Self-Adjoint Elliptic Operators and Manifold Decompositions Part II: Spectral Flow and Maslov Index

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Abstract

This the second part of a three-part investigation of the behavior of certain analytical invariants of manifolds that can be split into the union of two submanifolds. In Part I we studied a splicing construction for low eigenvalues of self-adjoint elliptic operators over such a manifold. Here we go on to study parameter families of such operators and use the previous "static" results in obtaining results on the decomposition of spectral flows. Some of these "dynamic" results are expressed in terms of Maslov indices of Lagrangians. The present treatment is sufficiently general to encompass the difficulties of zero-modes at the ends of the parameter families as well as that of "jumping Lagrangians." In Part III, we will compare infinite- and finite-dimensional Lagrangians and determinant line bundles and then introduce "canonical perturbations" of Lagrangian subvarieties of symplectic varieties. We shall then use this information to study invariants of 3-manifolds, including Casson's invariant.

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1. Introduction and Statements of Main Theorems

This is the second part of a three-part investigation of spectral flow, finite- and infinite-dimensional settings for Lagrangians, canonical perturbations of Lagrangian subvarieties, and applications to invariants of 3-manifolds, including Casson's invariant. In the present, primarily analytic part, we will use "static" results, recalled below, from Part I on the decomposition of the eigenspaces of low eigenvalues of a fixed operator. Here the focus will be the "dynamic" situation of a

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family of operators and the new analytic issues this entails. Some of our dynamic results are expressed here and in Part III in terms of Maslov indices of families of Lagrangians. The present treatment is sufficiently general to encompass the difficulties of zero-modes at the ends of the parameter families as well as those difficulties coming from the phenomena of "jumping Lagrangians." In Part III we shall compare infinite- and finite-dimensional Lagrangians and determinant line bundles and then introduce "canonical perturbations" of Lagrangian subvarieties of symplectic varieties; we shall then apply this information to the study of invariants of 3-manifolds, including Casson's invariant. A basic reference for the needed definitions and results with a list of numerous earlier sources on Maslov indices is [7]; these are reviewed in Section 4 below.

Below we prove decomposition theorems for the spectral flow of a smooth parameter family $\{D(u)\colon 0 \leq u \leq 1\}$ of first-order, self-adjoint elliptic operators $D(u)$ over a closed, smooth manifold M that is split into two pieces M_1 and M_2 by a smooth codimension-1 submanifold Σ:

$$M = M_1 \cup M_2, \qquad \Sigma = M_1 \cap M_2 = \partial M_1 = \partial M_2. \tag{1.1}$$

This treatment includes the general case when the family $D(u)$ has zero-modes at the ends.

As in Part I, we assume that $D(u)$ is of "Atiyah-Patodi-Singer type." That is, on a collar neighborhood $\Sigma \times [-1, +1]$ of $\Sigma = \Sigma \times 0$ in M, the operator $D(u)$ is of the special form

$$D(u) = \pi^* \sigma_u \left(\frac{\partial}{\partial s} + \pi^* \hat{D}(u) \right) \quad \text{on } \Sigma \times [-1, +1]. \tag{1.2}$$

Here s is the coordinate $[-1, +1]$, π is the projection of $\Sigma \times [-1, +1]$ onto Σ, σ_u is a bundle automorphism over Σ, and $\hat{D}(u)$ is a *self-adjoint* elliptic operator over Σ. More explicitly,

$$\begin{cases} D(u) : \Gamma(E) \to \Gamma(E), & E \text{ over } M \\ \hat{D}(u) : \Gamma(\hat{E}) \to \Gamma(\hat{E}), & \hat{E} \text{ over } \Sigma \\ \sigma_u : \hat{E} \to \hat{E} \text{ over } \Sigma \end{cases} \tag{1.3}$$

for bundles E and $\hat{E}$ with inner products and with $E \mid \Sigma \times [-1, +1]$ identified with $\pi^* \hat{E}$. $\Gamma(E)$ denotes the smooth sections of E, and similarly for $\Gamma(\hat{E})$.

In addition, we assume throughout that the kernel of D on $M_j(r)$, $j = 1, 2$, must be determined by its restriction to Σ; that is, the map

$$\ker D \to L^2(E \mid \Sigma), \quad \phi \to \phi \mid \Sigma \tag{1.4}$$

must be injective. This uniqueness property is is satisfied by the Dirac operator and the other natural geometric operators as in our applications [1]. The basic approach is to replace M by a stretched version $M(r)$ of the same manifold

$$M(r) = M_1 \cup \Sigma \times [-r, r] \cup M_2$$

obtained by first cutting M open along Σ and then regluing the pieces back to $\Sigma \times [-r, r]$ with $\Sigma \times (-r)$ and ∂M_1 identified and with $\Sigma \times (r)$ and ∂M_2 identified. Defining $E(r) \to M(r)$ by $E(r) \mid M_j = E \mid M_j$ and $E(r) \mid \Sigma \times [-r, r] = \pi^* \hat{E}$, the operator $D(u)$ on E over M naturally extends to define

$$D(u) = D(u)(M(r)) \colon \Gamma(E(r)) \to \Gamma(E(r)) \quad \text{over } M(r)$$

by setting $D(u)(M(r))$ on M_j as before and on $\Sigma \times [-r, r]$ by (1.2) again.

Similarly, we get operators $D(u)(j) = D(u)(M_j(\infty)) : \Gamma(E_j(\infty)) \to \Gamma(E_j(\infty))$ and bundles $E_j(\infty) \to M_j(\infty)$ for the manifolds with infinite cylindrical ends $M_1(\infty)$ and $M_2(\infty)$,

$$\begin{aligned} M_1(\infty) &= M_1 \cup \Sigma \times [0, \infty) \\ M_2(\infty) &= \Sigma \times (-\infty, 0] \cup M_2 \end{aligned} \tag{1.5}$$

obtained by attaching $\Sigma \times [0, \infty)$ to M_1 along $\Sigma \times 0 = \partial M_1$ and attaching $\Sigma \times (-\infty, 0]$ to M_2 along $\Sigma \times 0 = \partial M_2$. Here $E_j(\infty)$ over $\Sigma \times (\ell, m)$ equals $\pi^* \hat{E}$, and $D(M_j(\infty))$ is given by (1.2) over $\Sigma \times (\ell, m)$ again.

As proven in Part I, for r sufficiently large, all the eigenvalues λ of $D(u)(M(r))$ in the range $[-(1/r^2), +(1/r^2)]$ are exponentially small ($|\lambda| < \exp(-(\delta/4)r)$). Hence, we may fix $R_0 > 0$ such that:

$$\begin{cases} \text{For all } r \geq R_0,\ \pm(1/r^2) \text{ is not an} \\ \text{eigenvalue of } D(0)(M(r)) \text{ or of } D(1)(M(r)). \end{cases} \tag{1.6}$$

Since $+(1/r^2)$ is not an eigenvalue of $D(0)(M(r)), D(1)(M(r))$ for $r \geq R_0$, there is a well-defined $(+1/r^2)$-spectral flow of $D(u)(M(r)) : 0 \leq u \leq 1$. This counts with signs and multiplicities the number of eigenvalues of $D(u)(M(r)) : 0 \leq u \leq 1$ crossing $\lambda = +(1/r^2)$. (See Section 3 for a more explicit definition of spectral flow.) The main results of this paper give formulas for this $(+1/r^2)$-spectral flow ($r \geq R_0$) in terms of spectral flows of self-adjoint operators associated with the restrictions $D(u) \mid M_1$, $D(u) \mid M_2$, and a Maslov index term.

We now formulate these results in several different settings. The simplest setting is when the tangential operator $\hat{D}(u)$ on Σ has no zero-modes, and so we begin with this case:

$$\ker \hat{D}(u) = \{0\} \quad \text{for } 0 \leq u \leq 1\,. \tag{1.7}$$

By the continuity of the spectrum of $\hat{D}(u)$ (see [4], 17.1), we may choose $\delta > 0$ such that the spectrum of $\hat{D}(u)$ lies in $(-\infty, -\delta) \coprod (\delta, \infty)$ for all u ($0 \leq u \leq 1$).

In this situation, by the work of Muller [13] and Douglas and Wojciechowski [9], the self-adjoint extensions $D(u)(j)$ of $D(u)$ acting on the smooth L^2-sections of $E_j(\infty) \to M_j(\infty)$ has pure point spectrum of finite multiplicities and no essential spectrum in the range of eigenvalues λ with

$$-\delta/2 \leq \lambda \leq \delta/2\,. \tag{1.8}$$

In particular, assuming (1.7) and taking δ as above, we may choose $\varepsilon > 0$ so that $\varepsilon < \delta/2$ and $D(0)(j)$, $D(1)(j)$, $j = 1, 2$, have at most the eigenvalue $\lambda = 0$ for λ in the range $[-\varepsilon, +\varepsilon]$. Thus by the continuity of the eigenvalues of $D(u)(j)$ in the range $[-\delta/2, +\delta/2]$ we have a well-defined

$$(+\varepsilon)\text{-spectral flow of } D(u)(j)\colon 0 \leq u \leq 1\,.$$

This spectral flow is defined despite the essential spectrum outside the band $[-\delta/2, +\delta/2]$ and is independent of the choice of $\varepsilon > 0$ above.

THEOREM A. *If* $\ker \hat{D}(u) = 0$ *for* $0 \leq u \leq 1$, *and for* R_0, δ, *and* ε *chosen as above, and* $r \geq R_0$, *then*

$$\left[(+1/r^2)\text{-spectral flow of } D(u)(M(r))\colon 0 \leq u \leq 1 \text{ on } M(r)\right]$$

equals the sum $\sum_{j=1}^{2} \left[(+\varepsilon)\text{-spectral flow of } D(u)(j)\colon 0 \leq u \leq 1 \text{ on } M_j(\infty)\right]$.

Under assumption (1.7), the subspace $P_+(u)$, given by the L^2-closure of the span of the eigensections ϕ of $\hat{D}(u)\phi = \alpha\phi$ with $\alpha > 0$ varies continuously with respect to u. Hence, we may introduce the continuous family of operators over M_j induced from $D(u)$ on M:

$$D(u)(M_j)\colon L_1^2(E \mid M_j,\, P_+(u)) \to L^2(E \mid M_j)\,. \tag{1.9}$$

As in Part I, $L^2(E \mid M_j)$ are the L^2-sections of $E \mid M_j$ and $L_1^2(E \mid M_j, P_+(u))$ is the Sobolev L_1^2-completion of the space of smooth sections ψ of $E \mid M_j \to M_j$ such that $\psi \mid \partial M_j$ lies in $P_+(u) \subset L^2(\hat{E}) = L^2(E \mid \partial M_j)$.

As explained by Atiyah, Patodi, and Singer [2], these operators $D(u)(M_j)$ are Fredholm. By assumption $\ker \hat{D}(u) = 0$, so they are self-adjoint. By continuity of eigenvalues [4, 17.1], we can and do take $\delta > 0$, $\varepsilon > 0$, as above and so that $\varepsilon > 0$ satisfies the additional constraint

$$\begin{cases} D(0)(M_j),\ D(1)(M_j),\ j = 1, 2, \text{ have at most} \\ \lambda = 0 \text{ as an eigenvalue in the range } [-\varepsilon, +\varepsilon]\,. \end{cases} \tag{1.10}$$

With this choice of $\varepsilon > 0$, the

$$\left[(+\varepsilon)\text{-spectral flow of } D(u)(M_j)\colon 0 \leq u \leq 1 \text{ on } M_j\right]$$

is well-defined and independent of ε. It counts the number (with signs and multiplicity) of eigenvalues of $D(u) \mid M_j$ crossing $\lambda = +\varepsilon$.

Our second theorem relates the spectral flows of $D(u)(j)$ over $M_j(\infty)$ and of $D(u)(M_j)$ over M_j under the assumption (1.7).

THEOREM B. *If* $\ker \hat{D}(u) = 0$ *for* $0 \le u \le 1$, *then with* R_0, δ, *and* ε *chosen as above and for all* $r \ge R_0$ *the following equality holds:*

$$\left[(+\varepsilon)\text{-spectral flow of } D(u)(j)\colon 0 \le u \le 1 \text{ on } M_j(\infty)\right]$$
$$= \left[(+1/r^2)\text{-spectral flow of } D(u)(M_j)(r)\colon\ 0 \le u \le 1 \text{ on } M_j(r)\right].$$

By combining Theorems A and B we have, under assumption (1.7), a sum formula for the spectral flow of $D(u)(M(r))$ on $M(r)$ in terms of the spectral flows from two sides of the splitting. However, in general, the dimension of $\ker \hat{D}(u)$, $0 \leqq u \leqq 1$, may not be trivial and in fact may have discontinuous jumps as u varies. To treat this situation, we partition the parameter space $\{u : 0 \le u \le 1\}$ into subintervals $0 \le a_0 < a_1 < \cdots < a_n = 1$ such that over $[a_i, a_{i+1}]$ there are gaps in the spectra of $\hat{D}(u) : a_1 \le u \le a_{i+1}$. That is, there is a number $K_i \ge 0$ and $\delta > 0$ such that no eigenvalue λ of $\hat{D}(u)$ for any u with $a_i \le u \le a_{i+1}$ lies in the range $(K_i, K_i + \delta)$, $(-K_i - \delta, -K_i)$. Let $\mathscr{H}(u; K_i)$ denote the vector space spanned by the eigensections ϕ_j of $\hat{D}(u)\phi_j = \mu_j\phi_j$ with $|\mu_j| \le K_i$. By the spectral decomposition theorem, $\mathscr{H}(u; K_i)$ varies smoothly for $a_i \le u \le a_{i+1}$.

Let $P_+(u; K_i)$ and $P_-(u; K_i)$ denote the L^2-closure of the span of the eigensections ϕ_j with $\hat{D}(u)\phi_j = \mu_j\phi_j$ where $\mu_j > K_i$ and $\mu_j < -K_i$, respectively. Hence, there is a direct-sum decomposition

$$L^2(\hat{E}) = P_-(u; K_i) \oplus \mathscr{H}(u; K_i) \oplus P_+(u; K_i). \tag{1.11}$$

By choosing R_0 large we may ensure that for *each* of the operators $D(a_i)(M(r))$, $i = 0, 1, \ldots, n$, there are no eigenvalues $\pm 1/r^2$ for all $r \ge R_0$. With $r \ge R_0$, the $(+1/r^2)$-spectral flow of $D(u)(M(r))\colon 0 \le u \le 1$ is then the sum

$$\sum_{i=0}^{n-1} (+1/r^2)\text{-spectral flow of } [D(u)(M(r))\colon\ a_i \le u \le a_{i+1}].$$

Hence, it suffices to concentrate on a fixed subinterval $a_i \le u \le a_{i+1}$ in which the following property holds:

$$\begin{cases} \text{For all } a_i \le u \le a_{i+1},\ \hat{D}(u) \text{ has no eigenvalues in the} \\ \text{range } (K_i, K_i + \delta), (-K_i - \delta, -K_i) \text{ with } \delta > 0,\ K_i \ge 0. \end{cases} \tag{1.12}$$

Let $L_1(u)$ and $L_2(u)$ be Lagrangian subspaces in $\ker \hat{D}(u)$ as defined in (2.15) of Part I. In a similar fashion, the subspaces $L_1(u) \oplus [P_+(u) \cap \mathscr{H}(u; K_i)]$ and $L_2(u) \oplus [P_-(u) \cap \mathscr{H}(u; K_i)]$ are Lagrangians in $\mathscr{H}(u; K_i)$ for $a_i \leqq u \leqq a_{i+1}$. Our general spectral flow theorem can be stated with reference to *any choice* of smoothly varying Lagrangian pairs $\mathscr{L}_1(u), \mathscr{L}_2(u)$, $a_i \leqq u \leqq a_{i+1}$, that satisfy the endpoint condition:

$$\begin{cases} \mathscr{L}_1(u) = L_1(u) \oplus [P_+(u) \cap \mathscr{H}(u; K_i)] \text{ if } u = a_i, u = a_{i+1} \\ \mathscr{L}_2(u) = L_2(u) \oplus [P_-(u) \cap \mathscr{H}(u; K_i)] \text{ if } u = a_i, u = a_{i+1}. \end{cases} \tag{1.13}$$

For these choices of Lagrangians we may introduce the self-adjoint Fredholm operators (see Section 2)

$$(1.14)\qquad \begin{cases} D_1(u;\mathscr{L}_1(u))\colon L_1^2(E \mid M_1;\, \mathscr{L}_1(u) \oplus P_+(u;K_i)) \to L^2(E \mid M_1) \\ D_2(u;\mathscr{L}_2(u))\colon L_1^2(E \mid M_2;\, \mathscr{L}_2(u) \oplus P_-(u;K_i)) \to L^2(E \mid M_2) \end{cases}$$

by applying $D(u)$ to the L_1^2-closure of the smooth sections ψ of $E \mid M_j$, $j = 1,2$, whose restrictions $\psi \mid \partial M_j$ lie in the specified subspaces $\mathscr{L}_1(u) \oplus P_+(u;K_i)$ and $\mathscr{L}_2(u) \oplus P_-(u;K_i)$, respectively.

THEOREM C. *For the interval $a_i \le u \le a_{i+1}$ with $\hat{D}(u)$ satisfying condition* (1.12), *for K_i and R_0 as above and any choice of smoothly varying Lagrangians $\mathscr{L}_j(u)$ in $\mathscr{H}(u;K_i)$ satisfying the endpoint conditions* (1.13), *for all $r \ge R_0$ the $[(+1/r^2)$-spectral flow of $D(u)(M(r))\colon a_i \le u \le a_{i+1}]$ equals*

$$\sum_{j=1}^{2}[(\varepsilon')\text{-spectral flow of } [D_j(u;\mathscr{L}_j(u));\ a_i \le u \le a_{i+1}] \text{ on } M_j]$$

$$+\ \mathrm{Mas}\{(\mathscr{L}_1(u),\mathscr{L}_2(u))\colon a_i \le u \le a_{i+1}\} + \frac{1}{2}[\dim\ker \hat{D}(a_{i+1}) - \dim\ker \hat{D}(a_i)]$$

Here ε' is chosen so that the eigenvalues of $D_j(u;\mathscr{L}_j(u))$ at $u = a_i$ and a_{i+1} and in the band $[-\varepsilon',\varepsilon']$ contains at most the zero eigenvalue.

One may derive many variants of Theorem C by using different choices of $\mathscr{L}_j(u)$, the properties of the Maslov indices (see Section 3), and the following theorem relating the Maslov index directly to spectral flow.

We fix an operator D of Atiyah-Patodi-Singer type. Suppose $K \ge 0$ is chosen so that D enjoys the following property:

(1.15) Any L^2-solution of $D\psi = 0$ on $M_j(\infty)$, $j = 1,2$, that decays faster than $\exp(-K|s|)$ on $\Sigma \times [0,\infty)$ and $\Sigma \times (-\infty,0]$, respectively, vanishes identically on $M_j(\infty)$.

By assumption (1.4), such a $K \ge 0$ can always be found. Take $D(u) \equiv D$ for $0 \le u \le 1$, the constant family, and consider *any* smooth choice of Lagrangians $\mathscr{L}_1(u)$ and $\mathscr{L}_2(u)$ in the symplectic space $\mathscr{H}(u;K)$.

In this way we get a family $D_j(u;\mathscr{L}_j(u))$ of self-adjoint operators (defined by (1.14)) on M_j, taking $D(u) \equiv D$ for $0 \le u \le 1$ with $j = 1,2$. As explained in Section 2, there are Lagrangians $L_1(u,K)$ and $L_2(u,K)$ in $\mathscr{H}(u;K)$ such that $\ker D_j(u,\mathscr{L}_j(u)) \cong \mathscr{L}_j(u) \cap L_j(u,K)$.

THEOREM D. *For the constant family of operators $D(u) = D$, $0 \leqq u \leqq 1$, we choose K as in* (1.15) *and $\mathscr{L}_j(u)$, $0 \leqq u \leqq 1$, varying smoothly in $\mathscr{H}(u;K)$, and* *choose* $\varepsilon > 0$ *so that* $\lambda = 0$ *is the only eigenvalue in the range* $[-\varepsilon, +\varepsilon]$ *of* $D_j(0)$ *and* $D_j(1)$, $j = 1, 2$. *Then*

$$\{(+\varepsilon)\text{-spectral flow of } D_j(u; \mathscr{L}_j(u)) : 0 \leqq u \leqq 1\}$$

equals $\mathrm{Mas}\{(L_1(u; K), \mathscr{L}_1(u)); 0 \leqq u \leqq 1\}$ *for* $j = 1$ *and equals* $\mathrm{Mas}\{(\mathscr{L}_2(u), L_2(u; K)) : 0 \leqq u \leqq 1\}$ *for* $j = 2$.

In Section 2 we study the behavior of the Lagrangians $L_1(u)$ and $L_2(u)$, which may have jumps in $\mathscr{H}(u)$, and we also study their generalizations $L_1(u; K)$ and $L_2(u; K)$. In Section 3, we give an explicit definition of the ε-spectral flow and then obtain direct proofs of Theorems A and B. In Section 4, we review the needed properties of a mild generalization of the classical Maslov index for a path of Lagrangian pairs. These were derived in our paper on the Maslov index [7]. In Section 5 the results of Section 3 are used to prove Theorem D and then Theorem C in the special case when $\ker \hat{D}(u) = 0$, $0 \leqq u \leqq 1$. The general case of Theorem C is settled in Section 6.

The results in this paper have been announced in [8], and a general introduction with references and acknowledgments is given in Part I [6]. An earlier effort in this direction appeared in a paper of Yoshida [16], which presented some statements on spectral flow and manifold decomposition. Our work was motivated by applications of the present results to the Casson invariant of three-dimensional manifolds, some of which are treated in Part III. During the final stage of preparing the manuscript, we received the preprints of L. Nicolaescu [14] and B. Boos and K. Wojciechowski [5], in which related, though different, results in the setting of Dirac operators are discussed. From a technical viewpoint, their methods are quite different from ours. One of the reasons is that they have used special features of Dirac operators to overcome several technical difficulties, for example, by using the Kato selection criterion, whereas we address these issues differently. They have used an infinite-dimensional Lagrangian formulation, while we employ in Part II a finite-dimensional one. Moreover, we treat the difficult case of zero-modes at the ends. We reformulate our results in an infinite-dimensional setting in Section 8 below.[1] Various finite- and infinite-dimensional perspectives will be compared in the context of determinant line bundles in Part III.

2. Jumping Lagrangians $L_1(u)$ and $L_2(u)$ in $\mathscr{H}(u)$ via Symplectic Reduction

For a smooth parameter family $D(u)$, $a \leq u \leq b$, of first-order, self-adjoint elliptic operators on M of Atiyah-Patodi-Singer type (i.e., (1.2) holds), we have a pair of Lagrangians $L_1(u)$ and $L_2(u)$ in $\mathscr{H}(u) = \ker \hat{D}(u)$. As described in Part I, $L_j(u)$ has the following description: Consider the kernels of the operators defined

[1] We first heard of such a formulation from T. Mrowka.

by the restrictions $D(u) \mid M_j$

$$
(2.1) \qquad \begin{cases} \widetilde{D}(u)_1 \colon L_1^2(E \mid M_1, P_+(u) \oplus \mathcal{H}(u)) \to L^2(E \mid M_1) \\ \widetilde{D}(u)_2 \colon L_1^2(E \mid M_2, P_-(u) \oplus \mathcal{H}(u)) \to L^2(E \mid M_2) \\ D(u)_1 \colon L_1^2(E \mid M_1, P_+(u)) \to L^2(E \mid M_1) \\ D(u)_2 \colon L_1^2(E \mid M_2, P_-(u)) \to L^2(E \mid M_2) \end{cases}
$$

By [2] these are Fredholm operators, and the kernel of $D(u)_j$ is the restriction of L^2-solutions of $D(u)_j\psi = 0$ on $M_j(\infty)$, while the kernel of the $\widetilde{D}(u)_j$ is the restriction of extended L^2-solutions of $D(u)_j\psi = 0$ on $M_j(\infty)$. The extended L^2-solutions are those solutions of $D(u)\psi = 0$ on $M_j(\infty)$ that are of the form

$$
(2.2) \qquad \sum_{\mu_l \geq 0} a_l e^{-\mu_l s} \pi^* \phi_l \text{ and } \sum_{\mu_l \leq 0} a_l e^{-\mu_l s} \pi^* \phi_l, \text{ respectively,}
$$

with $\widehat{D}(u)\phi_l = \mu_l\phi_l$ on $\Sigma \times [0, \infty)$ and $\Sigma \times (-\infty, 0]$, respectively. Let $\pi[0]$ denote the orthogonal projection of $L^2(\hat{E}) = L^2(E \mid \Sigma)$ onto $\mathcal{H}(u) = \ker \widehat{D}(u)$. In view of (2.2) we have

$$
(2.3) \qquad L_j(u) = \{\pi[0](\psi \mid \partial M_j) \mid \psi \in \ker \widetilde{D}(u)_j\}
$$

as a subspace of $\mathcal{H}(u) = \ker \widehat{D}(u)$ (identifying ∂M_j with Σ).

In particular, we have a short exact sequence

$$
0 \to \nu_j(u) \overset{\text{def}}{=} \ker D(u)_j \hookrightarrow \ker \widetilde{D}(u)_j \xrightarrow{p} L_j(u) \to 0
$$

with $p(\psi) = \pi[0](\psi/\partial M_j)$. Here $\nu_j(u)$ consists of the restrictions to M_j of the L^2-solutions of $D(u)\psi = 0$ on $M_j(\infty)$. Moreover, $L_j(u)$ is a Lagrangian subspace of $\mathcal{H}(u)$ under the symplectic pairing $\{\alpha, \beta\} = (x, \sigma(u)y)_\Sigma$. Finally, recall from Part I that for any Lagrangian subspace $W \subset \mathcal{H}(u) = \ker \widehat{D}(u)$, by imposing the boundary conditions on $\psi \mid \partial M_j$ given by $P_+(u) \oplus W$ (respectively, $P_-(u) \oplus W$), we get self-adjoint Fredholm operators:

$$
\begin{aligned} &L_1^2(E \mid M_1; P_+(u) \oplus W) \to L^2(E \mid M_1) \\ &L_1^2(E \mid M_2; P_-(u) \oplus W) \to L^2(E \mid M_2) \end{aligned}
$$

by applying $D(u)$ to the domain.

These definitions and results have a natural generalization. Fix $K \geq 0$. Because $\sigma(u)\widehat{D}(u) = -\widehat{D}(u)\sigma(u)$ and $\sigma(u)^* = -\sigma(u)$ from the fact that $D(u)$ and $\widehat{D}(u)$ are self-adjoint and from (1.2), the subspace

$$
\mathcal{H}(u; K) = \operatorname{Span}\{\phi_j \colon |\mu_j| \leq K\}
$$

is a symplectic vector space under $\{\alpha, \beta\} = (\alpha, \sigma(u)\beta)_\Sigma$. Here $\{\phi_j\}$ is an orthonormal basis of eigensections of $\hat{D}(u)$ with

$$\hat{D}(u)\phi_j = \mu_j \phi_j . \tag{2.4}$$

There is an orthogonal decomposition

$$L^2(\hat{E}) = L^2(E \mid \Sigma) = P_+(u;K) \oplus \mathscr{H}(u;K) \oplus P_-(u;K) \tag{2.5}$$

with $P_+(u;K)$ and $P_-(u;K)$ constituting the L^2-completions of the span of ϕ_l with $\mu_l > K$ and $\mu_l < -K$, respectively. For $K = 0$ we regain $\mathscr{H}(u)$, $P_+(u)$, and $P_-(u)$. The symplectic vector spaces $\mathscr{H}(u;K)$ also have subspaces $L_j(u;K)$ generalizing $L_j(u)$.

From $D(u) \mid M_1$ and $D(u) \mid M_2$, we have the following operators with the prescribed boundary conditions:

$$\begin{aligned}
&\tilde{D}(u;K)_1 : L_1^2(E \mid M_1, P_+(u;K) \oplus \mathscr{H}(u;K)) \to L^2(E \mid M_1) \\
&D(u;K)_1 : L_1^2(E \mid M_1, P_+(u;K)) \to L^2(E \mid M_1) \\
&\tilde{D}(u;K)_2 : L_1^2(E \mid M_2, P_-(u;K) \oplus \mathscr{H}(u;K)) \to L^2(E \mid M_2) \\
&D(u;K)_2 : L_1^2(E \mid M_2, P_-(u;K)) \to L^2(E \mid M_2)
\end{aligned} \tag{2.6}$$

Note that for $K = 0$, we regain $\tilde{D}(u)_j$ and $D(u)_j$, $j = 1, 2$. Since $D(u)_j$, $j = 1, 2$, is Fredholm and the domains of the operators in (2.6) differ from the domain of $D(u)_j$ by finite-dimensional vector spaces, it follows that all the operators in (2.6) are Fredholm. By $K \geq 0$ and the description of $\ker D(u)_1$, $\ker D(u;K)_1$ consists of the restrictions of L^2-solutions ψ of $D(u)\psi = 0$ on $M_1(\infty)$ such that

$$\psi \mid \Sigma \times [0, \infty) = \sum_{u_l > K} a_l e^{-u_l s} \pi^* \phi_l , \quad s \geq 0 , \tag{2.7}$$

and $\ker D(u;K)_2$ consists of the restrictions of L^2-solutions ψ' of $D(u)\psi' = 0$ on $M_2(\infty)$ such that

$$\psi' \mid \Sigma \times (-\infty, 0] = \sum_{\mu_l < -K} e^{-\mu_l s} \pi^* \phi_l , \quad s \leq 0 , \tag{2.7'}$$

We set $\nu_1(u;K) = \ker D(u;K)_1$ and $\nu_2(u;K) = \ker D(u;K)_2$, generalizing $\nu_1(u)$ and $\nu_2(u)$. These are the restrictions of L^2-solutions that decay faster than $\exp(-K|s|)$ as $|s| \to +\infty$.

Let $\pi[K]$ denote the orthogonal projection of $\Gamma(\hat{E})$ onto $\mathscr{H}(u;K)$. Define the subspaces $L_1(u;K)$ and $L_2(u;K)$ in $\mathscr{H}(u,K)$ by the formula

$$\begin{aligned}
L_1(u;K) &= \{\pi[K](\psi \mid \partial M_1) \mid \psi \text{ in } \ker \tilde{D}(u;K)_1\} \\
L_2(u;K) &= \{\pi[K](\psi \mid \partial M_2) \mid \psi \text{ in } \ker \tilde{D}(u;K)_2\} .
\end{aligned} \tag{2.8}$$

By definition there are short exact sequences

$$(2.9)\qquad \begin{aligned} 0 \to \nu_1(u;K) \to \ker \widetilde{D}(u;K)_1 \xrightarrow{p_1} L_1(u;K) \to 0 \\ 0 \to \nu_2(u;K) \to \ker \widetilde{D}(u;K)_2 \xrightarrow{p_2} L_2(u;K) \to 0 \end{aligned}$$

with $p_j(\psi) = \pi[K](\psi \mid \partial M_j)$ $(\Sigma \equiv \partial M_1 \cong \partial M_2)$. Corresponding to the known properties of $L_1(u)$ and $L_2(u)$ in $\mathcal{H}(u)$ (the $K = 0$ case), we have the following:

PROPOSITION 2.1. *For $K \geq 0$*

(a) *$\widetilde{D}(u;K)_j$ and $D(u;K)_j$ with $j = 1, 2$ are Fredholm mappings with kernels consisting of smooth sections.*

(b) *The subspaces $L_j(u;K)$ are Lagrangian subspaces of the symplectic vector space $\mathcal{H}(u;K)$ under $\{\alpha, \beta\} = (\alpha, \sigma(u)\beta)_\Sigma$.*

Granting Proposition 2.1, it is natural to inquire about the relationship between the Lagrangians $L_j(u, K)$ in $\mathcal{H}(u;K)$ and $L_j(u)$ in $\mathcal{H}(u) = \mathcal{H}(u;0)$. This is best expressed in terms of two symplectic reduction mappings

$$(2.10)\qquad \rho_1, \rho_2 \colon \mathrm{Lag}(\mathcal{H}(u;K)) \to \mathrm{Lag}(\mathcal{H}(u)),$$

which carry Lagrangians in $\mathcal{H}(u;K)$ to Lagrangians in $\mathcal{H}(u)$.

Let $A_1(u;K)$ and $A_2(u;K)$ denote the span of the ϕ_j with $0 < \mu_j \leq K$ and $-K \leq \mu_j < 0$, respectively. These two subspaces are isotropic subspaces under $\{\cdot,\cdot\}$. We define ρ_1 and ρ_2 using the isotropic subspace $A_1(u;K)$ and $A_2(u;K)$, respectively. In view of the isomorphisms

$$\mathcal{H}(u;K) \cong A_1(u;K) \oplus \mathcal{H}(u) \oplus A_2(u;K)$$
$$[\text{annihilator of } A_j(u;K)]/A_j(u;K) \cong \frac{A_j(u;K) \oplus \mathcal{H}(u)}{A_j(u;K)} \cong \mathcal{H}(u)$$

the reduction mapping ρ_j sends Lagrangians in $\mathcal{H}(u;K)$ to Lagrangians in $\mathcal{H}(u)$. Here [annihilator of A_j] $= \{x \mid \{x, A_j\} = 0\}$.

The reduction mappings ρ_j are defined by

$$(2.11)\qquad \rho_j(\widehat{L}) = \left[\widehat{L} \cap (\text{annihilator of } A_j(u;K))\right] / \widehat{L} \cap A_j(u;K).$$

As explained in Guillemin and Sternberg's book [11], ρ_j carries Lagrangians to Lagrangians and is discontinuous only where $\dim(\widehat{L} \cap A_j)$ jumps. From this definition of ρ_j, it is not difficult to verify the following:

PROPOSITION 2.2. *ρ_j: $\mathrm{Lag}(\mathcal{H}(u;K)) \to \mathrm{Lag}(\mathcal{H}(u))$ sends the Lagrangian $L_j(u;K)$ to $L_j(u)$.*

Our Theorem C serves to avoid these difficulties. However, we may expect jump phenomena in $L_j(u)$ even when $\dim \mathcal{H}(u)$ is constant and $\mathcal{H}(u)$ varies smoothly. This is because for $K = 0$, the space $\nu_j(u)$ may be nontrivial. At these places a jump of $L_j(u)$ occurs by the theory of symplectic reduction.

2.1. Proof of Proposition 2.1

The four operators of (a) have domains differing by finite-dimensional subspaces from the domain of $D(u)_1$ and $D(u)_2$, respectively. Since these are Fredholm and defined by $D(u)$, the four operators are Fredholm. The smoothness of the kernel solutions now follows from standard elliptic methods. This proves (a).

As for (b), we use the self-adjointness of $D = D(u)$ over M to get the basic relation

$$(D\psi,\psi')_{M_1} - (\psi, D\psi')_{M_1} = -(\psi \mid \partial M_1,\ \sigma(u)\psi' \mid \partial M_1) \\ = -\{\psi \mid \partial M_1,\ \psi' \mid \partial M_1\} \tag{2.12}$$

for ψ and ψ' sections over M_1. As explained in Part I, this follows from integration since the symbol of $D \mid \Sigma \times [-r,r] = \pi^*\sigma(\frac{\partial}{\partial s} + \pi^*\widehat{D}(u))$ in the $\frac{\partial}{\partial s}$ direction is $\sigma(u)$. (Here $\Sigma \times [-1,0] \subset M_1$ with $\partial M_1 = \Sigma \times 0$.) The basic relation over M_2 is

$$(D\psi,\psi')_{M_2} - (\psi, D\psi')_{M_2} = +\{\psi \mid \partial M_2,\ \psi' \mid \partial M_2\} \tag{2.13}$$

for sections ψ and ψ' over M_2.

Applying (2.13) for ψ and ψ' in $\ker \widehat{D}(u;K)_1$ with $\phi = \pi[K]\psi$ and $\phi' = \pi[K]\psi'$, we have (by $\psi \mid \partial M_1$ and $\psi' \mid \partial M_1$ being elements of $P_+(u;K) \oplus \mathscr{H}(u;K)$),

$$0 = (D\psi,\psi')_{M_1} - (\psi, D\psi')_{M_1} = -\{\psi \mid \partial M_1, \psi' \mid \partial M_2\}_\Sigma \\ = -\{\pi[K](\psi|\partial M_1),\ \pi[K](\psi'|\partial M_2)\}_\Sigma \\ = -\{\phi,\phi'\}_\Sigma\,. \tag{2.14}$$

Thus $L_1(u;K)$ is an isotropic subspace in $\mathscr{H}(u;K)$. In particular,

$$\dim L_1(u;K) \le \frac{1}{2}\dim \mathscr{H}(u;K) \\ = \frac{1}{2}[\dim \mathscr{H}(u)] + \dim A_1(u;K)\,.$$

Hence, in order to complete the proof that $L_1(u;K)$ is Lagrangian, it will suffice to show that

$$\dim L_1(u;K) \ge \frac{1}{2}\dim \mathscr{H}(u) + \dim A_1(u;K)\,. \tag{2.15}$$

LEMMA 2.3. *The image of* $\widetilde{D}(u)_1 = \widetilde{D}(u;0)_1$ *in* $L^2(E \mid M_1)$ *is precisely the orthogonal complement* $\nu_1(u)$ *of* $\ker D_1(u)$.

This lemma is an immediate consequence of the assertion in [2] that $D(u)_1$ is exactly the adjoint of $\widetilde{D}(u)_1$.

Let $\{\phi_j\}_{1\leq j\leq a}$, with $\widehat{D}(u)\phi_j = \mu_j\phi_j$, be an orthonormal basis of $A_1(u;K)$. Then $\{\sigma(u)\phi_j\}_{1\leq j\leq a}$ is a basis for $A_2(u;K)$. Choose smooth sections ψ_j of $E \mid M_1$ with

$$\psi_j \mid \partial M_j = \sigma\phi_j .$$

Let $\{\bar{\psi}_k\}_{1\leq k\leq b}$ be a basis for $\nu_1(u)$. Thus $\bar{\psi}_k \mid \partial M_1 \in P_+(u)$ and $D(u)\bar{\psi}_k = 0$. Then using (2.13)

$$\begin{aligned}
&\left(\bar{\psi}_k, D\left(\sum_{j=1}^{N} c_j\psi_j\right)\right)_{M_1}\\
&= \left(D\bar{\psi}_k, \sum c_j\psi_j\right)_{M_1} + \left\{\bar{\psi}_k \mid \partial M_1, \sum c_j\psi_j \mid \partial M_1\right\}\\
&= \left\{\bar{\psi}_k \mid \partial M_1, \sum c_j\sigma\phi_j\right\}\\
&= \left\{\pi(0,K](\bar{\psi}_k \mid \partial M_1), \sum_{j=1}^{N} c_j\sigma\phi_j\right\} .
\end{aligned} \tag{2.16}$$

Here $\pi(0,K]$ is the orthogonal projection on $A_1(u;K)$. ($\pi[-K,0)$ is defined similarly).

In particular, $\left(\nu_1(u), D\left(\sum_{j=1}^{N} c_j\psi_j\right)\right)_{M_1} = 0$ imposes precisely $[\dim\nu_1(u) - \dim\nu_1(u,K)]$ conditions on the number $\{c_j\}$. Thus by Lemma 2.3, we may find at least $a - [\dim\nu_1(u) - \dim\nu_1(u;K)]$ linearly independent combinations $\{c_j\}$ with

$$D\left(\sum_{j=1}^{N} c_j\psi_j\right) = D(\tilde{\psi})$$

for some $\tilde{\psi} \mid \partial M_1 \in P_+(u) + \mathscr{H}(u)$. Note that

$$\left(\sum c_j\psi_j - \tilde{\psi}\right) \in \ker\widetilde{D}_1(u;K)$$

and $\pi[-K,0)(\sum c_j\psi_j - \tilde{\psi} \mid \partial M_1) = \sum c_j\sigma\phi_j$. This argument proves that the image of $L_1(u;K)$ in $\sigma(u)A_1(u;K) = A_2(u;K)$ under

$$\phi \mapsto \pi[-K,0)\phi$$

has dimension $\geq \{\dim[A_1(u;K)] - [\dim\nu_1(u) - \dim\nu_1(u;K)]\}$.

On the other hand, the extended L_2-solutions (by $L_1(u)$ Lagrangian in $\mathscr{H}(u)$) provide $\frac{1}{2}\dim\mathscr{H}(u) + [\dim\nu_1(u) - \dim\nu_1(u,K)]$ independent elements in $L_1(u,K)$ (via $\psi \mapsto \pi[K](\psi \mid \partial M_1)$), which project to 0 under $\phi \mapsto \pi[-K,0]\phi$.

By combining these two lower estimates we have proved

$$\dim L_1(u;K) \geq \frac{1}{2}\dim\mathscr{H}(u) + \dim A_+(u;K),$$

and so $L_1(u;K)$ is indeed Lagrangian and the inequalities are equalities.
As a conclusion we have also proven that

(2.17) $\pi[-K,0)L_1(u,K) = \{\omega \in A_2(u;K) \text{ such that } \{\omega, \nu_1(u) \mid \partial M_1\}_\Sigma = 0\}$

The case of $L_2(u)$ is similar:

(2.18) $\pi(0,K]L_2(u,K) = \{\omega \in A_1(u,K) \text{ such that } \{\omega, \nu_2(u) \mid \partial M_2\}_\Sigma = 0\}$

with $\pi(0,K]$ the orthogonal projection onto $A_1(u;K)$ in $L^2(\hat{E})$.

3. Definition of $(\varepsilon_1, \varepsilon_2)$-Spectral Flow and Proofs of Theorems A and B

For the proofs of Theorems A and B, it will be convenient to have an explicit, rigorous, and yet flexible definition of spectral flow. Let $D(t) : a \leqq t \leqq b$ be a one-parameter family of real, self-adjoint operators such that for some fixed $\delta > 0$ the total spectrum of $D(t)$ in the range of eigenvalues λ with $|\lambda| < \delta$ is finite-dimensional and has no essential spectrum. Furthermore, after taking into consideration multiplicities, these eigenvalues λ with $|\lambda| < \delta$ vary continuously with respect to t. Let ε_1 and ε_2 be real numbers with $|\varepsilon_1| < \delta$ and $|\varepsilon_2| < \delta$ such that ε_1 is not an eigenvalue of $D(a)$ and ε_2 is not an eigenvalue of $D(b)$. Intuitively, the $(\varepsilon_1, \varepsilon_2)$-spectral flow of $D(t) : a \leqq t \leqq b$ is the number, counted with sign and multiplicity, of eigenvalues λ of $D(t)$ in the range $|\lambda| < \delta$ that cross the line l joining (a, ε_1) to (b, ε_2). As in [2], the eigenvalues λ of $D(t)$ can be displayed as spectral curves $\{(t, \lambda_t) \mid a \leqq t \leqq b\}$ in $[a,b] \times (-\delta, \delta)$ and the spectral flow is the sum of intersection numbers of these curves with l. If $\varepsilon_1 = \varepsilon_2 = \varepsilon > 0$ as in Theorems A through D, then the line l is obtained by moving the x-axis upward to a horizontal line an ε-distance away. We simply refer to this $(+\varepsilon, +\varepsilon)$-spectral as the $(+\varepsilon)$-spectral flow. From the definition, it is easy to see that the $(+\varepsilon)$-spectral flow is additive with respect to path addition of operators. In contrast, this is no longer the case in the $(+\varepsilon, -\varepsilon)$-spectral flow that occurs in Floer's application of index theory [10]. Furthermore, in our application, $\varepsilon > 0$ is chosen so that at the endpoints the operators $D(a)$ and $D(b)$ have no eigenvalues in $[-\varepsilon, \varepsilon] - \{0\}$.
To give a precise definition of $(\varepsilon_1, \varepsilon_2)$-spectral flow, we need the following:

Definition 3.1.
(a) For any t_0 in $[a,b]$, a value λ_0 is called an *excluded value* of $D(t_0)$ if λ_0 is not an eigenvalue of $D(t_0)$.

(b) Given two excluded values ε_1 and ε_2 for $D(a)$ and $D(b)$, a *system of excluded values* of $D(t)$ (t_ℓ, λ_ℓ) from (a, ε_1) to (b, t_2) consists of a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$ together with values $\lambda_1, \ldots, \lambda_n$ in $(-\delta, \delta)$ such that λ_i is an excluded value of $D(t)$ for all t in $[t_{i-1}, t_i]$, $i = 1, \ldots, n$.

By the continuity of eigenvalues of $D(t)$ (see (17.1) in [4]), the excluded value λ_0 of $D(t_0)$ is also an excluded value of $D(t)$ for all t in a small-interval neighborhood

$[t_0 - \delta, t_0 + \delta]$ of t_0. From the compactness of $[a, b]$, it is easy to see that a system of excluded values (t_i, λ_i) always exists connecting two given values (a, ε_1) and (b, ε_2).

DEFINITION 3.2. Given a system of excluded values (t_ℓ, λ_ℓ), $\ell = 1, \ldots, n$, of $D(t)$, $a \leqq t \leqq b$, from ε_1 to ε_2, we define α_ℓ to be +1, 0, or -1 depending on whether $\lambda_{\ell+1} < \lambda_\ell$, $\lambda_{\ell+1} = \lambda_\ell$, or $\lambda_{\ell+1} > \lambda_\ell$, $\ell = 1, \ldots, n$, respectively. We also define $N(\ell)$ to be the number (counted with multiplicities) of eigenvalues λ of $D(a_\ell)$ between λ_ℓ and $\lambda_{\ell+1}$. Then

$$[(\varepsilon_1, \varepsilon_2)\text{-spectral flow of } D(t) : a \leqq t \leqq b] = \sum_{\ell=1}^{n-1} \alpha_\ell N(\ell). \tag{3.1}$$

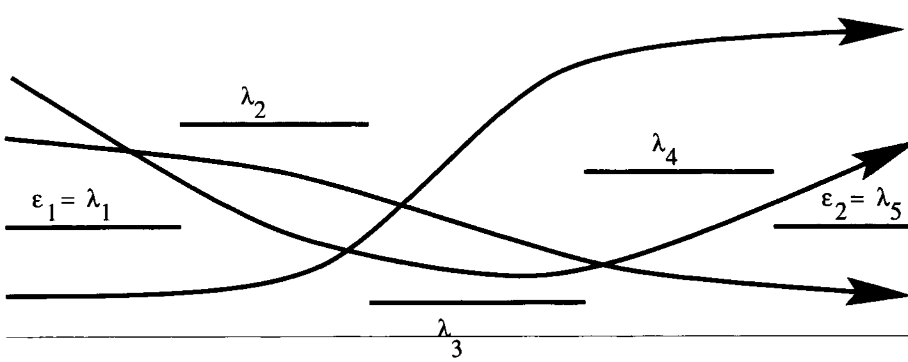

Figure 3.1

Observation I (Independence from Subdivision): Subdividing an interval $[t_\ell, t_{\ell+1}]$ by inserting a point t^*, $t_\ell < t^* < t_{\ell+1}$, and using the same excluded value λ on $[t_\ell, t^*]$ and $[t^*, t_{\ell+1}]$ yields a new system with the sum $\sum \alpha_\ell N(\ell)$ unchanged.

Observation II (Independence from Excluded Values): Taking a given interval $(t_\ell, t_{\ell+1})$, $\ell \neq 0$ or $n-1$, and replacing the given excluded value $\lambda_{\ell+1}$ of $D(t)$, $t_\ell \leqq t \leqq t_{\ell+1}$, by another excluded value μ of $D(t)$ yields a different system. By the continuity of eigenvalues,

$$\begin{aligned} &\{\text{number of eigenvalues of } D(t_\ell) \text{ between } \mu \text{ and } \lambda_{\ell+1}\} \\ &\quad = \{\text{number of eigenvalues of } D(t_{\ell+1}) \text{ between } \mu \text{ and } \lambda_{\ell+1}\}. \end{aligned} \tag{3.2}$$

Hence the new system has the same sum $\sum \alpha_\ell N(\ell)$ as the old because their difference is (± 1) times the difference between the two sides of (3.2).

The $(\varepsilon_1, \varepsilon_2)$-spectral flow of $D(t)$, $a \leqq t \leqq b$, is independent of the system $\{(t_\ell, \lambda_\ell)\}$ chosen. This fact follows from repeating the above two arguments and because any two systems are related to a cofinal one by inserting new points and changing the excluded values.

Observation III (Additivity): Let $\varepsilon_1 < \varepsilon_2 < \varepsilon_3$ be excluded values of $D(t)$ for $t = a, b, c,\ a < b < c$. Then

$$\begin{aligned}
&[(\varepsilon_1, \varepsilon_3)\text{-spectral flow of } D(t) : a \leqq t \leqq c] \\
&\quad = [(\varepsilon_1, \varepsilon_2)\text{-spectral flow of } D(t) : a \leqq t \leqq b] \\
&\qquad + [(\varepsilon_2, \varepsilon_3)\text{-spectral flow of } D(t) : b \leqq t \leqq c]
\end{aligned} \tag{3.3}$$

To verify the above, we only have to choose a system of excluded values of $D(t)$ on $[a, c]$ with $t_\ell = b$ and $\lambda_\ell = \varepsilon_2$ for some ℓ. The restriction of this system to $[a, b]$ and $[b, c]$ give excluded values on these intervals for which (3.3) is evident.

Observation IV (Homotopy Invariance): Let $D(t, s) : a \leqq t \leqq b,\ 0 \leqq \delta \leqq 1$, be a two-parameter family of real self-adjoint operators with finite-dimensional total spectra in the range of eigenvalues λ, $|\lambda| < \delta$, and with no essential spectrum in this range. Suppose the spectra of $D(t, s)$ in the range $|\lambda| < \delta$ vary continuously with respect to (s, t). Further suppose that $\varepsilon_1(s)$ and $\varepsilon_2(s)$ are continuous functions of s, $0 \leqq s \leqq 1$, with values in $(-\delta, \delta)$ such that $\varepsilon_1(s)$ and $\varepsilon_2(s)$ are excluded values of $D(a, s)$ and $D(b, s)$, respectively, for all s. Then

$$\begin{aligned}
&[(\varepsilon_1(0), \varepsilon_2(0))\text{-spectral flow of } D(t, 0) : a \leqq t \leqq b] \\
&\quad = [(\varepsilon_1(1), \varepsilon_2(1))\text{-spectral flow of } D(t, 1) : a \leqq t \leqq b].
\end{aligned} \tag{3.4}$$

Note that given s_0, $0 \leqq s_0 \leqq 1$, there exists a system of excluded values (t_i, λ_i), $i = 1, \ldots, n$, of $D(t, s_0)$ from $\varepsilon_1(s_0)$ to $\varepsilon_2(s_0)$. From the continuity of eigenvalues of $D(s, t)$, this system can also serve as a system for $D(t, s)$ when $|s - s_0|$ is sufficiently small. Furthermore, we can make sure that the multiplicities $N(\ell, s)$ are unchanged. Hence the $(\varepsilon_1(s), \varepsilon_2(s))$-spectral flow of $D(t, s) : a \leqq t \leqq b$ is locally constant as a function of s. The proof of (3.4) follows immediately.

From the above homotopy property of spectral flows, it is easy to see that our definition agrees with many others in the literature, for example, [2] and [4], which were defined in more restrictive settings.

3.1. Proof of Theorem A

By the continuity of eigenvalues and by the assumption $\ker \hat{D}(u) = 0$ for $0 \leqq u \leqq 1$, we may choose $\delta > 0$ such that $\hat{D}(u)$ has no eigenvalue in the range $(-\delta, +\delta)$ for $0 \leqq u \leqq 1$. From the work of Muller [13] and also Douglas-Wojciechowski [9], the self-adjoint extension $D(u)(j)$ of $D(u)|M_j(\infty)$, $j = 1, 2$, has total spectrum of finite dimension and no essential spectrum in the range of eigenvalues in $(-\delta/2, \delta/2)$. Since the eigenvalues vary continuously, we may choose $\varepsilon > 0$ and R_0 satisfying the condition on Theorem A of Part I with $|\varepsilon| < \delta/2,\ 1/R_0 < \varepsilon$.

Since ε is an excluded value of $D(0)(j)$ and $D(1)(j)$, the $(+\varepsilon)$-spectral flow of $D(u)(j) : 0 \leqq u \leqq 1$ is well-defined for $j = 1, 2$. Choosing a partition $0 = a_0 < a_1 < \cdots < a_n = 1$ of [0,1] and $|\lambda_\ell| < \delta/2$ such that (a_ℓ, λ_ℓ) are systems of excluded values for $D(u)(j) : 0 \leqq u \leqq 1,\ j = 1, 2$, we compute the $(+\varepsilon)$-spectral flow of $D(u)(j)$ by formula (3.1). The answer is $\sum_{\ell=1}^{n-1} \alpha_\ell N(\ell, j)$, where $\alpha_\ell = +1, 0$, or -1 and $N(\ell, j)$ equals the number of eigenvalues of $D(a_\ell)(j)$ between λ_ℓ and $\lambda_{\ell+1}$. Again by continuity of eigenvalues, we may choose $\delta' > 0$ with $|(|\lambda_\ell|) - \delta'| < \delta/2$ so that whenever λ is in the band $\lambda_\ell - \delta' \leqq \lambda \leqq \lambda_\ell + \delta'$, it is an excluded value of $D(u)(j)$ for all u, $a_{\ell-1} \leqq u \leqq a_\ell$ $(\ell = 1, \ldots, n-1,\ j = 1, 2)$.

Appealing to Theorem B of Part I and especially to the *uniform estimate* for R, we may choose $R_1 \geqq R_0$ such that for all $r \geqq R_1$, all eigenvalues λ of $D(u)(M(r))$ with $|\lambda| < \delta K/2$ are within k of an eigenvalue of $D(u)(j) \mid M_1(\infty)$ or $D(u) \mid M_2(\infty)$ for *all* u, $0 \leqq u \leqq 1$. This value k may be taken as small as required—in particular, smaller than δ'—by increasing R_1.

Thus for $r \geqq R_1, \lambda_\ell$ will not be an eigenvalue of $D(u)(M(r))$ for any u with $a_{\ell-1} \leqq u \leqq a_\ell$. In particular, the $(+\varepsilon)$-spectral flow of $D(u)(M(r))$, $0 \leqq u \leqq 1$, is well-defined and may be computed from the system of excluded values (a_ℓ, λ_ℓ). By Definition 3.2, this is given by $\sum_{\ell=1}^{n-1} \alpha_\ell N'(\ell)$, where α_ℓ is as above and $N'(\ell)$ is the number of eigenvalues of $D(a_\ell)(M(r))$ between λ_ℓ and $\lambda_{\ell+1}$.

Once again appealing to Theorem B of Part I, we may choose $R_2 \geqq R_1$ so that for each of the operators $D(a_\ell)(M(r))$, the eigenvalues of $D(a_\ell)(M(r))$, $r \geqq R_2$, in the range $\lambda_\ell, \lambda_{\ell+1}$ are exponentially close to those of $D(a_\ell)(1) \mid M_1(\infty)$ and $D(a_\ell)(2) \mid M_2(\infty)$ in the range $[\lambda_\ell, \lambda_{\ell+1}] - [\lambda_\ell - \delta', \lambda_\ell + \delta'] - [\lambda_{\ell+1} - \delta', \lambda_{\ell+1} + \delta']$. By taking R_2 large, we can make sure that those exponentially close eigenvalues do indeed lie in the band $(\lambda_\ell, \lambda_{\ell+1})$. Applying Theorem B of Part I one more time, we have $N'(\ell) = N(\ell, 1) + N(\ell, 2)$ for $r \geqq R_2$. Thus $\sum \alpha_\ell N'(\ell) = \sum \alpha_\ell N(\ell, 1) + \sum \alpha_\ell N(\ell, 2)$.

Finally, by Observation IV, the $\frac{1}{r^2}$-spectral flow of $D(u)(M(r)) : 0 \leqq u \leqq 1$ for $r \geqq R_3$ equals the $(+\varepsilon)$-spectral flow once $R_3 \geqq R_2$ is chosen so that there are no eigenvalues of $D(a)(M(r))$ and $D(b)(M(r))$ between ε and $\frac{1}{r^2}$. This holds for R_3 sufficiently large by Theorem B of Part I. Thus $\sum \alpha_\ell N'(\ell)$ is the $\frac{1}{r^2}$-spectral flow of $D(u)(M(r))$ for $r \geqq R_3$. The proof of Theorem A is complete.

3.2. Proof of Theorem B

Since the eigenvalues of the operators $D(u)(j) : 0 \leqq u \leqq 1$ vary continuously, we may, as before, form a system of excluded values $\{a_\ell, \lambda_\ell\}$. By using the uniform estimates of Theorem 9.2 in Part I, which relates the eigenvalues of $D(u)(j)$ on $M_j(u)$ to $D_j(u) \mid M_j(r)$, $j = 1, 2$, we may repeat the above argument to conclude Theorem B.

4. The Maslov Index

To proceed further, we need some of the basic properties of a mild generalization of the classical Maslov index. Because their proofs can be found in [7] together with references to numerous earlier sources and comparisons with other invariants, we simply cite them here. Let V be a finite-dimensional vector space with a nondegenerate, skew-symmetric, bilinear pairing

$$\{\cdot,\cdot\} : V \times V \to \mathbb{R}. \tag{4.1}$$

That is, $\{\cdot,\cdot\}$ is a symplectic pairing. Given such a symplectic structure, we may choose a complex structure J and a Hermitian pairing $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ on V with $\{\alpha,\beta\} = \operatorname{Re}\langle J\alpha,\beta\rangle = -\operatorname{Im}\langle\alpha,\beta\rangle$. A Lagrangian $L \subset V$ is a real subspace with $\dim_{\mathbb{R}} L_0 = n = \frac{1}{2}\dim_{\mathbb{R}} V$ such that $\{\cdot,\cdot\}$ vanishes on $L \times L$. Equivalently, L is the real span $L = \mathbb{R}\{e_1,\dots,e_n\}$ of an orthonormal basis $\{e_1,\dots,e_n\}$ of the Hermitian vector space $\{V, J, \langle\cdot,\cdot\rangle\}$. In particular, we obtain an action of the unitary group $U(n)$ on the space $\operatorname{Lag}(V)$ of Lagrangians by letting $L \to u \cdot L = \mathbb{R}\{ue_1,\dots,ue_n\}$, $u \in U(n)$. This action is transitive, and the isotropy subgroup with respect to a fixed Lagrangian L_0 is $O(n)$, and so $\operatorname{Lag}(V) \cong U(n)/O(n)$. Define

$$\Phi : \operatorname{Lag} V \to S^1 \tag{4.2}$$

by $\Phi(uL_0) = (\det u)^2$. It is well-known that Φ induces an isomorphism of fundamental groups. When we have a loop of Lagrangians, this isomorphism $\Phi_* : \pi_1(\operatorname{Lag} V) \to \mathbb{Z}$ provides us with the definition of Maslov index.

More generally, let $f(t) = (L_1(t), L_2(t))$, $a \leqq t \leqq b$, be a continuous family of pairs of Lagrangians $L_1(t)$ and $L_2(t)$ in V. As in [7], the Maslov index $\mu_V(f)$ is defined to be an integer with the following properties:

Property I (Affine Scale Invariance): Composing $f(t)$ with an affine map $\psi : t \mapsto kt + \ell$, $k > 0$, yields $\mu_V(f) = \mu_V(f \cdot \psi)$.

Property II (Deformation Relative to the Endpoints): If $f(s)(t) = (L_1(s,t), L_2(s,t))$, $0 \leqq s \leqq 1$, $a \leqq t \leqq b$, gives a continuous deformation of one family $f(0)$ of Lagrangian pairs to the other $f(1)$ so that $(L_1(s,a), L_2(s,a))$ and $(L_1(s,b), L_2(s,b))$ are independent of s, then $\mu_V(f(0)) = \mu_V(f(1))$.

Property III (Path Additivity): If $a \leqq x \leqq b$, then $\mu_V(f) = \mu_V(f|[a,x]) + \mu_V(f|[x,b])$.

Property IV (Symplectic Additivity): If $f : [a,b] \to (\operatorname{Lag} V)^2$ and $g : [a,b] \to (\operatorname{Lag} W)^2$ are continuous and $f \oplus g : [a,b] \to (\operatorname{Lag}(V \oplus W))^2$ is their direct sum, then $\mu_{V\oplus W}(f \oplus g) = \mu_V(f) \oplus \mu_W(g)$.

Property V (Symplectic Invariance): If $\psi_t : V \to V$ is a continuous family of symplectic automorphisms of V, then $\mu_V[\psi_t \cdot f(t) : a \leqq t \leqq b] = \mu_V(f)$.

Property VI (Normalization): Let V be the complex plane $\mathbb{C}$ with $\langle z, \omega\rangle = z\bar{\omega}$ and let

$$f(t) = (\mathbb{R}\{1\}, \mathbb{R}\{e^{it}\}), \quad -\pi/4 \leqq t \leqq \pi/4$$
$$g(t) = (\mathbb{R}\{1\}, \mathbb{R}\{e^{-it}\}), \quad -\pi/4 \leqq t \leqq \pi/4.$$

Then

$$\begin{aligned}\mu_V(f \mid [-\pi/4,\pi/4]) &= -\mu_V(g \mid [-\pi/4,\pi/4]) = 1\,,\\ \mu_V(f \mid [-\pi/4,0]) &= \mu_V(g \mid [0,\pi/4]) \quad = 0\,,\\ \mu_V(f \mid [0,\pi/4]) &= -\mu_V(g \mid [-\pi/4,0]) \quad = 1\,.\end{aligned}$$

Property VII (Nullity): If $L_1(t) \cap L_2(t)$ varies smoothly with respect to t and has constant dimension, then $\mu_V(f) = 0$.

Property VIII (Reparametrization Invariance): If $\psi : [c,d] \to [a,b]$ is a homeomorphism with $\psi(c) = a$, $\psi(d) = b$, $c < d$, and $a < b$, then $\mu_V(f \cdot \psi) = \mu_V(f)$.

Property IX (Symmetry and Reversal): Given $f(t) = (L_1(t), L_2(t)) : a \leqq t \leqq b$, we let $g(t) = (L_2(t), L_1(t)) : a \leqq t \leqq b$ and $h : [-b,-a] \to (\operatorname{Lag} V)^2$ be defined by $h(t) = (L_1(-t), L_2(-t))$. Then $\mu_V(g) = -\mu_V(f) + [h_{12}(a) - h_{12}(b)]$ and $\mu_V(g) = -\mu_V(f)$ where $h_{12}(t) = \dim_{\mathbb{R}}(L_1(t) \cap L_2(t))$.

Given three Lagrangians L_1, L_2, and L_3 in V, the triple Maslov index $\tau(L_1, L_2, L_3)$ is defined to be the signature of the quadratic forms

$$g(x,y,z) = \{x,y\} + \{y,z\} + \{z,x\}$$

on $L_1 \oplus L_2 \oplus L_3$.

Property X (Triple Maslov Index): Let $L_1(t)$, $L_2(t)$, and $L_3(t)$, $a \leqq t \leqq b$ be three continuous families of Lagrangians in V, and let

$$f(t) = (L_1(t), L_2(t)), \qquad g(t) = (L_2(t), L_3(t)), \qquad h(t) = (L_3(t), L_1(t))\,.$$

Then

$$\begin{aligned}&\frac{1}{2}\{\tau(L_1(b), L_2(b), L_3(b)) - \tau(L_1(a), L_2(a), L_3(a))\}\\ &\quad = \mu_V(f) + \mu_V(g) + \mu_V(h) + \frac{1}{2}\sum_{j<k}[h_{jk}(b) - h_{jk}(a)]\,,\end{aligned}$$

where $h_{jk}(t) = \dim L_j(t) \cap L_k(t)$.

Note: In [7] the Maslov index is defined only for smooth paths $(L_1(t), L_2(t))$. The extension to continuous paths as above is standard (e.g., smooth approximation).

5. Proofs of Theorem D and of Theorem C for the Case $\ker \widehat{D}(u) = 0$

Before proceeding with the proof of Theorem D, let us first clear up some notational matters. The operator $D(u) \mid M_j$ is constant under the assumption of Theorem D; therefore the symplectic vector $\mathcal{H}(u;K)$ and the Lagrangians $P_\pm(u;K)$ are independent of u. Nonetheless, we use the same notation and regard them as the fibers at u of corresponding vector bundles over $[0, 1]$. In addition, as the situations for $j = 1$ or 2 are more or less identical, we concentrate on $j = 1$ first.

Let $\mathscr{L}_1(u) : 0 \leqq u \leqq 1$ be a smooth one-parameter family of Lagrangians in $\mathscr{H}(u,K)$. The kernel of $D_1(u;\mathscr{L}_1(u))$ consists of smooth solutions Ψ to $D\Psi = 0$ for which $\Psi \mid \Sigma$ lies in $\mathscr{L}_1(u) \oplus P_+(u;K)$. Therefore, $\ker D_1(u;\mathscr{L}_1(u))$ is contained in the kernel of the Fredholm operator

$$\widetilde{D}(u;K)_1 : L^2(E \mid M_1,\ P_+(u;K) \oplus \mathscr{H}(u;K)) \to L^2(E \mid M_1).$$

By (2.9) and the choice of K in (1.15), it is clear, given our assumption (1.4), that under the restriction to Σ, $\ker D_1(u;\mathscr{L}_1(u))$ is mapped isomorphically onto $L_1(u;K) \cap \mathscr{L}_1(u,K)$,

(5.1) $$\ker D_1(u;\mathscr{L}_1(u)) \cong L_1(u,K) \cap \mathscr{L}_1(u).$$

Now, any smooth path of Lagrangians $\mathscr{L}_1(u)$, $0 \leqq u \leqq 1$, in $V = \mathscr{H}(u;K)$ can be smoothly deformed relatively to endpoints $\mathscr{L}_1(a)$ and $\mathscr{L}_1(b)$ to a composite of smooth paths on smaller intervals with one of the following properties:

(5.2a) $\mathscr{L}_1(u) : a \leqq u \leqq b$ with $L_1(u,K) \cap \mathscr{L}_1(u) = \{0\}$ for all u.

(5.2b) $\mathscr{L}_1(u) : a \leqq u \leqq b \cong L_1(u,K) \cap \mathscr{L}_1(u)$ of *fixed* dimension, say k, and varying smoothly with respect to u.

(5.2c) With respect to a symplectic basis $\{e_1, \ldots, e_n, \sigma e_1, \ldots, \sigma e_n\}$ of V, the fixed Lagrangian $L_1(u,K) = \mathbb{R}\{e_l : 1 \leqq l \leqq n\}$ and $\mathscr{L}_1(u)_k = e^{-\sigma u}\mathbb{R}\{e_l : 1 \leqq l \leqq k\} \oplus \sigma\mathbb{R}\{e_l : k < l \leqq n\}$, where k is fixed and u varies in the interval $[0, \pi/4]$.

(5.2d) The same as in (5.2c) except that we let u vary between $-\pi/4$ and 0.

By the additivity of $(+\varepsilon)$-spectral flow and the Maslov index, it suffices to prove Theorem D in each of the four cases of (5.2). In view of (5.1) and Properties II, III, and VI, both the $(+\varepsilon)$-spectral flow and the Maslov index vanish in cases (5.2a) and (5.2b). Hence Theorem D is proven in these cases. By splitting off the component $e^{-\sigma u}\mathbb{R}\{e_l\}$ in $\mathscr{L}_1(u)_k$, first for $l = 1$ and then for $l = 2, \ldots, k$, we can deform paths (5.2c) and (5.2d) into a composite of paths of the following two forms:

(5.3a) $\mathscr{L}_1(u)_k^{\#} : 0 \leqq u \leqq \pi/4$

(5.3b) $\mathscr{L}_1(u)_k^{\#} : -\pi/4 \leqq u \leqq 0$

where

$$\mathscr{L}_1(u)_k^{\#} = \mathbb{R}\{e_l : 1 \leqq l < k\} \oplus e^{-\sigma u}\mathbb{R}\{e_k\} \oplus \sigma\mathbb{R}\{e_l : k < l \leqq n\}.$$

Therefore, it will suffice to verify Theorem D for these special paths (5.3a) and (5.3b).

Given the symplectic basis e_l as above, we may choose smooth sections Ψ_l of $E \mid M_1$ such that $D\Psi_l = 0$ and the restriction $\Psi_l \mid \Sigma$ in $\mathscr{H}(u;K) \oplus P_+(u;K)$ projects onto e_l in $\mathscr{H}(u;K)$. After taking appropriate normalization of e_l, we may assume that $(\Psi_l, \Psi_l)_{M_1} = 1$. Because

$$L_1(u,K) \cap \mathscr{L}_1(u)_k^\# = \mathbb{R}\{e_l; 1 \leqq l < k\} \oplus m(u)$$

where $m(u) = 0$ for $0 < |u| \leqq \pi/4$ and $m(0) = \mathbb{R}\{e_k\}$, by (5.1) the kernel of $D_1(u;\mathscr{L}_1(u)_k^\#)$ for all $u > 0$ is spanned by Ψ_l, $1 \leqq l < k$. At $u = 0$, this kernel has a jump in dimension and is spanned by the sections Ψ_l, $1 \leqq l \leqq k$.

By the continuity of eigenvalues of $D_1(u;\mathscr{L}_1(u)_k^\#)$, we can choose $\varepsilon > 0$ and $\Delta u > 0$ so that $\lambda = 0$ is the only eigenvalue of $D_1(0;\mathscr{L}(0)_k^\#)$ for eigenvalue λ with $|\lambda| \leqq \varepsilon$, and $\lambda = \pm\varepsilon$ are excluded values of $D_1(u;\mathscr{L}_1(u)_k^\#)$ for all u with $|u| \leqq \Delta u$. Hence the sum of multiplicities of eigenvalues of $D_1(u;\mathscr{L}_1(u)_k^\#)$ in the range $[-\varepsilon, +\varepsilon]$ is k. By standard spectral decomposition, the projection $\Pi(u)$ onto the space spanned by the eigenvectors with eigenvalues in $[-\varepsilon, +\varepsilon]$ is smooth with respect to u. On the other hand, by construction we have

$$\mathbb{R}\{\Psi_l; 1 \leqq l < k\} \subseteq \text{Image of } \Pi(u), \tag{5.4}$$

where the dimension on the right-hand side equals k. Hence, the orthogonal complement of $\mathbb{R}\{\Psi_l : 1 \leqq l < k\}$ in Image of $\Pi(u)$ is one-dimensional and varies smoothly with respect to u. By (5.1) and by the orthogonality of eigenvectors of different eigenvalues, this one-dimensional subspace is spanned by a unit eigenvector $\Psi_k(u)$ with

$$D_1(u;\mathscr{L}_1(u)_k^\#)\Psi_k(u) = \lambda(u)\Psi_k(u) \tag{5.5}$$

for $0 \leqq |u| \leqq \pi/4$. In this manner, we have exhibited a smoothly varying eigensolution $\Psi_k(u)$, $|u| \leqq \Delta u$, with the property

$$\begin{aligned} D_1(u;\mathscr{L}_1(u)_k^\#)\Psi_k(u) &= \lambda(u)\Psi_k(u) \\ \Psi_k(0) &= \Psi_k \\ (\Psi_k(u), \Psi_k(u))_{M_1} &= 1 \end{aligned} \tag{5.6}$$

and Image of $\Pi(u) = \mathrm{Span}\{\Psi_l : 1 \leqq l < k\} \oplus \mathbb{R}\{\Psi_k(u)\}$.

Remark 5.1. Splitting off such a smoothly varying eigensection is crucial to our argument, given the known difficulty with multiplicity in general. We have employed this device of splitting off the eigenvectors one at a time to avoid these problems.

Recall from Part I the following result:

$$\begin{aligned} (Df,g)_{M_1} - (f,Dg)_{M_1} &= -\{f \mid \Sigma,\ g \mid \Sigma\} \\ &= -\{f \mid \Sigma,\ \sigma g \mid \Sigma\}, \end{aligned} \tag{5.7}$$

where f and g are sections of $E \mid M_1$. In particular, taking $f = \Psi_k$ and $g = \Psi_k(u)$, we obtain by (5.6)

$$-(\Psi_k, \lambda(u)\Psi_k(u))_{M_1} = -(e_k, \sigma\Psi_k(u) \mid \Sigma)_\Sigma$$

Differentiating and setting u to 0 yields

$$-\lambda'(0) = -\left(e_k, \sigma\left[\tfrac{d}{du}\Psi_k(u)\right]_{u=0} \mid \Sigma\right).$$

By the Lagrangian property of the solution space and the fact that $\{e_k, e_l\} = (e_k, \sigma e_l) = 0$, we can express $\Psi_k(u) \mid \Sigma$ as the sum $\sum a_l(u)e_l + b(u)e^{-\sigma u}e_k$, where $a_l(u)$ and $b(u)$ are smooth and $a_l(0) = 0$, $b(0) = 1$. Thus we have

$$\begin{aligned} &-\left(e_k, \sigma\left[\tfrac{d}{du}\Psi_k(u)\right]_{u=0} \mid \Sigma\right) \\ &= -\left(e_k, \sigma\left(\sum a_l'(0)e_l + b'(0)e_k\right)\right) - (e_k, \sigma(-\sigma \cdot b(0)e_k)) \\ &= (e_k, \sigma^2 e_k) \\ &= -(\sigma e_k, \sigma e_k) \end{aligned}$$

and so $\lambda'(0) = (\sigma e_k, \sigma e_k) > 0$. This argument proves the $(+\varepsilon)$-spectral flow of $D_1(u; \mathscr{L}_1(u)_k^\#)$ is $+1$ for the interval $[0, \Delta u]$ and 0 for $[-\Delta u, 0]$. On the other hand, since $e^{-\sigma u}\mathbb{R}\{e_k\}$ rotates the subspace $\mathbb{R}\{e_k\}$ in $\mathbb{R}\{e_k, \sigma e_k\}$ in the positive direction, the Maslov index of $(L_1(u, K), \mathscr{L}_1(u)_k^\#)$ is $+1$ on $[0, \Delta u]$ and 0 on $[-\Delta u, 0]$. This proves Theorem D for the case $j = 1$.

If we replace M_1 by M_2 in the above argument, we must replace $L_1(u; K)$ by $L_2(u; K)$ and D_1 by D_2, which uses $\mathscr{H}(u; K) \oplus P_-(u; K)$ as the boundary condition. For sections f and g of $E \mid M_2$, we obtain in place of (5.6) the equality

$$\begin{aligned} (Df, g)_{M_2} - (f, Dg)_{M_2} &= \{f \mid \Sigma, g \mid \Sigma\} \\ &= (f \mid \Sigma, \sigma g \mid \Sigma) \end{aligned}$$

because of the change in orientation. It follows that the $(+\varepsilon)$-spectral flow of $D_2(u; \mathscr{L}_2(u)_k^\#)$ is 0 on $[0, \Delta u]$ and -1 on $[-\Delta u, 0]$. From Section 4, we have

$$\begin{aligned} &\mathrm{Mas}[(e^{iu}\mathbb{R}, \mathbb{R}) : a \leqq u \leqq b] \\ &= \mathrm{Mas}[(\mathbb{R}, e^{-iu}\mathbb{R}) : a \leqq u \leqq b] \end{aligned}$$

where $\mathbb{R}$ is the Lagrangian subspace represented by the x-axis in $\mathbb{R}^2 = \mathbb{C}$. Applying this formula to evaluate $\mathrm{Mas}(\mathscr{L}_2(u)_k^\#, L_2(u; K))$, we obtain -1 on $[-\Delta u, 0]$ and 0 on $[0, \Delta u]$. This completes the proof of Theorem D.

5.1. Proof of Theorem C When $\ker_s \widehat{D}(u) = 0$

Let the constants K_i, ε, and R_o and subintervals $[a_i, a_{i+1}]$ be given as in the statement of Theorem C. Since we work on one interval at a time, we may assume that $[a,b] = [a_i, a_{i+1}]$ and $K = K_i$. In the case where $\ker \widehat{D}(u) = 0$, we have the subspace $L_j(u) = 0$ by definition, and so the Lagrangians $\mathscr{L}_j(u)$ are the same as $[P_\pm(u) \cap \mathscr{H}(u;\ K)]$ at the endpoints as $j = 1$ or $j = 2$. In fact, we may deform the path $\mathscr{L}_j(u)$ in $\mathscr{H}(u;K)$ relative to the ends to a composite of three smooth paths (with $\pm$ for $j = 1,\ j = 2$, respectively):

(5.8) $\Upsilon_j(u)_1$ in $\mathrm{Lag}(\mathscr{H}(u;K))$ connecting $P_\pm(a) \cap \mathscr{H}(a;K)$ to $P_\pm(a) \cap \mathscr{H}(a;K)$,

(5.9) the path $P_\pm(u) \cap \mathscr{H}(u;K)$, $a \leqq u \leqq b$, and

(5.10) $\Upsilon_j(u)_2$ in $\mathrm{Lag}(\mathscr{H}(b;K))$ connecting $P_\pm(b) \cap \mathscr{H}(b;K)$ to $P_\pm(b) \cap \mathscr{H}(b;K)$.

In other words, in the middle segment $\mathscr{L}_j(u) = P_\pm(u) = P_\pm(u) \cap \mathscr{H}(u;K)$, and at the two end segments the $\mathscr{L}_j(u)$ connect up with the given boundary condition.

By the additivity properties of the Maslov index and spectral flow, we can break down our calculation along the three paths described in (5.8) through (5.10). Since by Property VII $\mathrm{Mas}\{P_+(u) \cap \mathscr{H}(u;K),\ P_-(u) \cap \mathscr{H}(u;K)\} = 0$, we have

$$\mathrm{Mas}\{\mathscr{L}_1(u), \mathscr{L}_2(u)\} = \sum_{l=1}^{2} \mathrm{Mas}\{\Upsilon_1(u)_l, \Upsilon_2(u)_l\}.$$

On the other hand, by Theorem D,

$$\begin{aligned}\sum_{l=1}^{2}\sum_{j=1}^{2} &\left\{(+\varepsilon)\text{-spectral flow of } D_j(u;\Upsilon_j(u)_l)\right\} \\ &= \sum_{l=1}^{2} \mathrm{Mas}\{L_1(a_l;K), \Upsilon_1(u)_l\} + \mathrm{Mas}\{\Upsilon_2(u)_l, L_2(a_l;K)\}\end{aligned}$$

Here $a_1 = a$, $a_2 = b$. Therefore, calculating on M_1 and M_2, we have

$$\begin{aligned}\sum_{j=1}^{2} &\left\{(+\varepsilon)\text{-spectral flow of } D_j(u;\mathscr{L}_j(u)) + \mathrm{Mas}(\mathscr{L}_1(u),\ \mathscr{L}_2(u))\right\} \\ &= \sum_{j=1}^{2} (+\varepsilon)\text{-spectral flow of } D_j(u;P_\pm(u) \cap \mathscr{H}(u;K)) \text{ on } M_j \\ &\quad + \sum_{l=1}^{2} \mathrm{Mas}\{L_1(a_l;K), \Upsilon_1(u)_l\} + \mathrm{Mas}\{\Upsilon_2(u)_l, L_2(a_l;K)\} \\ &\quad + \sum_{l=1}^{2} \mathrm{Mas}\{\Upsilon_1(u)_l, \Upsilon_2(u)_l\}.\end{aligned} \tag{5.11}$$

The Maslov index terms in (5.11) sum to 0 because $\Upsilon_j(\text{initial point})_l = \Upsilon_j(\text{terminal point})_l$, and so, by Property X,

$$\begin{aligned}&\mathrm{Mas}(L_1(a_l,K),\ \Upsilon_1(u)_l) + \mathrm{Mas}(\Upsilon_1(u)_l, \Upsilon_2(u)_l) \\ &\quad = -\mathrm{Mas}(\Upsilon_2(u), L_1(a_l,K)).\end{aligned}$$

To obtain the desired sum, we add $\text{Mas}\{\Upsilon_2(u)_l, L_2(a_l;K)\}$ to the above; then by Properties IX and X, the right-hand side of the equality is

$$\begin{aligned} &- \text{Mas}\{\Upsilon_2(u)_l,\ L_1(a_l;K)\} + \text{Mas}\{\Upsilon_2(u)_l,\ L_2(a_l;K)\} \\ &= \text{Mas}\{L_1(a_l;K),\ \Upsilon_2(u)_l\} + \text{Mas}\{\Upsilon_2(u)_l,\ L_2(a_l;K)\} \\ &= \text{Mas}\{L_1(a_l;K),\ L_2(a_l;K)\} \\ &= 0. \end{aligned}$$

By definition, we have for $r \geqq R_0$

$$\begin{aligned} &\frac{1}{r^2}\text{-spectral flow of } D_j(u)(M_j(r)) \\ &\quad = \frac{1}{r^2}\text{-spectral flow of } D_j(u, P_\pm(u) \cap \mathcal{H}(u;K))(M_j(r)) \\ &\quad = (+\varepsilon)\text{-spectral flow of } D_j(u, P_\pm(u) \cap \mathcal{H}(u;K))(M_j). \end{aligned}$$

The last equation holds because $\ker D_j(u, P_\pm(u) \cap \mathcal{H}(u;K))(M_j(r))$ is independent of r, and these operators on $M_j(r)$ vary continuously with r. Thus, we may replace $\frac{1}{r^2}$ and $M_j(r)$ by ε and M_j, respectively. Hence, the sum of spectral flows in (5.11) is the same as

$$\sum_{j=1}^{2} \frac{1}{r^2}\text{-spectral flow of } D_j(u)M_j(r),$$

which by Theorems A and B equals $(1/r^2)$-spectral flow of $D(u)$ on $M(r)$ for $r \geqq R_0$. This proves Theorem C when $\ker \hat{D}(u) = 0$.

6. Proof of Theorem C in the General Case ($\ker \hat{D}(u)$ Varying)

The proof of the general case of Theorem C is carried out by a reduction to the situation treated above where $\ker \hat{D}(u) = 0$. We employ an explicit deformation of pseudodifferential operators that takes $\hat{D}$ to an operator with a vanishing kernel.

Let the operators D and $\hat{D}$ be fixed and given as in (1.2), and let $K \geqq 0$ be a constant chosen as before. Let L be a Lagrangian subspace in $\mathcal{H}(K)$, $\mathcal{H}(K) = \text{Span}\{\phi \mid \hat{D}\phi = \mu\phi,\ |\mu| < K\}$, and let $\{\phi_1, \dots, \phi_\ell\}$ be an orthonormal basis of L, $2\ell = \dim \mathcal{H}(K)$. Then the orthogonal projection π_L of $\Gamma(\hat{E})$ onto L is given by the kernel

$$\mathcal{K}_1(x,y) = \sum_{j=1}^{\ell} \phi_j(x) \otimes \phi_j(y)$$

or, in other words,

$$\pi_L(f)(x) = \sum_{j=1}^{\ell} \phi_j(x) \cdot (\phi_j(y), f(y))_\Sigma .$$

Similarly, the orthogonal projection of $\Gamma(\hat{E})$ onto JL has the kernel

$$\mathscr{K}_2(x,y) = \sum J\phi_j(x) \otimes J\phi_j(y).$$

This is because $J = \sigma \cdot (\sigma\sigma^*)^{-1/2}$ is an isometry and $\pi_{J \cdot L} = J \cdot \pi_L \cdot J^{-1}$.

Associated with L is a deformation $\hat{D}(L,s,\varepsilon)$ of the operator $\hat{D}$ defined by

$$\hat{D}(L,s,\varepsilon) = \hat{D} + s\varepsilon\pi_L - s\varepsilon\pi_{J\cdot L}, \quad 0 \leqq s \leqq 1. \tag{6.1}$$

Note that if ϕ is an eigenvector of $\hat{D}$, $\hat{D}\phi = \mu\phi$ with eigenvalue μ, $|\mu| \geqq K$, then ϕ continues to be an eigenvector of $\hat{D}(L,s,\varepsilon)$, $\hat{D}(L,s,\varepsilon)\phi = \mu\phi$, with the same eigenvalue. In other words, restricted to the subspace Span$\{\phi \mid \hat{D}\phi = \mu\phi,\ |\mu| \geqq K\}$, we have $\hat{D}(L,s,\varepsilon) = \hat{D}$. On the other hand, if ϕ_j is a linear combination $\sum_{\ell=1}^n a_\ell \phi_{j\ell}$ of eigenvectors $\phi_{j\ell}$, $\hat{D}\phi_{j\ell} = \mu_\ell\phi_{j\ell}$ with eigenvalues μ_ℓ in the range $|\mu_\ell| < K$, then

$$\hat{D}(L,s,\varepsilon)\phi_j = \sum_{\ell=1}^n a_\ell(\mu_\ell + \varepsilon s)\phi_{j\ell}$$
$$\hat{D}(L,s,\varepsilon)J\phi_j = \sum_{\ell=1}^n a_\ell(\mu_\ell - \varepsilon s)J\phi_{j\ell}$$

In particular, $\ker \hat{D}(L,1,K) = 0$, and so $\hat{D}(L,s,K)$ gives a deformation from $\hat{D}$ to $\hat{D}(L,1,K)$ with zero kernel. Note that $\hat{D}(L,1,K)$ is an elliptic, pseudodifferential operator with the same symbol as $\hat{D}$.

In a similar manner, we define a deformation $D(L,s,\varepsilon)$ of D. To simplify the discussion, we may assume that the imbedding $\Sigma \times [-1,1] \mapsto M$ extends to $\Sigma \times [-2,2] \mapsto M$ and $\Sigma \times [-r-1,\ r+1] \mapsto M(r)$ extends to $\Sigma \times [-r-2,\ r+2] \mapsto M(r)$. Let $h : M(r) \to \mathbb{R}$ be a smooth cutoff function such that

$$\begin{aligned} &h \mid M(r) - \Sigma \times [-r-1,\ r+1] \equiv 0 \\ &h(x,t) = \begin{cases} 1 & \text{for } (x,t) \text{ in } \Sigma \times [-r-1,\ r+1] \\ f(t+r+1) & \text{for } (x,t) \text{ in } \Sigma \times [-r-2,\ -r-1] \\ f(1-[t-r-1]) & \text{for } (x,t) \text{ in } \Sigma \times [r+1,\ r+2], \end{cases} \end{aligned} \tag{6.2}$$

where $f : [0,1] \to \mathbb{R}$ is a smooth, increasing function with $f'(t) \geqq 0$, $|f'(t)| \leqq 3$, $f \mid [0,\frac{1}{4}] \equiv 0$, $f \mid [3/4,1] \equiv 1$. Let $\mathscr{K}$ denote the finite kernel on $M(r)$ given by

$$\begin{aligned} &\mathscr{K} \mid M(r) - \Sigma \times [-r-1,\ r+1] \equiv 0 \\ &\mathscr{K} \mid \Sigma \times [-r-1,\ r+1] = h \cdot \pi^*(\mathscr{K}_1 - \mathscr{K}_2). \end{aligned} \tag{6.3}$$

Then the deformation $D(L,s,\varepsilon)$ is defined on $L^2(M(r),E)$ by the formula

$$D(L,s,\varepsilon) = D + s\varepsilon\mathscr{K}. \tag{6.4}$$

It is apparent that $D(L, s, t)$ is a self-adjoint, elliptic, pseudodifferential operator of first order and so enjoys all the standard properties of elliptic analysis. In addition, over the cylinder $\Sigma \times [-r, r]$

$$D(L, s, t) \mid \Sigma \times [-r, r] = \sigma \left(\frac{\partial}{\partial s} + \pi^* \cdot \hat{D}(L, s, \varepsilon) \right) \tag{6.5}$$

and so it is a deformation of D to $D(L, 1, K)$ with $\ker \hat{D}(L, 1, K) = 0$ via operators of Atiyah-Patodi-Singer type.

In the following, we will fix the cutoff function h throughout, and so $\hat{D}(L, s, \varepsilon)$ and $D(L, s, t)$ depend solely on L. They will be referred to as the *deformation associated with L*. If $L(u)$ varies smoothly in $\mathscr{H}(u, K)$, then we obtain accordingly a two-parameter family of operators $\hat{D}(L(u), s, \varepsilon), D(L(u), s, \varepsilon)$. As an application of these deformations, we give a proof of Theorem C under the following assumption:

(6.6a) Suppose, in addition to K being an excluded value for $\hat{D}(u)$, $a_i \leqq u \leqq a_{i+1}$, we have an excluded value K' for $\hat{D}(u)$, $a_i \leqq u \leqq a_{i+1}$, with $K' \geqq 2K_i$.

(6.6b) $\ker \hat{D}(a_i) = \ker \hat{D}(a_{i+1}) = 0$.

In this case, by the continuity of eigenvalues of $\hat{D}(u)$, we may choose $\delta > 0$ such that

$$\ker \hat{D}(u) = 0 \quad \text{for } a_i \leqq u \leqq a_i + \delta$$
$$\ker \hat{D}(u) = 0 \quad \text{for } a_{i+1} - \delta \leqq u \leqq a_{i+1}\,.$$

Now choose a smooth function $C : [a_i, a_{i+1}] \to \mathbb{R}$ such that $C(u) \geqq 0$, $C \mid [a_i,\ a_i + \delta/2] \equiv C \mid [a_{i+1} - \delta/2,\ a_{i+1}] \equiv 0$, $C \mid [a_i + \delta,\ a_{i+1} - \delta] \equiv 1$. Define a two-parameter family of operators on Σ by

$$\hat{D}(u, s) = \hat{D}(u) + C(u) s K(\mathscr{K}_1 - \mathscr{K}_2)\,.$$

At $s = 0$ we have the original family of operators $\hat{D}(u)$, and at $s = 1$ a new family $\hat{D}(u, 1)$ with zero kernel. Similarly, we have

$$D(u, s) = D(u) + C(u) \cdot sK\{h\pi^*(\mathscr{K}_1 - \mathscr{K}_2)\}\,,$$

which interpolates between $D(u)$ at $s = 0$ and the family $D(u, 1)$ at $s = 1$ with $\ker \hat{D}(u, 1) = 0$. Furthermore, throughout the deformation

$$D(a_i, s) \equiv D(a_i)\,, \quad D(a_{i+1}, s) \equiv D(a_{i+1})$$
and
$$\hat{D}(a_i, s) \equiv \hat{D}(a_i)\,, \quad \hat{D}(a_{i+1}, s) \equiv \hat{D}(a_{i+1})\,.$$

In Theorem C, all the spectral flows and Maslov indices remain unchanged when we replace the constant K by K' and the Lagrangian $\mathscr{L}_j(u)$ by $\mathscr{L}_j(u) \oplus [\mathscr{H}(u; K') \cap P_{\pm}(u; K)]$. Thus it suffices to treat this case. The advantage of using K' is that we have a well-defined elliptic boundary problem for $D(u, s)$ throughout the deformation. First, the spectra of $\hat{D}(u)$ with eigenvalues μ, $|\mu| \geqq K'$, is unchanged by $\hat{D}(u,s)$ and so K' is an excluded value of $\hat{D}(u,s)$, $0 \leqq s \leqq 1$. Second, let $\mathcal{H}(u,s,K')$ denote the symplectic vector space spanned by all the eigenvectors ϕ of $\hat{D}(u,s)$ with eigenvalues μ in the range $|\mu| \leqq K'$,

$$\mathcal{H}(u,s,K') = \mathrm{Span}\{\phi \mid \hat{D}(u,s)\phi = \mu\phi,\ |\mu| \leqq K'\}.$$

Then, even though the operator $\hat{D}(u,s)$ varies and along with it the eigenvalue μ for $\phi \in H(u,s)$, it keeps the symplectic space $\mathcal{H}(u,s,K')$ invariant. In particular, we can use the family of Lagrangians $\mathcal{L}_j(u) \oplus [\mathcal{H}(u;K') \cap P_\pm(u;K')]$ in $\mathcal{H}(u,s,K')$ to define the operators $D_j(u,s,\mathcal{L}_j(u))$ as in (1.14). Since these deformations $D_j(u,s,\mathcal{L}_j(u))$ and $D(u,s)(M(r))$ do not change the operators at $u = a_i$ or a_{i+1}, the spectral flows and Maslov indices are unchanged. In particular, in the proof of Theorem C, we can replace the families $D(u)(M(r)), D_j(u,\mathcal{L}_j(u))$ by their counterpart $D(u,1)(M(r)), D_j(u,1;L_j(u))$ at $s = 1$.

The advantage of using the family $D(u,1)$ is that $\ker \hat{D}(u,1) = 0$ for all $u \in [a_i, a_{i+1}]$. To be sure, $D(u,1)$ are not differential operators but belong to the larger class of pseudodifferential operators to which the analysis of Atiyah-Patodi-Singer extends easily. Therefore the argument of Part I remains valid, and the proof of the special case of Theorem C in Section 5 can be applied to $D(u,1)$ word for word. This proves Theorem C under the assumption (6.6).

Next we eliminate the first part of the assumption (6.6), thus proving Theorem C with only the assumption (6.6b). For each point $u_0 \in [a_i, a_{i+1}]$ we may choose a number $K'(u_0)$ that is an excluded value of $\hat{D}(u_0)$ with $K'(u_0) > 2K$. By the continuity of eigenvalues of $\hat{D}(u)$, this number $K'(u_0)$ is also an excluded value of $\hat{D}(u)$ for u lying in a small neighborhood of u_0. Since the interval $[a_i, a_{i+1}]$ is compact, there exists a subdivision $a_i = b_0 < b_1 < b_2 < \cdots < b_n = a_{i+1}$ of $[a_i, a_{i+1}]$ and numbers $K'(j) > 2K$ such that $K'(j)$ is an excluded value of $\hat{D}(u)$ for u in $[b_j, b_{j+1}]$. In other words, condition (6.6a) is satisfied for all the subintervals $[b_i, b_{i+1}]$.

Keeping everything fixed at a_i, a_{i+1}, we may smoothly deform the path of Lagrangians $\mathcal{L}_1(u)$ and $\mathcal{L}_2(u)$ to a new path that satisfies the $n+1$ constraint conditions

$$\mathcal{L}_1(u) = L_1(u) \oplus [P_+(u) \cap \mathcal{H}(u;K)]$$
$$\mathcal{L}_2(u) = L_2(u) \oplus [P_-(u) \cap \mathcal{H}(u;K)]$$

at each endpoint $u = b_0, \ldots, b_n$. Since this deformation leaves unchanged the quantities in the desired equation of Theorem C, we can replace the old path by the new one satisfying these additional constraints. Suppose we have $\ker \hat{D}(u) = 0$ for $u = b_0, \ldots, b_n$. Then by the previous argument Theorem C holds for each of the families $D(u) \mid [b_j, b_{j+1}]$, $j = 0, \ldots, n-1$, and by the additivity of the spectral flows and Maslov indices the assertion holds over the entire interval $[a_i, a_{i+1}]$.

Because of the assumption (6.6b), the failure of $\ker \hat{D}(u) = 0$ can occur only at some of the interior points b_j; then we subdivide the intervals $[b_{j-1}, b_j], [b_j, b_{j+1}]$ into four pieces $[b_{j-1},\ b_j - \varepsilon]$, $[b_j - \varepsilon, b_j]$, $[b_j, b_j + \varepsilon]$, and $[b_j + \varepsilon, b_{j+1}]$. There are rescaling diffeomorphisms $\rho : [b_{j-1}, b_j - \varepsilon] \to [b_{j-1}, b_j]$ and $\rho : [b_j + \varepsilon, b_{j+1}] \to [b_j, b_{j+1}]$ between the intervals, and via these diffeomorphisms we can introduce families of operators $D(\rho(u))$ over the intervals $[b_{j-1}, b_j - \varepsilon], [b_j + \varepsilon, b_j]$, which are basically the same operators as before. Now choose a Lagrangian L in the symplectic vector space $\mathscr{H}(b_j; K(j)')$. Then, as discussed before, there is a one-parameter family of operators $D(L, s)$ that brings $D(L, 0) = D(b_j)$ to the operator $D(L, 1)$ with $\ker \hat{D}(L, 1) = 0$. By letting $s = (u - b_j + \varepsilon)/\varepsilon$, $b_j - \varepsilon \leqq u \leqq b_j$, we obtain the deformation $D(L, (u - b_j + \varepsilon)/\varepsilon)$ as defined over $[b_j - \varepsilon, b_j]$. On the other hand, over $[b_j, b_j + \varepsilon]$ we reverse the deformation by considering $D(L, (-u + b_j + \varepsilon)/\varepsilon)$, $b_j \leqq u \leqq b_j + \varepsilon$, which brings $D(L, 1)$ back to $D(L, 0) = D(b_j)$. Repeating this process at all the interior points b_j, $j = 1, \ldots, n$, we obtain a new family of operators connecting $D(a_i)$ to $D(a_{i+1})$. Since the deformations over $[b_i - \varepsilon, b_i]$ and $[b_i, b_i + \varepsilon]$ are the reverse of each other, the spectral flows and Maslov indices introduced by them cancel out. As a result, all the quantities in the equation of Theorem C are the same for the new family. However, by introducing $D(L, 1)$ at b_j we have made sure that the condition (6.6b) is valid at these points. Hence, by the argument in the previous paragraph, Theorem C holds whenever we have the vanishing condition (6.6b) at the endpoints a_i, a_{i+1}.

To complete the proof of Theorem C, it remains only to address the situation when either $\ker \hat{D}(a_i)$ or $\ker \hat{D}(a_{i+1})$ is not zero. In fact, by subdividing $[a_i, a_{i+1}]$ and using the additivity property of both spectral flow and the Maslov index as before, it is enough to treat the situation in which only one of them is nonzero. Again by the reversal property of spectral flow and the Maslov index, we might as well assume that $\ker \hat{D}(a_i) \neq 0$ and $\ker \hat{D}(a_{i+1}) = 0$.

In addition to the above observations, the homotopy invariance and additivity properties of the spectral flows and Maslov indices allow us to deform the family of operators $D(u)$ and Lagrangian pairs $(\mathscr{L}_1(u), \mathscr{L}_2(u))$ into any prescribed position. In other words, it suffices to show that, given fixed operators $(D, \hat{D})$ as in (1.2) with $\ker \hat{D} \neq 0$, there exists a *single* family $(D(u), \hat{D}(u))$ of operators and Lagrangians $(\mathscr{L}_1(u), \mathscr{L}_2(u))$ such that Theorem C holds. To describe this family we choose $K > 0$ so that $\lambda = 0$ is the only eigenvalue λ of $\hat{D}$ with $|\lambda| \leqq K$. With respect to the symplectic space $\mathscr{H} = \ker \hat{D}$ and Lagrangian subspaces L_1 and L_2, we may choose a symplectic basis $\{\phi_j, \sigma\phi_j \mid 1 \leqq j \leqq n = \dim \mathscr{H}/2\}$ of $\mathscr{H}$ such that

(6.7)
(a) $\{\phi_j \mid 1 \leqq j \leqq a = \dim(L_1 \cap L_2)\}$ is a basis of $L_1 \cap L_2$
(b) $L_1 = \oplus\mathbb{R}\{\phi_j : 1 \leqq j \leqq n\}$
(c) $L_2 = \oplus\mathbb{R}\{\phi_j : 1 \leqq j \leqq a\} \oplus \mathbb{R}\{e^{-\sigma\alpha_j}\phi_j : a + 1 \leqq j \leqq n\}$

(see [7]). Here $e^{-\sigma\alpha_j}\phi_j$ stands for a counterclockwise rotation of ϕ_j by an angle α_j, $0 < \alpha_j < \pi$, and $-\sigma$ gives the complex structure. Take new angles β_j with $0 < \beta_j < \alpha_j$, $a + 1 \leqq j \leqq n$, and consider the Lagrangian

$$L = \oplus\mathbb{R}\{\phi_j : +1 \leqq j \leqq a\} \oplus \mathbb{R}\{e^{-\sigma\beta_j}\phi_j : a + 1 \leqq j \leqq n\}$$

in $\mathscr{H}$, which satisfies

(6.8)
(a) $L_1 \cap L = L_2 \cap L = L_1 \cap L_2$
(b) $L_1 \cap \sigma L = L_2 \cap \sigma L = \{0\}$.

As before, we have a one-parameter family of operators $(D(L,u),\hat{D}(L,u))$, $0 \leqq u \leqq 1$, associated with L. Define the operators $D(u)^*$ and $\hat{D}(u)^*$ by

$$D(u)^* = \begin{cases} D(L,u) & \text{for } 0 \leqq u \leqq \varepsilon \\ D(L,\varepsilon) & \text{for } \varepsilon \leqq u \leqq 1 \end{cases}$$

and

$$\hat{D}(u)^* = \begin{cases} \hat{D}(L,u) & \text{for } 0 \leqq u \leqq \varepsilon \\ \hat{D}(L,\varepsilon) & \text{for } \varepsilon \leqq u \leqq 1 . \end{cases}$$

PROPOSITION 6.1. *There exists* ε, $0 < \varepsilon < 1$, *and a continuous family of Lagrangian pairs* $(\mathscr{L}_1(u),\mathscr{L}_2(u)) : 0 \leqq u \leqq 1$ *in* $\mathscr{H}(u;K)$ *satisfying* (1.13) *so that Theorem* C *holds for the family of operators* $D(u)^*$ *and Lagrangian pairs* $(\mathscr{L}_1(u),\mathscr{L}_2(u))$.

We first take $\varepsilon < 1/2$ so that $\hat{D}(u)^*$ has no eigenvalue $\pm K$. Indeed, the restriction $\hat{D}(L,u) \mid L = Ku(Id)$ and $\hat{D}(L,u) \mid \sigma L = -Ku(Id)$. Hence $\hat{D}(L,u)$ has eigenvalues $\pm Ku$ in the band $[-K,K]$ and $\mathscr{H}(u;K) = \oplus\mathbb{R}\{\phi_j \mid 1 \leqq j \leqq n\}$.

Given a section Ψ of $E \to M(r)$ such that $D\Psi = 0$ on $M_1 \cup \Sigma \times [-r,r]$, we have the eigenexpansion:

$$\Psi \mid \Sigma \times [-r-2, r+2] = \pi^* \left(\sum a_j\phi_j + b_j\sigma\phi_j\right) + \tilde{\Psi} . \tag{6.9}$$

Here a_j and b_j are constants, $D\tilde{\Psi} = 0$, and $\tilde{\Psi} \mid \Sigma \times t$ has an expansion only involving eigensections ϕ_ℓ of $\hat{D}$, $\hat{D}\phi_\ell = \lambda_\ell\phi_\ell$, with $|\lambda_\ell| > K$. Recall that

$$D(L,u) = \begin{cases} D & \text{on } M_1(r) - \Sigma \times [-r-2,\ r+2] \\ \sigma\left(\frac{\partial}{\partial s} + \pi^*\hat{D}\right) + uKh_r(s)(\pi_L - \pi_{\sigma L}) & \text{on } \Sigma \times [-r-2,\ r] \end{cases}$$

with $h_r(s) = 0$ off $\Sigma \times [-r-3/2,\ r+3/2]$, $h_r(s) \equiv 1$ on $\Sigma \times [-r-1,\ r+1]$, $|h_r'(s)| \leqq 4$, and $h_r(s) \geqq 0$. It is straightforward to verify that if we define $\Psi(u)$ by

$$\psi(u) = \begin{cases} \tilde{\Psi} + \exp\left[-uK\int_{-r-2}^{s} h_r(t)\,dt\right]\pi_L\hat{\Psi} + \exp\left[uK\int_{-r-2}^{s} h_r(t)\,dt\right]\pi_{\sigma L}\hat{\Psi} \\ \qquad \text{on } \Sigma \times [-r-2,\ r+2] \\ \Psi \quad \text{on } M_1 , \end{cases}$$

then it satisfies $D(u)^*\Psi(u) = 0$ on $M_1 \cup \Sigma \times [-r-2,\ r+2]$. In a similar manner, we can alter the section Ψ on $M_2 \cup \Sigma \times [-r-2,\ r+2]$ so that it satisfies the equation $D(u)^*\psi(u) = 0$ there. Thus we have an explicit description of the Lagrangians $L_1(u,K)$ and $L_2(u,K)$:

$$\begin{aligned} L_1(u,K) &= \{\exp[-uKA]\pi_L \oplus \exp[uKA]\pi_{\sigma L}\}L_1 \\ L_2(u,K) &= \{\exp[uKA]\pi_L \oplus \exp[-uKA]\pi_{\sigma L}\}\, L_2 \end{aligned} \tag{6.10}$$

where $A = \int_{r+1}^{r+2} h_r(t)\,dt = \int_{-r-1}^{-r-2} h_r(t)\,dt$.

Note that by (6.8a) we have

$$L_1 \cap L_1(u,K) = L_2 \cap L_2(u,K) = L_1 \cap L_2 . \tag{6.11}$$

Since, at $u = 0$, $L_1(u,K) = L_1$ and $L_2(u,K) = L_2$, we can choose $\varepsilon > 0$ sufficiently small so that

$$L_1(u,K) \cap L_2(u,K) = L_1 \cap L_2 \tag{6.12}$$

for $0 \leqq u \leqq \varepsilon$. With ε chosen in this manner, we define $(\mathscr{L}_1(u), \mathscr{L}_2(u))$ to be

$$\begin{aligned} \mathscr{L}_1(u) &= L_1(u,K) \quad 0 \leqq u \leqq \varepsilon \\ \mathscr{L}_2(u) &= L_2(u,K) \quad 0 \leqq u \leqq \varepsilon , \end{aligned} \tag{6.13}$$

and for $\varepsilon \leqq u \leqq 1$ we define $(\mathscr{L}_1(u), \mathscr{L}_2(u))$ to be any smooth Lagrangians in $\mathscr{H}$ interpolating between $(L_1(\varepsilon,K), L_2(\varepsilon,K))$ at $u = \varepsilon$ and $(L,\mathscr{H})$ at $u = 1$. With respect to $D(u)^*$, the above choice of Lagrangians $(\mathscr{L}_1(u), \mathscr{L}_2(u))$ also satisfies the boundary condition (1.13), and so we are in a position to test Theorem C for this special example.

First we concentrate on the right-hand side of the equation:

$$\begin{aligned} &[\tfrac{1}{r^2}\text{-spectral flow of } D(u)^*(M(r)) : 0 \leqq u \leqq 1] \\ &\quad = \sum_{j=1}^{2} [\varepsilon'\text{-spectral flow of } D_j(u, \mathscr{L}_j(u)) : 0 \leqq u \leqq 1 \text{ on } M_j(r)] \\ &\qquad + \operatorname{Mas}[(\mathscr{L}_1(u), \mathscr{L}_2(u)) : 0 \leqq u \leqq 1] \\ &\qquad + \frac{1}{2}[\dim\ker \hat{D}(1)^* - \dim\ker \hat{D}(0)^*] . \end{aligned} \tag{6.14}$$

The spectral flows of the operators $\{D_j(u, \mathscr{L}_j(u)) : 0 \leqq u \leqq \varepsilon \text{ on } M_j(r)\}$ are clearly 0 since the zero-eigenspaces of these operators are of constant dimension and vary smoothly: Outside the band $|\lambda| > K$, they coincide with the old L^2-solutions while inside the band they belong to $L_j(u,K)$ and $\dim L_j(u,K) = \dim L_j$.

Now over the interval $\varepsilon \leqq u \leqq 1$ we apply Theorem D to study the spectral flows of $\{D_j(u, \mathscr{L}_j(u)) : \varepsilon \leqq u \leqq 1 \text{ on } M_j(r)\}$. Strictly speaking, a different choice of the constant $K > 0$ is required in order to satisfy (1.15). However, for the problem at hand we can always choose $K' > K$, enlarge the symplectic space from $\mathscr{H}(u;K)$ to $\mathscr{H}(u;K')$ and the Lagrangian $\mathscr{L}_j(u,K)$ to $\mathscr{L}_j(u,K) \oplus P_\pm(u;K) \cap \mathscr{H}(u;K')$. Since $D(u,K)^* \equiv D$ for $\varepsilon \leqq u \leqq 1$, the isotropy subspaces $P_\pm(u,K) \cap \mathscr{H}(u,K')$ are fixed subspaces and so do not enter into the corresponding calculation of Maslov indices. Thus, we have by Section 5 and Theorem C

$$\begin{aligned} &[(+\varepsilon')\text{-spectral flow of } D_j(u, \mathscr{L}_j(u)) : \varepsilon \leqq u \leqq 1] \\ &\quad = \begin{cases} \operatorname{Mas}[(L_1(\varepsilon,K), \mathscr{L}_1(u)) : \varepsilon \leqq u \leqq 1] & \text{for } j = 1 \\ \operatorname{Mas}[(\mathscr{L}_2(u), L_2(\varepsilon,K)) : \varepsilon \leqq u \leqq 1] & \text{for } j = 2 \end{cases} \end{aligned}$$

by (6.13) the intersection $\mathscr{L}_1(u) \cap \mathscr{L}_2(u)$ is constant for $0 \leqq u \leqq \varepsilon$, and so

$$\operatorname{Mas}[(\mathscr{L}_1(u), \mathscr{L}_2(u)) : 0 \leqq u \leqq \varepsilon] = 0.$$

Thus by path additivity of the Maslov index, the total contribution to the right-hand side of (6.14) is given by

$$\begin{aligned} &\operatorname{Mas}\{(L_1(\varepsilon, K), \mathscr{L}_1(u)) : \varepsilon \leqq u \leqq 1\} \\ (6.15) \qquad &+ \operatorname{Mas}\{(\mathscr{L}_2(u), L_2(\varepsilon, K)) : \varepsilon \leqq u \leqq 1\} \\ &+ \operatorname{Mas}\{(\mathscr{L}_1(u), \mathscr{L}_2(u)) : \varepsilon \leqq u \leqq 1\} - n. \end{aligned}$$

To simplify our discussion, we may choose $\mathscr{L}_j(u)$, $0 \leqq u \leqq \varepsilon$, to be of the following form:

$$\begin{aligned} \mathscr{L}_2(u) &= \oplus \mathbb{R}\{e^{-\sigma f_j(u)} \phi_j(u) : 1 \leqq j \leqq n\} \\ \mathscr{L}_1(u) &= \oplus \mathbb{R}\{e^{-\sigma g_j(u)} \phi_j(u) : 1 \leqq j \leqq n\}. \end{aligned}$$

Here $f_j(u)$ and $g_j(u)$ are smooth functions, chosen so that $L_1(1) = L_1$, $\mathscr{L}_2(1) = L_2$, $\mathscr{L}_1(\varepsilon) = L_1(u, K)$, and $\mathscr{L}_2(\varepsilon) = L_2(u, K)$. Since $L_1(\varepsilon, K)$, $L_2(\varepsilon, K)$, $\mathscr{L}_1(u)$, and $\mathscr{L}_2(u)$ can be decomposed into compatible sums of Lagrangians, we can break down the calculation of (6.15) into cases where the Lagrangians lie inside the symplectic space $\mathbb{R}\phi_j \oplus \mathbb{R}e^{-\sigma}\phi_j$, $j = 1, \ldots, n$, and then sum up the answers afterwards. In the case where $1 \leqq j \leqq a$, we take $\mathscr{L}_1(u) = \mathbb{R}\phi_j$ and $\mathscr{L}_2(u) = \mathbb{R}e^{-\sigma(\frac{u-\varepsilon}{1-\varepsilon})\pi/2}\phi_j$.

From the properties of the Maslov index described in Section 4, we get

$$\begin{aligned} &\operatorname{Mas}\{(L_1(\varepsilon, K), \mathscr{L}_1(u)) : \varepsilon \leqq u \leqq 1\} = \operatorname{Mas}\{\mathbb{R}\phi_j, \mathbb{R}\phi_j\} = 0 \\ &\operatorname{Mas}\{(\mathscr{L}_2(u), L_2(\varepsilon, K)) : \varepsilon \leqq u \leqq 1\} \\ &\quad = \operatorname{Mas}\{\mathbb{R}e^{-\sigma(\frac{u-\varepsilon}{1-\varepsilon})\pi/2}\phi_j,\ \mathbb{R}\phi_j : \varepsilon \leqq u \leqq 1\} = 0 \\ &\operatorname{Mas}\{(\mathscr{L}_1(u), \mathscr{L}_2(u) : \varepsilon \leqq u \leqq 1\} = \operatorname{Mas}\{\mathbb{R}\phi_j, \mathbb{R}e^{-\sigma(\frac{u-\varepsilon}{1-\varepsilon})\pi/2}\phi_j : \varepsilon \leqq u \leqq 1\} = 1 \end{aligned}$$

and so the contribution to (6.15) is $0 + 0 + 1 - 1 = 0$.

The situation for $a + 1 \leqq j \leqq n$ is more complicated and can be explained in terms of Figure 6.1 below:

To begin with, we have the Lagrangian lines $L_1 = \mathbb{R}\phi_j$ and $L_2 = \mathbb{R}e^{-\sigma\alpha_j}\phi_j$ in the symplectic plane $\mathbb{R}\phi_j \oplus \mathbb{R}(\sigma\phi_j)$. Since $L = \mathbb{R}e^{-\sigma\beta_j}\phi_j$ with $0 < \beta_j < \alpha_j$, it can be represented by a line in the interior angle of L_1, L_2, while $(-\sigma)L$ lies in the exterior. According to (6.10), $L_1(\varepsilon, K)$ is obtained from L_1 by applying a symplectic automorphism that contracts in the L-direction and expands in the $(-\sigma)L$-direction. Thus $L_1(\varepsilon, K)$ lies between L_1 and σL, and similarly $L_2(\varepsilon, K)$, obtained from the inverse symplectic automorphism, lies between L_2 and L. As for $\mathscr{L}_1(u)$ and $\mathscr{L}(u)$, they are represented by arrowed lines that can be thought of as two motions of Lagrangian lines from $L_1(\varepsilon, K)$ to L and from $L_2(\varepsilon, K)$ to σL.

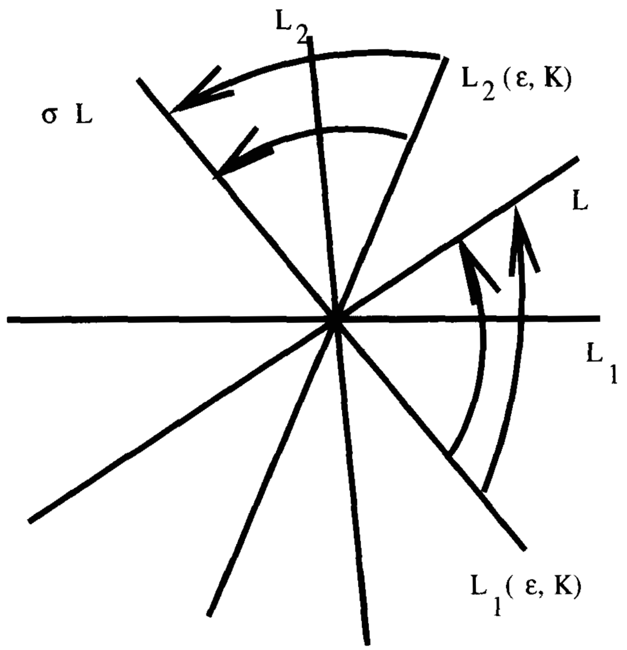

Figure 6.1

From the above diagram, it is not difficult to see that

$$\begin{aligned}
&\operatorname{Mas}\{(\mathscr{L}_1(u), \mathscr{L}_2(u)) : \varepsilon \leqq u \leqq 1\} = 0 \\
&\operatorname{Mas}\{(L_1(\varepsilon, K), \mathscr{L}_1(u)) : \varepsilon \leqq u \leqq 1\} = 1 \\
&\operatorname{Mas}\{(\mathscr{L}_2(u), L_2(\varepsilon, K)) : \varepsilon \leqq u \leqq 1\} = 0,
\end{aligned}$$

and hence they again contribute to $0 + 1 + 0 - 1 = 0$ in (6.15). As we go through $j = 1, \ldots n$, the right-hand side of (6.14) equals 0.

Next we examine the left-hand side of (6.14). As the operators $D(u)^* = D(L, \varepsilon)$ stay constant for $\varepsilon \leqq u \leqq 1$, the corresponding spectral flow over the interval $[\varepsilon, 1]$ is 0, and so

$$\begin{aligned}
&\left\{\frac{1}{r^2}\text{-spectral flow of } D(u)^*(M(r)) : 0 \leqq u \leqq 1\right\} \\
&= \left\{\frac{1}{r^2}\text{-spectral flow of } D(L, u)(M(r)) : 0 \leqq u \leqq \varepsilon\right\}.
\end{aligned}$$

To complete the proof, we have to show this last spectral flow is 0. For this, we appeal to the analysis employed in the proof of Theorem A in Part I. The key step is to reformulate the splicing construction

$$\Phi_r(u) : V_1 \oplus V_2 \oplus (L_1 \cap L_2) \to \Gamma(E(r))$$

so that it works for the family of operators $D(L,u)$, $0 \leqq u \leqq \varepsilon$, and satisfies the inequality of Lemma 4.1 of Part I with $\delta = K$ and for all $u \in [0,\varepsilon]$.

From the argument in Part I, it follows that for $r \geqq R_0$ the zero-modes can be computed in terms of the image of $\Phi_r(u)$. Since $V_1 \oplus V_2 \oplus (L_1 \cap L_2)$ stays constant, we obtain the desired vanishing property for the corresponding spectral flow.

Given an element $(\alpha,\beta,\gamma) \in (V_1 \oplus V_2 \oplus (L_1 \cap L_2))$ we have by definition a matching pair (Ψ_1,Ψ_2) of solutions $D\Psi_j = 0$ on $M_j(\infty)$ with common value $\hat{\phi}\varepsilon \ker \hat{D}$. Over $\Sigma \times [0,\infty)$ and over $\Sigma \times (-\infty, 0]$, respectively, we write Ψ_j as the sum $\pi^*\hat{\phi} + \hat{\Psi}_j$, $j = 1,2$, where $\hat{\Psi}_j$ is an L_2-solution of $D\hat{\Psi}_j = 0$ and $\hat{\Psi}_j \mid \Sigma \times t$ has an eigenexpansion with eigenvalues $\lambda > K$ for $j = 1$ and $\lambda < K$ for $j = 2$. Now define a section $\Psi_1(u)$ over $M_1 \cup \Sigma \times [-r-2,\ r+2]$ by letting $\Psi_1(u) = \Psi_1$ on M_1 and

$$\Psi_1(u)(x,s) = \hat{\Psi}_1(x,s) + \left\{ \begin{array}{c} \exp\left[-Ku\int_{-r-2}^{s} h_r(t)\,dt\right]\pi_L \\ \oplus \exp\left[Ku\int_{-r-2}^{s} h_r(t)\,dt\right]\pi_{\sigma L} \end{array} \right\} \pi^*\hat{\phi}.$$

It is easy to see that $\Psi_1(u)$ is a solution of $D(u,K)\Psi_1(u) = 0$ on $M_1 \cup \Sigma \times [-r-2,\ r+2]$. Note that if we let $B_r = \int_{-r+2}^{r+2} h_r(t)\,dt$, then for $r + 3/2 \leqq s \leqq r+2$

$$\Psi_1(u)(x,s) = \hat{\Psi}_1(x,s) + \left\{ \begin{array}{c} \exp[-B_rKu]\pi_L \\ \oplus \exp[B_rKu]\pi_{\sigma L} \end{array} \right\} \pi^*\hat{\phi}.$$

Since by assumption $L \cap L_1 = L_1 \cap L_2$, we have $\hat{\phi}$ in $L_1 \cap L_2$, and so

$$\left\{ \begin{array}{c} \exp(-B_rKu)\pi_L \\ \oplus \exp(B_rKu)\pi_{\sigma L} \end{array} \right\} \pi^*\hat{\phi} = \exp(-B_rKu)\pi^*\hat{\phi}.$$

Thus, we obtain a matching solution on $\Sigma \times [r+3/2,\ r+2]$ given by $\Psi_1(u)$, $\exp[-B_rKu]\Psi_2$, in the range $-K$ to $+K$. As in Part I, we can splice these two sections together to define $\Phi_r(u)(\alpha,\beta,\gamma)$, replacing $\hat{\Psi}_2$ by $\exp[-B_rKu]\hat{\Psi}_2$. It is not difficult to see that all the estimates in Part I work for this variant of the splicing construction as well.

More explicitly, we may form a spliced solution $\Phi_{r,u}(\Psi_1,\Psi_2)$ over $M(r)$ as follows:

$$\Phi_{r,u}(\Psi_1,\Psi_2) \mid (M_1 - \Sigma \times [-r-2,\ -r]) = \Psi_1$$
$$\Phi_{r,u}(\Psi_1,\Psi_2) \mid (M_2 - \Sigma \times [r,\ r+2]) = \exp(-B_rKu)\Psi_2$$
$$\Phi_{r,u}(\Psi_1,\Psi_2) \mid \Sigma \times [-r-2,\ r+2]) = A + B$$

where

$$A = \exp\left(-Ku\int_{-r-2}^{s} h_r(t)\,dt\right)\pi^*\hat{\phi}$$

and B equals the result of splicing $\hat{\Psi}_1(x,s)$ with $\exp(-B_rKu)\hat{\Psi}_2(x,s)$ as in Part I (2.18). There we regard $\hat{\Psi}_1$ as in $M_1(\infty) = M_1 \cup \Sigma \times [0,\infty)$; here $\hat{\Psi}_1$ is regarded as over $M_1(r) = M_1 \cup \Sigma \times [-r,+r] \subset M_1(\infty)$, and similarly for $\exp(-B_rKu)\hat{\Psi}_2$.

The essential point of this construction is that $D(u,K)\Psi_{r,u}(\Psi_1,\Psi_2)$ vanishes outside $\Sigma \times [-1,+1]$; inside $\Sigma \times [-1,+1]$ it equals $\sigma(\frac{\partial}{\partial r} + \pi^* s\hat{D})B$. B involves only the eigenvectors of $\hat{D}$ with value λ satisfying $|\lambda| \geqq K$; we immediately get (using $\delta = K$) the inequality

$$\text{(6.16)} \qquad \|\mathbf{P}_r\mathbf{\Phi}_{r,u}(\alpha,\beta) - \mathbf{\Phi}_{r,u}(\alpha,\beta)\| \leqq \exp\left(-\frac{\delta r}{4}\right)\|\Phi_{r,u}(\alpha,\beta)\|$$

for matching solutions (α,β). This becomes merely a restatement of Section 4 of Part I.

Now the inequality (6.16) proves that in the range $|\lambda| \leqq K/2$ and for some R (independent of u), there are at least $a + L' + R'$ eigenvalues λ of $D(u,K)$ on $M(r)$ with $|\lambda| \leqq \exp(-\frac{\delta r}{4})$ for all $r \geqq R$. Indeed, the projections of the spliced solution provide these solutions. Here L' and R' are the numbers of L^2-solutions of D on $M_1(\infty)$ and $M_2(\infty)$, respectively.

In order to complete the analysis, we must now reread all of the proof of Theorem A of Part I in this new context. The arguments proceed with minor changes. As a result, we may conclude that there is an R (*independent* of u) such that the number of eigensolutions of $D(u,K)$ on $M(r)$ for $r \geqq R$ in the range $|\lambda| < 1/r^2$ is precisely $a + L' + R'$. We leave this to the careful reader.

The conclusion of this analysis is that for $r \geqq R$, the $(1/r^2)$-spectral flow of $D(u,K) : 0 \leqq u \leqq \varepsilon$ is 0 as claimed. This completes the verification of Proposition 6.1.

7. Averaged Spectral Flow and Averaged Maslov Index

In Theorem C, the $(+1/r^2)$-spectral flow of a family of operators $\{D(u) : a \leqq u \leqq b\}$ in the situation of a manifold decomposition $M = M_1 \cup M_2$ is expressed as a sum of two spectral flows $D_j(u, \mathscr{L}_j(u))$ on M_j, a Maslov index $\mathrm{Mas}\{(L_1(u), L_2(u)) : a \leqq u \leqq b\}$, and a dimension correction term $\frac{1}{2}[\dim\ker\hat{D}(b) - \dim\ker\hat{D}(a)]$. In this section, we give a different formulation of Theorem C that has the advantage of eliminating the last correction term. To accomplish this, we introduce the *averaged spectral flow* and the *averaged Maslov index*.

As in Section 3, we consider a one-parameter family of real, self-adjoint operators $\{D(u) : a \leqq u \leqq b\}$ such that for some fixed $\delta > 0$ the spectrum of $D(u)$ in the range of eigenvalues λ with $|\lambda| < \delta$ is finite-dimensional and has no essential spectrum. Choose $\varepsilon > 0$ so that $\varepsilon < \delta$ and, for the operators $D(a)$ and $D(b)$ at the two ends, there are no eigenvalues in the range $[-\varepsilon, \varepsilon]$ except the zero eigenvalue. Define the averaged spectral flow by taking the average of the $(+\varepsilon)$-spectral flow and the $(-\varepsilon)$-spectral flow; that is,

$$\begin{aligned}(7.1)\qquad &\text{Averaged spectral flow of } \{D(u) : a \leqq u \leqq b\}\\ &= \frac{1}{2}\Big[(+\varepsilon)\text{-spectral flow of } \{D(u) : a \leqq u \leqq b\}\\ &\quad + (-\varepsilon)\text{-spectral flow of } \{D(u) : a \leqq u \leqq b\}\Big]\end{aligned}$$

From the above definition, it is easy to see that the averaged spectral flow of $\{D(u) : a \leqq u \leqq b\}$ enjoys many of the properties of the $(\varepsilon_1, \varepsilon_2)$-spectral flows in Section 3, such as additivity and homotopy invariance. Moreover, it is independent of the choice of ε used in its definition.

In the situation where $\{D(u) : a \leqq u \leqq b\}$ is a smooth family of first-order elliptic operators on M, we have the operator

$$\tilde{D} = \pi^* \left(\frac{\partial}{\partial u} + \pi^* D(u)\right)$$

defined over the product manifold $M \times [a, b]$. By imposing the boundary condition as in [2], this operator becomes Fredholm and has a well-defined index, Index $\tilde{D}$. Since $\frac{\partial}{\partial u}$ points inward into $M \times [a, b]$ at $u = a$ and outward at $u = b$, this index can be identified with the $(-\varepsilon, +\varepsilon)$-spectral flow of $\{D(u) : a \leqq u \leqq b\}$. (See [7] for an extensive discussion of this point.) Thus, in this case the Atiyah-Patodi-Singer index theorem [2] asserts the equality:

$$\begin{aligned}(7.2)\qquad &(-\varepsilon, \varepsilon)\text{-spectral flow}\{D(u) : a \leqq u \leqq b\}\\ &= \int_M \mathscr{L}_0(x)\, dx - \frac{1}{2}[-\eta(D(b)) + \eta(D(a))]\\ &\quad - \frac{1}{2}[+h(b) + h(a)]\end{aligned}$$

where $\eta(D(\cdot))$ is the eta invariant of $D(\cdot)$ and $h(\cdot)$ is the dimension $\dim\ker D(\cdot)$ of the zero-modes of $D(\cdot)$ (cf. [7], pp. 160–161).

Note that the $(+\varepsilon)$-spectral flow, $(-\varepsilon)$-spectral flow, and $(-\varepsilon, \varepsilon)$-spectral flow differ from each other by the dimension counts $h(a)$ and $h(b)$ at the two ends. In particular,

$$\begin{aligned}(7.3)\qquad &(-\varepsilon, +\varepsilon)\text{-spectral flow}\{D(u) : a \leqq u \leqq b\}\\ &= -h(a) + [(+\varepsilon)\text{-spectral flow}\{D(u) : a \leqq u \leqq b\}]\end{aligned}$$

$$\begin{aligned}(7.4)\qquad &(-\varepsilon, \varepsilon)\text{-spectral flow}\{D(u) : a \leqq u \leqq b\}\\ &= -h(b) + [(-\varepsilon)\text{-spectral flow}\{D(u) : a \leqq u \leqq b\}]\end{aligned}$$

and after taking the average of (7.3) and (7.4) we have

$$\begin{aligned}&(-\varepsilon,\varepsilon)\text{-spectral flow}\{D(u) : a \leqq u \leqq b\}\\ (7.5)\qquad &= -\frac{1}{2}[h(a)+h(b)] + [\text{averaged spectral flow of } \{D(u) : a \leqq u \leqq b\}]\end{aligned}$$

In view of (7.5), the Atiyah-Patodi-Singer formula (7.2) can be expressed as:

$$\begin{aligned}&\text{Averaged spectral flow of } \{D(u) : a \leqq u \leqq b\}\\ (7.6)\qquad &= \int_M \mathscr{L}_0(x)\,dx + \frac{1}{2}[\eta(D(b)) - \eta(D(a))]\end{aligned}$$

In other words, we can conveniently absorb the dimension correction term by using the averaged spectral flow. We can use these flows to reformulate Theorem C as follows:

THEOREM F. *Let K_i, R_0, $\mathscr{L}$, and $\mathscr{H}(u, K_i)$ be given as in Theorem C. Then for all $r \geq R_0$, the averaged $\frac{1}{r^2}$-spectral flow of $\{D(u)(M(r)) : a_i \leqq u \leqq a_{i+1}\}$ equals*

$$\begin{aligned}(7.7)\qquad &\sum_{j=1}^{2} \text{Averaged spectral flow of } \{D_j(u; \mathscr{L}_j(u)) : a_i \leqq u \leqq a_{i+1}\}\\ &+ \text{averaged Mas}\{(\mathscr{L}_1(u), \mathscr{L}_2(u)) : a_i \leqq u \leqq a_{i+1}\}.\end{aligned}$$

The definition of averaged Maslov index (cf. A-Mas$\{\cdot\}$ in [7]) is an averaged spectral flow; see below.

To simplify our notation in proving Theorem F, we consider the above data K_i, R_0, $\mathscr{L}_j(u)$, and $\mathscr{H}(u, K_i)$ as defined over a single interval $[a, b]$—that is, $a_i = a$, $a_{i+1} = b$—and drop the subscript i throughout the proof.

Proof of Theorem F: According to formula (7.1) in [7], the Maslov index $\text{Mas}\{(\mathscr{L}_1(u), \mathscr{L}_2(u)) : a \leqq u \leqq b\}$ has a description as the $(+\varepsilon)$-spectral flow of a family of operators $\{D(\mathscr{L}_1(u), \mathscr{L}_2(u))\}$. Thus, it is more appropriate to refer to the Maslov index term in Theorem C as the $(+\varepsilon)$-Maslov index; that is, we rewrite it as

$$\begin{aligned}&+\frac{1}{r^2}\text{-spectral flow of } \{D(u)(M(r)) : a \leqq u \leqq b\}\\ (7.8)\qquad &= \sum_{j=1}^{2} (+\varepsilon)\text{-spectral flow of } \{D(u; \mathscr{L}_j(u)) : a \leqq u \leqq b\}\\ &+ (+\varepsilon)\text{-Mas}\{(\mathscr{L}_1(u), \mathscr{L}_2(u)) : a \leqq u \leqq b\}\\ &+ \frac{1}{2}[\dim\ker \hat{D}(b) - \dim\ker \hat{D}(a)]\end{aligned}$$

By reversing the sign or by taking the average, we can define the $(-\varepsilon)$-Maslov index and the averaged Maslov index:

$$\begin{aligned}(-\varepsilon)\text{-Mas}\{(\mathscr{L}_1(u),\mathscr{L}_2(u)) : a \leqq u \leqq b\} \\ = (-\varepsilon)\text{-spectral flow of } \{D(\mathscr{L}_1(u),\mathscr{L}_2(u)) : a \leqq u \leqq b\}\end{aligned} \tag{7.9}$$

$$\begin{aligned}&\text{averaged Mas}\{(\mathscr{L}_1(u),\mathscr{L}_2(u)) : a \leqq u \leqq b\} \\ &\quad = \text{averaged spectral flow of } \{D(\mathscr{L}_1(u),\mathscr{L}_2(u)) : a \leqq u \leqq b\} \\ &\quad = \frac{1}{2}[(+\varepsilon)\text{-Mas}\{(\mathscr{L}_1(u),\mathscr{L}_2(u)) : a \leqq u \leqq b\} \\ &\qquad - (-\varepsilon)\text{-Mas}\{(\mathscr{L}_1(u),\mathscr{L}_2(u)) : a \leqq u \leqq b\}]\end{aligned} \tag{7.10}$$

It is not difficult to see that the $(\pm\varepsilon)$-Maslov indices are related to each other by the dimension of the intersections of Lagrangians, $\mathscr{L}_1(a) \cap \mathscr{L}_2(a)$ and $\mathscr{L}_1(b) \cap \mathscr{L}_2(b)$, at the two ends:

$$\begin{aligned}&(+\varepsilon)\text{-Mas}\{(\mathscr{L}_1(u),\mathscr{L}_2(u)) : a \leqq u \leqq b\} \\ &\quad = (-\varepsilon)\text{-Mas}\{(\mathscr{L}_1(u),\mathscr{L}_2(u)) : a \leqq u \leqq b\} \\ &\qquad + \dim(\mathscr{L}_1(a) \cap \mathscr{L}_2(a)) - \dim(\mathscr{L}_1(b) \cap \mathscr{L}_2(b))\end{aligned} \tag{7.11}$$

As we switch $(+)$ to $(-)$ there is, for each of the spectral flows of (7.8), a corresponding dimension correction term. Explicitly, for r large,

$$\begin{aligned}&\left(+\frac{1}{r^2}\right)\text{-spectral flow of } \{D(u)(M(r)) : a \leqq u \leqq b\} \\ &\quad = \left(-\frac{1}{r^2}\right)\text{-spectral flow of } \{D(u)(M(r)) : a \leqq u \leqq b\} \\ &\qquad + \dim\left\{\text{eigenmodes in band } \left[-\frac{1}{r^2},\frac{1}{r^2}\right] \text{ for } D(a)(M(r))\right\} \\ &\qquad - \dim\left\{\text{eigenmodes in band } \left[-\frac{1}{r^2},\frac{1}{r^2}\right] \text{ for } D(b)(M(r))\right\}\end{aligned} \tag{7.12}$$

$$\begin{aligned}&(+\varepsilon)\text{-spectral flow of } \{D(u;\mathscr{L}_j(u)) : a \leqq u \leqq b\} \\ &\quad = (-\varepsilon)\text{-spectral flow of } \{D(u;\mathscr{L}_j(u)) : a \leqq u \leqq b\} \\ &\qquad + \dim\left\{\text{extended } L^2\text{-solutions of } D(a) \mid M_j\right\} \\ &\qquad - \dim\left\{\text{extended } L^2\text{-solutions of } D(b) \mid M_j\right\}\end{aligned} \tag{7.13}$$

Substituting (7.11), (7.12), and (7.13) into (7.8), we have for r large,

$$
\begin{aligned}
&\left(-\frac{1}{r^2}\right)\text{-spectral flow}\{D(u)(M(r)) : a \leqq u \leqq b\} \\
&\quad = \sum_{j=1}^{2} (-\varepsilon)\text{-spectral flow}\{D(u;\mathscr{L}_j(u)) : a \leqq u \leqq b\} \\
&\quad + (-\varepsilon)\text{-Mas}\{(\mathscr{L}_1(u), \mathscr{L}_2(u)) : a \leqq u \leqq b\} \\
&\quad + \frac{1}{2}[\dim\ker \hat{D}(b) - \dim\ker \hat{D}(a)] \\
&\quad + [\dim(\mathscr{L}_1(a)\cap\mathscr{L}_2(a)) - \dim(\mathscr{L}_1(b)\cap\mathscr{L}_2(b))] \qquad (7.14) \\
&\quad + \sum_{j=1}^{2}[\dim\{\text{extended } L^2\text{-solutions of } D(a) \mid M_j\} \\
&\qquad\qquad - \dim\{\text{extended } L^2\text{-solutions of } D(b) \mid M_j\}] \\
&\quad - \dim\left\{\text{eigenmodes in band } \left[-\frac{1}{r^2}, \frac{1}{r^2}\right] \text{ for } D(a)(M(r))\right\} \\
&\quad + \dim\left\{\text{eigenmodes in band } \left[-\frac{1}{r^2}, \frac{1}{r^2}\right] \text{ for } D(b)(M(r))\right\}
\end{aligned}
$$

By [6], we know that after stretching M to $M(r)$ for r sufficiently large, $r \geq R_0$, the dimension of the low eigenmodes at $u = a, b$ can be computed by

$$
\begin{aligned}
&\dim\left\{\text{eigenmodes in band } \left[-\frac{1}{r^2}, \frac{1}{r^2}\right] \text{ for } D(a)(M(r))\right\} \\
&\quad = \sum_{j=1}^{2} \dim \text{ of } L^2\text{-solutions of } D(a) \mid M_j + \dim(L_1(a)\cap L_2(a))
\end{aligned}
$$

and

$$
\begin{aligned}
&\dim\left\{\text{eigenmodes in band } \left[-\frac{1}{r^2}, \frac{1}{r^2}\right] \text{ for } D(b)(M(r))\right\} \\
&\quad = \sum_{j=1}^{2} \dim\{L^2\text{-solutions of } D(b) \mid M_j\} + \dim(L_1(b)\cap L_2(b))
\end{aligned}
$$

Substituting these into (7.13) we have

$$
\begin{aligned}
&\left(-\frac{1}{r^2}\right)\text{-spectral flow of } \{D(u)(M(r)) : a \leqq u \leqq b\} \\
&\quad = \sum_{j=1}^{2} (-\varepsilon)\text{-spectral flow of } \{D(u;\mathscr{L}_j(u)) : a \leqq u \leqq b\} \qquad (7.15) \\
&\quad + (-\varepsilon)\text{-Mas}\{(\mathscr{L}_1(u), \mathscr{L}_2(u)) : a \leqq u \leqq b\} \\
&\quad - \frac{1}{2}[\dim\ker \hat{D}(b) - \dim\ker \hat{D}(a)]
\end{aligned}
$$

The proof of Theorem F now follows by taking the average of (7.8) and (7.15).

8. Formulation in Terms of Infinite Lagrangians

In the above formulation of Theorem C, we first had to partition the interval $[a, b]$ into $a = a_0 < a_1 < \cdots < a_n = b$, choose K_i so that over each subinterval $[a_i, a_{i+1}]$ it was not an eigenvalue of $\hat{D}(u)$ for $a_i \leqq u \leqq a_{i+1}$, and then inside the smoothly varying symplectic vector space $\mathscr{H}(u, K_i) : a_i \leqq u \leqq a_{i+1}$, choose Lagrangians $L_j(u; K_i)$ that satisfy additional conditions at each endpoint a_i. It will be convenient in applications to have a formulation in which the reference to the partition and to the K_i does not appear. This formulation will be developed in this section. The main reformulation is Theorem G.

For this purpose we consider infinite-dimensional Lagrangians $\mathscr{L}_j(u)$ in $L^2(\hat{E})$ with the following properties:

(8.1) For each u in $[a, b]$ there is a neighborhood $N(u)$ and a constant $K(u)$ such that $K(u)$ is not an eigenvalue of $\hat{D}(\nu)$ for ν in $N(u)$.

(8.2) $\mathscr{L}_j(\nu) = L_j(\nu; K) \oplus P_\pm(u; K)$ for ν in $N(u)$ where $K = K(u)$, the $L_j(\nu, K)$ are to be Lagrangian subspaces of $\mathscr{H}(\nu, K)$, and $P_\pm(\nu, K)$ is defined as in (1.11) with the sign $\pm$ the same as $(-1)^{j+1}$.

(8.3) The family of finite-dimensional Lagrangians $\{L_j(\nu; K(u)) : \nu \in N(u)\}$ is a smoothly varying family.

(8.4) For $u = a, b$ we have

$$\mathscr{L}_j(u) = L_j(u) \oplus P_\pm(u)$$

where the $L_j(u)$ are given by the limiting values of the extended L^2-solutions as in (1.13).

We refer to infinite familes of Lagrangians that satisfy conditions (8.1), (8.2), and (8.3) as *proper restricted infinite Lagrangians*. Note that $P_\pm(u)$ can be regarded as a polarization of the Hilbert space $L^2(\hat{E})$. Then the proper restricted infinite Lagrangians differ from the appropriate $P_\pm(u)$ by finite-dimensional subspaces and can be compared with the much more general infinite-dimensional Lagrangians considered in [16].

In view of its definition, it is apparant that the notions of Maslov index and averaged Maslov index extend easily to proper families of restricted infinite Lagrangians. Note again that the first Lagrangian differs by a finite subspace from $P_-(u)$, while the second Lagrangian differs by a finite subspace from $P_+(u)$.

Using the proper restricted Lagrangians $\mathscr{L}_j(u)$, $a \leqq u \leqq b$, as boundary conditions, we obtain self-adjoint elliptic operators

$$D(u; \mathscr{L}_j(u)) : L_1^2(E \mid M_j; \mathscr{L}_j(u)) \longrightarrow L^2(E \mid M_j)$$

as in (1.14). In addition, for the pair of proper restricted Lagrangians $(\mathscr{L}_1(u), \mathscr{L}_2(u))$ with complementary polarizations, the Maslov index $\mathrm{Mas}\{(\mathscr{L}_1(u), \mathscr{L}_2(u)) :$

$a \leqq u \leqq b\}$ and also the averaged Maslov index is well-defined. In fact, in view of (8.2), choosing finite intervals $\{[a_i, a_{i+1}]\}$ with $a = a_0 < a_1 < \cdots < a_n = b$ subordinate to the covering of $[a,b]$ by the open sets $N(u)$, we can break the calculation into those over the subintervals $[a_i, a_{i+1}]$ and so get the formula

$$\begin{aligned} &\operatorname{Mas}\{(\mathscr{L}_1(u), \mathscr{L}_2(u)) : a \leqq u \leqq b\} \\ &= \sum_{i=0}^{n-1} \operatorname{Mas}\{(L_1(u;K_i), L_2(u,K_i)) : a_i \leqq u \leqq a_{i+1}\} \end{aligned} \tag{8.5}$$

In terms of these we may state Theorem G as follows:

THEOREM G. *Let* $\{\mathscr{L}_j(u) : a \leqq u \leqq b\}$ *be two smooth families of proper restricted Lagrangians satisfying* (8.1), (8.2), (8.3), *and* (8.4). *Then for r sufficiently large the* $+\frac{1}{r^2}$-spectral flow of $\{D(u)(M(r))$ on $M(r) : a \leqq u \leqq b\}$ *equals*

$$\sum_{j=1}^{2} (+\varepsilon)\text{-spectral flow of } D_j(u, \mathscr{L}_j(u))$$
$$+ \operatorname{Mas}\{(\mathscr{L}_1(u), \mathscr{L}_2(u)) : a \leqq u \leqq b\} + \frac{1}{2}[\dim\ker \hat{D}(b) - \dim\ker \hat{D}(a)]$$

Here ε is chosen so that there are no eigenvalues for $\hat{D}(a)$ or for $\hat{D}(b)$ in the interval $[-\varepsilon, +\varepsilon]$ except the zero eigenvalues.

Of course, analogously to Theorem F, we could also formulate the above in terms of averaged spectral flow. For that theorem we would not have any zero-mode corrections.

Proof of Theorem G: By compactness of the interval, we may choose a partition $\{[a_i, a_{i+1}]\}$ subordinate to the open covering by the $\{N(u)\}$'s. Each interval has a K_i for which the end points are not the eignevalues is not an eigenvalue of $\hat{D}(\nu)$ for ν in that interval. Now we wish to use additivity over these intervals.

The difficulty is that the Lagrangians $\mathscr{L}_j(u)$ are not necessarily of the required form $L_j(a_i) \oplus P_\pm(a_i)$ at the endpoints $u = a_i, a_{i+1}$ of these intervals. However, because

$$\mathscr{H}(a_i, K_i) = \mathscr{H}(a_i) \oplus [P_+(a_i) \cap \mathscr{H}(a_i, K_i)] \oplus [P_\pm(a_i) \cap \mathscr{H}(a_i, K_i)]$$

we can easily deform the Lagrangians to the required form. Because both sides of the equation of Theorem G are invariant under continuous deformation, we might as well assume that the proper restricted infinite Lagrangians $\mathscr{L}_j(u)$ are already of the form $L_j(a_i) \oplus P_\pm(a_i)$ at each $u = a_i$.

In a similar manner, we can choose ε sufficiently small so that the only eigenvalues of the finite set of operators $\{\hat{D}(a_i)\}$ in the range $[-\varepsilon, +\varepsilon]$ consist of the zero-modes. Hence, for r sufficiently large, we have for each i, by Theorem C, that the $(+\frac{1}{r^2})$-spectral flow of $\{D(u)(M(r))$ over $M(r) : a_i \leqq u \leqq a_{i+1}\}$ equals

$$\begin{aligned}\sum_{j=1}^{2}&(+\varepsilon)\text{-spectral flow of } \{D_j(u, L_j(u, K_i)) : a_i \leqq u \leqq a_{i+1}\}\\ &= \operatorname{Mas}\{(\mathscr{L}_1(u), \mathscr{L}_2(u)) : a \leqq u \leqq b\}\\ &\quad + \frac{1}{2}[\dim\ker \hat{D}(a_{i+1}) - \dim\ker \hat{D}(a_i)]\end{aligned} \tag{8.6}$$

In view of equation (8.5), Theorem G follows by additivity.

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