is the triple $(x_1^N, x_2^N, x_3^N)$ defined by

$$\begin{aligned} x_1^N &= e_r = c_1 + \dots + c_r, \qquad x_2^N = c_1 + c_2 + \dots + c_{n_2} - c_{n_2+1} - \dots - c_r, \\ x_3^N &= c_1 + \dots + c_{n_1} - c_{n_1+1} - \dots - c_{n_3} + c_{n_3+1} + \dots + c_{n_4} \\ &\qquad - ic_{n_4+1} - \dots - ic_{n_5} + ic_{n_5+1} + \dots + ic_r. \end{aligned}$$

For this triple, one has

$$n_{1,2,3} = n_1, \quad n_{1,2} = n_2, \quad n_{1,3} = n_1 + n_4 - n_3, \quad n_{2,3} = n_1 + n_3 - n_2,$$

and

$$\iota(x_1^N, x_2^N, x_3^N) = n_5 - n_4 - (r - n_5) = 2n_5 - n_4 - r.$$

**Theorem 5.2.** *If $\mathcal{D}$ is an irreducible bounded symmetric domain of tube-type, then any triple in $S$ is conjugate to one and only one of the standard triples.*

*Proof.* For the standard triples we have

$$n_1 = n_{1,2,3}, \quad n_2 = n_{1,2}, \quad n_3 = n_{2,3} + n_2 - n_1 = n_{2,3} + n_{1,2} - n_{1,2,3}, \tag{5.1}$$

$$n_4 = n_{1,3} + n_3 - n_1 = n_{1,3} + n_{2,3} + n_{1,2} - 2n_{1,2,3}, \tag{5.2}$$

and

$$n_5 = \tfrac{1}{2}(\iota(x_1^N, x_2^N, x_3^N) + n_4 + r) = \tfrac{1}{2}(\iota(x_1^N, x_2^N, x_3^N) + r + n_{1,3} + n_{2,3} + n_{1,2} - 2n_{1,2,3}). \tag{5.3}$$

Since the numbers $n_{1,2,3}$, $n_{1,2}$, $n_{2,3}$, $n_{3,1}$ and the Maslov index are $G$-invariant, it follows that, for different values of $N$, the corresponding standard triples are not conjugate under $G$.

To show, conversely, that each triple $(e, f, h) \in S \times S \times S$ is conjugate to a standard triple, we first use Theorem 3.1 to see that we may without loss of generality assume that $(e, f, h)$ is contained in the torus

$$T := \Big\{ \sum_{j=1}^{r} \lambda_j c_j \mid (\forall j)\ |\lambda_j| = 1 \Big\}$$

defined by the Jordan frame $(c_1, \ldots, c_r)$. It is the Shilov boundary of the polydisk

$$\Delta^r := \Big\{ \sum_{j=1}^{r} \lambda_j c_j \mid (\forall j)\ |\lambda_j| < 1 \Big\}.$$

We write

$$e = \sum_{j=1}^{r} \xi_j^e c_j, \quad f = \sum_{j=1}^{r} \xi_j^f c_j, \quad \text{and} \quad h = \sum_{j=1}^{r} \xi_j^h c_j.$$

From Remark 1.2 it follows that every element of $\mathrm{Aut}(\Delta^r)_0 \cong \mathrm{PSU}_{1,1}(\mathbb{C})^r$ is the restriction of an element of $\mathrm{Aut}(\mathcal{D})_0$, because

$$\mathfrak{g}_{c_1} + \cdots + \mathfrak{g}_{c_r} \cong \mathfrak{su}_{1,1}(\mathbb{C})^r = \mathbf{L}(\mathrm{Aut}(\Delta^r))$$

is a subalgebra of $\mathfrak{g} = \mathbf{L}(G)$. We may therefore assume that $\xi_j^e = 1$ for each $j$. Let

$$n_2 := |\{j \mid \xi_j^e = \xi_j^f\}| = |\{j \mid \xi_j^f = 1\}|.$$

Since each permutation of the set $\{c_1, \ldots, c_r\}$ is induced by an element of $K$, which acts transitively on the set of Jordan frames, we may without loss of generality assume that

$$f = c_1 + \cdots + c_{n_2} - c_{n_2+1} - \ldots - c_r$$

because the $\mathrm{Aut}(\Delta)_0$-orbits in $\mathbb{T} \times \mathbb{T}$ are represented by $(1, 1)$ and $(1, -1)$ (Example 4.2).

Let $n_1 := |\{j \mid \xi^e_j = \xi^f_j = \xi^h_j\}|$ and write

$$n_4 := |\{j \mid \xi^e_j = \xi^f_j \text{ or } \xi^e_j = \xi^h_j \text{ or } \xi^f_j = \xi^h_j\}|$$

for the number of components in which at least two elements of $\{e, f, h\}$ have the same entries. Then $h$ has precisely $n_1$ entries 1 among the first $n_2$, and we may without loss of generality assume that they arise in position $j = 1, \ldots, n_1$. We may likewise assume that the components of $e, f$ and $h$ are mutually different for $j > n_4$. Then the entries of $h$ in positions $n_1+1, \ldots, n_2$ can be moved by elements of the group $\mathrm{Aut}(\Delta)_0^{n_2-n_1}$ acting on these components to $-1$. For $j \in \{n_2+1, \ldots, n_4\}$ the $j$th component of $h$ equals either 1 or $-1$. Moving the 1-entries with some element of $K_e$ permuting $\{c_1, \ldots, c_r\}$ to the rightmost positions, we get entries $-1$ for $j = n_1+1, \ldots, n_3$ for some $n_3$ satisfying $n_2 \leqslant n_3 \leqslant n_4$. For $j > n_4$ we then have $\mathrm{Im}\, \xi^h_j \neq 0$ and, after permuting the Jordan frame, we may assume that for some $n_5 \geqslant n_4$ we have $\mathrm{Im}\, \xi^h_j < 0$ for $j = n_4+1, \ldots, n_5$ and $\mathrm{Im}\, \xi^h_j > 0$ for $j > n_5$. We finally use elements of $\mathrm{Aut}(\Delta)_0$ fixing 1 and $-1$ to move each entry with negative imaginary part to $-i$ and the others to $i$ (see Example 4.2). This proves that each triple is conjugate to a standard triple. □

**Remark 5.3.** In Theorem 5.2 we have classified the $G$-orbits in the space of triples in $S$ by the set of all 5-tuples $N = (n_1, n_2, n_3, n_4, n_5) \in \{0, \ldots, r\}$ satisfying the monotonicity condition

$$n_1 \leqslant n_2 \leqslant n_3 \leqslant n_4 \leqslant n_5.$$

The description the standard triples shows that each such tuples arises via (5.1)–(5.3). We claim that for the 5-tuple

$$(r_0, r_1, r_2, r_3, d) := \big(n_{1,2,3}, n_{1,2}, n_{2,3}, n_{3,1}, \iota(x^N_1, x^N_2, x^N_3)\big)$$

of integers we then have

(P1) $0 \leqslant r_0 \leqslant r_1, r_2, r_3 \leqslant r$.
(P2) $r_1 + r_2 + r_3 \leqslant r + 2r_0$.
(P3) $|d| \leqslant r + 2r_0 - (r_1 + r_2 + r_3)$.
(P4) $d \equiv r + r_1 + r_2 + r_3 \bmod 2$.

In fact, (P1) is clear,

$$r_1 + r_2 + r_3 = n_4 + 2r_0 \leqslant r + 2r_0,$$

$$|d| = |n_5 - n_4 - (r - n_5)| \leqslant n_5 - n_4 + r - n_5 = r - n_4 = r + 2r_0 - r_1 - r_2 - r_3,$$

and

$$d = n_5 - n_4 - (r - n_5) \equiv n_4 + r \equiv r + r_1 + r_2 + r_3 \mod 2.$$

Suppose, conversely, that $(r_0, r_1, r_2, r_3, d) \in \mathbb{Z}^5$ satisfies (P1)–(P4). We then define

$$n_1 := r_0, \quad n_2 := r_1, \quad n_3 := r_2 + r_1 - r_0, \quad n_4 := r_3 + r_2 + r_1 - 2r_0,$$

and

$$n_5 = \tfrac{1}{2}(d + r_3 + r_2 + r_1 + r) - r_0.$$

Then (P4) implies $n_5 \in \mathbb{Z}$. From (P1/2) we immediately get $0 \leqslant n_1 \leqslant n_2 \leqslant n_3 \leqslant n_4 \leqslant r$. Further (P3) leads to $|d| \leqslant r - n_4$, and $n_4 \leqslant n_5$ follows from

$$2n_5 = d + r_3 + r_2 + r_1 + r - 2r_0 = d + r + n_4 \geqslant r + n_4 - (r - n_4) = 2n_4.$$

This is turn implies $n_5 = \frac{1}{2}(r + d + n_4) \leqslant r$.

Conditions (P1)–(P4) are well known conditions describing the classification of triples of Lagrangian subspace of symplectic vector spaces ([KS90]).

## 6. Classification of orbits in $S \times S$

In this section we describe how the classification of $G$-orbits in $S \times S$ can be derived from the Bruhat decomposition of $G$ (resp., the description of the orbits of the maximal parabolic subgroup $G_e$ in $G$ with $G/G_e \cong S$).

Throughout this section we assume $\mathcal{D}$ to be irreducible. Let $(c_1, \ldots, c_r)$ be a Jordan frame and put

$$\varepsilon_k = c_1 + \cdots + c_k - c_{k+1} - \cdots - c_r \quad \text{for} \quad k = 0, \ldots, r.$$

Moreover, let $e = c_1 + \cdots + c_r = \varepsilon_r$, and observe that $\varepsilon_0 = -e$. The vector space

$$\mathfrak{a} = \bigoplus_{j=1}^{r} \mathbb{R}c_j$$

is a maximal flat in $V$ in the sense of Loos [Lo77] and can be thought of as a Cartan subspace in the tangent space of $\mathcal{D}$ at the origin. The corresponding vector fields form a Cartan subspace of $\mathfrak{p}$. Denoting by $\gamma_j$ the $j$th coordinate in $\mathfrak{a}$ with respect to the basis $(c_1, c_2, \ldots, c_r)$, it is known that the (restricted) roots of $(\mathfrak{g}, \mathfrak{a})$ are

$$\pm\gamma_j \pm \gamma_k, \pm 2\gamma_j, \quad 1 \leqslant j \neq k \leqslant r,$$

and, in addition, $\pm\gamma_j, 1 \leqslant j \leqslant r$, in the nontube-type case. We choose as positive Weyl chamber in $\mathfrak{a}$ the one defined by the inequalities

$$\gamma_1 \geqslant \cdots \geqslant \gamma_r \geqslant 0,$$

so that the corresponding simple roots are

$$\gamma_1 - \gamma_2, \ \ldots, \gamma_{r-1} - \gamma_r, \ \gamma_r.$$

The Weyl group $W$ is isomorphic to the semidirect product $S_r \ltimes \mathbb{Z}_2^r$, where $S_r$ acts by permutation of the coordinates $\gamma_j$, and the $j$th factor $\mathbb{Z}_2$ acts by changing the sign of the $j$th coordinate.

The stabilizer $G_e$ of the point $e \in S$ is known to be a maximal parabolic subgroup (see Section I). It is the standard parabolic subgroup associated to the subset

$$\Theta = \{\gamma_1 - \gamma_2, \ldots, \gamma_{r-1} - \gamma_r\}$$

of the set of simple roots. The subgroup $W^\Theta$of $W$ generated by the reflections associated to the roots in $\Theta$ is just $S_r$, and double cosets in $W^\Theta \backslash W / W^\Theta$ correspond to orbits of $S_r$ in $\mathbb{Z}_2^r$, which are characterized by their number of sign changes. In particular, this shows that the elements $\varepsilon_j, 0 \leqslant j \leqslant r$, form a set of representatives of the $W^\Theta$-orbits in $W.e$.

**Theorem 6.1.** *There are $r+1$ orbits of $G$ in $S \times S$. A set of representatives of these orbits is given by the pairs $(e, \varepsilon_k), 0 \leqslant k \leqslant r$.*

*Proof.* As $G$ acts transitively on $S$, any orbit of $G$ in $S \times S$ meets the subset $\{e\} \times S$. So the statement amounts to show that a $G_e$-orbit in $S$ contains $\varepsilon_k$ for some $k, 0 \leqslant k \leqslant r$. By Bruhat's theory, the orbits of the parabolic subgroup $G_e$ of $G$ are in one-to-one correspondence with the $W^\Theta$-double cosets in $W$. In view of the preceding discussion, this shows the result. □

**Remark 6.1.** The open orbit in $S$ under the $G_e$-action (the big Bruhat cell) corresponds to the point $-e$ and is nothing but the set of all points in $S$ transversal to $e$.

**Definition 6.2.** For $(x, y) \in S \times S$ we define their *transversality index* $\mu(x, y)$ to be the unique number $k \in \{0, \dots, r\}$ such that $(x, y)$ belongs to the $G$ orbit of $(e, \varepsilon_k)$. Clearly, the transversality index is invariant by the action of $G$, and two pairs are conjugate if and only if they have the same transversality index. Moreover, a pair $(x, y)$ is transversal if and only if its transversality index is 0.

**Theorem 6.2.** *A pair $(x, y) \in S \times S$ has transversality index $k$ if and only if the face $F(x, y)$ generated by $x$ and $y$ has rank $k$.*

*Proof.* For $0 \leqslant k \leqslant r$ let $e_k = c_1 + \ldots + c_k$. Then the face generated by $e$ and $\varepsilon_k$ is

$$\mathrm{Face}(e, \varepsilon_k) = (e_k + V_0(e_k)) \cap \overline{\mathcal{D}},$$

which has rank $k$. As any pair in $S \times S$ is conjugate to one of the pairs $(e, \varepsilon_k)$, the theorem follows immediately. □

## 7. Appendix: Bounded symmetric domains and tube-type domains

In this Appendix we briefly review the relation between *bounded symmetric domains* and *positive Hermitian Jordan triple systems* on one hand, and the relation between *bounded symmetric domains of tube-type* and *Euclidean Jordan algebras* on the other hand. Main references are [Lo77] for (Hermitian) Jordan triples and [FK94] for (Euclidean) Jordan algebras.

A *Hermitian Jordan triple system* $V$ is a finite-dimensional complex vector space, together with a map $\{\cdot,\cdot,\cdot\} : V \times V \times V \longrightarrow V$, such that $\{x, y, z\}$ is complex linear in $x$ and $z$, conjugate linear in $y$, and such that

$$\{x, y, z\} = \{z, y, x\} \tag{JT1}$$

and

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\} \tag{JT2}$$

hold for all $a, b, x, y, z \in V$.

For $x, y \in V$ denote by $x \Box y$ the linear endomorphism of $V$ defined by

$$(x \Box y)\, z = \{x, y, z\}$$

and by $Q(x)$ the conjugate linear endomorphism of $V$ defined by $Q(x)z = \{x, z, x\}$. Define the *trace form* $B$ on $V$ by $B(x, y) = \operatorname{tr}(x\Box y)$. The Jordan triple system $V$ is said to be *nondegenerate* if, as a sesquilinear form, $B$ is nondegenerate. If this is the case, then $B$ is Hermitian (i.e., $B(x, y) = \overline{B(y, x)}$ for all $x, y \in V$). If, moreover, $B$ is *positive definite*, then $V$ is said to be a *positive Hermitian Jordan triple system.*

Let $V$ be a positive Hermitian Jordan triple system. An element $c \in V$ is said to be a *tripotent* if $\{c, c, c\} = c$. For a tripotent $e \in V$ let $V_j := V_j(e)$ denote the $j$-eigenspace of the operator $2e\Box e$. Then we obtain the corresponding *Peirce decomposition of* $V$:

$$V = V_0 \oplus V_1 \oplus V_2$$

([Lo77, Theorem 3.13]).

There is a (partial) order relation on tripotents. For two tripotents $c, d \in V$, we define $c \prec d$ if there exists a tripotent $c'$, such that:

(i) $c\Box c' = 0$ (orthogonality of $c$ and $c'$);
(ii) $d = c + c'$.

A nonzero tripotent is said to be *primitive* if it is minimal among nonzero tripotents for this order. Any tripotent $c$ can be written as a sum of pairwise orthogonal primitive tripotents, say $c = c_1 + \cdots + c_k$. The number $k$ of primitive tripotents in such a decomposition of $c$ depends only on $c$ and is called the *rank of c.*

A *Jordan frame* of $V$ is a maximal family $(c_1, \ldots, c_r)$ of orthogonal primitive tripotents. All Jordan frames have the same number of elements called the *rank of* $V$. For any Jordan frame $(c_1, \ldots, c_r)$, the sum $e = \sum_{j=1}^{r} c_j$ is a maximal tripotent of $V$, and all maximal tripotents are obtained this way.

One of the main results in the theory of positive Hermitian Jordan triple systems is the *spectral theorem.*

**Proposition 7.1.** *For any $x \in V$, there exists a Jordan frame $(c_1, \ldots, c_r)$ and positive real numbers $\lambda_j, 1 \leqslant j \leqslant r$, such that*

$$x = \sum_{j=1}^{r} \lambda_j c_j. \tag{A.1}$$

*The $\lambda_j$ are unique up to a permutation.*

The identity (A.1) is called a *spectral decomposition* of $x$. The $\lambda_j$ are called the *eigenvalues* of $x$. The largest eigenvalue is the *spectral norm* of $x$, denoted by $|x|$. As notation suggests, the map $x \mapsto |x|$ is a complex Banach norm on $V$.

**Theorem 7.2.** *The unit ball of $(V, |\cdot|)$ is a bounded symmmetric domain. Conversely, any bounded symmetric domain is holomorphically equivalent to such a unit ball.*

There is a subclass of symmetric bounded domains, the domains of *tube-type.* They are associated to a subclass of positive Hermitian Jordan triple systems, obtained by complexification from *Euclidean Jordan algebras.*

A *Euclidean Jordan algebra E* is a real finite-dimensional Euclidean vector space $E$ with an inner product $\langle\cdot,\cdot\rangle$, a bilinear map $E \times E \longrightarrow E$ and an element $e \in E$ such that

$$xy = yx, \qquad ex = x, \quad x^2(xy) = x(x^2y), \quad \text{and} \quad \langle xy, z\rangle = \langle y, xz\rangle$$

for all $x, y, z \in E$. Let $V = E^{\mathbb{C}}$ be the complexification of $E$, and extend the Jordan product from $E$ in a $\mathbb{C}$-bilinear way to $V$. Denote by $z \mapsto \overline{z}$ the conjugation of $V$ with respect to $E$. For $x, y, z \in V$, let

$$\{x, y, z\} := (x\overline{y})z + x(\overline{y}z) - \overline{y}(xz). \tag{A.2}$$

This endows $V$ with a structure of positive Hermitian Jordan triple systems. The element $e$ is a tripotent of $V$. It satisfies $e\Box e = \mathrm{id}_E$, so that $V_0(e) = \{0\}$ (hence $e$ is a maximal tripotent), $V_1(e) = \{0\}$ and $V = V_2(e)$.

Among positive Hermitian Jordan triple systems, those coming from Euclidean Jordan algebras are characterized by this last property. Let $V$ be a positive Hermitian Jordan triple system, and let $e$ be a maximal tripotent. By maximality of $e$, $V_0(e) = \{0\}$. Assume further that $V_1(e) = \{0\}$, so that $V = V_2(e)$. Now $Q(e)$ is a conjugate linear involution of $V$. Its fixed points set $E = \{x \in V \mid Q(x) = x\}$ is a real vector space. For $x, y \in E$, define

$$xy = \{x, e, y\}. \tag{A.3}$$

With the product defined by (A.3) and the inner product induced by $B$, $E$ is then a Euclidean Jordan algebra, $V$ is the complexification of $E$ and the Jordan triple product on $V$ can be recovered by formula (A.2) from the Jordan algebra product on $E$.

An element $c \in E$ is called an idempotent if $c^2 = c$. A Jordan frame in $E$ is a maximal set of orthogonal minimal idempotents. The number of elements in a Jordan frame is equal to $r$, the rank of the Jordan algebra $E$, and if $(c_1, \ldots, c_r)$ is a Jordan frame, then $e = c_1 + \ldots + c_r$. A tripotent $c$ for the associated triple Jordan system structure on $V$ is of the form $c = \sum_{j=1}^r \lambda_j c_j$, for a certain Jordan frame $(c_1, \ldots, c_r)$ of $E$ and for each $j, 1 \leqslant j \leqslant r$, $|\lambda_j| = 1$ or $\lambda_j = 0$. A maximal tripotent $z$ of $V$ is of the form with $x = \sum_{j=1}^r \lambda_j c_j$, with $|\lambda_j| = 1$ for all $j, 1 \leqslant j \leqslant r$, so that $z$ as an element of the complex Jordan algebra $V$ is invertible, and satisfies $\overline{z} = z^{-1}$.

The corresponding bounded symmetric domain is described as before by

$$\mathcal{D} = \{z \in V \mid |z| < 1\}.$$

The domain $\mathcal{D}$ can be shown to be holomorphically equivalent to a tube domain. If $E^+$ is the interior of the *cone of squares* of $E$, then, by the Cayley transform $C_e$, the domain $\mathcal{D}$ is mapped to

$$\mathcal{D}^C = C_e(\mathcal{D}) = \{v \in V \mid \mathrm{Re}(v) > 0\} = E^+ \oplus iE.$$

The domain $\mathcal{D}^C$ is a tube domain in $V$, which is the justification for calling $\mathcal{D}$ a bounded symmetric domain of tube-type.

The description of maximal tripotents of $V$ we gave supra shows that the Shilov boundary can be described as

$$S = \{z \in V \mid \overline{z} = z^{-1}\}. \tag{A.4}$$

Hence the Shilov boundary $S$ is a totally real submanifold of $V$ with $\dim_{\mathbb{R}} S = \dim_{\mathbb{C}} V$. This last condition is another characterization of bounded symmetric domains of tube-type inside the family of bounded symmetric domains. In fact if $\mathcal{D}$ is a bounded symmetric domain, then its Shilov boundary $S$ is a real submanifold of $V$, and its dimension satisfies

$$\dim_{\mathbb{R}} S \geqslant \frac{1}{2} \dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V.$$

Equality is obtained if and only if $\mathcal{D}$ is of tube-type.

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