Self-Adjoint Elliptic Operators and Manifold Decompositions Part I: Low Eigenmodes and Stretching

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Abstract

This paper is the first of a three-part investigation into the behavior of analytical invariants of manifolds that can be split into the union of two submanifolds. In this article, we will show how the low eigensolutions of a self-adjoint elliptic operator over such a manifold can be studied by a splicing construction. This construction yields an approximated solution of the operator whenever we have two L^2 -solutions on both sides and a common limiting value of two extended L^2 -solutions. In Part II, the present analytic "Mayer-Vietoris" results on low eigensolutions and further analytic work will be used to obtain a decomposition theorem for spectral flows in terms of Maslov indices of Lagrangians. In Part III after comparing infinite- and finite-dimensional Lagrangians and determinant line bundles and then introducing "canonical perturbations" of Lagrangian subvarieties of symplectic varieties, we will study invariants of 3-manifolds, including Casson's invariant. © 1996 John Wiley & Sons, Inc.

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1. Introduction

Let *M* be a closed, oriented, smooth manifold that is decomposed into the union of two submanifolds M_1, M_2 by a codimension-1, oriented submanifold Σ ,

(1.1)
$$M = M_1 \cup M_2, \qquad \Sigma = M_1 \cap M_2 = \partial M_1 = \partial M_2.$$

Communications on Pure and Applied Mathematics, Vol. XLIX, 825–866 (1996) © 1996 John Wiley & Sons, Inc. CCC 0010-3640/96/080825-42 The object of this paper is to study a decomposition of eigenspaces with small eigenvalues of a self-adjoint elliptic operator D on M into the contributions from M_1 and M_2 together with a resonance term created by the interaction of these eigenspaces along Σ (see Theorem A). A phenomenon well-known to topologists is the decomposition of cohomology of M by the Mayer-Vietoris sequence of $(M; M_1, M_2)$, which is related to resonance terms in Appendix B. Another example of the resonance term is the background noise (low-frequency modes) created sometimes by a long corridor connecting two chambers. In our treatment, these resonance terms arise naturally as the intersection of two Lagrangian subspaces in a symplectic vector space.

In this primarily analytical part, we will concentrate on first-order, *self-adjoint*, elliptic differential operators D on the space $\Gamma(E)$ of smooth sections of a real vector bundle $E \to M$,

$$(1.2) D: \Gamma(E) \to \Gamma(E),$$

which are of the Atiyah-Patodi-Singer type [1] near Σ . More precisely, under an identification $i: \Sigma \times [-1, 1] \to M$ of the cylinder $\Sigma \times [-1, 1]$ with a neighborhood of $\Sigma, i(\Sigma \times 0) = \Sigma$, and an identification of $E \mid \Sigma \times [-1, 1]$ with the pullback $\pi^* \hat{E}$ of a vector bundle \hat{E} over Σ via the projection $\pi: \Sigma \times [-1, 1] \to \Sigma$, we can write D over $\Sigma \times [-1, 1]$ as a sum

(1.3)
$$D = (\pi^* \sigma) \circ \left(\frac{\partial}{\partial s} + \pi^* \hat{D}\right).$$

Here \hat{D} is a self-adjoint, elliptic operator on the space $\Gamma(\hat{E})$ of smooth sections of \hat{E} ,

(1.4)
$$\hat{D}: \Gamma(\hat{E}) \to \Gamma(\hat{E}),$$

s is the coordinate of [-1, 1], $\sigma : \hat{E} \to \hat{E}$ is a bundle isomorphism, and $\pi^* \sigma$ is the pullback of σ to $E \mid \Sigma \times [-1, 1]$. In short, in the neighborhood of Σ , we are in the same setting as [1].

The basic approach is to replace M by a stretched version M(r) of the same manifold:

(1.5)
$$M(r) = M_1 \cup \Sigma \times [-r, r] \cup M_2$$

obtained by first cutting M open along Σ and then gluing the pieces back to $\Sigma \times [-r, r]$ with the end $\Sigma \times (-r)$ and ∂M_1 identified and with $\Sigma \times (r)$ and ∂M_2 identified. As we stretch M, the above operator D can also be extended to produce an operator D = D(M(r)),

(1.6)
$$D: \Gamma(E(r)) \to \Gamma(E(r))$$

over M(r). In fact, for manifolds

(1.7)
$$M_1(\infty) = M_1 \cup \Sigma \times [0, \infty)$$
 and $M_2(\infty) = \Sigma \times (-\infty, 0] \cup M_2$

obtained by attaching the infinite cylinder $\Sigma \times [0, \infty)$ to M_1 along $\partial M_1 = \Sigma \times 0$ and attaching $\Sigma \times (-\infty, 0]$ to M_2 along $\partial M_2 = \Sigma \times 0$, similar bundles $E_j(\infty) \to M_j(\infty)$ and operators $D : \Gamma(E_j(\infty)) \to \Gamma(E_j(\infty))$ exist (see Section 1).

The key to this approach is to relate the L^2 -solutions of $D : \Gamma(E_j(\infty)) \rightarrow \Gamma(E_j(\infty))$ to the low eigenspaces of the operator D = D(M(r)). Let V_j , j = 1, 2, denote the space of L^2 -sections ϕ of the bundle $E_j(\infty)$ satisfying $D\phi = 0$. Then a general technique called *splicing* allows us to put two such sections $(\phi_1, \phi_2), \phi_1 \in V_1, \phi_2 \in V_2$, together to form a section $\Phi(\phi_1, \phi_2, 0)$ in $\Gamma(E(r))$. By making estimates of lower eigenspaces of $\Gamma(E(r))$, we will show that these spliced sections $\Phi(\phi_1, \phi_2, 0)$ represent contributions to the low eigenmode problem on M from the two pieces M_1, M_2 . To obtain the complete picture, we also have to consider the extended L^2 -solutions of D on $\Gamma(E_j(\infty))$ that take on limiting values at infinity.

Let \mathcal{H} denote the kernel of the operator \hat{D} on $\Gamma(\hat{E})$. In Proposition 2.2 we will show that \mathcal{H} naturally inherits the structure of a symplectic vector space and that the limits of the extended L^2 -solutions in $\Gamma(E_j(\infty))$ give rise to a Lagrangian subspace L_j in \mathcal{H} . An element α lying in the intersection $L_1 \cap L_2$ of these two Lagrangian subspaces can be regarded as a pair of sections on $M_1(\infty), M_2(\infty)$ that match at infinity. The general form of the splicing construction takes into consideration not only ϕ_1, ϕ_2 but also the element α and produces a section $\Phi_r(\phi_1, \phi_2, \alpha)$ in $\Gamma(E(r))$. Thus we have

$$\Phi_r: V_1 \oplus V_2 \oplus (L_1 \cap L_2) \to \Gamma(E(r))$$

$$(\phi_1, \phi_2, \alpha) \mapsto \Phi_r(\phi_1, \phi_2, \alpha),$$

and the subspace $\Phi_r(0 \oplus 0 \oplus L_1 \cap L_2)$ gives the resonance term at the beginning of our discussion.

THEOREM A. Let N(r, K) denote the number of linearly independent eigenvectors ψ_i of $D\psi_i = \lambda_i \psi_i$ on M(r) with $|\lambda_i| < K$. Let sp(r, K) denote the span of these eigenvectors ψ_i . Given $\varepsilon > 0$, there exists an R such that for $r \ge R$ we have the following:

- (1.8) $N(r, r^{-(1+\varepsilon)}) = \dim V_1 + \dim V_2 + \dim L_1 \cap L_2.$
- (1.9) Let $\delta = \min\{|u| : u \neq 0 \text{ is an eigenvalue of } \hat{D} \text{ over } \Sigma\}$. Then the equality of numbers $N(r, \exp(-\frac{\delta}{4}r)) = N(r, r^{-(1+\varepsilon)})$ and the equality of two subspaces $\operatorname{sp}(r, \exp(-\frac{\delta r}{4})) = \operatorname{sp}(r, r^{-(1+\varepsilon)})$ hold.
- (1.10) Denote the projection of $\Gamma(E(r))$ onto the subspace $\operatorname{sp}(r, \exp(-\frac{\delta r}{4}))$ by \mathbf{P}_r . Then $\|\Phi_r(x) - \mathbf{P}_r \Phi_r(x)\| < \exp(-\frac{\delta r}{4})\|x\|$.

Roughly, the above theorem says that small eigenvalues (in the range $r^{-(1+\varepsilon)}$) of D on M(r) are exponentially small ($\sim \exp(-\frac{br}{4})$) and the corresponding eigensections are exponentially close to a splicing construction. In particular, if $V_1 \oplus$ $V_2 \oplus (L_1 \cap L_2) = 0$, then for large r there exists no zero mode on M(r), ker D = 0. A related result for manifolds with cylindrical ends is presented in Appendix A. In the special case that $\mathcal{H} = \ker \hat{D} = 0$, we present a much stronger result, Theorem B. By the work of Muller [19], chapter VI, and Douglas and Wojciechowski [14], section 6, if $\mathcal{H} = \ker \hat{D} = 0$, then the self-adjoint extension D(j) of D acting on smooth L^2 -sections of $E_j(\infty)$ over $M_j(\infty)$ (M_j with the infinite cylinder attached, that is,

(1.11)
$$D(j) : \{\text{smooth } L^2 \text{-sections of } E_j(\infty)\} \to \{\text{smooth } L^2 \text{-sections of } E_i(\infty)\})$$

has pure point spectrum of finite multiplicity in the range of eigenvalues λ with

(1.12)
$$-\frac{\delta}{\sqrt{2}} \leq \lambda \leq +\frac{\delta}{\sqrt{2}}.$$

However, D(j) has essential spectrum for $|\lambda| \ge \delta$.

(1.13) Let $0 \le k < \delta/\sqrt{2}$ and $\varepsilon > 0$ be chosen such that neither D(1) nor D(2) has any eigenvalues in the range $(k, k + \varepsilon]$. Denote by $\{\lambda_{j1}, \ldots, \lambda_{jn(j)}\}$ the eigenvalues of D(j) over $M_j(\infty)$ counted with multiplicities in the range of eigenvalues λ with

$$-k \leq \lambda \leq k$$

and let $V_i(k)$ be the span of the associated eigenvectors.

Define K, L in terms of the bundle automorphism σ as in (4.5) and choose k, ε so that

$$(1.14) k + \varepsilon \le \frac{\delta K}{2}$$

As before, there is a natural splicing construction

(1.15)
$$\Phi_r: V_1(k) \oplus V_2(k) \to \Gamma(E(r) \text{ over } M(r)).$$

THEOREM B. Suppose $\mathcal{H} = \ker \hat{D} = 0$ and k, ε are chosen as in (1.13) and (1.14). Let N(r, t), sp(r, t) be defined as in Theorem A. Then there is an R > 0, depending only on $k, \varepsilon, K, L, \delta$ such that for all $r \ge R$ we have

(1.16)
$$N(r, k + (\varepsilon/2)) = \dim[V_1(k) \oplus V_2(k)].$$

(1.17) Let P_r be the orthogonal projection of $\Gamma(E(r))$ onto the span of the eigenvectors of D on E(r) with eigenvalues λ satisfying

$$|\lambda - \lambda_{j\alpha}| \leq \exp\left(-\frac{\delta r}{4}\right)$$

for some $j, \alpha, 1 \leq \alpha \leq n(j)$. Then the equality of two subspaces $sp(r, k + (\varepsilon/2)) = \mathbf{P}_r sp(r, k + (\varepsilon/2))$ holds.

(1.18) For $(x, y) \in V_1(k) \oplus V_2(k)$,

$$\|\Phi_r(x,y)-\mathbf{P}_r\Phi_r(x,y)\| \leq N\left[\exp(-\frac{\delta r}{4})\right] \cdot \|(x,y)\|$$

(1.19) with N depending only on $k, \varepsilon, K, L, \delta$. $\mathbf{P}_r \Phi_r$ is an isomorphism onto $\operatorname{sp}(r, k + (\varepsilon/2))$.

Roughly, the above theorem says that the eigenvalues of D on M(r) for $r \ge R$ to an exponentially small error in r in the range $[-(k + \varepsilon/2), +(k + \varepsilon/2)]$ consists of the eigenvalues

$$\{\lambda_{j1},\ldots,\lambda_{jn(j)}\}, \qquad j=1,2,$$

coming from D(j) on $M_j(\infty)$ for j = 1 or j = 2. Moreover, the multiplicities correspond, and this correspondence holds *uniformly* in r depending only on k, ε , K, L, δ —not on any other properties of D(1) or D(2).

In the sequel, Theorems A and B together with the general techniques of controlling low eigensections will be used in proving a similar decomposition of spectral flow. Let $\{D(u)\}_{0 \le u \le 1}$ be a smooth family of operators of the Atiyah-Patodi-Singer type. That is, over $\Sigma \times [-1, 1]$, the operator D(u) can be written as a sum $\pi^* \sigma \circ (\frac{\partial}{\partial s} + \pi^* \hat{D}(u))$ where $\hat{D}(u)$ is a smooth family over Σ and σ is independent of u. Then, after stretching $\Sigma \times [-1, 1] \subset M$ sufficiently long, we will show in Part II [5] under suitable hypotheses that the spectral flow of $\{D(u)\}_{0 \le u \le 1}$ is a sum of two spectral flows from M_1, M_2 and a Maslov index term from Σ . (Our paper [7] develops the Maslov index and needed results from it; these are briefly reviewed in Part II.) Crucial to such results on spectral flow will be further analytic work in Part II comparing low eigenvalues of operators on a manifold and on its pieces by achieving estimates that are uniform in the u parameter. Our treatment is sufficiently general to encompass the difficulties of zero modes at the ends of the parameter families as well as that of "jumping" Lagrangians.

While investigating the above topics, we benefited from the paper of Yoshida [26], who wrote of a decomposition of spectral flow for the particular operators in dimension 3 relevant to Floer homology. The statements of some analytic lemmas (viz. Propositions 5.3, 5.4, 6.1, and 6.2) in the present paper have been adapted from that paper, but both statements and proofs should be compared.

Theorem B was suggested by the results of Douglas and Wojciechowski. The uniform estimates of Theorem B will be crucial to us in Part II.

In Part III we will compare finite- and infinite-dimensional Lagrangian settings and the uniform estimates involved in relating them. By viewing infinite Lagrangians constructed from the graphs of Hilbert-Schmidt operators, we will relate them to sections of determinant line bundles. Our methods will be used there to introduce the technique of "canonical perturbations" of Lagrangian subvarieties of symplectic varieties to achieve transversality in a controlled fashion. We apply our methods there to the study of invariants of 3-manifolds, including Casson's invariant [6]. It is worth pointing out the relation between our study of small eigenvalues under stretching of manifolds and several well-known, important results on the elliptic operators of pinched manifolds, stretched manifolds, and adiabatic limits. As explained before, we have heavily employed the context and methods of [1] on manifolds with boundaries. Several results of Cheeger, in particular his studies of elliptic operators on manifolds with conical singularities, incorporated the idea on how limits arise under pinching. He studied the role that the choices of Lagrangians in the solution space of Dirac operators play on these limiting processes and applied them in η -invariant problems (cf. [9], [10], and [11]). In addition, there are the deep studies of adiabatic limits of η -invariants of fibrations in [3], [4], [10], [12], [13], [17], and [23].

Mrowka [18] in his important thesis considered Mayer-Vietoris-type decompositions in the context of gauge theory. His work includes some hard, nonlinear generalizations analogous to results of the present paper.

2. Symplectic Structures, Self-Adjoint Operators, and Splicing

Throughout this paper, our setting is as follows: M is a closed, oriented manifold, and

a smooth embedding with $\Sigma = i(\Sigma \times 0)$ a codimension-1, closed, oriented submanifold of M. This submanifold Σ splits M into two compact submanifolds M_1, M_2 with

(2.2) $M = M_1 \cup M_2, \qquad M_1 \cap M_2 = \partial M_1 = \partial M_2 = i(\Sigma \times 0),$ $\Sigma \times [-1,0] = i(\Sigma \times [-1,0]) \subset M_1,$ $\Sigma \times [0,1] = i(\Sigma \times [0,1]) \subset M_2.$

Defined over M and Σ are six more objects $D, E, \hat{D}, \hat{E}, \sigma, \phi$. First,

$$(2.3) D: \Gamma(E) \to \Gamma(E)$$

is a first-order, elliptic operator on the space $\Gamma(E)$ of smooth sections of a real vector bundle $E \to M$. This operator D is self-adjoint with respect to an inner product $(\cdot, \cdot)_M$ on $\Gamma(E)$ induced from a metric on E.

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Second,

$$(2.4) \qquad \qquad \hat{D}: \Gamma(\hat{E}) \to \Gamma(\hat{E})$$

is a first-order, elliptic operator on the space $\Gamma(\hat{E})$ of smooth sections of a real vector bundle $\hat{E} \to \Sigma$. Again this is self-adjoint with respect to the induced inner product $(\cdot, \cdot)_{\Sigma}$ on $\Gamma(\hat{E})$.

Third,

$$(2.5) \qquad \qquad \sigma: \hat{E} \to \hat{E}$$

is a bundle isomorphism of \hat{E} over Σ . Let $\pi : \Sigma \times [-1, 1] \to \Sigma$ denote the projection and $\pi^* \hat{E}$ the pullback of \hat{E} via π . Then

(2.6)
$$\phi: \pi^* \hat{E} \to E \mid \Sigma \times [-1, 1]$$

is a bundle isometry preserving the obvious metrics on these bundles. (Here $E \mid \Sigma \times [-1, 1]$ is understood as the pullback of E via (2.1)).

Finally,

(2.7)
$$D \mid \Sigma \times [-1, 1]$$
 equals $(\pi^* \sigma) \circ \left(\frac{\partial}{\partial s} + \pi^* \hat{D}\right)$

where $\pi : \Sigma \times [-1, 1] \to \Sigma$ is the projection, π^* is the pullback of operators via π , and the symbol *s* is the coordinate of [-1, 1] in $\Sigma \times [-1, 1]$. We also use the identifications in (2.1) and (2.6) in the above formula.

The self-adjointness of D and \hat{D} implies the following:

PROPOSITION 2.1. Let $D, E, \hat{D}, \hat{E}, \sigma, \phi$ be defined as above. Then

- (i) $\sigma^* = -\sigma$ and $\sigma \hat{D} = -\hat{D}\sigma$.
- (ii) There exists a nondegenerate symplectic pairing on $\Gamma(\hat{E})$ defined by

$$\{f,g\}=(f,\sigma g)_{\Sigma}.$$

(iii) Let $\mathcal{H} = \ker \hat{D}$. Then the restriction of the pairing in (ii) to \mathcal{H} is a nondegenerate, symplectic pairing.

For operators of Dirac type, for example, the natural geometrical operators, one has the simplification that $\sigma^2 = -1$; this characteristic, however, is not assumed in our treatment.

Next, we will construct two Lagrangian subspaces L_1, L_2 in the symplectic space \mathcal{H} arising from the limiting values of extended L^2 -solutions on $M_1(\infty)$, $M_2(\infty)$. Here the noncompact manifolds

(2.8)
$$\begin{aligned} M_1(\infty) &= M_1 \cup \Sigma \times [0,\infty), \qquad \partial M_1 = \Sigma \times 0, \\ M_2(\infty) &= \Sigma \times (-\infty,0] \cup M_2, \qquad \partial M_2 = \Sigma \times 0, \end{aligned}$$

are defined as in (1.7). To begin, we recall some basic constructions in [1]. Let $\{\phi_k\}$ be a complete orthonormal basis of eigensolutions of \hat{D} with $\hat{D}\phi_k = \mu_k\phi_k$. Let

(2.9)
$$P_{+} = L^{2} \text{-closure of } \{\phi_{k} \mid \mu_{k} > 0\} \text{ in } L^{2}(E), \\ P_{-} = L^{2} \text{-closure of } \{\phi_{k} \mid \mu_{k} < 0\} \text{ in } L^{2}(\widehat{E}).$$

Then, from the spectral decomposition theorem, there exists a decomposition of $L^2(\hat{E})$ into an orthogonal sum

$$L^2(\hat{E}) = \mathcal{H} \oplus P_+ \oplus P_- .$$

In a natural manner, the vector bundle E gives rise to vector bundle E(r) over M(r). The procedure is to consider over M_j the bundle $E \mid M_j$ and over $\Sigma \times [-r, r]$ the bundle $\pi^* \hat{E}$ and glue these two pieces together by the same gluing data as in E. Similarly, there are bundles $E_1(\infty)$ and $E_2(\infty)$ over $M_1(\infty)$ and $M_2(\infty)$ with the property that $E_j(\infty) \mid M_j$ is the same as $E \mid M_j$ and

 $E_1(\infty) \mid \Sigma \times [0,\infty) = \pi^* \hat{E}, \qquad E_2(\infty) \mid \Sigma \times (-\infty,0] = \pi^* \hat{E}.$

The operator $D: \Gamma(E) \rightarrow \Gamma(E)$ can also be extended to produce operators

(2.10)
$$D = D(r) : \Gamma(E(r)) \to \Gamma(E(r)),$$
$$D = D_j(\infty) : \Gamma(E_j(\infty)) \to \Gamma(E_j(\infty)), \quad j = 1, 2,$$

over E(r) and $E_j(\infty)$; over the submanifolds M_1, M_2 , they are the same as $D \mid M_1, D \mid M_2$, and over the cylinders $\Sigma \times [-r, r], \Sigma \times [0, \infty), \Sigma \times (-\infty, 0]$, they are defined by the formula $\pi^* \sigma \circ (\frac{\partial}{\partial s} + \pi^* \hat{D})$. Note that the manifold M(r) of (1.5) is decomposed by $\Sigma \times 0 \hookrightarrow M(r)$ into two pieces as follows:

(2.11)
$$\begin{array}{l} M_1(r) = M_1 \cup \Sigma \times [0,r], \qquad \partial M_1(r) = \Sigma \times r, \\ M_2(r) = \Sigma \times [-r,0] \cup M_2, \qquad \partial M_2(r) = \Sigma \times (-r), \end{array}$$

and likewise the bundle E(r) as the union of bundles $E_1(r) \rightarrow M_1(r)$ and $E_2(r) \rightarrow M_2(r)$. Let $L^2(E_j(r))$ denote the space of L^2 -sections of $E_j(r)$, and let $L^2_1(E_j(r))$ denote the Sobolev space of L^2_1 -sections. Given a subspace V in $L^2(\hat{E})$, let $L^2_1(E_j(r); V)$ denote the Sobolev L^2_1 completion of the smooth sections ϕ such that $\phi \mid \partial M_j(r) = \phi \mid \Sigma$ lies in V. In particular, by letting $V = P_+, P_-, P_+ \oplus \mathcal{H}, P_- \oplus \mathcal{H}$, we have four Sobolev spaces $L^2_1(E_1(r), P_+), L^2_1(E_1(r), P_+ \oplus \mathcal{H}), L^2(E_2(r), P_-)$, and $L^2(E_2(r), P_- \oplus \mathcal{H})$. Moreover, as shown in [1], the closures of the operator D induce Fredholm operators on these spaces:

(2.12)
$$D_{1}: L_{1}^{2}(E_{1}(r); P_{+}) \to L^{2}(E_{1}(r));$$
$$\tilde{D}_{1}: L_{1}^{2}(E_{1}(r); P_{+} \oplus \mathcal{H}) \to L^{2}(E_{1}(r)) \quad \text{over } M_{1}(r).$$
$$D_{2}: L_{1}^{2}(E_{2}(r); P_{-}) \to L^{2}(E_{2}(r));$$
$$\tilde{D}_{2}: L_{1}^{2}(E_{2}(r); P_{-} \oplus \mathcal{H}) \to L^{2}(E_{2}(r)) \quad \text{over } M_{2}(r).$$

The kernels of these operators also have their counterparts in $L^2(E_j(\infty))$.

Define V_j to be the space of L^2 -solutions of D = 0 over $M_j(\infty)$,

(2.13)
$$V_{j} = \{ \phi \in L_{1}^{2}(E_{j}(\infty)) \mid D\phi = 0 \},$$

and \tilde{V}_j to be the space of extended L^2 -solutions. More precisely, the latter \tilde{V}_j is the space of pairs $(\phi, \hat{\phi})$ with the following properties:

- (1) ϕ is a section of $E_i(\infty)$ with $D\phi = 0$.
- (2) $\hat{\phi}$ is a section of \hat{E} with $\hat{D}\hat{\phi} = 0$.
- (3) $\phi \mid M_j$ is square integrable.
- (4) $\phi \mid \Sigma \times (\pm \infty, 0] \pi^* \hat{\phi}$ is square integrable $(-\infty \text{ for } j = 2, +\infty \text{ for } j = 1)$.

The above spaces V_j and \tilde{V}_j are finite-dimensional and can be compared with the kernels of the Fredholm operators in (2.12) because of the following theorem of Atiyah-Patodi-Singer [1]:

THEOREM 2.2. Let V_j , \tilde{V}_j , $L_1^2(E_1(r), P_+)$, $L_1^2(E_1(r), P_+ \oplus \mathcal{H})$, $L_1^2(E_2(r), P_-)$, and $L_1^2(E_2(r), P_- \oplus \mathcal{H})$ be defined as above. Then

(i) An element $(\phi, \hat{\phi})$ of \tilde{V}_1 has a representative of the form

$$\boldsymbol{\phi} \mid \boldsymbol{\Sigma} \times [0,\infty) = \pi^* \boldsymbol{\hat{\phi}} + \sum_{\mu_k > 0} c_k e^{-\mu_k s} \pi^* \boldsymbol{\phi}_k$$

where $\phi - \pi^* \hat{\phi}$ over $\Sigma \times [0, \infty)$ is square integrable and $\phi, \hat{\phi}$ are C^{∞} . Similarly, an element $(\psi, \hat{\psi})$ of \tilde{V}_2 takes the form $\psi \mid \Sigma \times (-\infty, 0] = \pi^* \hat{\psi} + \sum_{\mu_k < 0} c_k e^{-\mu_k s} \pi^* \phi_k$ here. Moreover, $\psi, \hat{\psi}$ are C^{∞} .

(ii) The restriction maps

$$V_{1} \rightarrow L_{1}^{2}(E_{1}(r), P_{+}) \qquad \phi \mapsto \phi \mid M_{1}(r)$$

$$\tilde{V}_{1} \rightarrow L_{1}^{2}(E_{1}(r), P_{+} \oplus \mathscr{H}) \qquad (\phi, \hat{\phi}) \mapsto \phi \mid M_{1}(r)$$

$$V_{2} \rightarrow L_{1}^{2}(E_{2}(r), P_{-}) \qquad \psi \mapsto \psi \mid M_{2}(r)$$

$$\tilde{V}_{2} \rightarrow L_{1}^{2}(E_{2}(r), P_{-} \oplus \mathscr{H}) \qquad (\psi, \hat{\psi}) \mapsto \psi \mid M_{2}(r)$$

define isomorphisms $V_1 \cong \ker D_1$, $\tilde{V}_1 \cong \ker \tilde{D}_1$, $V_2 \cong \ker D_2$, and $\tilde{V}_2 \cong \ker \tilde{D}_2$.

From the above theorem, there are natural homomorphisms

(2.14)
$$P_1: \tilde{V}_1 \to \mathcal{H} = \ker \hat{D} \qquad (\phi, \hat{\phi}) \mapsto \hat{\phi}$$
$$P_2: \tilde{V}_2 \to \mathcal{H} = \ker \hat{D} \qquad (\psi, \hat{\psi}) \mapsto \hat{\psi}$$

of \tilde{V}_j into \mathcal{H} . Denote by L_j the image of P_j . Then elements in L_j represent the limiting values of the extended L^2 -solutions of D on $M_j(\infty)$.

PROPOSITION 2.3. Let L_j , j = 1, 2, be defined as above. Then L_j is a Lagrangian subspace in the symplectic vector space \mathcal{H} .

The operators in (2.12) are not self-adjoint, and the remedy is to introduce the boundary conditions $P_1 \oplus \sigma L_1$ and $P_+ \oplus \sigma L_2$, respectively, in $L_1^2(E_2(r), P_- \oplus \sigma L_1)$ and $L_1^2(E_1(r), P_+ \oplus \sigma L_2)$ (see [24], [25], and [26]). The closure of D provides us with Fredholm operators

$$\mathcal{D}_1 : L_1^2(E_1(r), P_+ \oplus \sigma L_1) \to L^2(E_1(r))$$
$$\mathcal{D}_2 : L_1^2(E_2(r), P_- \oplus \sigma L_2) \to L^2(E_2(r))$$

PROPOSITION 2.4. Let \mathcal{D}_j , j = 1, 2, be defined as above. Then they are selfadjoint Fredholm operators with ker $\mathcal{D}_j = V_j$.

We conclude this section with the *splicing construction*. Let \mathcal{W} denote the space of pairs of "matching" extended L^2 -space. That is, \mathcal{W} is the Hilbert space of pairs $(\phi, \hat{\phi}), (\psi, \hat{\psi})$ such that

(2.15) The pairs $(\phi, \hat{\phi})$ and $(\psi, \hat{\psi})$ are extended L^2 -solutions of $D(\cdot) = 0$ over $M_1(\infty)$ and $M_2(\infty)$ and as smooth sections $\hat{\phi} = \hat{\psi}$ over Σ .

Choose a smooth, nondecreasing function $\rho(t)$, $0 \le t \le 1$ such that

(2.16)
$$\rho(t) = \begin{cases} 0 & 0 \le t \le \frac{1}{4} \\ 1 & \frac{3}{4} \le t \le 1 \end{cases}$$

and $\left|\frac{d\rho}{dt}\right| \leq 4$. For a matching pair of solutions $(\phi, \hat{\phi}), (\psi, \hat{\psi})$ in \mathcal{W} , we define $h = \Psi_r((\phi, \hat{\phi}), (\psi, \hat{\psi}))$ as a section of E(r) by the formula

(2.17)

$$h \mid M_{1} = \phi, \qquad h \mid M_{2} = \psi,$$

$$(h \mid \Sigma \times [-r, -1])(x, s) = \phi(x, s + r),$$

$$(h \mid \Sigma \times [-1, 0])(x, s)$$

$$= [\rho(-s) \cdot (\phi - \pi^{*}\hat{\phi})(x, s + r)] + (\pi^{*}\hat{\phi})(x, s + r),$$

$$(h \mid \Sigma \times [0, 1])(x, s)$$

$$= [\rho(s) \cdot (\psi - \pi^{*}\hat{\psi})(x, s - r)] + \pi^{*}\hat{\psi}(x, s - r),$$

$$(h \mid \Sigma \times [1, r])(x, s) = \psi(x, s - r).$$

It is not difficult to see that there exists an exact sequence

$$0 \to V_1 \oplus V_2 \xrightarrow{\beta} \mathcal{W} \xrightarrow{\alpha} L_1 \cap L_2 \to 0$$

where the map $\alpha : \mathcal{W} \to L_1 \cap L_2$ is defined by sending a matching pair $[(\phi, \hat{\phi}), (\psi, \hat{\psi})]$ to $\hat{\phi} = \hat{\psi}$ in $L_1 \cap L_2$, and the map $\beta : V_1 \oplus V_2 \to \mathcal{W}$ is defined by sending $\phi \oplus \psi$ to $[(\phi, 0), (\psi, 0)]$.

PROPOSITION 2.5.

(i) Given elements $(\phi_1, \hat{\phi}_1) \in \tilde{V}_1$ and $\phi_2 \in V_1$, the limit, $(\phi_1, \phi_2)_{M_1(\infty)} = \lim_{r \to \infty} \{(\phi_1 \mid M_1, \phi_2 \mid M_1) + (\phi_1 \mid \Sigma \times [0, r], \phi_2 \mid \Sigma \times [0, r])\}$ always exists, and the assignment $(\phi_1, \hat{\phi}_1) \times \phi_2 \rightarrow (\phi_1, \phi_2)_{M_1(\infty)}$ gives a bilinear pairing on $\tilde{V}_1 \times V_1$. Furthermore, when restricted to the subspace $V_1 \times V_1$, this pairing becomes a symmetric, positive definite pairing.

(ii) Similarly, the limit

$$(\psi_1,\psi_2)_{M_2(\infty)} = \lim_{r \to \infty} \{ (\psi_1 \mid M_2, \psi_2 \mid M_2) + (\psi_1 \mid \Sigma \times [-r,0], \psi_2 \mid \Sigma \times [-r,0]) \}$$

gives rise to a bilinear pairing on $\tilde{V}_2 \times V_2$ that is symmetric and positive definite over $V_2 \times V_2$.

An immediate consequence of Proposition 2.5 is that for each element χ in $L_1 \cap L_2$ there exists a unique matching pair $[(\phi_1(\chi), \chi), (\phi_2(\chi), \chi)]$ in \mathcal{W} such that $\phi_j(\chi)$ is perpendicular to the subspace V_j with respect to the pairing $(\cdot, \cdot)_{M_j(\infty)}$. In other words, \mathcal{W} is decomposed into a sum $V_1 \oplus V_2 \oplus (L_1 \cap L_2)$.

Finally, the required splicing construction

$$(2.18) \qquad \Phi_r: V_1 \oplus V_2 \oplus L_1 \cap L_2 \to \Gamma(E(r))$$

is defined by the formula

$$\Phi_r(\phi,\psi,\chi) = \Psi_r((\phi,0),(0,0)) + \Psi_r((0,0),(\psi,0)) + \Psi_r[(\phi_1(\chi),\chi),(\phi_2(\chi),\chi)].$$

Remark 2.6. In Part II, we will generalize the decomposition of $L^2(\hat{E})$ studied in this paper to a decomposition, for $k \ge 0$,

$$L^{2}(\hat{E}) = \mathcal{H}(k) \oplus P_{+}(k) \oplus P_{-}(k),$$

where the summands are the spans of the eigenspaces with eigenvalues in [-k, +k], $(+k, \infty)$, and $(-\infty, -k)$, respectively. We will also produce corresponding Lagrangian subspaces in $\mathcal{H}(k)$ generalizing L_1 and L_2 , given above for k = 0. These generalized Lagrangians yield by a symplectic reduction process the above L_1 and L_2 .

3. Results on L^2 -Solutions and Lagrangians

3.1. Proof of Proposition 2.1

By the self-adjoint property of the operators D and \hat{D} , we have

(3.1)
$$\pi^* \sigma \circ \left(\frac{\partial}{\partial s} + \pi^* \hat{D}\right) = D = D^* = \left(-\frac{\partial}{\partial s} + \pi^* \hat{D}\right) \circ \pi^* \sigma^*$$

over $\Sigma \times [-1, 1]$. Comparing the coefficients of $\frac{\partial}{\partial s}$ on both sides of the above equation, it follows that $\sigma = -\sigma^*$, and hence from the rest of this equation $\sigma \circ \hat{D} = -\hat{D} \circ \sigma$. This proves Proposition 2.1(i).

Now from the first equality $\sigma = -\sigma^*$, we have

$$(f, \sigma g)_{\Sigma} = (\sigma^* f, g)_{\Sigma} = (-\sigma f, g)_{\Sigma} = -(g, \sigma f)_{\Sigma};$$

in other words, if we define $\{f, g\}$ to be $(f, \sigma g)_{\Sigma}$, we obtain a skew-symmetric pairing in $\Gamma(\hat{E})$. Since the original pairing (\cdot, \cdot) in $\Gamma(\hat{E})$ is nonsingular and σ is an isomorphism, the skew-symmetric pairing $\{\cdot, \cdot\}$ is also nonsingular or, in other words, a symplectic pairing as asserted in Proposition 2.1(ii).

From the second equality $\sigma \circ \hat{D} = -\hat{D} \circ \sigma$, it is clear that σ preserves the subspace $\mathcal{H} = \ker \hat{D}$. Since (\cdot, \cdot) is positive definite, its restriction to H is nonsingular, and the proof of Proposition 2.1(iii) follows from the same argument as before.

3.2. Proof of Proposition 2.3

In order to prove Proposition 2.3, we need the following:

LEMMA 3.1. Let f and g be two smooth sections in $\Gamma(E \mid M_j)$. Then

(3.2)
$$(Df,g)_{M_j} - (f,Dg)_{M_j} = \{f \mid \partial M_j, g \mid \partial M_j\}\varepsilon_j$$

with $\varepsilon_j = 1$ for j = 2 and $\varepsilon_j = -1$ for j = 1.

Since the proof is identical for j = 1 or j = 2, we will concentrate on the case j = 2. In this case Lemma 3.1 is but a restatement of a theorem in the Palais treatment of the Atiyah-Singer index theorem [20]. It is immediately proved by using D, \hat{D} self-adjoint and then integrating by parts.

We will concentrate on M_j for j = 2, since the case for j = 1 is similar. For extended L^2 -solutions $(\phi_1, \hat{\phi}_1), (\phi_2, \hat{\phi}_2)$ over $M_2(\infty)$, we may apply (3.2) to the submanifold $M_2(r) = \Sigma \times [-r, 0] \cup M_2$,

(3.3)

$$0 = (D\phi_1, \phi_2)_{M_2(r)} - (\phi_1, D\phi_2)_{M_2(r)} \\
= \{\phi_1 \mid \partial M_2(r), \phi_2 \mid \partial M_2(r)\} \\
= \left\{ \hat{\phi}_1 + \sum_{\mu_k < 0} c_k(1)\phi_k, \, \hat{\phi}_2 + \sum_{\mu_\ell < 0} c_\ell(2)\phi_\ell \right\}.$$

In the last equation we use Theorem 2.2 to obtain the eigenexpansion of $\phi_j \mid \partial M_2(r)$; that is,

(3.4)
$$\begin{aligned} \phi_1 \mid \partial M_2(r) &= \hat{\phi}_1 + \sum_{\mu_k < 0} c_k(1)\phi_k \\ \phi_2 \mid \partial M_2(r) &= \hat{\phi}_2 + \sum_{\mu_\ell < 0} c_\ell(2)\phi_\ell \,. \end{aligned}$$

Since $\sigma \circ \hat{D} = -\hat{D} \circ \sigma$, we have

$$\hat{D}(\sigma\phi_{\ell}) = -\mu_{\ell}(\sigma\phi_{\ell})$$

and so

(3.6)
$$\{\phi_k, \phi_\ell\} = (\phi_k, \sigma\phi_\ell) = 0$$

whenever $\mu_k < 0$ and $\mu_\ell \leq 0$ or $\mu_k \leq 0$ and $\mu_\ell < 0$. In particular, in the above expansion (3.3)

$$0 = \left\{ \hat{\phi}_1 + \sum_{\mu_k < 0} c_k(1)\phi_k, \hat{\phi}_2 + \sum_{\mu_\ell < 0} c_\ell(2)\phi_\ell \right\}$$

= { $\hat{\phi}_1, \hat{\phi}_2$ }.

This proves that L_2 is an isotropic subspace in H.

To prove that L_2 is the maximal isotropy subspace (i.e., $\dim L_2 = \frac{1}{2} \dim \mathcal{H}$), we recall the following assertion in [1]. Let $D : \Gamma(E) \to \Gamma(F)$ be an operator of Atiyah-Patodi-Singer type near ∂M_2 , and let D^* denote its formal adjoint. Then, by imposing the nonlocal boundary condition $P_- \oplus \mathcal{H}$ as in Theorem 2.2, corresponding to D^* there is defined the operator

$$D_2^{\#}: L^2_1(F, P_- \oplus \mathcal{H}) \to L^2(E)$$

This last operator $D_2^{\#}$ has a finite-dimensional kernel consisting of all extended L^2 -solutions ψ , $D_2^*\psi = 0$, and also gives rise to a subspace $L_2^{\#}$ in ker \hat{D} consisting of the limits $\hat{\psi}$ of extended L^2 -solutions of D^* . Then, from [1], we recall the following result:

THEOREM 3.2. Let L_2 and $L_2^{\#}$ be defined as above. Then dim L_2 + dim $L_2^{\#}$ = dim ker \hat{D} .

In our application, we set $D = D^*$, $D_2^{\#} = \tilde{D}_2(M_2)$, $L_2^{\#} = L_2$, and so $2 \dim L_2 = \dim \mathcal{H}$ as claimed. Applying a similar analysis to M_1 , the proof of Proposition 2.3 follows.

Because the proof of Proposition 2.4 is a more straightforward application of [1], we will omit the details here. Related discussions on these operators can be found in [24] and [25].

3.3. Proof of Proposition 2.5

Let $(\psi_1, \hat{\psi}_1)$ be an element in \tilde{V}_2 , and let ψ_2 be an element in V_2 . Consider their eigenexpansions over $\Sigma \times (-\infty, 0]$,

(3.7)
$$\psi_j \mid \Sigma \times (-\infty, 0] = \pi^* \hat{\psi}_j + \sum_{\mu_k < 0} c_k(j) e^{-\mu_k s} \pi^* \psi_k, \quad j = 1, 2.$$

For j = 2, the section ψ_j is square integrable and so $\hat{\psi}_2 = 0$. In general, the section $\psi_j \mid \Sigma \times (-\infty, 0] - \pi^* \hat{\psi}_j$ is square integrable, and we have

$$A_{j} = \|\psi_{j} - \pi^{*} \hat{\psi}_{j}\|^{2} = \sum_{\mu_{k} < 0} |c_{k}(j)|^{2} \int_{-\infty}^{0} e^{-2\mu_{k}s} ds$$
$$= \sum_{\mu_{k} < 0} \frac{|c_{k}(j)|^{2}}{2|\mu_{k}|}.$$

On the other hand, the inner product $(\psi_1 \mid M_2(r), \psi_2 \mid M_2(r))$ can be written (by $\hat{\psi}_2 = 0$) as

$$(\psi_{1} - \pi^{*}\hat{\psi}_{1} \mid \Sigma \times [-r, 0], \psi_{2} \mid \Sigma \times [-r, 0]) + (\psi_{1} \mid M_{2}, \psi_{2} \mid M_{2})$$

$$= \left(\sum_{\mu_{k} < 0} c_{k}(1)c_{k}(2) \int_{-r}^{0} e^{-2\mu_{k}s} ds\right) + (\psi_{1} \mid M_{2}, \psi_{2} \mid M_{2})$$

$$= \sum_{\mu_{k} < 0} c_{k}(1)c_{k}(2) \left[\frac{1 - e^{2\mu_{k}r}}{|2\mu_{k}|}\right] + (\psi_{1} \mid M_{2}, \psi_{2} \mid M_{2})$$

$$= \sum_{\mu_{k} < 0} \frac{c_{k}(1)c_{k}(2)}{|2\mu_{k}|} - \sum_{\mu_{k} < 0} \frac{c_{k}(1)c_{k}(2)}{|2\mu_{k}|} e^{2\mu_{k}r} + (\psi_{1} \mid M_{2}, \psi_{2} \mid M_{2})$$

By the Schwarz inequality,

$$\sum_{\mu_k < 0} \left| \frac{c_k(1)c_k(2)}{2\mu_k} \right| \leq \sqrt{\sum_{\mu_k < 0} \frac{|c_k(1)|^2}{2|\mu_k|}} \cdot \sum_{\mu_k < 0} \frac{|c_k(2)|^2}{2|\mu_k|} = \sqrt{A_1 A_2},$$

and so the first sum in (3.8) converges. Also,

$$\sum_{\mu_k<0} \left| \frac{c_k(1)c_k(2)}{2\mu_k} e^{2\mu_k r} \right| \leq e^{-2\delta r} \left(\sum_{\mu_k<0} \left| \frac{c_k(1)c_k(2)}{2\mu_k} \right| \right)$$
$$\leq e^{-2\delta r} \sqrt{A_1 A_2} ,$$

where δ (as in Theorem A) is the smallest $|\mu_k|$ as μ_k goes through nonzero eigenvalues. Consequently, the second sum in (3.8) approaches 0 as r tends to ∞ . From a comparison of the expression in (3.8) with the corresponding inner product $(\psi_1, \psi_2)_{M_2(\infty)}$, it follows immediately that

$$\begin{aligned} & (\psi_1, \psi_2)_{M_2(\infty)} \\ &= \lim_{r \to \infty} \{ (\psi_1 \mid M_2, \psi_2 \mid M_2) + (\psi_1 \mid \Sigma \times [-r, 0], \ \psi_2 \mid \Sigma \times [-r, 0]) \} \end{aligned}$$

exists. Moreover, if $\hat{\psi}_1 = 0$, then $(\psi_1, \psi_2)_{M_2(\infty)} = \lim_{r \to \infty} (\psi_1, \psi_2)_{M_2(r)}$ is the standard inner product on the space of L^2 -sections on $E_2(\infty)$. This proves Proposition 2.5.

4. Producing Eigensolutions by Splicing

In this section, we prove statement (1.10) and part of (1.9) in Theorem A.

LEMMA 4.1. Let δ denote the minimum of |u| for nonzero eigenvalues u of \hat{D} , and let \mathbf{P}_r denote the projection of $\Gamma(E(r))$ onto the subspace $\operatorname{sp}(r; \exp(-\frac{\delta r}{4}))$. For $r \geq R \geq 2$ and R sufficiently large, the composite

(4.1)
$$\mathbf{P}_r \circ \Phi_r : V_1 \oplus V_2 \oplus (L \cap L_2) \to \Gamma(E(r)) \to \operatorname{sp}\left(r, \exp\left(-\frac{\delta r}{4}\right)\right)$$

is an injection.

For an element (α, β, γ) in $V_1 \oplus V_2 \oplus (L_1 \cap L_2)$, the inequality

(4.2)
$$\|\mathbf{P}_r \circ \Phi_r(\alpha, \beta, \gamma) - \Phi_r(\alpha, \beta, \gamma)\| \leq \exp\left(-\frac{\delta r}{4}\right) \|\Phi_r(\alpha, \beta, \gamma)\|$$

holds.

To prove (4.1), we need the following:

LEMMA 4.2. For an element (α, β, γ) in $V_1 \oplus V_2 \oplus (L_1 \cap L_2)$ and for $r \ge R \ge 2$, and R sufficiently large, the inequality

(4.3)
$$\|D\Phi_r(\alpha,\beta,\gamma)\| \leq \exp\left(-\frac{\delta r}{2}\right) \|\Phi_r(\alpha,\beta,\gamma)\|$$

holds.

The bundle automorphism $\tau = \sigma^* \sigma : \hat{E} \to \hat{E}$ induces on each fiber $\hat{E}_x, x \in \Sigma$, a self-adjoint isomorphism $\tau_x : E_x \to E_x$. Since $\tau = \sigma^* \sigma$, the linear automorphisms $\tau_x = \sigma_x^* \sigma_x$ are positive definite with respect to the inner product on E_x . Let k_x and ℓ_x denote, respectively, the smallest and largest eigenvalue of τ_x ; that is,

(4.4)
$$k_x = \min\{\langle v, \tau_x(v) \rangle \mid ||v|| = 1, v \in E_x\}$$
$$\ell_x = \max\{\langle v, \tau_x(v) \rangle \mid ||v|| = 1, v \in E_x\}$$

Clearly the assignments $x \mapsto k_x$, $x \mapsto \ell_x$ define two continuous positive functions $k: \Sigma \to \mathbb{R}^+$, $\ell: \Sigma \to \mathbb{R}^+$ on Σ and hence positive extremum values. Define positive numbers K > 0 and L > 0 to be the square roots of these extremum values

(4.5)
$$K = \sqrt{\min\{k(x) \mid x \in \Sigma\}}$$
$$L = \sqrt{\max\{\ell(x) \mid x \in \Sigma\}}$$

In particular, for ϕ a section of $\pi^* \hat{E}$ over $\Sigma \times [a, b]$, we have

(4.6)
$$K \|\phi\|_{\Sigma \times [a,b]} \le \|\sigma\phi\|_{\Sigma \times [a,b]} \le L \|\phi\|_{\Sigma \times [a,b]}$$

To prove Lemma 4.2, first of all, given an element γ in $L_1 \cap L_2$, there exist unique elements $(\gamma_1, \hat{\gamma}_1) \in \tilde{V}_1$ and $(\gamma_2, \hat{\gamma}_2) \in \tilde{V}_2$ such that $\hat{\gamma}_1 = \hat{\gamma}_2 = \gamma$ and $(\gamma_1, V_1)_{M_1(\infty)} = (\gamma_2, V_2)_{M_2(\infty)} = 0$. Using (γ_1, γ_2) we can construct from (α, β) a matching pair $(\alpha + \gamma_1, \gamma), (\beta + \gamma_2, \gamma)$ of extended L^2 -solutions in \mathcal{W} . Applying the formula (2.17) to this matching pair $(\alpha + \gamma_1, \gamma), (\beta + \gamma_2, \gamma)$, we obtain the explicit description of $\Phi_r(\alpha, \beta, \gamma)$.

Note from (2.17) the section $D\Phi_r(\alpha, \beta, \gamma)$ over M(r) vanishes everywhere except on $\Sigma \times [-1, 0]$ and $\Sigma \times [0, 1]$. Over the first piece, we have

(4.7)
$$\|D\Phi_{r}(\alpha,\beta,\gamma) | \Sigma \times [-1,0]\|^{2} = \left\| \left(\frac{\partial \rho}{\partial s}(-s) \right) \cdot \sigma[\alpha + \gamma_{1} - \pi^{*}\gamma] | \Sigma \times [-1,0] \right\|^{2} \le 16 \|\sigma[\alpha + \gamma_{1} - \pi^{*}\gamma] | \Sigma \times [-1,0]\|^{2} \le 16 L^{2} \|\alpha + \gamma_{1} - \pi^{*}\gamma| | \Sigma \times [-1,0]\|$$

Here we are identifying $\Sigma \times [-1,0]$ in $\Sigma \times [-r,0]$ from $M_1(r)$ and from M(r)with $\Sigma \times [r-1,r]$ in $M_1(\infty)$. The identification is: $(\chi, s) \to (\chi, s+r)$ of $\Sigma \times [-r,0]$ in $M_1(r)$ with $\Sigma \times [0,r]$ in $M_1(\infty)$.

Let $\zeta = \alpha + \gamma_1 - \pi^* \gamma \mid \Sigma \times [0, \infty)$ have eigenexpansion

$$\zeta = \sum_{\mu_k>0} C(k) e^{-\mu_k s} \pi^* \phi_k \,.$$

Thus

$$\|\zeta | \Sigma \times s \|^2 = \sum_{\mu_k > 0} |C(k)|^2 e^{-2\mu_k s}$$

and by (4.7) we have

$$\|D\Phi_{\nu}(\alpha,\beta,\gamma) \mid M_{1}(r)\|^{2} \leq 16L^{2} \int_{r-1}^{r} \sum_{\mu_{k}>0} |C(k)|^{2} e^{-2\mu_{k}s} ds \leq 16L^{2} e^{-2\delta(r-1)} \left(\sum_{\mu_{k}>0} |C(k)|^{2} \left(\frac{1-e^{-2\mu_{k}}}{2\mu_{k}} \right) \right)$$

Since $\zeta \mid \Sigma \times s$ in P_+ , and γ in \mathcal{H} , we have

$$\|\alpha + \gamma_1 \mid \Sigma \times s\|^2 = \|\zeta \mid \Sigma \times s\|^2 + \|\pi^* \gamma \mid \Sigma \times s\|^2 \ge \|\zeta \mid \Sigma \times s\|^2.$$

This allows us to estimate $\|\Phi_{\gamma}(\alpha, \beta, \gamma) \| M_1(\gamma) \|^2$ as

$$\begin{split} \|\Phi_r(\alpha,\beta,\gamma) \mid M_1(r)\|^2 &\geq \|\alpha+\gamma_1\mid \Sigma \times [0,r-1]\|_{M_1(\infty)}^2 \\ &\geq \|\zeta\mid \Sigma \times [0,r-1]\|_{M_1(\infty)}^2 \\ &= \int_0^{r-1} \sum_{\mu_k > 0} |C(k)|^2 e^{-2\mu_k s} \, ds \\ &= \sum_{\mu_k > 0} \left(\frac{1-e^{-2\mu_k(r-1)}}{2\mu_k}\right) |C(k)|^2 \\ &\geq \sum_{\mu_k > 0} |C(k)|^2 \left(\frac{1-e^{-2\mu_k}}{2\mu_k}\right) \, . \end{split}$$

There are similar inequalities for $||D\Phi_r| |M_2(r)||^2$, $||\Phi_r| |M_2(v)||^2$. Adding these yields the inequality

$$\|D\Phi_r(\alpha,\beta,\gamma)\|_{\mathcal{M}(r)}^2 \leq 16L^2 e^{-2\delta(r-1)} \cdot \|\Phi_r(\alpha,\beta,\gamma)\|_{\mathcal{M}(r)}^2$$

If $r \ge R$ with $R \ge 1$ and $4L \cdot e^{-(\delta/2)R} \cdot e^{\delta} \le 1$, then we obtain

$$\|D\Phi_r(\alpha,\beta,\gamma)\|_{M(r)} \leq \exp\left(\frac{-\delta r}{2}\right) \|\Phi_r(\alpha,\beta,\gamma)\|_{M(r)}$$

for $r \ge R$ as desired.

We are now in a position to prove Lemma 4.1. Let $\Phi_r(\alpha, \beta, \gamma) = \sum d_\mu \tilde{\phi}_\mu$ be the eigenexpansion of $\Phi_r(\alpha, \beta, \gamma)$ in terms of an orthonormal basis $\{\tilde{\phi}_\mu\}$ of eigensolutions of $D\tilde{\phi}_\mu = \mu \tilde{\phi}_\mu$ over M(r). Then we have

(4.8)
$$\|D\Phi_r\|_{M(r)}^2 = (D\Phi_r, D\Phi_r) = (\Sigma \mu d_\mu \tilde{\phi}_\mu, \Sigma \mu d_\mu \tilde{\phi}_\mu) = \Sigma \mu^2 (d_\mu)^2,$$

and so

$$\sum_{\mu \ge \exp(-\frac{\delta r}{4})} (d_{\mu})^2 \le \exp\left(+\frac{\delta r}{2}\right) \left(\sum_{\mu \ge \exp(-\frac{\delta r}{4})} \mu^2 d_{\mu}^2\right) \le \exp\left(+\frac{\delta r}{2}\right) \|D\Phi_r\|^2.$$

Note that under the projection $\mathbf{P}: \Gamma(E(r)) \to \operatorname{sp}(r, \exp(-\frac{\delta r}{4}))$ projects on the low eigenmodes, and so $\|\mathbf{P}_r \Phi_r(\alpha, \beta, \gamma) - \Phi_r(\alpha, \beta, \gamma)\|^2$ is precisely the expression

$$\sum_{\mu \geqq \exp(-\frac{\delta r}{4})} (d_{\mu})^2 \, .$$

Using the estimate of (4.3), we have

$$\|\mathbf{P}_{r}\Phi_{r}(\alpha,\beta,\gamma)-\Phi_{r}(\alpha,\beta,\gamma)\|^{2} \leq \exp\left(\frac{\delta r}{2}\right)\|D\Phi_{r}(\alpha,\beta,\gamma)\|^{2}$$
$$\leq \exp\left(-\frac{\delta r}{2}\right)\|\Phi_{r}(\alpha,\beta,\gamma)\|^{2}$$

and therefore (4.2).

Furthermore, if $\mathbf{P}\Phi_r(\alpha,\beta,\gamma) = 0$, then the inequality (4.3) gives us

$$\|\Phi_r(\alpha,\beta,\gamma)\| \leq \exp\left(-\frac{\delta r}{4}\right) \|\Phi_r(\alpha,\beta,\gamma)\|$$

This is impossible for $r \ge R > 0$ unless $\Phi_r(\alpha, \beta, \gamma) = 0$ or, in other words, $\mathbf{P}_r \circ \Phi_r$ is injective. This completes the proof of Lemma 4.1.

5. Estimates of Low Eigenmodes of a Stretched Manifold

In this section, we establish estimates concerning the behavior of low eigenmodes of M(r) over its cylindrical submanifold $\Sigma \times [-r, r]$. These estimates will be used in the next section to show that there are no eigenmodes in the range $r^{-(1+\varepsilon)}$ other than those lying in the image of $\mathbf{P}_r \circ \Phi_r$, already discussed in (4.1).

Recall that we have the inclusion $i: \Sigma \times [-1, 1] \rightarrow M$, and after stretching M to M(r) we have the inclusion $\Sigma \times [-r-1, r+1]$ into M(r) with

(5.1)
$$X_1 = \Sigma \times [-r - 1, -r] \quad \text{in } M_1, \\ X_2 = \Sigma \times [r, r+1] \qquad \text{in } M_2,$$

(5.2)
$$\frac{\partial M_1 = \Sigma \times (-r) \quad \text{in } M(r),}{\partial M_2 = \Sigma \times (+r) \quad \text{in } M(r).}$$

To understand the low eigenmodes over $\Sigma \times [-r - 1, r + 1]$, we need explicit description of eigensections of \hat{D} over Σ . Define

(5.3)
$$\mathcal{H}_{\mu} = \{ \phi \in \Gamma(\hat{E}) \mid \hat{D}\phi = \mu\phi \}$$

to be the subspace of eigensections of \hat{D} with eigenvalue μ . Since $\sigma \circ \hat{D} = -\hat{D} \circ \sigma$, the automorphism σ switches \mathcal{H}_{μ} to $\mathcal{H}_{-\mu}$ and maps \mathcal{H}_0 to \mathcal{H}_0 . Denote by τ the composite automorphism $\tau = \sigma^* \sigma = -\sigma^2$. Then τ commutes with \hat{D} and keeps all eigenspaces \mathcal{H}_{μ} invariant.

PROPOSITION 5.1. There exists a complete orthonormal basis of C^{∞} -eigensections $\{\phi_k \mid k = 1, 2, ...\}$ of \hat{D} with $\hat{D}\phi_k = \mu_k\phi_k$. In addition, they satisfy the following conditions:

- (5.4) For each integer k, there is a real positive number $\lambda_k > 0$ with $\tau \phi_k = -\sigma^2 \phi_k = (\lambda_k)^2 \phi_k$.
- (5.5) For each eigenvalue μ of \hat{D} , let $N(\mu)$ denote the set of integers k with $\mu_k = \mu$. Then $\{\phi_k \mid k \in N(\mu)\}$ forms an orthonormal basis of the finitedimensional vector space \mathcal{H}_{μ} .

- (5.6) For each eigenvalue μ of \hat{D} , the set $\{\frac{\sigma\phi_k}{\lambda_k} \mid k \in N(\mu)\}$ forms an orthonormal basis of $\mathcal{H}_{-\mu}$. For $\mu > 0$, this basis coincides with $\{\phi_k \mid k \in N(-\mu)\}$.
- (5.7) Denote by 2N the dimension of the symplectic vector space \mathcal{H}_0 . The first 2N elements $\{\phi_k \mid 1 \leq k \leq 2N\}$ of our system form an orthonormal basis of \mathcal{H}_0 . Moreover,

$$\phi_{k+N} = \frac{\sigma \phi_k}{\lambda_k}$$

for k = 1, ..., N, and the set $\{\phi_1, ..., \phi_N, (-\phi_{N+1}/\lambda_1), ..., (-\phi_{2N}/\lambda_N)\}$ forms a symplectic basis for \mathcal{H}_0 .

The proof of the above proposition follows directly from the fact that τ commutes with \hat{D} , and so these operators can be simultaneously diagonalized. The eigenvectors ϕ_k can be chosen to be C^{∞} because of the known regularity theorem for elliptic, self-adjoint operators. As for (5.7), we first choose a Lagrangian subspace $L \subset \mathcal{H}_0$ invariant under the operator $\tau = -\sigma^2$. Then we choose $\{\phi_1, \ldots, \phi_N\}$ to be an orthonormal basis of eigenvectors of τ on L.

PROPOSITION 5.2. Let K > 0 be defined as in (4.5). Then, for the λ_k 's in (5.4), the inequality $|\lambda_k| \ge K$ is satisfied.

The proof of Proposition 5.2 is immediate in view of the following:

(5.8)

$$\lambda_k^2 = (\phi_k, \tau \phi_k)_{\Sigma} = \int_{\Sigma} (\phi_k(x), \tau_x \phi_k(x))_x d \operatorname{vol}_x$$

$$\equiv K^2 \int_{\Sigma} (\phi_k(x), \phi_k(x))_x d \operatorname{vol}_x$$

$$= K^2 (\phi_k, \phi_k)_{\Sigma}$$

$$= K^2.$$

Let ψ be a smooth section of the pullback $\pi^* \hat{E}$ over $\Sigma \times [-r-1, r+1]$. Then, using the orthonormal basis $\{\phi_k\}$ of Proposition 5.1, we can expand ψ uniquely as

(5.9)
$$\psi = \Sigma A_k(s) \pi^* \phi_k$$

where $A_k(s) = (\psi \mid \Sigma \times s, \phi_k)_{\Sigma \times s}$ are smooth functions. We will denote by ψ_0 the sum

$$\psi_0 = \sum_{1 \le k \le 2N} A_k(s) \pi^* \phi_k$$

of the terms $A_k(s)\pi^*\phi_k$, $1 \le k \le 2N$, in the expansion and refer to this sum as the 0-mode part of ψ .

Recall from (5.7) that the sections $\{\phi_k \mid 1 \leq k \leq 2N\}$ form an orthonormal basis of \mathcal{H}_0 . Since the subspace $\mathcal{H}_{\mu}, \mu \neq 0$, is orthogonal to \mathcal{H}_0 , we have

$$(5.10) \qquad \qquad (\boldsymbol{\phi}_{\ell}, \boldsymbol{\phi}_{k})_{\Sigma} = 0,$$

for $k \in N(\mu)$ and $1 \le \ell \le 2N$ when $\mu \ne 0$. In particular, given two sections ψ and ϕ of $\pi^* \hat{E}$, we can express their inner product

(5.11)
$$(\psi, \phi)_{\Sigma} = (\psi_0, \phi_0)_{\Sigma} + (\psi - \psi_0, \phi - \phi_0)_{\Sigma}$$

as the sum of the inner product (ψ_0, ϕ_0) of the 0-mode components and the inner product $(\psi - \psi_0, \phi - \phi_0)$ of their orthogonal complements.

In terms of expansion (5.9), the solutions of the differential equation $D\psi = \pi^* \sigma \circ (\frac{\partial}{\partial s} + \pi^* \hat{D})\psi = \lambda \psi$ can be written explicitly. Our main concern is the situation where an eigenvalue λ lies within the bound of δK , $|\lambda| < \delta K$, where δ and K are defined as before (see (1.9) and (4.5)). The method of separation of variables yields the following:

PROPOSITION 5.3. For $|\lambda| < \delta K$, a solution of the differential equation $D\psi = \lambda \psi$ can be written as a sum $\psi_0 + \psi_R + \psi_L$. The first term is of the form

(5.12)
$$\psi_{0} = \sum_{1 \leq k \leq N} A_{k} \left[\cos\left(\frac{\lambda s}{\lambda_{k}}\right) \pi^{*} \phi_{k} - \sin\left(\frac{\lambda s}{\lambda_{k}}\right) \pi^{*} \phi_{k+N} \right] \\ + \sum_{1 \leq k \leq N} B_{k} \left[\cos\left(\frac{\lambda s}{\lambda_{k}}\right) \pi^{*} \phi_{k+N} + \sin\left(\frac{\lambda s}{\lambda_{k}}\right) \pi^{*} \phi_{k} \right]$$

with A_k and B_k constants, and ϕ_k and ϕ_{k+N} as in (5.7). The second and third terms ψ_R and ψ_L are given by

(5.13)
$$\psi_R = \sum_{\mu_k > 0} A_k e^{-p(k)s} \pi^*(\psi_k^+),$$

which decreases exponentially to the right, and

(5.14)
$$\psi_L = \sum_{\mu_k > 0} B_k e^{p(k)s} \pi^*(\psi_k^-),$$

which decreases exponentially to the left. In these formulas, the terms p(k) and ψ_k^{\pm} stand for

(5.15)
$$p(k) = \sqrt{(\mu_k)^2 - (\lambda/\lambda_k)^2} > 0,$$

(5.16)
$$\psi_k^+ = \left[(\mu_k + p(k))\phi_k + (\lambda/\lambda_k^2)(\sigma\phi_k)\right]$$

(5.17)
$$\psi_k^- = \lambda \phi_k + (\mu_k + p(k))\sigma \phi_k.$$

Conversely, all sections of the form

(5.18)
$$\psi = \psi_0 + \psi_R + \psi_L,$$

where ψ_0 , ψ_R , and ψ_L are given as in (5.12), (5.13), and (5.14), respectively, constitute solutions to the equation $D\psi = \lambda \psi$. For $\lambda = 0$, $\psi_k^+ = 2\mu_k \phi_k$ and $\psi_k^- = 2\mu_k \sigma \phi_k$.

The expressions in (5.12), (5.13), and (5.14) indicate that ψ_0 corresponds to the 0-mode part of ψ , the term ψ_R decreases exponentially to the right, and ψ_L to the left. More precisely, we have the following:

PROPOSITION 5.4. Let ψ be a smooth section of $\pi^* \hat{E}$ satisfying the equation $D\psi = \lambda \psi$ with $|\lambda| < \delta K$. Let $\psi = \psi_0 + \psi_R + \psi_L$ be the decomposition of ψ as in Proposition 5.3. Then, for $r\sqrt{\delta^2 - (\lambda/K)^2} \ge 1$ and $0 \le t \le 2r$, we have

(5.19)
$$\begin{aligned} \|\psi_R \mid \Sigma \times (-r+t)\| \\ &\leq \exp(-\sqrt{\delta^2 - (\lambda/K)^2} t) \|\psi_R \mid \Sigma \times (-r)\| \end{aligned}$$

(5.20)
$$\begin{aligned} \|\psi_L \| \Sigma \times (r-t)\| \\ \leq \exp(-\sqrt{\delta^2 - (\lambda/K)^2} t) \|\psi_L \| \Sigma \times (r) \| \end{aligned}$$

$$\leq \exp(-\gamma \delta - (\lambda/K) t) \|\psi_L + Z \times (t)\|$$

(5.21)
$$\|\psi_R | \Sigma \times (-r)\|^2 \leq 2(\|\psi | X_1\|^2 + \|\psi | X_2\|^2)$$

(5.22)
$$\|\psi_L \mid \Sigma \times (r)\|^2 \leq 2(\|\psi \mid X_1\|^2 + \|\psi \mid X_2\|^2)$$

We also have the following estimates on the 0-mode part ψ_0 of ψ :

PROPOSITION 5.5. Let ψ be a smooth section of $\pi^* \hat{E}$ satisfying the equation $D\psi = \lambda \psi$ with $|\lambda| < \delta K$, and let ψ_0 denote the 0-mode part of ψ . Then for $|\lambda|(r+1) \leq (\frac{\pi}{2})K$ and $-r-1 \leq s$, $s' \leq r+1$, we have

 $(5.23) ||(\psi_0 | \Sigma \times s) - (\psi_0 | \Sigma \times s')|| \le (|\lambda|/K) \cdot |s - s'| \min(||\psi| | X_1||, ||\psi| | X_2||).$

Here we regard $\psi_0 \mid \Sigma \times s$ and $\psi_0 \mid \Sigma \times s'$ as sections of \hat{E} .

For the proof of (5.20), we observe that $\sqrt{\delta^2 - (\lambda/K)^2}$ is positive and denote it by α . Then from the definition $p(k) \ge \alpha$, and

(5.24)
$$(\psi_L \mid \Sigma \times (r-t)) = \sum_{\mu_k > 0} e^{(p(k)(r-t))} \pi^* (B_k \psi_k^-).$$

Since by (5.3), (5.14), and (5.17) the inner product $(\psi_k, \psi_\ell) = 0$ for $k \neq \ell$, with $\mu_k > 0$, $\mu_\ell > 0$, we have

(5.25)
$$\|\psi_L \mid \Sigma \times (r-t)\|^2 = \sum_{\mu_k > 0} e^{2p(k)(r-t)} \cdot \|B_k \psi_k^-\|^2$$
$$\leq \exp(-2\alpha t) \Sigma e^{2p(k)r} \cdot \|B_k \psi_k^-\|^2$$
$$= \exp(-2\alpha t) \|\psi_L \mid \Sigma \times (r)\|^2$$

The proof of (5.19) follows from a similar argument.

As for (5.22), we have

$$\|\psi \mid \Sigma \times s\|^2 = \|\psi_0 \mid \Sigma \times s\|^2 + \|\psi_R + \psi_L \mid \Sigma \times s\|^2$$

(see (5.11)) and hence

(5.26)
$$\begin{aligned} \|\psi \mid \Sigma \times s\|^2 \\ & \cong \|\psi_L \mid \Sigma \times s\|^2 + \|\psi_R \mid \Sigma \times s\|^2 + 2(\psi_R \mid \Sigma \times s, \ \psi_L \mid \Sigma \times s) \\ & \cong \|\psi_L \mid \Sigma \times s\|^2 - 2|(\psi_L \mid \Sigma \times s, \ \psi_R \mid \Sigma \times s)|. \end{aligned}$$

From (5.13) and (5.14) and the formula with $\mu_k > 0$, $\mu_\ell > 0$:

$$(\psi_k^+,\psi_\ell^-) = \begin{cases} 0 & \text{for } k = \ell \\ 2\lambda(\mu_k + p(k)) & \text{for } k = \ell, \end{cases}$$

it follows that for $-r - 1 \leq s \leq r + 1$

(5.27)
$$|(\psi_L | \Sigma \times s, \psi_R | \Sigma \times s)| = \left| \sum_{\mu_k > 0} 2A_k B_k \lambda(\mu_k + p(k)) \right| = |(\psi_L | \Sigma \times r, \psi_R | \Sigma \times r)| \le ||\psi_L | \Sigma \times r|| \cdot ||\psi_R | \Sigma \times r|| \le ||\psi_L | \Sigma \times r|| \cdot ||\psi_R | \Sigma \times (-r)|| \exp(-2\alpha r).$$

By setting t = 2r and $\alpha = \sqrt{\delta^2 - (\lambda/K)^2}$, the last inequality is a consequence of (5.19). On the other hand, if we repeat this last stage of the argument using (5.20), we obtain the inequality

(5.28)
$$\begin{aligned} \|\psi_L \mid \Sigma \times r\| &\leq \exp(-\alpha(s-r)) \|\psi_L \mid \Sigma \times s\| \\ &\leq \|\psi_L \mid \Sigma \times s\| \end{aligned}$$

whenever $r \leq s \leq r + 1$.

Combining inequalities (5.26), (5.27), and (5.28), it is not difficult to see that

(5.29)
$$\begin{aligned} \|\psi \mid X_2\|^2 \\ & \geq \int_r^{r+1} \{ \|\psi_L \mid \Sigma \times s\|^2 - 2|\langle \psi_L \mid \Sigma \times s, \psi_R \mid \Sigma \times s)| \} ds \\ & \geq \|\psi_L \mid \Sigma \times r\|^2 - 2\|\psi_L \mid \Sigma \times r\| \cdot \|\psi_R \mid \Sigma \times (-r)\| \exp(-2\alpha r) \end{aligned}$$

and similarly

(5.30)
$$\|\psi \mid X_1\|^2 \ge \|\psi_R \mid \Sigma \times (-r)\|^2 - 2\|\psi_R \mid \Sigma \times (-r)\| \cdot \|\psi_L \mid \Sigma \times r\| \exp(-2\alpha r)$$

Adding the two inequalities (5.29) and (5.30) and using the inequality $2AB \le A^2 + B^2$ for $A = ||\psi_L| \Sigma \times r||$, $B = ||\psi_R| \Sigma \times (-r)||$, we have

(5.31)
$$\|\psi \| X_1 \|^2 + \|\psi \| X_2 \|^2 \ge (A^2 + B^2)[1 - 2\exp(-2\alpha r)].$$

The condition $\alpha r \ge 1$ implies that

$$1-2\exp(-2\alpha r)\geq \frac{1}{2},$$

and so inequality (5.31) yields both (5.21) and (5.22). This completes the proof of Proposition 5.4.

We now turn to the proof of Proposition 5.5. Identifying $\Sigma \times s$ and $\Sigma \times s'$ with Σ , and using the eigenexpansion of (5.12), we have

(5.32)
$$\begin{aligned} \|(\psi_0 \mid \Sigma \times s) - (\psi_0 \mid \Sigma \times s')\|^2 \\ &= 4 \sum [\sin(\lambda/2\lambda_k)(s-s')]^2 (|A_k|^2 + |B_k|^2) \end{aligned}$$

For $-r-1 \leq s$, $s' \leq r+1$, the term $|(\lambda/2\lambda_k)(s-s')|$ is smaller than $\pi/2$ because

$$|(\lambda/2\lambda_k)(s-s')| \leq (|\lambda|/\lambda_k)(r+1) \leq \frac{|\lambda|(r+1)}{K}r \leq \pi/2.$$

Therefore the corresponding sine term $|\sin(\lambda/2\lambda_k)(s-s')|$ can be estimated:

$$2|\sin(\lambda/2\lambda_k)(s-s')| \leq 2|(\lambda/2\lambda_k)(s-s')|$$
$$\leq \left|\frac{\lambda}{K}(s-s')\right|.$$

This gives us

$$\|(\psi_0 \mid \Sigma \times s) - (\psi_0 \mid \Sigma \times s')\|^2 \leq \left|\frac{\lambda}{K}(s-s')\right|^2 \cdot \Sigma(|A_k|^2 + |B_k|^2).$$

On the other hand, by (5.12) and the orthonormality of $\{\phi_k\}$, the sum $\Sigma(|A_k|^2 + |B_k|^2)$ is smaller than $\|\psi \| X_2 \|^2$:

$$\|\psi \mid X_2\|^2 \ge \int_r^{r+1} \|\psi_0 \mid \Sigma \times s\|^2 \, ds = \|\psi_0 \mid \Sigma \times [r, r+1]\|^2$$
$$= \sum_{k=1}^N \int_r^{r+1} (|A_k|^2 + |B_k|^2) \, ds$$
$$= \sum_{k=1}^N (|A_k|^2 + |B_k|^2) = \|\psi_0 \mid \Sigma \times s\|^2$$

The proof of Proposition 5.5 follows.

Let ψ be an eigensection of E(r) satisfying $D\psi = \lambda \psi$ with $|\lambda| < \frac{\delta K}{2}$. Then the above can be used to estimate $\|\psi | \Sigma \times [a, b]\|$ for $-r \leq a < b \leq r$. By (5.19) and (5.20), we have for $-r \leq s \leq r$,

(5.33)

$$\|\psi \mid \Sigma \times s\|^{2} = \|\psi_{0} \mid \Sigma \times s\|^{2} + \|(\psi_{L} + \psi_{R}) \mid \Sigma \times s\|^{2}$$

$$\leq \|\psi_{0} \mid X_{2}\|^{2} + \|\psi_{L} \mid \Sigma \times s\|^{2} + 2|(\psi_{L}, \psi_{R})_{\Sigma \times s}|$$

$$+ \|\psi_{R} \mid \Sigma \times s\|^{2}$$

$$\leq \|\psi_{0} \mid X_{2}\|^{2} + 2\|\psi_{R} \mid \Sigma \times s\|^{2} + 2\|\psi_{L} \mid \Sigma \times s\|^{2}$$

$$\leq \|\psi \mid X_{2}\|^{2} + 2\{\exp[-\delta(r - s)]\}\|\psi_{L} \mid \Sigma \times r\|^{2}$$

$$+ 2\{\exp[-\delta(r + s)]\}\|\psi_{R} \mid \Sigma \times (-r)\|^{2}$$

$$\leq \mathbf{N} + 4\{\exp[-\delta(r - s)] + \exp[-\delta(r + s)]\}\mathbf{N}$$

$$\leq 9\mathbf{N}$$

where $\mathbf{N} = [\|\psi \| X_1 \|^2 + \|\psi \| X_2 \|^2]$. Integrating both sides, we obtain

(5.34)
$$\|\psi | \Sigma \times [a,b] \|^2 \leq 9[\|\psi | X_1\|^2 + \|\psi | X_2\|^2] \cdot |a-b|$$

for any pair of (a, b) between -r and r. This last estimate will play a crucial role in Section 6.

6. Convergence Results

To prove Theorem A, we will draw a contradiction from the assumption that there are more eigensections ψ on M(r) than those predicted by (1.8). This step will be accomplished by establishing certain convergence results (see Propositions 6.1 and 6.2) on a sequence of eigensections of $D : \Gamma(E_2(r)) \to \Gamma(E_2(r))$ over $M_2(r)$. In a sense, these results can be regarded as refinements of analogous results appearing in [26].

PROPOSITION 6.1. Regard $M_2(r)$ as the submanifold $\Sigma \times [-r, 0] \cup M_2$ imbedded in $M_2(\infty) = \Sigma \times (-\infty, 0] \cup M_2$ in the obvious manner. Let $\{\psi(j) \mid 1 \leq j < \infty\}$ be a sequence of C^{∞} -sections of the bundle $E_2(r)$ with the properties that

$$(6.1) D\psi(j) = \lambda_j \psi(j) ,$$

the set $\{\|\psi_j \mid M_2(r)\| \mid 1 \leq j \leq \infty\}$ is bounded, and $\lim \lambda_j = \lambda$.

If we denote by $\bar{\psi}(j)$ the restriction of $\psi(j)$ to $M_2(r-1)$, then there exists a subsequence $\{\bar{\psi}(j')\}$ of $\{\bar{\psi}(j)\}$ that converges to a C^{∞} -section $\bar{\psi}(\infty)$ of the bundle $E_2(r-1)$ with $D\bar{\psi}(\infty) = \lambda \bar{\psi}(\infty)$.

PROPOSITION 6.2. Let $r_1 \leq r_2 \leq \cdots \leq r_j \cdots$ be a monotonically increasing sequence of numbers with $\lim r_j = \infty$.

(6.2) Suppose $\{\psi(j)\}$ is a sequence of nontrivial eigensections of $E(r_j)$ such that $D\psi(j) = \lambda_j \psi_j$ with $|\lambda_j| < \delta K$. Then

$$\|\psi(j) \mid M_1\|^2 + \|\psi(j) \mid M_2\|^2 \neq 0.$$

(6.3) Suppose the sequence in (6.2) is normalized so that

 $\|\psi(j) \mid M_1\|^2 + \|\psi(j) \mid M_2\|^2 = 1$

and $\lim \lambda_j = \lambda$ and $|\lambda| < \delta K$. Then there exists a subsequence $\{\psi(j')\}$ such that $\{\psi(j') \mid M_1\}$ and $\{\psi(j') \mid M_2\}$ converge, respectively, to C^{∞} -eigensections $\psi_1(\infty)$ and $\psi_2(\infty)$ of $D\phi = \lambda\phi$ over M_1 and M_2 .

- (6.4) Suppose the limit λ in (6.3) equals 0. Then $\psi_1(\infty)$ and $\psi_2(\infty)$ are, respectively, the restrictions of extended L^2 -solutions $\psi_1(\infty)^{\#}$ and $\psi_2(\infty)^{\#}$ of $D\phi = 0$ over $M_1(\infty)$ and $M_2(\infty)$. In particular, this means that $[\psi_1(\infty) \mid \partial M_1]$ is a section in $P_+ \oplus L_1$ and $[\psi_2(\infty) \mid \partial M_2]$ in $P_- \oplus L_2$.
- (6.5) Suppose in (6.4) the limit $\lim_{j\to\infty} |\lambda_j| r_j = 0$. Then the 0-mode part $[\psi_1(\infty) \mid \partial M_1]_0$, $[\psi_2(\infty) \mid \partial M_2]_0$ of the limiting sections $[\psi_1(\infty) \mid \partial M_1]$, $[\psi_2(\infty) \mid \partial M_2]$ are equal in $\mathcal{H}_0 = \ker \hat{D}$. That is, they lie in $L_1 \cap L_2$.

An immediate consequence of (6.5) is that the sequence of bounded eigensections $\{\psi(j)\}$ in (6.3) has a convergent subsequence $\{\psi(j')\}$ whose limiting sections $\psi_1(\infty) = \lim_{j'} \psi(j') \mid M_1$ and $\psi_2(\infty) = \lim_{j'} \psi(j') \mid M_2$ represent matching pairs in \mathcal{W} (see (2.15)).

For the proof of Proposition 6.1, we pick a C^{∞} -function β on $M_2(r)$ such that $0 \leq \beta \leq 1$, $\beta \equiv 1$, on a neighborhood of $M_2(r-1)$ in $M_2(r)$ and $\beta \equiv 0$ on a neighborhood of $\partial M_2(r)$. By considering $M_2(r)$ as a submanifold in M(r), we have for each j a C^{∞} -section $\eta(j)$ of E(r) that equals $\beta \cdot \psi(j)$ over $M_2(r)$ and becomes 0 over $M(r) - M_2(r)$.

For k, l integers with $l \ge k \ge 0$, denote by a_k the norm of $(\ell - k)$ -fold commutator $[D[D, \dots [D, \beta]]$; that is,

$$a_k = \max_{\|x\|=1} ([D, [D, \dots [D, \beta]]]x).$$

This a_k is finite as D is a first-order operator. Then, it follows from standard interior elliptic estimates (see [2], theorem 5, p. 236) that

(6.6)
$$\|\eta(j)\|_{\ell}^{2} \leq C_{\ell} \left(\|\psi(j)\|^{2} + \sum_{k=1}^{\ell} \sum_{a=0}^{k} |\lambda_{j}|^{2a} {k \choose a}^{2} a_{k}^{2} \|\psi(j)\|^{2} \right)$$

In particular, for ℓ fixed, the sequence $\{\|\eta(j)\|_{\ell}\}$ is bounded.

Next, we recall the Sobolev lemma:

(6.7)
$$L^{2}_{\lfloor\frac{n}{2}\rfloor+1+\ell}(E(r)) \subseteq C^{\ell}(E(r))$$
$$\bigcap_{\ell} L^{2}_{\ell}(E(r)) = C^{\infty}(E(r))$$

where *n* stands for the dimension of M(r), and $L^2_{\lfloor \frac{n}{2} \rfloor + 1 + \ell}(E(r))$ is the Hilbert space of sections of E(r) with respect to the Sobolev $(\lfloor \frac{n}{2} \rfloor + 1 + \ell)$ -norm, and $C^{\ell}(E(r))$ is the space of ℓ -fold differentiable sections of E(r).

In addition, we also have the *Rellich lemma*: For $\ell' < \ell$, the inclusion

$$L^2_\ell(E(r)) \to L^2_{\ell'}(E(r)),$$

is compact. That is, any sequence bounded in the Sobolev norm $\|\cdot\|_{\ell}$ contains a strongly convergent subsequence in $L^2_{\ell'}(E(r))$.

We apply these standard facts to $\eta(j)$ with $\ell = \ell' + 1$ and $\ell' \ge \left[\frac{n}{2}\right] + 1 + \alpha$. Then $\{\eta(j)\}$ is bounded in the Sobolev norm $\|\cdot\|_{\ell}$ and therefore contains a strongly convergent subsequence $\{\eta(j')\}$ in the Sobolev norm $\|\cdot\|_{\ell'}$ to $\eta(\infty)$ in $L^2_{\lfloor\frac{n}{2}\rfloor+1+\alpha}(E(r))$. This limit $\eta(\infty)$ belongs to the C^{α} -class, $\alpha \ge 1$, and with the limit of L^2 -norm $\lim_{j'\to\infty} \|\eta(j') - \eta(\infty)\| = 0$.

Let $\bar{\psi}(\infty)$ denote the restriction of $\eta(\infty)$ to $M_2(r-1)$, $\bar{\psi}(\infty) = \eta(\infty) \mid M_2(r-1)$. Then $\bar{\psi}(\infty)$ belongs to C^{α} -class and

(6.8)
$$\begin{aligned} [\|D\bar{\psi}(\infty) - \lambda\bar{\psi}(\infty)\|_{M_{2}(r-1)}]^{2} \\ &= \|D(\bar{\psi}(\infty) - \bar{\psi}(j')) + \lambda'_{j}(\bar{\psi}(\infty) - \bar{\psi}(j')) + (\lambda - \lambda'_{j})\bar{\psi}(\infty)\|^{2} \\ &\leq \|D(\bar{\psi}(\infty) - \bar{\psi}(j'))\|^{2} + |\lambda'_{j}|^{2}\|\bar{\psi}(\infty) - \bar{\psi}(j')\|^{2} \\ &+ |\lambda - \lambda'_{j}|^{2}\|\bar{\psi}(\infty)\|^{2} . \end{aligned}$$

On the right-hand side of (6.8), the first term $[\|D\bar{\psi}(\infty) - \bar{\psi}(j')\|_{M_2(r-1)}]^2$ approaches 0 as $j' \to \infty$ because

$$\begin{split} \| D(\bar{\psi}(\infty) - \bar{\psi}(j')) \|_{M_2(r-1)} \|_2^2 &\leq \| \| D(\eta(\infty) - \eta(j')) \|_{M_2(r)} \|_2^2 \\ &\leq C [\| \eta(\infty) - \eta(j') \|]_1^2 \,. \end{split}$$

As for the second term of (6.8), we have

$$[\|\bar{\psi}(\infty) - \bar{\psi}(j')\|_{M_2(r-1)}]^2 \leq [\|\eta(\infty) - \eta(j')\|]^2.$$

Since $\lim \lambda'_i = \lambda$, the third term also converges to 0, and so by (6.5) we have

$$\|D\bar{\psi}(\infty)-\lambda\bar{\psi}(\infty)\|_{M_2(r-1)}=0.$$

To begin with, the section $\bar{\psi}(\infty)$ is of C^1 -class, but by the equation $D\bar{\psi}(\infty) = \lambda \bar{\psi}(\infty)$, it must be of C^2 -class, and therefore inductively of C^∞ -class. In addition, the convergence of $\eta(j')$ to $\eta(\infty)$ for all Sobolev norms $\|\cdot\|_{\ell}$ implies the convergence of $\bar{\psi}(j')$ to $\bar{\psi}(\infty)$ in $L^2_{\ell}(E(r-1))$. This proves Proposition 6.1.

We now turn to the proof of (6.2). Suppose the section $\psi(j)$ is nontrivial but its restriction to M_2 has trivial norm $\|\psi(j) \mid M_2\| = 0$. Then by the C^{∞} property of $\psi(j)$, it is identically 0 over the open set $\Sigma \times (r, r+1)$ in M(r); that is,

 $\psi(j) \mid \Sigma \times (r, r+1) \equiv 0$. Since the eigenvalues λ_j are bounded by δK , $|\lambda_j| < \delta K$, we can apply (5.12) to the cylinder $\Sigma \times [r, r+1]$ and conclude that the coefficients A_k and B_k in the expansion of $\psi(j)$ are equal to 0 for all k, $A_k \equiv B_k \equiv 0$. Over the whole cylinder $\Sigma \times [-r, r]$, the vanishing of these coefficients A_k and B_k implies the vanishing of the section $\psi_j \mid \Sigma \times [-r, r] \equiv 0$. Thus

(6.9)
$$\|\psi(j) \| M_1 \|^2 + \|\psi(j) \| M_2 \|^2 = \|\psi(j)\|^2 \neq 0$$

as claimed. If $\|\psi(j) \mid M_2\| \neq 0$, then (6.2) is immediate.

From (5.34) it follows that under the normalization condition in (6.3), the inequality

(6.10)
$$\|\psi(j) \mid M_1 \cup \Sigma \times [-r_j, -r_j+1]\|^2 + \|\psi(j) \mid \Sigma \times [r_j-1, r_j] \cup M_2\|^2 \leq 19$$

holds. That is, $\|\psi(j) | M_1(1)\|^2 + \|\psi(j)| M_2(1)\|^2 \leq 19$. Therefore, by (6.1), we may choose a subsequence $\{\psi(j')\}$ from $\{\psi(j)\}$ such that $\{\psi(j') | M_1\}$ and $\{\psi(j') | M_2\}$ converge strongly in the Sobolev norm $\|\cdot\|_{\ell}$ to, respectively, C^{∞} -section $\psi_1(\infty)$ and $\psi_2(\infty)$ with $D\phi = \lambda\phi$ over M_1 and M_2 . This proves the assertion in (6.3).

The estimates of (5.19) and (5.20) provide us with the inequalities

(6.11)
$$\begin{aligned} \|\psi(j)_{R} \mid \Sigma \times r_{j}\|^{2} &\leq 2 \exp\left(-4\sqrt{\delta^{2} - (\lambda_{j}/K)^{2}}r_{j}\right) \\ \|\psi(j)_{L} \mid \Sigma \times (-r_{j})\|^{2} &\leq 2 \exp\left(-4\sqrt{\delta^{2} - (\lambda_{j}/K)^{2}}r_{j}\right) \end{aligned}$$

By taking the limit of convergent subsequences $\psi(j') \mid M_1$ and $\psi(j') \mid M_2$ as $j \to \infty$, we obtain $\psi_1(\infty)$ and $\psi_2(\infty)$ defined over M_1 and M_2 , respectively. The above inequalities imply $\|\psi_2(\infty)_R \mid \partial M_2\| = \|\psi_1(\infty)_L \mid \partial M_1\| = 0$, and so

$$\psi_1(\infty) \mid \partial M_1 = [\psi_1(\infty)_0 + \psi_1(\infty)_R] \mid \partial M_1,$$

$$\psi_2(\infty) \mid \partial M_2 = [\psi_2(\infty)_0 + \psi_2(\infty)_L] \mid \partial M_2.$$

To prove (6.4), it remains to show that $\psi_1(\infty)_R$ is contained in the subspace P_+ and $\psi_2(\infty)_L$ in P_- . By (5.14) and (5.16), each eigensection $\psi(j)_R$ over $\Sigma \times [-r_j, -r_j + 1]$ has the following expansion:

(6.12)
$$\begin{bmatrix} \psi(j)_{R} \mid \Sigma \times [-r_{j}, -r_{j}+1] \end{bmatrix} = \left\{ \sum_{\mu_{k}>0} A_{k} e^{-p(k)s} (\mu_{k}+p_{k}) \phi_{k} \right\} + \left\{ \sum_{\mu_{k}>0} A_{k} e^{-p(k)s} (\lambda_{j}'/\lambda_{k}) (1/\lambda_{k}) \sigma \phi_{k} \right\}$$

Let $\psi(j)_{R,+}$ denote the first sum in (6.12) and $\psi(j)_{R,-}$ denote the second sum. Then $\psi(j)_{R,+}$ is contained in P_+ and $\psi(j)_{R,-}$ in P_- . Note that from the definition ϕ_k and

 $\frac{1}{\lambda_k}\sigma\phi_k$ are both of length 1, while $\mu_k + p(k)$ is approximately $2\mu_k$ and $|\lambda_j/\lambda_k^2|$ is small for $|\lambda'_j|$ small. Comparing the two terms $\psi_{R,+}$ and $\psi_{R,-}$, we have

(6.13)
$$\|\psi(j)_{R,-}\|_{\partial M_1} \leq \left(\frac{|\lambda'_j|}{K\delta}\right) \|\psi(j)_{R,+}\|_{\partial M_1}$$

because $|\lambda'_j|/|\lambda_k \mu_k| \leq |\lambda'_j|/K\delta$. Recall from (5.21) that $||\psi(j)_{R,+}| \Sigma \times (-r_j)||$ is bounded by $\sqrt{2}$. The conclusion from this inequality is clear: As *j* tends to ∞ , we have $|\lambda'_j| \to 0$ and so in the limit the term $\psi_1(\infty)_{R,-}$ becomes 0, or, in other words, the term $\psi_1(\infty)_R | \partial M_1$ is contained in P_+ . Since the situation for $\psi(j)_L$ over M_2 is analogous to $\psi(j)_R$ over M_1 , we conclude that $\psi_2(\infty) | \partial M_2$ is contained in $\mathcal{H}_0 + P_-$ by a similar argument.

By the above boundary conditions together with $D\psi_1(\infty) = D\psi_2(\infty) = 0$, the work of Atiyah-Patodi-Singer (see Theorem 2.2) shows that $\psi_1(\infty) | \partial M_1$ is contained in $L_1 \oplus P_+$ and $\psi_2(\infty) | \partial M_2$ in $L_2 \oplus P_-$. This proves (6.4).

Finally, by Proposition 5.5, we have

$$\|[\psi(j)]_0 \mid \Sigma \times r_j - [\psi(j)]_0 \mid \Sigma \times (-r_j)\| \leq 2r_j(|\lambda'_j|/K)$$

Under the hypothesis in (6.5), we can let $j \to \infty$ and obtain

$$\|[\psi_2(\infty)]_0 \mid \partial M_2 - [\psi_1(\infty)]_0 \mid \partial M_1\| = 0.$$

This completes the proof of Proposition 6.2.

7. Proof of Theorem A: Splitting Low Eigenspaces into Three Summands

Let q denote the sum dim V_1 + dim V_2 + dim $L_1 \cap L_2$. By Lemma 4.1, when r is large there are at least q nonvanishing orthogonal eigensolutions to

$$D\phi = \lambda\phi$$
 on $M(r)$

with $|\lambda| < \exp(-\delta r/4)$. Moreover, the mapping $\mathbf{P}_r \circ \Phi_r$ of the sum $V_1 + V_2 + (L_1 \cap L_2)$ into $\operatorname{sp}(r, \exp(-\delta r/4))$ is a monomorphism.

In order to complete the proof of Theorem A, we need but show that for r large there are at most q orthogonal eigensolutions to $D(\phi) = \lambda \cdot \phi$ on M(r) with $|\lambda| < (1/r^{1+\varepsilon})$. This fact will be shown in this section.

Let $\mathscr{R}: C^{\infty}(E(r)) \longrightarrow L^2(E(r) \mid M_1) \oplus L^2(E(r) \mid M_2)$ denote the restriction of the space $C^{\infty}(E(r))$ of smooth sections of E(r) over the closed manifold M(r) to the L^2 -sections on the two sides $M_1(r), M_2(r)$ of Σ , and let $\tilde{N}(r)$ denote the image of $\mathscr{N}(r, r^{-(1+\varepsilon)})$ under \mathscr{R} . Then Proposition 6.2 implies that

$$\sup\{\operatorname{dist}(\psi, \mathcal{W}); \ \psi \in \overline{N}(r), \ \|\psi\| = 1\}$$

approaches 0 as r tends to ∞ . In terms of Kato's gap δ (see p. 197 of [16] for the definition) this means

$$\lim_{r\to\infty}\delta(N(r),\mathcal{W})=0.$$

As is well-known (Corollary 2.6 of [16]), if U and V are two finite-dimensional subspaces of a Hilbert space and if $\delta(U, V) < 1$, then dim $U \leq \dim V$. Thus for sufficiently large r we have dim $\tilde{N}(r) \leq \dim \mathcal{R}(\Psi_r(W)) = \dim W$. Since the restriction \mathcal{R} is injective, we conclude that

$$\dim \mathcal{N}(r^{-(1+\varepsilon)}) \leq \dim \mathcal{W}.$$

This proves our claim.

8. Proof of Theorem B: The Case ker $\hat{D} = \mathcal{H} = 0$

The proof of Theorem B is parallel to that of Theorem A. Here we may use results (see (1.12)) of Muller [19] and Douglas and Wojciechowski [14].

Let ϕ be a smooth L^2 -solution of $D\phi = \lambda\phi$ on $M_1(\infty) = M_1 \cup \Sigma \times [0, \infty)$ with $\|\phi\|_{M_1(\infty)} = 1$ and $|\lambda| < \delta K/2$. Since ϕ is L^2 , in the decomposition of Proposition 5.3 we have

(8.1)
$$\phi \mid \Sigma \times [0,\infty) = \phi_R = \sum A_k e^{-p(k)s} \pi^*(\psi_k^+)$$

as in (5.13). Consequently, the estimates of Proposition 5.4 are easy to apply for such ϕ in $V_1(k)$.

For example, for $s \ge 0$ (assuming $r \ge 2$) we obtain by (5.19) and (5.21) (with [-r, +r] replaced by [0, 2r] and [-1, +1] with r = 0, respectively)

(8.2)

$$\|\phi \mid \Sigma \times s\|$$

$$\leq \exp(-\sqrt{\delta^2 - (\lambda/K)^2}s)\|\phi \mid \Sigma \times 0\|$$

$$\leq \sqrt{2}\exp(-(\delta/2)s)\sqrt{\|\phi \mid \Sigma \times [-1,0]\|^2 + \|\phi \mid \Sigma \times [0,1]\|^2}$$

$$\leq 2\exp(-(\delta/2)s)$$

and also

(8.3)
$$\|\phi | \Sigma \times [s, \infty)\|^2 \leq \int_s^\infty 4 \exp(-\delta t) dt \leq (4/\delta) \exp(-\delta s).$$

The splicing map

(8.4)
$$\Phi_r: V_1(k) \oplus V_2(k) \to \Gamma(E(r))$$

is defined for $(\phi, \psi) \in V_1(k) \oplus V_2(k)$ by the familiar formula (2.16) with the convention that $\hat{\phi} = \hat{\psi} = 0$.

The map Φ_r makes for but a small change in norms. For example, if again

(8.5)
$$\phi \in V_1(k), \quad D\phi = \lambda \phi \quad \text{on } M_1(\infty), \quad \|\phi\| = 1,$$

then by (8.3) and the definition of Φ_r

(8.6)

$$0 \leq \|\phi\|^{2} - \|\Phi_{r}(\phi, 0)\|^{2}$$

$$= \int_{r-1}^{r} \left[1 - \rho(r-s)^{2}\right] \|\phi | \Sigma \times t\|^{2} dt + \int_{r}^{\infty} \|\phi | \Sigma \times t\|^{2} dt$$

$$\leq (4/\delta) \exp(-\delta(r-1))$$

with a similar result for unit eigenvectors in $V_2(k)$.

In order to verify inequality (1.18) of Theorem B, we start with an eigenvector ϕ of D on $M_1(\infty)$ satisfying (8.5). Let \mathbf{P}_{λ} be the projection onto the span of the eigenvectors of D on M(r) with eigenvalues in the range

$$\lambda \pm \exp\left(-\frac{\delta r}{4}\right) \, .$$

As in (4.6) we may expand $\Phi_r(\phi, 0)$ in terms of the eigensolutions $\tilde{\phi}_u$ of D on M(r) $(D\tilde{\phi}_n = \mu\tilde{\phi}_n)$ $\Phi_r(\phi, 0) = \Sigma d_u \tilde{\phi}_u$

As in (4.6),
$$||(D - \lambda)\Phi_r(\phi, 0)||^2 = \Sigma(u - \lambda)^2 (d_u)^2$$
; hence

$$\|\Phi_r(\phi, 0) - \mathbf{P}_{\lambda} \Phi_r(\phi, 0)\|^2 = \sum_{|u-\lambda| \ge \exp(-\delta r/4)} (d_u)^2$$
$$\le \exp\left(\frac{\delta v}{2}\right) \sum_{(u-\lambda) \ge \exp(-\delta r/4)} (u-\lambda)^2 (d_u)^2$$
$$\le \exp\left(\frac{\delta r}{2}\right) \|(D-\lambda) \Phi_r(\phi, 0)\|^2$$

But $(D-\lambda)\Phi_r(\phi, 0)$ vanishes off of $\Sigma \times [-1, 0)$ in M(r), and on $\Sigma \times [-1, 0]$ it equals $\sigma \frac{d\rho(-s)}{ds}\phi(x, s+r)$. Hence, by (8.2) and (4.5)

$$\|(D-\lambda)\Phi_r(\phi,0)\| \leq \left\|\sigma \frac{d\rho(-s)}{ds}\phi(x,s+r) \mid \Sigma \times [-1,0]\right\|$$
$$\leq (L \cdot 2) \cdot 2\exp(-(\delta/2)(r-1)).$$

In particular, we obtain

(8.7)
$$\|\Phi_r(\phi,0) - \mathbf{P}_r \Phi_r(\phi,0)\| \leq \|\Phi_r(\phi,0) - P_\lambda \Phi_r(\phi,0)\| \\ \leq N_1 \exp(-(\delta/4) \cdot r)$$

with $N_1 = 4L \exp(\delta/2)$ since the orthogonal projection \mathbf{P}_{λ} has image in \mathbf{P}_r .

Let $\{\phi_{jk}\}$, $1 \leq k \leq n(j)$, denote orthonormal eigenvectors of D with ϕ_{jk} being L^2 and smooth,

$$D\phi_{j,k} = \lambda_{jk}\phi_{j,k}$$
 on $M_j(\infty)$,

and $|\lambda_{jk}| \leq k$ so that $\{\phi_{jk}\}$ is an orthonormal basis for $V_j(k)$. Then (8.7) holds for each of $\phi_{j,k}$. If

$$x=\sum a_k\phi_{1,k}, \qquad y=\sum b_\ell\phi_{2,\ell},$$

then

(8.8)
$$\|\Phi_{r}(x, y) - \mathbf{P}_{r}\Phi_{r}(x, y)\| \leq \sum |a_{k}| \|\Phi_{r}(\phi_{1,k}, 0) - \mathbf{P}_{r}\Phi_{r}(\phi_{1,k}, 0)\| + \sum |b_{\ell}| \|\Phi_{r}(0, \phi_{2,\ell}) - \mathbf{P}_{r}\Phi_{r}(0, \phi_{2,\ell})\| \leq N_{1} \exp\left(-\frac{\delta r}{4}\right) (\Sigma |a_{k}| + \Sigma |b_{\ell}|) \leq (n(1) + n(2))N_{1} \exp\left(-\frac{\delta r}{4}\right) \|(x, y)\|,$$

proving inequality (1.18).

In a similar manner, the inequality (8.6) holds for each $\phi_{j,k}$ and in toto yields

$$0 \leq ||(x, y)||^{2} - ||\Phi_{r}(x, y)||^{2}$$

= $\sum a_{k}^{2} + \sum b_{\ell}^{2} - \sum a_{k}^{2} ||\Phi_{r}(\phi_{1,k}, 0)||^{2} - \sum b_{\ell}^{2} ||\Phi_{r}(\phi_{2,\ell}, 0)||^{2}$
 $\leq \left(\sum a_{k}^{2} + \sum b_{\ell}^{2}\right) (4/\delta) \exp(-\delta(r-1))$
= $||(x, y)||^{2} (4/\delta) \exp(-\delta(r-1)).$

If we choose $R_1 \ge 2$, with $(4/\delta) \exp(-\delta(R_1 - 1)) \le 5/9$ and

$$(n(1) + n(2))(N_1)^2 \exp(-\delta R_1) \le 1/3$$

then $r \ge R_l$ will imply $\|\Phi_r(x, y) - \mathbf{P}_r \Phi_r(x, y)\| \le (1/3) \|(x, y)\|$ and

$$0 \leq \|\Phi_r(x, y)\| \leq (2/3)\|(x, y)\|^2;$$

that is, $\|\mathbf{P}_r \Phi_r(x, y)\|^2 \ge (1/3) \|x, y\|^2$. In particular, for $r \ge R_1$, $\mathbf{P}_r \Phi_r$ is a monomorphism. This is half of statement (1.19).

In order to complete the proof of Theorem B, we must show that any eigenvector of D on M(r) (r large) with eigenvalue λ satisfying

$$|\lambda| \leq k + (\varepsilon/2)$$

is in the image of $V_1(k) \oplus V_2(k)$ under $\mathbf{P}_r \Phi_r$. This is to be proved for all $r \ge R$ for some R. We argue by contradiction, taking $r \ge R$ and assuming that the

orthogonal complement of $\mathbf{P}_r \Phi_r(V_1(k) \oplus V_2(k))$ in $\operatorname{sp}(r, k + (\varepsilon/2))$ is not empty. That is, we may choose $\phi \in \Gamma(E(r))$ with

$$\|\phi\| = 1$$
, $(\phi, \mathbf{P}_r \Phi_r(\phi_{1,j}, 0)) = (\phi, \mathbf{P}_r \Phi_r(0, \phi_{2,k})) = 0$

for all $1 \le j \le n(1)$, $1 \le k \le n/2$; moreover, ϕ has an expansion

(8.9)
$$\phi = \sum_{\ell=1}^{T_r} a_\ell \phi_\ell \quad \text{in } \Gamma(E(r))$$

with $\sum (a_{\ell})^2 = 1$ and ϕ_{ℓ} eigenvectors of D on M(r) with

(8.10)
$$D\phi_{\ell} = \lambda_{\ell}\phi_{\ell}, \qquad \|\phi_{\ell}\| = 1, \qquad (\phi_{\ell}, \phi_{m}) = 0, \\ \ell \neq m, \qquad |\lambda_{\ell}| \leq k + (\varepsilon/2).$$

Here $T_r = N(r, k + (\varepsilon/2))$.

Now $\phi_{\ell} \mid \Sigma \times [-r, r] = (\phi_{\ell})_L + (\phi_{\ell})_R$ since $\mathcal{H} = \ker \hat{D} = 0$, so we have for $-1 \leq s \leq +1$ the estimates from Proposition 5.4:

$$\|\phi_{\ell} \mid \Sigma \times s)\|_{\Sigma} \leq \|(\phi_{\ell})_{L} \mid \Sigma \times s)\| + \|(\phi_{\ell})_{R} \mid \Sigma \times s)\|$$

$$\leq \exp(-\sqrt{\delta^{2} - (\lambda/k)^{2}}(r+s))\|(\phi_{\ell})_{L} \mid \Sigma \times (-r)\|$$

$$+ \exp(-\sqrt{\delta^{2} - (\lambda/k)^{2}}(r-s))\|(\phi_{\ell})_{R} \mid \Sigma \times (r)\|$$

$$\leq \exp\left(-\frac{\delta}{2}(r-1)\right)$$

$$\times \left[\sqrt{2}\sqrt{\|(\phi_{\ell})_{L} \mid \Sigma \times [-r-1,r+1]\|^{2}}\right]$$

$$+ \sqrt{2}\sqrt{\|\phi_{\ell})_{k} \mid \Sigma \times [r-1,r+1]^{2}}$$

$$\leq 4\exp\left(-(\delta/2)(r-1)\right)$$

by $\|(\phi_\ell)_\alpha \mid \Sigma \times [a, b]\| \leq \|\phi_\ell\|_{M(r)} \leq 1$, with $\alpha = L$ or R. Define as in (2.17) a decomposition mapping

(8.12)
$$\psi_r: \Gamma(E(r) \text{ over } M(r)) \to \Gamma(E_1(\infty)) \oplus \Gamma(E_2(\infty))$$

by $\psi_r(\phi) = (h_1, h_2)$ with $h_j \mid M_j = h_j$

$$(h_1 \mid \Sigma \times [0, r-1])(x, s) = h(x, s-r) (h_1 \mid \Sigma \times [r-1, r])(x, s) = \rho(r-s)h(x, s-r) h_1 \mid \Sigma \times [r, \infty) = 0$$

$$(h_2 \mid \Sigma \times [-(r-1), 0])(x, s) = h(x, s+r) (h_2 \mid \Sigma \times [-(r-1), -r])(x, s) = \rho(r+s) h(x, s+r) h_2 \mid \Sigma \times [-\infty, -r] = 0$$

Again ψ_r does not change the norm much on the span of the ϕ_ℓ 's. For example, we use (8.11) to obtain the inequality

(8.13)

$$0 \leq \|\phi_{\ell}\|^{2} - \|\Psi_{r}(\phi_{\ell})\|^{2}$$

$$= \int_{-1}^{0} (1 - \rho(-s)^{2}) \|\phi_{\ell}| \Sigma \times s\|^{2} ds + \int_{0}^{1} (1 - \rho(s)^{2}) \|\phi_{\ell}(\Sigma \times s)\|^{2} ds$$

$$\leq 32 \exp(-\delta(r-1)) .$$

For $\ell \neq m$, $(\phi_{\ell}, \phi_m)_{M(r)} = 0$ also, while

$$(\Psi_r(\boldsymbol{\phi}_\ell), \Psi_r(\boldsymbol{\phi}_m))_{M_j(\infty)} = (\boldsymbol{\phi}_\ell, \boldsymbol{\phi}_m)_{M_j(r)} + A_j + B_j$$

with $|A_j| = |(\Psi_r(\phi_\ell) - \phi_\ell, \Psi_r(\phi_m))_{M_j(r)}|$, and so

$$|A_{1}| \leq ||\Psi_{r}(\phi_{\ell}) - \phi_{\ell}| |M_{1}(r)|| \cdot ||\Psi_{r}(\phi_{m})| \Sigma \times [-1, 0]||$$

$$\leq \left[\int_{-1}^{0} (1 - \rho(-s))^{2} ||\phi|| \Sigma \times s ||^{2} ds\right]^{1/2} \cdot 4 \exp(-(\delta/2) (r - 1))$$

$$\leq 16 \exp(-\delta(r - 1)) .$$

Similarly,

$$B_j = (\Psi_r(\phi_\ell) - \phi_\ell, \Psi_r(\phi_m) - \phi_m)_{M_j(\infty)}$$

with estimates

$$A_j \leq 16 \exp(-\delta(r-1)), \qquad |B_j| \leq 16 \exp(-\delta(r-1)),$$

for j = 1 and j = 2.

Taking $r \ge R_1$ and $\ell \ne m$ (whence $(\phi_\ell, \phi_m)_{M(r)} = 0$), we obtain

(8.14)
$$|(\Psi_r(\phi_\ell), \Psi_r(\phi_m))| \leq |(\phi_\ell, \phi_m)_{M(r)}| + \sum_{1}^{2} (|A_j| + |B_j|) \\ \leq 64 \exp(-\delta(r-1)) .$$

With (8.13) and (8.14) in hand, we can deduce from (8.9) the desired inequality:

$$0 \leq \|\phi\|^{2} - \|\Psi_{r}(\phi)\|^{2} \leq \sum_{\ell} a_{\ell}^{2} \sum_{\ell} a_{\ell}^{2} \|\Psi_{r}(\phi_{\ell})\|^{2} - \sum_{\ell \neq m} a_{\ell} a_{m}(\Psi_{r}(\phi_{\ell}), \Psi_{r}(\phi_{m})) \leq \sum_{\ell} a_{\ell}^{2} (1 - \|\Psi_{r}(\phi_{\ell})\|^{2}) + \sum_{\ell \neq m} \left(\sum_{p} a_{p}^{2}\right) |(\Psi_{r}(\phi_{\ell}), \Psi_{r}(\phi_{m})| \leq 32 \exp(-\delta(r-1)) + (T_{r})^{2} \cdot 64 \exp(-\delta(r-1)) .$$

Now since ϕ is orthogonal to $\Phi_r(\phi_{1,j}, \phi_{2,k})$, we get by assumption

$$(\Psi_r(\phi), (\phi_{1,j}, \phi_{2,k}))_{M_1(\infty) \coprod M_2(\infty)} = (\phi, \Phi_r(\phi_{1,j}\phi_{2,j})) = 0.$$

Therefore, $\Psi_r(\phi)$ is orthogonal to the subspace $V_1(k) \oplus V_2(k)$ spanned by the eigenspaces of value λ with $|\lambda| \leq k$, and hence $|\lambda| \leq k + \varepsilon$ by the choice of ε . Consequently, by the results of Muller [19] and of Douglas and Wojciechowski [14]:

(8.16)
$$\frac{\|D\Psi_r(\phi)\|}{\|\Psi_r(\phi)\|} > k + \varepsilon.$$

We may now estimate $D\Psi_r(\phi)$ easily:

$$\|D\phi\|_{M(r)}^2 = \sum (a_\ell \lambda_\ell)^2 \leq \max(\lambda_\ell)^2 \leq (k + \varepsilon/2)^2$$

while $||D\Psi_r(\phi)||^2 = ||D\phi||^2_{M(r)} + C_1 + C_2$ with

$$C_{1} = \int_{-1}^{0} \left(- \|D\phi | \Sigma \times s\|^{2} + \left\|\rho(-s)D\phi + \frac{\partial\rho(-s)}{\partial s}\sigma\phi | \Sigma \times s\right\|^{2} \right) ds$$

 C_2 has a similar expression for the interval [0,1]. By (8.11),

$$\begin{split} \sqrt{C_1} &\leq \left| \left| D\phi \mid \Sigma \times [-1,0] \right| \right| + \left| \left| \rho(-s) D\phi \mid \Sigma \times [-1,0] \right| \right| \\ &+ \left| \left| \frac{\partial \rho(-s)}{\partial s} \sigma\phi \mid \Sigma \times [-1,0) \right| \right| \\ &\leq 2 \sum |a_\ell| ||D\phi_\ell| \Sigma \times [-1,0]| \\ &+ 2K \sum |a_\ell| ||\phi_\ell| \Sigma \times [-1,0]| | \\ &\leq (2T_r(k+\varepsilon/2)+2KT_r) \cdot 4 \exp\left(-\frac{\delta}{2}(r-1)\right), \end{split}$$

and similarly for $\sqrt{C_2}$. Hence,

(8.17)
$$||D\Psi_r(\phi)||^2 \leq (k + \varepsilon/2)^2 + 128(T_r)^2((k + \varepsilon/2) + K)^2 \exp(-\delta(r-1))$$

By comparing (8.17) and (8.15) we obtain

(8.18)
$$\frac{\|D\Psi_r(\phi)\|^2}{\|\Psi_r(\phi)\|^2} \le \frac{(k+\varepsilon/2)^2 + 128(T_r)^2(k+\varepsilon/2+K)^2\exp(-\delta(r-1))}{1-[32+64(T_r)^2]\exp(-\delta(r-1))}$$

if $r \ge R_2$. This inequality almost contradicts (8.16). However, it is conceivable that $T_r = N(r, k + \varepsilon/2)$ grows so rapidly with r that no contradiction arises. We will now preclude this possibility.

Define a > 0 by $[(k + 2\varepsilon/3)^2/(k + \varepsilon)^2] = 1 - 2a$. Choose $R_3 \ge R_2$ with the properties that

$$(k + \varepsilon/2)^2 + 128a^{-1} \cdot (n(1) + n(2))(k + \varepsilon/2 + K^2) \exp(-\delta(R_3 - 1)) \le (k + (2\varepsilon/3))^2$$

and

$$1 - [32 + 64 \cdot a^{-1}(n(1) + n(2))^2] \exp(-\delta(R_3 - 1)) \ge \frac{(k + \frac{2\varepsilon}{3})^2}{(k + \varepsilon)^2}$$

By these inequalities, for any $r \ge R_3$, if in addition $T_r = N(r, k + \varepsilon/2) \le a^{-1}(n(1) + n(2))$, then by (8.18) we get

$$\frac{\|D\Psi_r(\phi)\|^2}{\|\Psi_r(\phi)\|^2} \leq (k+\varepsilon)^2$$

This last inequality contradicts (8.16). In view of this, we arrive at a contradiction unless for $r \ge R_3$ we have

(8.19)
$$T_r = N(r, k + \varepsilon/2) \ge a^{-1}(n(1) + n(2)).$$

In order to complete the argument we will use the following lemma with $V = sp{\phi_{\ell}} = sp(r, k + \varepsilon/2)$ and $W = \Phi_r(V_1(r) \oplus V_2(r))$.

LEMMA 8.1. Let $\{\phi_\ell\}_{1 \le \ell \le T}$ be an orthonormal basis of a real vector space V with inner product, of dimension T. Let W be a subspace of dimension n. Then there is a basis element ϕ_ℓ with

$$\|\mathbf{P}\phi_{\ell}\|^2 \leq n/T$$

where **P** is the orthogonal projection of ϕ_{ℓ} onto W.

Proof: Let $\{f_m: 1 \le m \le n\}$ be an orthonormal basis of W. Then expanding f_m gives

$$f_m = \sum_{\ell} \langle \phi_{\ell}, f_m \rangle \phi_{\ell}$$

Moreover,

$$n = \sum_{m} ||f_{m}||^{2} = \sum_{m,\ell} \langle \phi_{\ell}, f_{m} \rangle^{2}$$
$$= \sum_{\ell} ||\mathbf{P}\phi_{\ell}||^{2}$$

since $\mathbf{P}\phi_{\ell} = \sum_{m} \langle \phi_{\ell}, f_{m} \rangle f_{m}$. For the *T* nonnegative numbers $\|\mathbf{P}\phi_{\ell}\|_{1 \le \ell \le T}$ to add up to *n*, at least one must be $\le (n/T)$.

Now we apply Lemma 8.1 to $V = \operatorname{sp}(r, k + \varepsilon/2)$ of dimension $T_r = N(r, k + \varepsilon/2)$ and $W = \Phi_r(V_1(r) \oplus V_2(r))$ with $r \ge R_3$. By the above, necessarily $T_r \ge$

 $a^{-1}(n(1) + n(2))$ in order to avoid a contradiction. We choose a unit eigenvector ϕ_{ℓ} , $\|\phi_{\ell}\| = 1$, $D\phi_{\ell} = \lambda\phi_{\ell}$, $|\lambda| \leq (k + \varepsilon/2)$, with $\|\mathbf{P}\phi_{\ell}\|^2 \leq (n(1) + n(2))/T_r$. Hence, $\|\mathbf{P}\phi_{\ell}\|^2 \leq a$.

By definition $(\Psi_r(\phi_\ell), (\phi_{1,j}, \phi_{2,1})) = (\phi_\ell, \Phi_r(\phi_{1,j}, \phi_{2,k}))$, so if $\mathbf{P}\Psi_r(\phi_\ell)$, $\mathbf{P}\phi_\ell$ denotes the orthogonal projections of $\Psi_r(\phi_\ell)$, ϕ_ℓ on $V_1(k) \oplus V_2(k)$, $\Phi_r(V_1(k) \oplus V_2(k))$, respectively, then

$$\|\mathbf{P}\Psi_r(\phi_\ell)\|^2 = \|\mathbf{P}\phi_\ell\|^2 \leq a.$$

Now by (8.13), $0 \le 1 - \|\Psi_r(\phi_\ell)\|^2 \le 32 \exp(-\delta(r-1))$. Hence, with $x = [\Psi_r(\phi_\ell) - \mathbf{P}\Psi_r(\phi_\ell)]$

(8.20)
$$||x||^2 \ge [1 - 32 \exp(-\delta(r-1))] - a$$

Since **P** is the orthogonal projection on all eigenvectors of $D(1) \oplus D(2)$ with values in the range [-k, +k], we necessarily have

(8.21)
$$\|Dx\|^{2} = \|D\Psi_{r}(\phi_{\ell})\|^{2} - \|\mathbf{P}\Psi_{r}(\phi_{\ell})\|^{2} \\ \leq \|D\Psi_{r}(\phi_{\ell})\|^{2}.$$

This last expression was estimated in (8.17). The factor T_r occurred since for a general unit vector ϕ in sp $(r, k + \varepsilon/2)$ one must use $\phi = \sum a_\ell \phi_\ell$ with $\sum_{\ell=1}^{T_r} |a_\ell| \leq T_r$ by $\sum |a_\ell|^2 = 1$. In our case, a special eigenvector ϕ_ℓ is used. Hence, the inequality (8.17) is improved to read in this case

(8.22)
$$||D\Psi_r(\phi_\ell)||^2 \leq (k + \varepsilon/2)^2 + 128(k + \varepsilon/2 + K)^2 \exp(-\delta(r-1))$$

with T_r absent. By combining (8.18) and (8.19) we get

(8.23)
$$||Dx||^2 \leq (k + \varepsilon/2)^2 + 128(k + \varepsilon/2 + K)^2 \exp(-\delta(r-1)).$$

Choose $R_4 \ge R_3$ with

$$(k+\varepsilon/2)^2+128/k+\varepsilon/2+K)^2\exp(-\delta(R_4-1)) \leq \left(k+\frac{2\varepsilon}{3}\right)^2$$

and

$$32\exp(-\delta(R_4-1))\leq a.$$

Then for $r \ge R_4$ by (8.19) and (8.20)

$$\frac{\|Dx\|^2}{\|x\|^2} \le \frac{(k+2\varepsilon/3)^2}{1-2a} = (k+\varepsilon)^2$$

by the definition of *a*. This last inequality contradicts (8.16), thus proving that for $r \ge R_4$, we necessarily have $\mathbf{P}_r \Phi_r : V_1(r) \oplus V_2(r) \to \operatorname{sp}(r, k + \varepsilon/2)$ as an isomorphism. This completes the proof of Theorem B.

Appendix A: Manifolds with Cylindrical Ends

In this appendix, we consider the low eigensections of D(r) on

$$M_1(r) = M_1 \cup \Sigma \times [0, r],$$

with the identification $\partial M_1 \cong \Sigma \times 0$ as r increases. Let $L \subset \ker \hat{D} = \mathcal{H}$ be a choice of Lagrangian subspace. Recall that

$$D_1: L_1^2(E(r), P_+ \oplus L) \to L^2(E_1(r))$$

is a self-adjoint operator on $M_1(r)$ with kernel consisting of the subspace V(L) of extended L^2 -solutions of D(r) with limiting values in L. Analogous to Theorem A we have the following result for the manifold $M_1(r)$:

THEOREM A.1. Given an $\varepsilon > 0$, there is an R > 0 such that, for all $r \ge R$, if ϕ is an eigensection for D_1 with eigenvalue λ satisfying $|\lambda| < r^{-(1+\varepsilon)}$, then $\lambda = 0$ and ϕ is an element in V(L).

The proof is similar to and, in fact, easier than that for Theorem A because the splicing construction already yields a zero eigensection. For these reasons, we omit the details.

Correspondingly, there is an analogue of Theorem B in this setting. Let ker $\hat{D} = 0$ and δ , λ , ε be chosen as in (1.14). There is the natural restriction mapping

$$\Gamma(E|M_1(\infty)) \rightarrow \Gamma(E|M_1(r))$$
.

The proof is again parallel to that of M(r).

Let N(r, t), sp(r, t) be as in Theorem A except for replacing M(r) by $M_1(r)$. We then have the following result:

THEOREM A.2. Suppose $H = \ker \hat{D} = 0$, and k and ε are chosen as in Theorem A. Then there is an R > 0 depending only on $k, \varepsilon, K, \delta$ such that for all $r \ge R$ we have

(A.1)
$$N(r, k + (\varepsilon/2)) = \dim(V_1(k) \oplus V_2(k))$$

(replacing M(r) by $M_1(r)$).

(A.2) Let \mathbf{P}_r be the orthogonal projection of $\Gamma(E|M_1(r))$ onto the span of eigenvectors of D_1 on $E|M_1(r)$ with eigenvalue λ satisfying

$$|\lambda - \lambda_{j\alpha}| \leq \exp\left(-\frac{\delta r}{4}\right)$$

for some j and α , $1 \leq \alpha \leq n(1)$. Then the equality of subspaces $\operatorname{sp}(r, k + \varepsilon/2) = \mathbf{P}_r S_p(r, k + (\varepsilon/2))$ holds.

(A.3) For χ in $V_1(k)$, then

$$\|(\chi|M_1(r)) - \mathbf{P}_r(\chi|M_1(r))\| \leq N \cdot \exp\left(-\frac{\delta r}{4}\right) \|\chi\|$$

with N depending only on k, ε , K, L, and δ . (A.4) $\mathbf{P}_r(\chi)$ is an isomorphism of $V_1(k)$ onto $\operatorname{sp}(r, k + (\varepsilon/2))$.

Appendix B: The Mayer-Vietoris Sequence

As is well-known, the Mayer-Vietoris sequence for the cohomology of the triad $(M, M_1, M_2, \Sigma = M_1 \cap M_2)$ takes the form

(B.1)
$$\xrightarrow{\rightarrow} H^*(M_1) \oplus H^*(M_2) \to H^*(\Sigma) \xrightarrow{\circ} H^{*+1}(M)$$
$$\xrightarrow{\rho} H^{*+1}(M_1) \oplus H^{*+1}(M_2) \to H^{*+1}(\Sigma) \to$$

Using the Riemannian metric on Σ , we can, by the usual method of Hodge theory, identify $H^*(\Sigma)$ with the space of harmonic forms

(B.2)
$$H^*(\Sigma) \cong \ker d \cap \ker \delta$$

In particular, $H^*(\Sigma)$ has an inner product $\langle \cdot, \cdot \rangle$ defined on harmonic forms by

(B.3)
$$\langle \alpha, \beta \rangle = \int_{\Sigma} \alpha \wedge *\beta$$

Using (B.1) and the inner product $\langle \cdot, \cdot \rangle$, we can rewrite $H^*(M)$ as a sum

(B.4)
$$H^*(M) = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus (L_1 \cap L_2) \oplus (L_1^{\perp} \cap L_2^{\perp})$$

where $\mathcal{V}_j = \operatorname{image}\{H^*(M_j, \Sigma) \to H^*(M_j)\}, L_j = \operatorname{image}\{H^*(M_j) \to H^*(\Sigma)\}$, and L_j^{\perp} = the orthogonal complement of L_j in $H^*(\Sigma)$. Note that

$$L_1^{\perp} \cap L_2^{\perp} \cong \operatorname{Im} \delta \subset H^*(M)$$

and also that

$$(B.5) \qquad \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus L_1 \cap L_2 \cong \{(x, y) \in H^*(M_1) \oplus H^*(M_2) \mid (x|\Sigma) = (y|\Sigma)\}$$

is isomorphic to the image of ρ .

The above decomposition is related to Theorem A for the operator $D = d + \delta$. By the Hodge theorem,

(B.6)
$$H^*(M(r)) = \ker D \text{ on } M(r)$$

and by [1]

(B.7)
$$\mathcal{V}_{i} \cong L^{2}$$
-solution space of D on $M_{i}(\infty)$.

Our theorem asserts that for r large

(B.8)
$$H^*(M(r)) \cong \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus (\mathcal{L}_1 \cap \mathcal{L}_2)$$

where \mathcal{L}_j is the Lagrangian subspace inside the symplectic vector space ker \hat{D} . Here, under the identification

(B.9)
$$\Lambda^*(T(\Sigma \times R^1)) \cong \pi^*(\Lambda^*T\Sigma) \oplus \pi^*(\Lambda^*T\Sigma) \wedge ds,$$

the operator \hat{D} becomes $(\pm (d - \delta), \pm (d - \delta))$ on $A^*(\Sigma) \oplus A^*(\Sigma)$, two copies of the forms on Σ . Thus we have

(B.10)
$$\ker \hat{D} \cong H^*(\Sigma) \oplus H^*(\Sigma).$$

LEMMA B.1. With the notation as before, we have the following:

- (a) $\mathscr{L}_i = L_i \oplus *L_i$ in $H^*(\Sigma) \oplus H^*(\Sigma)$.
- (b) $*L_i$ coincides with the orthogonal complement L_i^{\perp} in $H^*(\Sigma)$.
- (c) $\{(\alpha, \beta), (x, y)\} = \int_{\Sigma} (\alpha \wedge y \beta \wedge x).$

Granting this formula, our formula (B.8) becomes the same as the Mayer-Vietoris decomposition (B.4) for M(r). Again by Hodge theory, we have

$$\dim(\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{L}_1 \cap \mathcal{L}_2) = \dim H^*(M(r))$$
$$= \dim(\ker D \text{ on } M(r))$$

and so we get the following result:

THEOREM B.2. There exists $R \ge 1$ such that for $r \ge R$, any λ -eigenvalue for $d + \delta$ on M(r) with $|\lambda| < r^{-(1+\varepsilon)}$ is necessarily the 0-eigenvalue $\lambda = 0$.

The above theorem should be compared with a result of Cheeger: For the Laplacian operator $\Delta = (d + \delta)^2$ on functions, the smallest nonzero eigenvalue can be estimated from below (see [8]).

Proof of Lemma B.1(b): If dim $\Sigma = 4n$, it is well-known that the restriction $H^{2n}(M) \to H^{2n}(\Sigma)$ has an image of rank equal to $\frac{1}{2}$ the dimension of $H^{2n}(\Sigma)$. See, for example, Hirzebruch's book [15] (p. 85). This argument goes over equally well for the total restriction mapping

$$H^*(M_i) \to H^*(\Sigma)$$

in general. Thus,

(B.11)
$$\dim L_1 = (1/2) \dim H^*(\Sigma).$$

Now $\langle L_j, *L_j \rangle = \int_{\Sigma} L_j \wedge *(*L_j) = \pm \int_{\Sigma} L_j \wedge L_j = 0$, since $\Sigma = \partial M$ and cohomology classes in $L_j \cap L_j$ are the restrictions of classes in M. Thus $*L_j \subset L_j^{\perp}$. By dim $L_j = \dim *L_j$, and dim $L_j^{\perp} = \dim H^*(\Sigma) - \dim L_j = \dim L_j$ (by (B.11)), it follows that dim $*L_j = \dim L_j^{\perp}$, and so

$$*L_j = L_j^\perp$$
.

Proof of Lemma B.1(a): If ϕ is an extended L^2 -solution of $D\phi = 0$ on $M_1(\infty)$, then

$$\boldsymbol{\phi} \mid \boldsymbol{\Sigma} \times [0,\infty) = \pi^* \boldsymbol{\phi}_0 + \sum_{\mu_k > 0} c_k \, e^{-\mu_k s} \, \pi^* \boldsymbol{\phi}_k$$

with $\hat{D}\phi_0 = 0$. Here $\phi_0 = (\phi', \phi'')$ is a pair of forms on Σ that pulls back to

$$\pi^*\phi_0 \stackrel{\text{def.}}{=} \pi^*\phi' + \pi^*\phi'' \wedge ds.$$

The equation $\hat{D}\phi_0 = 0$ means ϕ', ϕ'' are harmonic forms. Now $\Delta \phi = (d + \delta)^2 \phi = 0$, so by decomposition by type we have

$$d\phi = \delta\phi = 0 \quad (\delta = \pm *d*).$$

In particular, $*\phi$ is another extended L^2 -solution with

$$*\phi \mid \Sigma \times [0,\infty) = *\pi^* \phi_0 + \sum_{\mu_k > 0} c_k \, e^{-\mu_k s} *\pi^* \phi_s , \\ *\pi^* \phi_0 = \pm \pi^* (\hat{*}\phi'') + \pi^* (\hat{*}\phi') \wedge \, ds$$

 $(\hat{*} = \text{star operator for } \Sigma)$, and similarly for $*\pi^*\phi_k$.

Hence, the harmonic forms

$$\phi', *\phi''$$

represent cohomology classes in $H^*(\Sigma)$ that are the restrictions of classes in $H^*(M)$. That is,

$$L_1 \oplus (*L_1) \subset \mathscr{L}_1 \subset H^*(\Sigma) \oplus H^*(\Sigma)$$

since $(\phi', \phi'') = (\phi', \pm *(*\phi''))$. By (B.11)

$$\dim L_1 \oplus *L_1 = 2\dim L_1 = \dim H^*(\Sigma)$$

while because \mathscr{L}_1 is Lagrangian, dim $\mathscr{L}_1 = \frac{1}{2} \dim(H^*(\Sigma) \oplus H^*(\Sigma))$. Hence $L_1 \oplus *L_1 = \mathscr{L}_1$ as claimed, and similarly for \mathscr{L}_2 .

Finally, Lemma B.1(c) is a direct computation using that $\sigma(\phi', \phi'') = (\pm \phi'', \pm \phi')$ for ϕ', ϕ'' of pure type and using Proposition 2.1.

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