Self-Adjoint Elliptic Operators and Manifold Decompositions Part III: Determinant Line Bundles and Lagrangian Intersection

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1 Introduction

The theory of spectral flows developed in the series [10, 11, 12], and the present paper has a wide range of applications to important geometric operators on compact manifolds. To present our results on spectral flow and manifold decomposition, the present paper develops a theory of determinant line bundles and infinite-dimensional Lagrangians associated to self-adjoint elliptic operators on compact manifolds. The trace-class properties of these infinite Lagrangians established here and the precise uniform estimates relating them to finite Lagrangians are crucial for such a determinant line bundle approach to analytical questions. As an application, we elucidate the Walker's and other generalizations of Casson's $\mathrm{SU}(2)$ representation theoretic invariant of 3-manifolds in terms of the η-invariant of certain Dirac operators. This is carried out by introducing the technique of "canonical perturbations" of singular Lagrangian subvarieties in symplectic geometry.

At the end of Part II of this series, we obtained a formulation of the spectral flow of a family of self-adjoint elliptic operators $D(u) : L^2(E) \to L^2(E)$ in terms

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of infinite-dimensional restricted Lagrangians. (These restricted Lagrangians are constructed as the sum of Lagrangian subspaces in a finite eigenband together with a standard complement outside the eigenband.) More precisely, in the situation when the underlying manifold is decomposed into two pieces $M = M_1 \cup M_2$, there are two natural choices of infinite Lagrangians with, as we will show, trace-class properties:

$$L_j(u) = \text{the } L^2\text{-closure of } \phi \mid \partial M_j \text{ where } \phi \text{ is a } C^\infty\text{-solution of } D(u) = 0$$

in the L^2-completion, $L^2(\hat{E})$, of the space of sections of the bundle $\hat{E}$, where $\hat{E}$ is the restriction of E to the common boundary, $\partial M_1 = \partial M_2$. After specifying a K-eigenband $\mathcal{H}(u, K)$, there are two other restricted Lagrangians

$$L_j(u, K) \oplus P_\pm(u, \pm K)\,, \quad j = 1, 2,$$

defined in an analogous manner. In the present treatment, using the results of R. Seeley [19], we investigate conditions under which these (potentially jumping) Lagrangians $L_j(u, K)$ can vary continuously and also approximate uniformly the infinite ones. In addition, we prove quite generally that these infinite, restricted Lagrangians can be constructed from the graphs of trace-class operators and that they give rise to sections of the determinant line bundle associated to the family of operators. In our applications to 3-manifold invariants, these analytically constructed determinant sections will be compared with Walker's geometrical treatment.

This passage from geometry (e.g., counting intersection numbers and twistings of framings) to analysis (e.g., η-invariants) is achieved by appealing to the Atiyah-Patodi-Singer index theorem and the interpretation of spectral flows geometrically as Maslov indices. Our geometric setting demands the full strength of our analysis in [11, 12] (as well as our earlier survey paper on the Maslov index [10], which was written to conveniently address the Maslov aspects of our problem).

The theory of the Casson invariant has been of considerable interest, in part because it is related to the Euler characteristic of the Floer homology groups of 3-manifolds. Casson originally defined his integer-valued invariant for integral homology spheres by roughly counting (with signs) the irreducible $\mathrm{SU}(2)$-representations of the fundamental groups [1]. Subsequently, several different extensions of $\mathrm{SU}(2)$-invariants to more general 3-manifolds have been proposed and studied. We consider those by K. Walker [24], which takes on fractional values, and a different integer-valued generalization developed for a special class of rational homology spheres by C. Boyer and A. Nicas [7]. The Walker generalization has a transformation formula for rational surgeries of M that generalizes the presently known combinatorial properties of the Casson invariant. In this paper we will give a general definition of a pair of integer-valued invariants that agree with the Boyer-Nicas invariant whenever the latter is defined; these invariants are quite natural in view of its relation with Fukaya-Floer homology (see [18] and Section 8 for details). In particular, the present study will yield computations and comparisons of all these invariants in terms of analytically defined data such as the η-invariants and the dimension of the 0-modes of elliptic operators.

To give a more precise description of the aforementioned results, let us recall the general framework for such representation theoretic invariants. Let M be a closed, oriented 3-manifold, and let $M = W_1 \cup W_2$ be a Heegaard decomposition of M into the union of two handles, W_1 and W_2, along a separating surface Σ. Then in the space $R = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SU}(2))/\mathrm{SU}(2)$ of $\mathrm{SU}(2)$-representations of the fundamental group $\pi_1(\Sigma)$ up to conjugacy, there are two subspaces Q_1 and Q_2 defined, by those representations extendible to $\pi_1(W_1)$ and $\pi_1(W_2)$, respectively. These Q_1 and Q_2 are singular (Lagrangian) real algebraic varieties, and if we restrict our attention to the irreducible representations, we obtain the nonsingular smooth strata:

$$(1.1)\qquad \begin{aligned} R - S &= \mathrm{Hom}(\pi(\Sigma), \mathrm{SU}(2))_{\mathrm{irred}}/\mathrm{SU}(2)\,, \\ Q_j^- &= \mathrm{Hom}(\pi_1(W_j), \mathrm{SU}(2))_{\mathrm{irred}}/\mathrm{SU}(2)\,, \quad j = 1, 2, \end{aligned}$$

of these varieties R and Q_j. In the case when M is an integral homology sphere, Casson defined his invariant

$$(1.2)\qquad \lambda(M) = \frac{1}{2}\Sigma_{P \in Q_1^- \cap Q_2^-} \mathrm{sign}(P)$$

as one-half of the intersection number of Q_1^- and Q_2^-. More precisely, under his assumption on M, $Q_1 \cap Q_2$ consists of irreducible points in $Q_1^- \cap Q_2^-$ together with the *isolated* intersection point given by the trivial representation. By perturbing Q_j into transverse position by a motion compactly supported in $R - S$, we obtain a finite number of intersection points in $Q_1^- \cap Q_2^-$ in $R - S$, which are then counted with signs as in (1.2).

In [7], Boyer and Nicas generalized Casson's invariant to 3-manifolds M for which the fundamental group $\pi_1(M)$ of M is *cyclically finite*. Under this assumption, the subspaces Q_1 and Q_2 may have nontrivial intersection points along the singular strata of R, corresponding to the $U(1)$-representations ρ of $\pi_1(M)$; however, the Zariski tangent spaces at ρ,

$$(1.3)\qquad (TQ_j)_\rho = H^1(W_j, \mathrm{Ad}\,\rho)$$

of the Q_j ($j = 1, 2$), are transverse to each other. In particular, these singular intersection points are isolated from $Q_1^- \cap Q_2^-$, and so, after perturbing the Q_j as before, the formula

$$(1.4)\qquad \lambda_{\mathrm{B.N.}}(M) = \Sigma_{P \in Q_1^- \cap Q_2^-} \mathrm{sign}(P)$$

gives a well-defined invariant with no contributions from the reducible representations. (It is now conventional to drop the $\frac{1}{2}$ in the above original formula (1.2) of Casson.)

In [24], Walker provided a different generalization that takes values in the rationals. For M a rational homology 3-sphere, the Walker invariant $\lambda_{\mathrm{W}}(M)$ counts not only the intersection number $\Sigma_{P \in Q_1^- \cap Q_2^-}$ sign(P) after a perturbation of these subspaces Q_1 and Q_2 but also an additional fractional correction term $I(\rho)$ (see Section 7 for details) for each reducible representation ρ of $\pi_1(M)$

$$\lambda_W(M) = \Sigma_{P \in Q_1^- \cap Q_2^-} \operatorname{sign}(P) + \Sigma_\rho I(\rho) \tag{1.5}$$

where the second sum is over the reducible representations ρ of $\pi_1(M)$ into $\mathrm{SU}(2)$ up to conjugacy.

The two invariants $\lambda_{\mathrm{B.N.}}(M)$ and $\lambda_{\mathrm{W}}(M)$ are apparently different because of these correction terms $I(\rho)$, and our theory provides an explanation and calculation of this discrepancy. (To simplify our discussion, we have taken out the normalization factor, whereas the definition of Walker's invariant in [24] is the above sum (1.5) divided by the order $|H_1(M, Z)|$ of the first homology group of M.)

THEOREM A *Let M be a rational homology 3-sphere with cyclically finite fundamental group. Let $\lambda_{\mathrm{B.N.}}(M)$, $\lambda_{\mathrm{W}}(M)$, and $I(\rho)$ be defined as above. Then the difference $\lambda_{\mathrm{B.N.}}(M) - \lambda_{\mathrm{W}}(M)$ equals $-\frac{1}{2}$ times the signature defect* $\mathrm{Def}(M_{ab} \to M)$ *where $M_{ab} \to M$ is the abelian covering of M associated to the homomorphism*

$$\pi_1(M) \to H_1(M, Z) \to H_1(M, Z)/\textit{modulo order-}2 \textit{ elements.}$$

THEOREM B *Let M be as in Theorem* A *and let $\lambda_{\mathrm{B.N.}}(M)$, $\lambda_{\mathrm{W}}(M)$, and $I(\rho)$ be defined as above. Let ρ be the sum $\sigma \oplus \sigma^{-1}$ of a $U(1)$-representation $\sigma : \pi_1(M) \to U(1)$ and its complex conjugate σ^{-1}. Then*

$$I(\rho) = -\frac{\rho(M, \sigma^2)}{2}, \tag{1.6}$$

where $\rho(M, \sigma^2)$ stands for the ρ-invariant of M associated to the representation σ^2.

The first theorem follows from the second as will be explained in Section 7. Note that Walker's corrections $I(\rho)$ in general depend on the perturbation that renders Q_1^- and Q_2^- transverse at ρ, but in the cyclically finite case there is no need for any perturbation since the appropriate Lagrangians are already transverse. Theorem B identifies this canonical $I(\rho)$ as a reduced η-invariant.

In Section 8, using our method of canonical perturbations in symplectic geometry, we will introduce and study two other integer-valued invariants, $\lambda(M)_R$ and $\lambda(M)_L$, which are extensions of Casson's invariant to all rational homology spheres. Both of them coincide with the Boyer-Nicas invariant $\lambda_{\mathrm{B.N.}}(M)$ for the class of rational homology spheres satisfying the cyclically finite condition. We then extend our two main theorems, Theorems A and B, to the general (non-cyclically finite) case, thus elucidating the relationships between these different extensions of Casson's invariant to all rational homology spheres.

More precisely, we construct the "canonical" right-handed (R) and, alternatively, left-handed (L) symplectic deformations to render Q_1^- and Q_2^- transverse at any reducible representation ρ. After making these perturbations, the formulae $\sum_{P \in Q_1^- \cap Q_2^-} \operatorname{sign}(P)$ yield, respectively, our pair of integer-valued invariants $\lambda(M)_R$ and $\lambda(M)_L$. As it turns out, not only is the Walker correction $I(\rho)$ for these perturbed Lagrangians well-defined, but it can be calculated explicitly. Our treatment of Theorems A and B in this general case includes a study of the average left- and right-handed invariants, which takes on half-integer values and occurs naturally in certain formulae.

Elsewhere [8, 9] we have introduced an $\mathrm{SU}(n)$-invariant of all oriented 3-manifolds that takes on fractional values, generalizing Casson's and Walker's invariant for $n = 2$. The present results raise the question of formulating analogues of Theorems A and B in more general settings.

For the geometrically arising analytical problems considered here, the pathologies of "jumping Lagrangians" and "jumping boundary conditions" are inevitable, and their treatment requires the delicate analysis and the corresponding geometric reasoning of the present paper. For this reason, in Section 2 we give a general study of these infinite Lagrangians associated to a self-adjoint elliptic operator over a compact manifold with boundary using the results of our papers [11, 12]. Our concern is to define the determinant line bundle and its section (coming from these Lagrangians). The main technical point is the trace-class property of the associated Lagrangians (see Proposition 2.4) and the *uniform* estimates (see Proposition 2.5). Their proofs are carried out in Section 3. In Section 4 we extend the Maslov index into a Hermitian setting, using the complex determinant line bundle of Quillen. In Section 5, explicit formulae are given for the operator B^{ev}, which has been the main tool in studying the gauge theory of 3-manifolds. With this preparation, we turn in Section 6 to the application to the Casson invariant and its generalization by Walker, which requires the Maslov index in the Hermitian setting. The analysis of the Walker correction of a reducible connection, $I(\rho)$, is carried out in Section 7. In Section 8 we present our extension of the Boyer-Nicas invariant to all rational homology spheres and our general formula for the difference from the Walker invariant and the analogues of Theorems A and B in this general setting.

2 Determinants and Infinite-Dimensional Lagrangians

Let $\{D(u) : 0 \le u \le 1\}$ be a smooth, one-parameter family of first-order, self-adjoint elliptic operators $D(u)$ over a closed, smooth, oriented manifold M that is split into two pieces, M_1 and M_2, by a codimension-1 submanifold Σ:

$$M = M_1 \cup M_2 \,, \qquad \Sigma = M_1 \cap M_2 = \partial M_1 = \partial M_2 \,. \tag{2.1}$$

As in Parts I and II, we assume that $D(u)$ is of "Atiyah-Patodi-Singer type." That is, on a collar neighborhood $\Sigma \times [-1, 1]$ of $\Sigma = \Sigma \times 0$ in M, the operator $D(u)$ is of the special form

$$D(u) = \pi^* \sigma_u \left(\frac{\partial}{\partial s} + \pi^* \hat{D}(u) \right) \quad \text{on } \Sigma \times [-1, 1] \,. \tag{2.2}$$

Here s is the coordinate $[-1,1]$, π is the projection of $\Sigma \times [-1,1]$ onto Σ, σ_u is a bundle automorphism over Σ, and $\hat{D}(u)$ is a *self-adjoint* operator over Σ. More explicitly,

$$
\begin{aligned}
D(u) &: \Gamma(\mathbb{E}) \longrightarrow \Gamma(\mathbb{E})\,, && \mathbb{E} \text{ over } M, \\
\hat{D}(u) &: \Gamma(\hat{\mathbb{E}}) \longrightarrow \Gamma(\hat{\mathbb{E}})\,, && \hat{\mathbb{E}} \text{ over } \Sigma, \\
\sigma_u &: \hat{\mathbb{E}} \longrightarrow \hat{\mathbb{E}}\,, && \hat{\mathbb{E}} \text{ over } \Sigma,
\end{aligned}
\tag{2.3}
$$

for bundles $\mathbb{E}$ and $\hat{\mathbb{E}}$ with inner products, with $\mathbb{E} \mid \Sigma \times [-1,1]$ identified with $\pi^*\hat{\mathbb{E}}$, and $\Gamma(\mathbb{E})$ denoting the smooth sections of the bundle $\mathbb{E}$, and similarly for $\Gamma(\hat{\mathbb{E}})$.

In Part II we established a method for computing the spectral flow $\mathrm{SF}\{D(u) : 0 \leq u \leq 1\}$ of the family of operators $D(u)$ via the spectral flow decomposition theorems. The object of this section and the next is to reformulate these results in terms of determinant line bundles using some natural, infinite-dimensional Lagrangians provided by our setting. Improving upon the method of Part II in choosing arbitrary Lagrangians inside a finite eigenband, this new approach leads directly to our applications in later sections.

Because $D(u)$ and $\hat{D}(u)$ are self-adjoint, by proposition 2A of Part I, the bundle automorphisms σ_u and operator $\hat{D}(u)$ satisfy the formulae

$$
\sigma_u^* = -\sigma_u\,, \quad \sigma_u \hat{D}(u) = -\hat{D}(u)\sigma_u\,. \tag{2.4}
$$

To simplify our discussion, we will assume in addition that

$$
(\sigma_u)^2 = -\mathrm{Id}\,, \quad 0 \leq u \leq 1. \tag{2.5}
$$

Note (2.5) is the same as requiring σ_u to be orthogonal, $\sigma_u\sigma_u^* = \mathrm{Id}$, and is satisfied by a large number of geometric operators, e.g., the Dirac operator coupled to connections. In view of (2.5), the bundle automorphism $J_u = -\sigma_u$ can be interpreted as defining a complex structure on $\hat{\mathbb{E}}$. Associated to this complex structure, there is a unique Hermitian inner product $\langle\cdot,\cdot\rangle_u$ on each fiber of $\hat{\mathbb{E}}$ such that $\mathrm{Re}\langle\cdot,\cdot\rangle_u = (\cdot,\cdot)$. In a similar manner, $\Gamma(\mathbb{E})$ becomes a complex vector space under J_u with Hermitian inner product $\langle\cdot,\cdot\rangle_u$. Following the conventions of [11, 12], we have

$$
\mathrm{Re}\langle f,g\rangle_u = (f,g)_u\,, \qquad -\mathrm{Im}\langle f,g\rangle_u = (f,\sigma_u g)_u := \{f,g\}_u\,, \tag{2.6}
$$

where $\{\cdot,\cdot\}_u$ is a symplectic pairing on $\Gamma(\hat{E})$.

2.1 Definition of the Determinant Line Bundle (à la Quillen)

Our first chore is to review the determinant and Lagrangian aspects of our setting.

In view of (2.4), the *real*, first-order, elliptic operator $\hat{D}(E)$ may be regarded as a *complex* operator $\hat{D}_J(u) = \hat{D}(u)$

$$
\hat{D}_J(u) : (\Gamma(\hat{E}), -J_u) \to (\Gamma(\hat{E}), J_u) \tag{2.7}
$$

where the complex structure on the range is given by J_u as above but on the domain by $-J_u$. Following [14, 21], we form the complex determinant line bundle $\det(\hat{D}_J)$ over $[0,1]$ for this family of complex operators $\hat{D}_J(u)$. More explicitly, a local coordinate chart of $\det(\hat{D}_J)$ is obtained as follows: Fix $K > 0$ and define $\mathcal{H}(u;K)$ by

$$\mathcal{H}(u;K) = \operatorname{span}\{\phi_j : \hat{D}(u)\phi_j = \lambda_j\phi_j,\ |\lambda_j| \le K\}\,. \tag{2.8}$$

By elliptic regularity, $\mathcal{H}(u;K)$ is a finite-dimensional subspace in $\Gamma(\hat{E})$. Let $V(K)$ denote the set of u such that $\hat{D}(u)$ does not have $\pm K$ as its eigenvalues. The spectral decomposition theorem assures us that $V(K)$ is an open set in $[0,1]$ and that $\mathcal{H}(u;K)$ varies smoothly for u in $V(K)$. Indeed, this theorem also provides a smooth family of projections $\Pi[K]$ from the L^2-sections, $L^2(\hat{E})$ of $\hat{E}$ onto $\mathcal{H}(u;K)$. Since $\hat{D}(u)$ skew-commutes with $J_u = -\sigma_u$, the subspace $\mathcal{H}(u;K)$ is invariant under J_u and hence can be viewed as complex subspaces with respect to the two complex structures $(\Gamma(\hat{E}), \pm J_u)$. For u in $V(K)$ we introduce the tensor product

$$\left(\bigwedge^{\max}[\mathcal{H}(u;K), -J_u]\right)^* \otimes \left(\bigwedge^{\max}[\mathcal{H}(u;K), +J_u]\right) \tag{2.9}$$

where $\bigwedge[\cdot]^*$ is the top exterior power of the complex vector space $[\cdot]$. As u varies over $V(K)$, these tensor products form a smoothly varying, complex, one-dimensional space and hence a complex line bundle $\det(\hat{D}_J)$ with fiber

$$\det(\hat{D}_J)(u) = \left(\bigwedge^{\max}[\mathcal{H}(u;K), -J_u]\right)^* \otimes \left(\bigwedge^{\max}[\mathcal{H}(u;K), +J_u]\right). \tag{2.10}$$

The tensor product (2.9) depends on the choice of K, and so for u in $V(K) \cap V(K')$, $K < K'$, we must identify the two corresponding tensor products together. Note that $\mathcal{H}(u;K')$ has an orthogonal sum decomposition

$$\mathcal{H}(u;K') = \mathcal{H}(u;K) \oplus [\mathcal{H}(u;K)^\perp \cap \mathcal{H}(u;K')]\,. \tag{2.11}$$

Let $\{\phi_1, \ldots, \phi_N\}$ be an orthogonal basis of eigensections of $\hat{D}(u)$ such that

$$\hat{D}(u)\phi_j = \lambda_j\phi_j \quad \text{with } K < \lambda_j < K'\,;$$

then these vectors form a basis for the complex space $\mathcal{H}(u;K)^\perp \cap \mathcal{H}(u;K')$. In particular, as

$$\begin{aligned}\bigwedge^{\max}[\mathcal{H}(u;K'), \pm J_u] = {} & \bigwedge^{\max}[\mathcal{H}(u;K), \pm J_u] \\ & \otimes \bigwedge^{\max}[\mathcal{H}(u;K)^\perp \cap \mathcal{H}(u;K'), \pm J_u]\end{aligned} \tag{2.12}$$

we can identify

$$\left(\bigwedge^{\max}[\mathcal{H}(u;K),-J_u]\right)^* \otimes \left(\bigwedge^{\max}[\mathcal{H}(u;K),+J_u]\right)$$

with $(\bigwedge^{\max}[\mathcal{H}(u;K'),-J_u])^* \otimes (\bigwedge^{\max}[\mathcal{H}(u;K'),+J_u])$ via the mapping

$$a\otimes b \to \left[a\wedge\left(\bigwedge_{j=1}^{N}(\phi_j)^*\right)\right]\otimes\left[b\wedge\left(\bigwedge_{j=1}^{N}(\phi_j)\right)\right].$$

This last mapping does not depend on the choices of the $\{\phi_j\}$: Changing $\{\phi_j\}$ by an orthogonal matrix A results in the multiplication by

$$(\det A)(\det A^{\mathsf{T}}) = \det(A\cdot A^{\mathsf{T}}) = 1\,.$$

Thus we can identify $\det(\hat{D}_J) \mid V(K)$ with $\det(\hat{D}_J) \mid V(K')$ over the overlap $V(K)\cap V(K')$ in a canonical manner and obtain the definition of Quillen's determinant line bundle $\det(\hat{D}_J)$ over the interval $[0,1]$.

Note that given a complex linear map

$$R : (\mathcal{H}(u;K),-J_u) \longrightarrow (\mathcal{H}(u;K),J_u)\,,$$

there is a well-defined element

$$\det(R) = \left(\bigwedge^{N}\phi_j\right)^*\otimes\left(\bigwedge^{N}R(\phi_j)\right) \tag{2.13}$$

in $(\bigwedge^{\max}[\mathcal{H}(u;K),-J_u])^*\otimes\bigwedge^{\max}[\mathcal{H}(u;K),J_u])$. In addition, if $\{R(u) \mid V(K)\}$ is a smooth family of such mappings, then $\det R(u)$ gives rise to a smooth section of $\det(\hat{D}_J \mid V(K))$.

2.2 Lagrangians and Determinants

Let L be a Lagrangian subspace of the Hermitian vector space

$$(\mathcal{H}(u,K),J_u,\langle,\rangle_u)\,.$$

From the definition of Lagrangian, there exists an orthonormal complex basis $\{f_j : 1\le j\le n\}$ of $\mathcal{H}(u;K)$ such that $\langle f_j,f_k\rangle_u = \delta_{jk}$ and L is the real vector space spanned by the f_j's, i.e., $L=\operatorname{span}_R\{f_j\}$. In particular,

$$\mathcal{H}(u;K) = L\oplus\sigma_u L \tag{2.14}$$

is a real orthogonal decomposition into the sum of L and

$$\sigma_u L = \operatorname{span}\{\sigma_u f_j : 1\le j\le n\}\,.$$

Define the reflection $R(L)$ by the formula

$$\begin{aligned} R(L) &= f \quad \text{if } f\in L\,,\\ R(L) &= -g \quad \text{if } g\in\sigma_u L\,. \end{aligned} \tag{2.15}$$

Since the decomposition (2.11) is an orthogonal decomposition, $R(L)$ is an orthogonal transformation. On the other hand, from the relations $(\sigma_u)^2 = -\mathrm{Id}$ and $J_u = -\sigma_u$, it follows that

$$R(L)(-J_u f_j) = R(L)(\sigma_u f_j) = -\sigma_u f_j = J_u f_j\,,$$
$$R(L)(-J_u(\sigma_u f_j)) = R(L)(-f_j) = -f_j = J_u(-\sigma_u f_j) = J_u R(L)(\sigma_u f_j)\,.$$

Consequently, $R(L)(-J_u) = J_u R(L)$. Since a real, orthogonal, complex linear map such as $R(L)$ is also unitary, this proves the following:

PROPOSITION 2.1 *Let L be a Lagrangian subspace of $\mathcal{H}(u;K)$ and $R(L)$ be defined as in* (2.15). *Then*

(a) *the reflection $R(L)$ is a complex linear map*

$$R(L) : (\mathcal{H}(u;K), -J_u) \longrightarrow (\mathcal{H}(u;K), J_u)$$

with respect to the specified complex structures $\pm J_u$ on $\mathcal{H}(u;K)$, and

(b) *$R(L)$ is a unitary transformation with respect to the Hermitian structure $\overline{\langle\cdot,\cdot\rangle_u}$ on $(\mathcal{H}(u;K), -J_u)$ and $\langle\cdot,\cdot\rangle_u$ on $(\mathcal{H}(u;K), J_u)$.*

We find this approach most natural and effective for our purpose in this paper. Of course, we will explain next how the other points of view are equivalent.

Combining Proposition 2.1 with (2.13), we can assign to a Lagrangian L in $\mathcal{H}(u;K)$ an element

$$s(L) = \det R(L) \tag{2.16}$$

which is of unit length in the complex line

$$\left(\bigwedge^{\max}[\mathcal{H}(u;K), -J_u]\right)^* \otimes \left(\bigwedge^{\max}[\mathcal{H}(u;K), J_u]\right) = (\det \hat{D}_J)(u)\,.$$

Given a smoothly varying family of Lagrangians $L(u)$ in the symplectic vector bundle $\{\mathcal{H}(u;K) : u \in V(K)\}$, this assignment $u \mapsto s(L(u))$ yields a smooth section of $(\det \hat{D}_J)$ over $V(K)$. Furthermore, for $K \le K'$, $u \in V(K) \cap V(K')$, and $\{\phi_j : 1 \le j \le m\}$ chosen to span $\mathcal{H}(u;K)^\perp \cap \mathcal{H}(u;K')$ as in (2.12), then both $L \oplus \mathrm{span}_R\{\phi_j : 1 \le j \le n\}$ and $L \oplus \mathrm{span}_R\{\sigma_u\phi_j : 1 \le j \le m\}$ are Lagrangians in $\mathcal{H}(u;K')$. Under the above identification of $(\det \hat{D}_J) : V(K)$ with $(\det \hat{D}_J) : V(K')$ on the overlapping coordinate charts $V(K) \cap V(K')$, we have

$$\begin{aligned} &s(L \oplus \mathrm{Span}_R\{\phi_j : 1 \le j \le m\}) = s(L)\,,\\ &s(L \oplus \mathrm{Span}_R\{\sigma_u\phi_j : 1 \le j \le m\}) = (-1)^m s(L)\,. \end{aligned} \tag{2.17}$$

For definiteness, let us fix an orthonormal basis $\{\phi_j : 1 \le j \le n\}$ for the Hermitian vector space $(\mathcal{H}(u;K), J_u, \langle\cdot,\cdot\rangle_u)$ and compare some approaches to determinant sections.

Any Lagrangian L in $\mathcal{H}(u;K)$ can be written in the form

$$L = A \cdot L_0$$

where A is a unitary transformation of $\mathcal{H}(u;K)$ and L_0 is the reference Lagrangian $L_0 = \operatorname{span}\{\phi_j : 1 \leq j \leq n\}$. If we identify this last group of unitary transformations with $U(n)$ via the basis ϕ_j,

$$\varphi\colon U(n) \to U[\mathcal{H}(u;K), J_u, \langle \cdot, \cdot \rangle_u]\,, \tag{2.18}$$

then the above element A is well-defined in $U(n)$ modulo the right multiplication by elements in $O(n)$, the real orthogonal group. In other words, we have an isomorphism

$$\begin{aligned} \operatorname{Lag}\{\mathcal{H}(u,k)\} &\longrightarrow U(n)/O(n)\,, \\ L &\longrightarrow A/O(n)\,, \end{aligned} \tag{2.19}$$

between $\operatorname{Lag}\{\mathcal{H}\}$, the space of Lagrangians, and $U(n)/O(n)$, the homogeneous space. The latter can in turn be embedded into $U(n)$ via the mapping

$$U(n)/O(n) \longrightarrow U(n)\,, \quad A/O(n) \to A \cdot A^{\mathsf{T}}\,. \tag{2.20}$$

Combining (2.18), (2.19), and (2.20), we send $L = A \cdot L_0$ to

$$A \cdot A^{\mathsf{T}} = A \cdot (\bar{A})^{-1} = \varphi(A) R(L_0) \varphi(A)^{-1} R(L_0)^{-1} = R(L) \cdot R(L_0)^{-1}\,.$$

Here the second identity follows because complex conjugation is given by conjugation with $R(L_0)$. Thus we have the standard embedding

$$\operatorname{Lag}\{\mathcal{H}(u;K)\} \hookrightarrow U[\mathcal{H}(u;K), J_u, \langle \cdot, \cdot \rangle_u] \tag{2.21}$$

given by sending L to $R(L) \cdot R(L_0)^{-1}$. In particular, if we compose the last mapping with the determinant map

$$\det(R(L) \cdot R(L_0)^{-1}) = [\det A]^2\,,$$

we recover the standard mapping

$$[\det]^2 : \operatorname{Lag}\{\mathcal{H}(u;K)\} \longrightarrow S^1$$

in the geometric definition of Maslov index in [10, section 5].

Another approach in [12, section 5] is to assign to a Lagrangian L in $\mathcal{H}(u;K)$ the element $\beta(L) = (\bigwedge_{j=1}^{N} g_j)^{\otimes 2}$ in $\bigwedge^{\max}[\mathcal{H}(u;K), J_u]^{\otimes 2}$. Here $\{g_j\}$ is an orthonormal basis of $\mathcal{H}(u;K)$ such that $L = \operatorname{span}\{g_j\}$. Note we can identify the dual, $(\mathcal{H}(u;K), -J_u)^*$, with $[\mathcal{H}(u;K), J_u]$ and so

$$\bigwedge^{\max}[\mathcal{H}(u;K), -J_u]^* \otimes \bigwedge^{\max}[\mathcal{H}(u;K), J_u]$$

is isomorphic to $\bigwedge^{\max}[\mathcal{H}(u;K), J_u]^{\otimes 2}$. Under this last isomorphism, $\beta(L)$ becomes the above $s(L)$ of (2.16). Hence, our present approach also coincides with the treatment in [12].

2.3 Infinite-Dimensional Lagrangians and Determinants

Let $(\mathcal{H}, J, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space with complex structure J and Hermitian inner product $\langle\cdot,\cdot\rangle$. In this infinite-dimensional setting, a Lagrangian L is by *definition* a closed subspace of $\mathcal{H}$ such that L is the L^2-closure of $\operatorname{span}\{f_j\}$ where $\{f_j\}$ is a complete orthonormal basis of the Hilbert space $(\mathcal{H}, J, \langle\cdot,\cdot\rangle)$. Here $\operatorname{span}\{\cdot\}$ stands for the span of the vectors listed. It is easy to see that our definition agrees with the more conventional one: A Lagrangian L is a closed subspace in $\mathcal{H}$ such that the symplectic form

$$\{f, g\} = -\operatorname{Im}\langle f, g\rangle = (f, -Jg)$$

vanishes on L and L is maximal with respect to this property. As in the finite-dimensional case, we may fix a reference Lagrangian L_0 in $\mathcal{H}$; then any other Lagrangian can be written in the form

$$L = A \cdot L_0$$

where A is a unitary transformation of the Hilbert space $\mathcal{H}$, $A \in U(\mathcal{H})$. It follows that we have an isomorphism

$$\begin{aligned} U(\mathcal{H})/O(L_0) &\longrightarrow \operatorname{Lag}(\mathcal{H})\,, \\ A &\longmapsto A \cdot L_0\,, \end{aligned} \tag{2.22}$$

from the quotient of the unitary group $U(\mathcal{H})$ modulo the orthogonal group $O(\mathcal{H})$ onto the Grassmannian $\operatorname{Lag}(\mathcal{H})$ of Lagrangians in $\mathcal{H}$.

As explained in [22, p. 206], the determinant $\det A$ of a linear transformation $A : \mathcal{H} \to \mathcal{H}$ is well-defined if A is the sum of the identity and a trace-class operator. Therefore, it is natural to restrict our attention to the subspace $\operatorname{Lag}(\mathcal{H})_{\mathrm{res}}$ in $\operatorname{Lag}(\mathcal{H})$ defined by

$$\operatorname{Lag}(\mathcal{H})_{\mathrm{res}} = \{A \cdot L_0 : A - \mathrm{Id} \text{ is trace-class}\}\,. \tag{2.23}$$

A Lagrangian as in (2.23) is referred to as a *restricted* Lagrangian; $\operatorname{Lag}(\mathcal{H})_{\mathrm{res}}$ can be identified with a subspace of the infinite-dimensional Grassmannian $\operatorname{Gr}(\mathcal{H})$ of A. Pressley and G. Segal [20]. Over this last space, there exists a well-defined determinant line bundle that can be compared with Quillen's line bundle over the space of Fredholm operators [20, p. 116].

In our application, the underlying Hilbert space is $(L^2(\hat{\mathbb{E}}), J_u, \langle\cdot,\cdot\rangle_u)$ where $L^2(\hat{\mathbb{E}})$ is the L^2-completion of the space of smooth sections with $J_u, \langle\cdot,\cdot\rangle_u$ the extensions of the corresponding structures on $\Gamma(\hat{\mathbb{E}})$. Following R. Seeley [19], we consider the subspace $L_j(u)_\infty$ in the smooth sections $\Gamma(\hat{\mathbb{E}})$ defined by

$$L_j(u)_\infty = (\phi \mid \partial M_j) \quad \begin{array}{l}\text{such that } \phi \text{ is a smooth section of } E \mid M_j \\ \text{satisfying } D(u)\phi = 0\end{array} \tag{2.24}$$

and its closure $L_j(u)$ in $L^2(\hat{\mathbb{E}})$. Our first general result in the infinite setting is:

PROPOSITION 2.2 (a) *There is an orthogonal decomposition of* $\Gamma(\hat{E})$ *onto the sum*

$$\Gamma(\hat{E}) = L_j(u)_\infty \oplus \sigma_u L_j(u)_\infty$$

for $j = 1, 2$ *and any* $u \in [0,1]$.

(b) *There exists a complete orthonormal basis* $\{f_j\}$ *for* $(L^2(\hat{E}), J_u, \langle\cdot,\cdot\rangle_u)$ *with* f_j *in* $L_j(u)_\infty$ *such that* $L_j(u)$ *is the* L^2*-closure of* $\operatorname{span}\{f_j\}$. *In particular,* $L_j(u)$ *is a Lagrangian subspace in* $L^2(\hat{E})$.

The proof is given in Section 3. It follows from combining the results of [19] with a result of Booss and Wojciechowski [6].

We denote by $P_+(u;K)$ and $P_-(u;K)$ the L^2-closure of the space of eigensections ϕ_l with $\hat{D}\phi_l = \lambda_l\phi_l, \lambda_l > K$ and $\lambda_l < K$, respectively (cf. (2.5) of Part II). When K is included as a possible eigenvalue (i.e., $\lambda_l \geq K$ or $\lambda_l \leq K$), we will use the notation $P_+(u,K]$ and $P_-(u;K]$, respectively. Recall from section 2 of Part II that for any $K \geq 0$

$$\begin{aligned} L_1(u) \cap P_+(u;-K] &= L_1(u)_\infty \cap P_+(u;-K]\,, \\ L_2(u) \cap P_-(u;K] &= L_2(u)_\infty \cap P_-(u;K]\,, \end{aligned} \tag{2.25}$$

are both finite-dimensional subspaces in $L^2(\hat{E})$ whose orthogonal projections into $\mathcal{H}(u;K)$,

$$\mathcal{H}(u;K) := \operatorname{span}\phi_l \quad \text{with } \lambda_l \text{ in } [-K,K]\,,$$

namely,

$$\begin{aligned} L_1(u;K) &:= \Pi[K](L_1(u) \cap P_+(u;-K])\,, \\ L_2(u;K) &:= \Pi[K](L_2(u) \cap P_-(u;K])\,, \end{aligned} \tag{2.26}$$

are Lagrangian subspaces in $\mathcal{H}(u;K)$. As in (2.8) of Part II, the symbol $\Pi[K]$ stands for the projection of $L^2(\hat{E})$ onto $\mathcal{H}(u;K)$.

Now there are two pairs of infinite-dimensional Lagrangians:

$$\begin{aligned} &L_1(u) \quad \text{and} \quad L_1(u;K) \oplus P_-(u;K)\,, \\ &L_2(u) \quad \text{and} \quad L_2(u;K) \oplus P_+(u;K)\,, \end{aligned} \tag{2.27}$$

which are in an explicit sense close to each other. We will make this last statement precise and use this knowledge to construct sections $s_j(u)$ of the determinant line bundle $\det(\hat{D}_J)$, encoding the relative position of the $L_j(u)$ in $L^2(\hat{E})$. Towards this goal, the first step is to prove the following:

PROPOSITION 2.3 *With the notation as above, there exists a* K^* *such that the Lagrangian subspaces* $L_j(u;K)$ *in* $\mathcal{H}(u;K)$ *vary smoothly with* $u \in V(K)$ *whenever* $K > K^*$. *In particular, the section* $s(L_j(u;K))$ *varies smoothly in bundle* $\det(\hat{D}_J) \mid V(K)$ *whenever* $K > K^*$.

The second step is to relate the pairs of Lagrangians in (2.27). Analogous to (2.21) in the case of finite-dimensional Lagrangians, we have embeddings

$$\begin{aligned} \operatorname{Lag}(L^2(\hat{\mathbb{E}})) &\hookrightarrow U(L^2(\hat{\mathbb{E}}), J_u, \langle\cdot,\cdot\rangle_u))\,, \\ L &\mapsto R(L)\cdot R(L_0)^{-1}\,, \end{aligned} \tag{2.28}$$

of the space $\operatorname{Lag}(L^2(\hat{\mathbb{E}}))$ of Lagrangians in $L^2(\hat{\mathbb{E}})$ to the unitary group of $L^2(\hat{\mathbb{E}})$. Here again L_0 is a reference Lagrangian. By choosing L_0 to the given by

$$L_0 := L_j(u;K)\oplus P_\pm(u;\mp K)\,,$$

its relation with $L_j(u;K)$ can be examined via the unitary map $R(L)\cdot R(L_0)^{-1}$ as in the following:

PROPOSITION 2.4 (a) *For $K>0$ and $u\in[0,1]$, the unitary transformations*

$$\begin{aligned} A_1(u;K) &:= R[L_1(u)]\cdot\{R[L_1(u;K)\oplus P_-(u;-K)]\}^{-1} \\ A_2(u;K) &:= R[L_2(u)]\cdot\{R[L_2(u;K)\oplus P_+(u;+K)]\}^{-1} \end{aligned} \tag{2.29}$$

have the property that $[A_j(u;K)-\mathrm{Id}]$ is trace-class, $j=1,2$. In particular, $\det A_j(u;K)$ is defined.

(b) *There exists a $K^*>0$ such that for $K>K^*$ the determinant $\det A_j(u;K)$ above varies smoothly as a function of u in $V(K)$ to S^1.*

(c) *For any $\varepsilon>0$, there exists $K(\varepsilon)>K^*$ such that*

$$\|\det A_J(u;K)-1\|<\varepsilon \quad \textit{for all } K>K(\varepsilon) \textit{ and } u\in V(K)\,.$$

In view of Proposition 2.4(a), we have elements $s_j(u)$ in $\det(\hat{D}_J)(u)$ defined by

$$\begin{aligned} s_1(u) &= [\det A_1(u;K)]s(L_1(u:k))\,, \\ s_2(u) &= (-1)^{\dim_{\mathbb{C}}\mathcal{H}(u;K)}[\det A_2(u;K)]s(L_2(u;K))\,. \end{aligned} \tag{2.30}$$

By the remarks in (2.17), it follows that the $s_j(u)$ in $\det(\hat{D}_J)$ are well-defined independently of the choice of K. For $K>K^*$ and K^* as given by Propositions 2.3 and 2.4, both $\det A_j(u;K)$ and $s(L_j(u;K))$ vary smoothly over $V(K)$. As the collection of these $V(K)$'s cover $[0,1]$, we see that the s_j form a smooth section of $\det(\hat{D}_J)$. In fact, Proposition 2.4 gives a uniform approximation of s_j by $(\pm1)s(L_j(u;K))$ over $V(K)$. To sum up these conclusions from Propositions 2.3 and 2.4, we have the following:

PROPOSITION 2.5 *Let s_j be the sections of $\det(\hat{D}_J)\mid V(K)$ over $V(K)$ for $K>K^*$ defined by* (2.30)*. Then*

(a) *these sections are smooth and compatible over $V(K)\cap V(K)$ for $K>K^*$, $K'>K^*$;*

(b) *for any $\varepsilon>0$, there exists $K(\varepsilon)$ such that for all $u\in[0,1]$, we have*

$$\begin{aligned} \|s_1(u)-s(L_1(u;K))\| &< \varepsilon\,, \\ \|s_2(u)-(-1)^{\dim_{\mathbb{C}}\mathcal{H}(u;K)}s(L_2(u;K))\| &< \varepsilon\,, \end{aligned} \tag{2.31}$$

for all K and u with $K > K(\varepsilon), u \in V(K)$. That is, the sections $s_j(u)$ can be uniformly approximated by the sections coming from the finite Lagrangians.

There remains one more issue: the behavior of the infinite Lagrangians $L_j(u)$ as we stretch M_1 and M_2, that is, as we replace M_1 and M_2 by $M_1(r) = M_1 \cup (\Sigma \times [0, r])$ and $M_2(r) = M_2 \cup (\Sigma \times [-r, 0])$, respectively. (Here the boundary $\Sigma = \partial M_1$ of M_1 is glued to $\Sigma \times 0$ in $\Sigma \times [0, r]$; similarly, the boundary $\Sigma = \partial M_2$ is glued to $\Sigma \times 0$ in $\Sigma \times [-r, 0]$.) Let $L_j(u; M_j(r))$ denote the infinite-dimensional Lagrangian in $L^2(\hat{E})$ obtained in the same way as $L_j(u)$ except for replacing M_j by $M_j(r)$ and ∂M_j by the boundary $\partial M_j(r) = \Sigma \times (\pm r)$. As we stretch r to ∞, we have an open manifold denoted by $M_j(\infty)$. All these $M_j(r)$'s have a unique operator $D(u)$ of Atiyah-Parodi-Singer type (2.1), which restricted to M_j gives back our operator $D(u)$ on M_j.

PROPOSITION 2.6 *Let the space of L^2-solutions of the equation $D(u)\phi = 0$ on the manifold $M_j(\infty)$ be denoted by $\nu_j(u)$. Suppose that $\nu_1(u) = \nu_2(u) = 0$; then*

$$\begin{aligned} &\lim_{r\to\infty} L_1(u; M_1(r)) = L_1(u; 0) \oplus P_-(u; 0)\,, \\ &\lim_{r\to\infty} L_2(u; M_1(r)) = L_2(u; 0) \oplus P_+(u; 0)\,. \end{aligned} \tag{2.32}$$

The proofs of Propositions 2.2 through 2.6 will be given in the next section.

3 Proofs of Propositions 2.2 Through 2.6

According to Seeley [19], suppose we are given a self-adjoint elliptic operator $D(u) : \Gamma(\mathbb{E}) \to \Gamma(\mathbb{E})$ over a closed manifold $M = M_1 \cup M_2$ as in our setting. Suppose, in addition, that

$$D(u) \text{ is invertible (i.e., } \ker D(u) = 0). \tag{3.1}$$

Then Seeley proves that $\Gamma(\hat{\mathbb{E}})$ is the direct sum

$$\Gamma(\hat{\mathbb{E}}) = L_1(u)_\infty \oplus L_2(u)_\infty \tag{3.2}$$

and similarly for the L^2-completions

$$L^2(\hat{\mathbb{E}}) = L_1(u) \oplus L_2(u)\,. \tag{3.3}$$

Furthermore, in [19], Seeley gives an explicit formula for the projection

$$\Pi(L_1(u)) : L^2(\hat{\mathbb{E}}) \longrightarrow L_1(u) \tag{3.4}$$

onto $L_1(u)$ associated to the decomposition. That is, $\Pi(L_1(u))(a + b) = a$ for $a \in L_1(u), b \in L_2(u)$.

Now given M_j, we can form the double $\widetilde{M_j}$ of M_j by gluing two copies of $M_j \cup (\Sigma \times [-1, +1])$ along this collar neighborhood of its boundary. Here the point (x, s) in the first collar is glued to the point $(x, -s)$ in the second collar. In a similar manner but using the bundle automorphism $\pi^*(\sigma_u)$ in addition to the reflection above, we construct a bundle $\widetilde{E}_j \to M_j$. As in [6], from the Atiyah-Patodi-Singer condition (2.2) on $D(u)$ and from the restriction $D_j(u) = D(u) \mid M_j : \Gamma(\mathbb{E} \mid M_j) \longrightarrow \Gamma(\mathbb{E} \mid M_j)$, we may construct an operator of Atiyah-Patodi-Singer type

$$\widetilde{D}_j(u) : \Gamma(\widetilde{E}_j) \longrightarrow \Gamma(\widetilde{E}_j) \tag{3.5}$$

over the double $\widetilde{M}_j$. This construction of Booss and Wojciechowski [6] has the following properties (j is fixed):

(3.6a) The smooth sections of $\widetilde{E}_j \to \widetilde{M}_j$ are in one-to-one correspondence with pairs of sections (f_1, f_2) of $\mathbb{E} \mid [M_j \cup (\Sigma \times [-1,1])]$ such that $f_2(x,s) = \sigma_u f_1(x,-s)$ over $\Sigma \times [-1,1]$.

(3.6b) $\widetilde{D}_j(u)(f_1, f_2) = (D_j(u) f_1, D_j(u) f_2)$,

(3.6c) $\ker \widetilde{D}_j(u) = 0$.

In particular, the hypothesis of Seeley's theorem is satisfied. Since the Lagrangians in question are, respectively, $L_j(u)_\infty$ and $\sigma_u L_j(u)_\infty$, this proves that $\Gamma(\hat{\mathbb{E}})$ is the direct sum $L_j(u)_\infty \oplus \sigma_u L_j(u)_\infty$, and its L^2-completion is the sum $L_j(u) \oplus \sigma_u L_j(u)$ as in Proposition 2.2(a).

Recall from Part I the formula

$$(D_j(u)f, g)_{M_j} - (f, D_j(u))_{M_j} = \pm\{f \mid \Sigma,\ g \mid \Sigma\}_u$$

for two smooth sections f and g of $E \mid M_j$. In particular,

$$\{v_1, v_2\}_u \equiv \mathrm{Re}\langle v_1, -\sigma v_2\rangle_u \equiv \mathrm{Im}\langle v_1, v_2\rangle_u = 0$$

for v_1 and v_2 in $L_j(u)_\infty$. Thus the above decompositions are real orthogonal. Furthermore, if $\{\phi_l\}$ is a complete orthonormal basis for $L_j(u)$ with respect to the real part of the Hermitian inner product, then $0 = \{\phi_k, \phi_l\} = -\mathrm{Im}\langle\phi_k, \phi_l\rangle_u$ and $\mathrm{Re}\langle\phi_k, \phi_l\rangle_u = \delta_{k,l}$. Since by Seeley's result applied to $\widetilde{M}_j$, $\Gamma(\hat{\mathbb{E}}) = L_j(u)_\infty \oplus \sigma_u L_j(u)_\infty$, it also follows that these $\{\phi_l\}$ form a complete orthonormal basis of the complex Hilbert space $(L^2(\hat{\mathbb{E}}), J_u, \langle\cdot,\cdot\rangle_u)$. Thus $L_j(u)$ is indeed a Lagrangian as claimed in Proposition 2.2.

As a corollary of the above discussion, the projection

$$\Pi(L_j(u)) : L^2(\hat{\mathbb{E}}) \to L_j(u) \hookrightarrow L^2(\hat{\mathbb{E}})$$

is an orthogonal projection. Thus $\Pi(\sigma_u L_j(u)) = \mathrm{Id} - \Pi(L_j)$ and the reflection $R(L_j(u))$ about $L_j(u)$ satisfies the formula

$$R(L_j(U)) = -\mathrm{Id} + 2\Pi(L_j(u)) \tag{3.7}$$

To prove Propositions 2.3 through 2.6, we need Seeley's explicit description of the projection $\Pi(L_j(u))$ in terms of $\widetilde{D}_j(u)$ over the double $\widetilde{M}_j$. Since the case $j = 2$ is similar, we concentrate on the $j = 1$ situation. In the double $\widetilde{M}_1$, we have the submanifold $\Sigma \times [-1, +1]$, which contains $\Sigma \times [-1, 0]$ already in M_1. (Recall that M_1 has boundary $\Sigma \times 0$ in our convention.)

For $s_0 \in [-1,1]$, we denote by $R_{s_0} : \Sigma = \Sigma \times s_0 \hookrightarrow \widetilde{M}_1$ the inclusion of the slice $\Sigma \times s_0$ into $\widetilde{M}_1$. As is well-known, R_{s_0} induces a restriction map

$$R_{s_0} : L^2_k(\widetilde{E}) \longrightarrow L^2_{k-1/2}(\hat{\mathbb{E}}) \quad \text{for } k > \frac{1}{2} \tag{3.8}$$

from the Sobolev space $L^2_k(\widetilde{E})$ with k^{th}-order derivative to $L^2_{k-1/2}(\hat{\mathbb{E}})$ where the order is dropped by $\frac{1}{2}$. Since these are Hilbert spaces, there is also the dual map

$$R^*_{s_0} : L^2_k(\hat{\mathbb{E}}) \longrightarrow L^2_{k+1/2}(\widetilde{E})\,. \tag{3.9}$$

For smooth sections Ψ of $\widetilde{E}$, we have $R^*_{s_0}(\Psi) = \Pi^*(\Psi)\cdot\delta(s-s_0)$ where $\delta(s-s_0)$ is the delta function distribution supported on the slice $\Sigma \times s_0$ and Π is the projection of $\Sigma \times [-1,0]$ onto Σ.

Since $\widetilde{D}_1(u)$ is invertible on $\widetilde{M}_1$, we consider, following Seeley [19], the composite

$$S(U)\Psi = \tau[\widetilde{D}_1(u)]^{-1}R^*_0(J_u\Psi)$$

where τ is the restriction map from $\widetilde{M}_1$ to M_1 and Ψ is a smooth section of $\widetilde{E}$. In [19] it is shown that $S(u)$ extends to a continuous linear operator

$$S(u) : L^2(\hat{\mathbb{E}}) \longrightarrow L^2_{1/2}(\widetilde{E} \mid M_1) \tag{3.10}$$

which maps the subspace $\Gamma(\hat{\mathbb{E}})$ of smooth sections of $\hat{\mathbb{E}}$ bijectively onto the space $\ker(D(u) \mid \Gamma(\widetilde{E} \mid M_1))$ (i.e., smooth sections ϕ of $\widetilde{E} \mid M_1$ with $D(u)\phi = 0$). Moreover, the limit

$$T(u) = \lim_{s\to 0, s<0} R_s S(u)$$

exists and gives the desired projection

$$\Pi(L_1(u)) : L^2(u) \longrightarrow L^2(u)$$

mentioned before.

This last assertion can be seen as follows: Given ψ in $\Gamma(\hat{\mathbb{E}})$, by (3.3) we write ψ as the sum $f + \sigma_u g$ where $f = F \mid \Sigma \times 0$, $g = G \mid \Sigma \times 0$, $D(u)F = D(u)G = 0$, and F and G are, respectively, smooth sections of $\widetilde{E}$ over the first copy $M_1 \hookrightarrow \widetilde{M}_1$ and of $\widetilde{E}$ over the second copy of $M_1 \hookrightarrow \widetilde{M}_1$. Then

$$S(u)\psi = \tau(\widetilde{D_1(u)})^{-1}\Pi^*(-\sigma_u(f + \sigma g))\delta(s-0) = \tau(F, -G) = F$$

where (F, G) is the section over $\widetilde{E}$ defined by these sections over the first and second copies of M_1 in $\widetilde{M}_1$. Hence, $T(u)(\psi) = f$.

Now suppose we are given ψ in $P_-(u;K)$. We may write ψ in the form

$$\psi = \sum_{\lambda_l > K} a_l \sigma_u \phi_l \tag{3.11}$$

where the ϕ_l's comprise a complete orthonormal basis of eigenvectors of $\hat{D}(u)$, $\hat{D}(u)\phi_l = \lambda_l\phi_l$, and $\sum |a_l|^2 < \infty$. (Note that $\hat{D}(u)\sigma_u\phi_l = -\lambda_l\phi_l$ here.) Then on $\Sigma \times [-1, 0]$ in M_1, there is the well-defined L^2-section

$$\psi^{\#} = \sum_{\lambda_l > K} a_l e^{\lambda_l s}\Pi^*(\sigma_u\phi_l) \tag{3.12}$$

with the property $D(u)\psi^{\#} = 0$. Let $\rho : M_1 \to [0, \infty)$ be the smooth cutoff function defined by

$$\rho \mid M_1 - \Sigma \times (-\tfrac{3}{4}, 0] \equiv 0\,, \qquad \rho \mid \Sigma \times [-\tfrac{1}{4}, 0] \equiv 1\,,$$
$$\rho(x, s) = h(s) \quad \text{for } (x, s) \in \Sigma \times [-1, 0]\,,$$

where $h(s) \geq 0$ on $[0, 1]$, $h \mid [-1, -\frac{3}{4}] \equiv 0$, $h \mid [-\frac{1}{4}, 0] \equiv 1$, and $|h'(s)| \leq 4$. Using this cutoff function, we can define a section $\psi^{\#\#}$ on $\widetilde{M}_1$ by the formula

$$\psi^{\#\#} = \begin{cases} \rho \cdot \psi^{\#} & \text{on } \Sigma \times [-1, 0], \\ 0 & \text{on } \widetilde{M}_1 - \Sigma \times [-\frac{3}{4}, 0]. \end{cases} \tag{3.13}$$

Clearly, $\psi^{\#\#}$ has a jump discontinuity at $s = 0$, and in the sense of distributions

$$(\widetilde{D_1(u)})(\psi^{\#\#}) = \Pi^*(-\sigma_u\psi)\delta(s - 0) + (dh/ds)\Pi^*(\sigma_u\psi)$$

with the second term a smooth section $\mathbf{m} := (dh/ds)\Pi^*(\sigma_u\psi)$ supported on $\Sigma \times [-\frac{3}{4}, -\frac{1}{4}]$. Consequently,

$$\begin{aligned} \widetilde{D_1(u)}\Big[\psi^{\#\#} - (\widetilde{D_1(u)})^{-1}(\mathbf{m})\Big] &= \Pi^*(J_u\psi)\delta(s - 0) \\ &= R_0^*(J_u\psi)\,. \end{aligned} \tag{3.14}$$

That is, for ψ in $P_-(u; K)$, we have

$$S(u)\psi = \tau\psi^{\#\#} - \tau((\widetilde{D_1(u)})^{-1}(\mathbf{m})\,,$$

and so we have

$$T(u)\psi - \psi = -R_0\tau(\widetilde{D_1(u)})^{-1}(\mathbf{m})\,.$$

Since $\widetilde{D_1(u)}$ is invertible and continuous, we have, by the continuity of spectra, a constant $\Lambda > 0$ such that $\widetilde{D_1(u)}$ has no eigenvalue in the band $(-\Lambda, \Lambda)$, or in other words $||(\widetilde{D_1(u)})^{-1}|| \leq 1/\Lambda$. Hence, we have

$$\begin{aligned} \|T(u)\psi - \psi\|^2 &\leq \|R_0\|^2\|\tau\|^2\left(\frac{1}{\Lambda}\right)^2\|\mathbf{m}\|^2 \\ &\leq \|R_0\|^2(1/\Lambda)^2(4)^2\|\psi^{\#}|\Sigma \times [-\tfrac{3}{4}, -\tfrac{1}{4}]\|^2\,. \end{aligned}$$

On the other hand, by the exponential decay of $\psi^{\#}$ in (3.13),

$$\|\psi^{\#}|\Sigma \times [-\tfrac{3}{4}, -\tfrac{1}{4}]\|^2 \leq \frac{1}{2}\|\psi^{\#}|\Sigma \times (-\tfrac{1}{4})\|^2$$

$$= \frac{1}{2} \sum_{\lambda_l > K} |a_l e^{-\frac{1}{4}\lambda_l}|^2$$
$$\leq e^{-\frac{K}{2}} \sum |a_l|^2 \,.$$

This yields the following basic estimate:

PROPOSITION 3.1 *Let Λ be chosen (as is possible) so that the family $\{\widetilde{D_1(u)} : 0 \leq u \leq 1\}$ of operators on $\widetilde{M}_1$ does not have eigenvalues in the range $[-\Lambda, \Lambda]$. Then*

(a) $\|\Pi(L_1(u))\psi - \psi\| \leq 4\|R_0\| \left(\frac{1}{\Lambda}\right) e^{-\frac{K}{4}} \|\psi\|$*for ψ in $P_-(u; -K)$ and*

(b) $\|\Pi(L_1(u))\psi\| \leq 4\|R_0\| \left(\frac{1}{\Lambda}\right) e^{-\frac{K}{4}} \|\psi\|$*for ψ in $P_+(u; K)$.*

In (b) we use the fact that $\Pi(\sigma_u(u)) = \mathrm{Id} - \Pi(L_1(u))$, and so

$$\|\Pi(L_1(u))\psi\| = \|(\mathrm{Id} - \Pi(L_1(u))\psi - \psi\| \leq 4\|R_0\| \left(\frac{1}{\Lambda}\right) e^{-K/4} \|\psi\|$$

for ψ in $P_+(u; K)$.

PROPOSITION 3.2 *Let $K > 0$ be chosen so that $4\|R_0\|(1/\Lambda)e^{-K/4} \leq \frac{1}{2}$. Then for $u \in [0, 1]$, the orthogonal projection $\Pi[K] : L^2(\hat{E}) \to P_-(u; K)$ is an injection when it is restricted to $L_1(u)$.*

The proof follows immediately from Proposition 3.1(b) because if $\psi \in L_1(u)$ and $\Pi[-\infty, K]\psi = 0$, then $\psi \in P_+(u; K)$ and so $\|\psi\| \leq \frac{1}{2}\|\psi\|$, which shows $\psi = 0$.

PROPOSITION 3.3 *Let $K > 0$ be chosen so that $4\|R_0\|(1/\Lambda)e^{-K/4} \leq \frac{1}{2}$. Let $\Pi(L_1(u; K))$ denote the orthogonal projection of $\mathcal{H}(u; K)$ onto the finite Lagrangian subspace $L_1(u; K)$. Then, for $\psi \in \mathcal{H}(u; K)$, we have*

$$\|\Pi(L_1(u))\psi - \Pi(L_1(u; K))\psi\| \leq 2\|\psi\|$$

PROOF: Note there is an orthogonal decomposition

$$\mathcal{H}(u; K) = L_1(u; K) \oplus \sigma_u L_1(u; K) \,.$$

Hence, we can write $\psi \in \mathcal{H}(u; K)$ as a sum $f + \sigma_u g$ where f and g are in $L_1(u; K)$ and $\Pi(L_1(u; K))\psi = f$. By the definition of $L_1(u; K)$ in (2.25), there exist $f^\#$ and $g^\#$ in $P_+(u; K)$ such that both $f + f^\#$ and $g + g^\#$ are in $L_1(u)$.

A straightforward computation shows

$$\|f^\#\|^2 = -\{f + f^\#, (\mathrm{Id} - \Pi(L_1(u))\sigma_u f^\#\} \,,$$

and by Proposition 3.1(b)

$$\|(\mathrm{Id} - \Pi(L_1(u))\sigma_u f^\#\| \leq 4\|R_0\|(1/\Lambda)e^{-K/4}\|f^\#\| \,.$$

Hence, $\|f^{\#}\| \leq 4\|R_0\|(1/\Lambda)e^{-K/4}\|f + f^{\#}\|$. Also, by the Schwarz inequality and the assumption on K, the right-hand side is less than

$$4\|R_0\|\frac{1}{\Lambda}e^{-K/4}\|f\| + \frac{1}{2}\|f^{\#}\|\,.$$

Hence we get an estimate for $\|f^{\#}\|$ in terms of $\|f\|$. Similarly, we get the estimate $\|g^{\#}\| \leq 8\|R_0\|(1/\Lambda)e^{-K/4}\|g\|$. On the other hand,

$$\begin{aligned}
&\Pi(L_1(u))\psi - \Pi(L_1(u;K))\psi \\
&\quad = \Pi(L_1(u))(f + f^{\#} + \sigma(g + g^{\#})) - \Pi(L_1(u))(f^{\#} + \sigma g^{\#}) - f \\
&\quad = (f + f^{\#}) - \Pi(L_1(u))f^{\#} - \Pi(L_1(u))(\sigma g^{\#}) - f\,,
\end{aligned}$$

and so

$$\begin{aligned}
\|\Pi(L_1(u))\psi - \Pi(L_1(u;K)\psi\| &\leq \|f^{\#}\| + \|\Pi(L_1(u)f^{\#}\| + \|\Pi(L_1(u))(\sigma g^{\#})\| \\
&\leq 2(\|f^{\#}\| + \|g^{\#}\|) \\
&\leq 16\|R_0\|\frac{1}{\Lambda}e^{-\frac{K}{4}}(\|f\| + \|g\|) \\
&= 16\|R_0\|\frac{1}{\Lambda}e^{-\frac{K}{4}}\|\psi\|
\end{aligned}$$

as claimed. ☐

We are now in a position to compare the projections $\Pi(L_1(u))$ and $\Pi(L_1(u;K) \oplus P_-(u;K))$.

PROPOSITION 3.4 *Let $K > 0$ be chosen so that $\|R_0\|(1/\Lambda)e^{-K/4} < \frac{1}{2}$. Then for $\psi \in \mathcal{H}(u;K)$ we have*

(a) $\|\Pi(L_1(u))\psi - \Pi(L_1(u;K) \oplus P_-(u;-K))\psi\| \leq 2\|\psi\|$ *and*
(b) $\|R(L_1(u)) \cdot [R(L_1(u;K) \oplus P_-(u;-K))]^{-1}\psi - \psi\| \leq 12\|\psi\|$.

On the other hand, when ψ_l is an eigensolution of $\hat{D}(u)$, $\hat{D}(u)\psi_l = \lambda_l\psi_l$ with $\lambda_l > K$ or $\lambda_l < -K$, we have

(c) $\|\Pi(L_1(u))\psi_l - \Pi(L_1(u;K) \oplus P_-(u;-K))\psi_l\| \leq 4\|R_0\|(1/\Lambda)e^{-|\lambda_l|/4}\|\psi_l\|$ *and*
(d) $\|R(L_1(u)) \cdot [R(L_1(u;K) \oplus P_-(u;-K))]^{-1}\psi_l - \psi_l\| \leq$ $24\|R_0\|(1/\Lambda)e^{-|\lambda_l|/4}\|\psi_l\|$.

PROOF: (a) is a reformulation of Proposition 3.3, while (c) is a reformulation of Proposition 3.1 with K being replaced by $|\lambda_l|$. As in (3.7), given two Lagrangians L_+ and L_-, we have $R(L_\pm) = -\mathrm{Id} + 2\Pi(L_\pm)$ where $\Pi(L_\pm)$ are the orthogonal projections onto $L_\pm$, respectively. Then

$$\begin{aligned}
R(L_+)R(L_-) &= (-\mathrm{Id} + 2\Pi(L_+))(-\mathrm{Id} + 2\Pi(L_-)) \\
&= \mathrm{Id} - 2\Pi(L_+) - 2\Pi(L_-) + 4\Pi(L_+)\Pi(L_-)
\end{aligned}$$

$$= \mathrm{Id} + 2(\Pi(L_+) - \Pi(L(_-)) + 4\Pi(L_+)(\Pi(L_-) - \Pi(L_+))\,.$$

Hence,

$$\|R(L_+)R(L_-)^{-1}\psi - \psi\| \le 6\|\Pi(L_+) - \Pi(L_-)\|\,.$$

By letting $L_+ = L_1(u)$ and $L_- = L_1(u;K) \oplus P_-(u;-K)$, we obtain (b) and (d).

As is well-known [13], the number $N(\lambda)$ of eigenmodes with $|\lambda_l| \le N$ has an asymptotic of the form $c_o\lambda^{(n-1)/2} + O(\lambda^{(n-2)/2})$. On the other hand, when we apply $[R(L_1(u))\; R(L_1(u;K) \oplus P_-(u;-K))^{-1} - \mathrm{Id}]$ to ψ_l, by the above it decays at an exponential rate $e^{-|\lambda_l|/4}$. Comparing these two asymptotics, it follows that $[R(L_1(u))R(L_1(u;K) \oplus P_-(u;-K))^{-1} - \mathrm{Id}]$ is trace-class for all K with

$$\|R_0\|\frac{1}{\Lambda}e^{-\frac{|\lambda_l|}{4}} \le \frac{1}{2}\,.$$

Since a change in K only changes $\Pi(L_1(u;K) \oplus P_-(u;-K))$ by a finite operator, this trace-class property holds for all K. This proves Proposition 2.4(a). ☐

Note that the above argument is independent of u. In particular, the determinant $\det[R(L_1(u))(R(L_1(u;K) \oplus P_-(u;-K))^{-1}]$ is well-defined and approximates 1 in a uniform manner.

Remaining unmatched left paren in 2nd line above Prop. 3.5. Please fix.

PROPOSITION 3.5 (a) *The projection* $\Pi(-\infty,0) : L^2(\hat{E}) \to P_-(u;0)$ *maps* $L_1(u)$ *onto the subspace* $\{\phi \in P_-(u;0) : \{\phi,\nu_1(u)\} = 0\}$ *in* $P_-(u;0)$ (*i.e., elements annihilated under the symplectic pairing* $\{\cdot,\cdot\}$ *by* $\nu_1(u) = L_1(u) \cap P_+(u;0)$).

(b) *Suppose* $K > 0$ *is chosen so that* $4\|R_0\|(1/\Lambda)e^{-K/4} \le \frac{1}{2}$. *Then the projection* $\Pi(-\infty,-K)\colon L^2(\hat{E}) \to P_-(u;-K)$ *maps* $L_1(u)$ *onto* $P_-(u;-K)$.

PROOF: Since $L_1(u)$ is isotropic, it is easy to see that $\Pi(-\infty,0)L_1(u)$ is contained in $\phi \in P_-(u;0)$ with $\{\phi,\nu_1(u)\} = 0$. On the other hand, $L_1(u) \cap P_+(u;-K]$ is finite-dimensional and consists of smooth sections, and its image under $\Pi(-\infty,0)$ is the subspace $\{\phi : \phi$ lies in $\mathcal{H}(u;K)$ and $\{\phi,\nu_1(u)\} = 0\}$ by lemma 2.5 of Part II. As we vary K, the last subspace forms a dense subspace in $\{\phi \in P_-(u;0) : \{\phi,\nu_1(u)\} = 0\}$. This proves (a) and the fact that the smooth sections in $\{L_1(u) \cap P_+(u;-K) : K > 0\}$ are dense in $L_1(u)$. As for (b), we consider the linear map

$$\begin{aligned} \nu_1(u) &\longrightarrow \mathbb{R}\,,\\ \psi &\longmapsto \{\sigma_u\phi_l,\psi\}\,, \end{aligned}$$

where ϕ_l is a fixed eigensection of $\hat{D}(u)$, $\hat{D}(u)\phi_l = \lambda_l\phi_l$ with $\lambda_l > K$ (i.e., $\sigma_u\phi_l \in P_-(u;-K)$). By Proposition 3.2, $L_1(u)$ maps injectively into $P_-(u;K)$ and so does $\nu_1(u)$ into $\mathcal{H}(u;K) \cap P_+(u;0)$. Hence there exists a unique element $\alpha \in \mathcal{H}(u;K)$ such that

$$\{\sigma_u\phi_l,\psi\} = \{\alpha,\psi\} \quad \text{for all } \psi \in \nu_1(u).$$

It follows that $\{\sigma_u\phi_l - \alpha, \nu_1(u)\} = 0$, and we can choose β in $L_1(u) \cap P_+(u;-K)$ so that $\sigma_u\phi_l - \alpha - \beta = \gamma$ where γ lies in $P_+(u;K)$. In particular, $\sigma_u\phi_u = \Pi(-\infty,-K)\beta$, which proves (b).

In view of Proposition 3.5, the composite

$$L^2(\hat{E}) \xrightarrow{S(u)} L^2(\hat{E}) \xrightarrow{\Pi(-\infty,-K)} P_-(u;-K)$$

is onto when $4\|R_0\|(1/\Lambda)e^{K/4} \le \frac{1}{2}$. For $u \in V(K)$, both $S(u)$ and the projection $\Pi(-\infty,-K)$ vary smoothly with respect to u, and so does their composition. The kernel of this composition can be identified with the finite-dimensional Lagrangian $L_1(u;K)$ under the projection $\Pi[-K,K] : L^2(\hat{E}) \to \mathcal{H}(u;K)$. Thus $L_1(u;K)$ over $V(K)$ represents a smooth family of Lagrangians in the smooth symplectic vector bundle $\{\mathcal{H}(u;K) : u \in V(K)\}$ if K satisfies the above inequality. Consequently, there exists K^*; for $K > K^*$, we have $4\|R_0\|(1/\Lambda)e^{K/4} \le \frac{1}{2}$, and the determinant $\det(L_1(u;K) \oplus P_-(u;-K))$ is a smooth section of $\det(\hat{D}_J \mid V(K))$. By the uniform approximation results of Proposition 3.4, the sections $s_1(u)$ and $s_2(u)$ are also smooth. Putting these results together, we obtain the proof of Propositions 2.3 and 2.4.

We now turn to the proof of Proposition 2.6. Since by assumption $\nu_1(u) = 0$ and since $L_1(u) \cap \sigma_u L_1(u) = 0$, we can apply the results of Part I to conclude that for r sufficiently large, the operator $\widetilde{D_1(u)}$ on $\widetilde{M}_1(r)$ does not have any eigenvalue in the range $[-1/r^2, 1/r^2]$.

Let $|\lambda_0|$ be the lowest nonzero eigenvalue of $\hat{D}(u)$ on Σ. In the argument proving Proposition 3.4, we constructed a cutoff function $h(s)$ on the interval $[-1,0]$ and used it to define sections over $\Sigma \times [-1,1]$ in $\widetilde{M}_1$. For the case of $\widetilde{M}_1(r)$ we have the product $\Sigma \times [-1,r]$ embedded into the double $\widetilde{M}_1(r)$. In this case we may replace the cutoff function $h(s)$ by $h^*(s)$ defined by $h^*(s) = h(s)$ for $s \in [-1,0]$ and $h^*(s) = 1$ for $s \in [0,r]$. Since the eigensections of $D(u)$ on $M_1(r)$ decay exponentially, we have a simple improvement of the estimates of Proposition 3.4. Rerunning that analysis gives the following result: For $\psi \in P_-(u;M_1(r))$, then

$$\|\Pi(L_1(u;M_1(r))\psi - \psi\| \le 4\|R_0\|_{\widetilde{M}_1(r)} \frac{1}{\Lambda(r)} e^{-|\lambda_0|\frac{r+1}{4}} \|\psi\| \,.$$

Since

$$\|R_0\|_{\widetilde{M}_1(r)} \le \|R_0\|_{\widetilde{M}_1}$$

and $\Lambda(r) \le 1/r^2$ for r sufficiently large, we can make sure that the right-hand side of the inequality is less that $e^{-|\lambda_0|r/8}$ and, in particular, that

$$\lim_{r\to\infty} \|\Pi(L_1(u;M_1(r))\psi - \psi\| = 0 \,.$$

Repeating the same argument for $\widetilde{M_2(r)}$, the proof of Proposition 2.6 is complete. $\square$

4 Maslov Index in a Hermitian Setting

Let X be a compact, smooth, oriented manifold with boundary Σ. Let $\Sigma \times [-1,0]$ be a collar neighborhood of Σ in W with the coordinates t of $[-1,0]$ pointing outward. In particular, $\Sigma \times 0$ is identified with $\Sigma = \partial X$. Over X, there is a smooth *complex* vector bundle $\mathbf{E}$ that is endowed with a Hermitian structure $\langle \cdot, \cdot \rangle_{\mathbf{E}}$. That is, smoothly varying over each fiber $\mathbf{E}_x$, $x \in W$, we have a Hermitian pairing $\langle \cdot, \cdot \rangle_{\mathbf{E}_x} : \mathbf{E}_x \otimes \mathbf{E}_x^* \to \mathbb{C}$ whose real part $(\cdot, \cdot)_{\mathbf{E}} = \mathrm{Re}\langle \cdot, \cdot \rangle_{\mathbf{E}}$ gives us a Riemannian metric structure. This Hermitian structure induces a corresponding product structure $\langle \cdot, \cdot \rangle$ on the space $\Gamma(\mathbf{E})$ of C^∞-sections on $\mathbf{E}$.

Similar to the real setting, we consider a first-order elliptic operator $D\colon \Gamma(\mathbf{E}) \to \Gamma(\mathbf{E})$ that is *complex linear and self-adjoint* $D = D^*$ with respect to the Hermitian structure $\langle \cdot, \cdot \rangle_{\mathbf{E}}$. In addition, this Hermitian self-adjoint operator D satisfies the Atiyah-Patodi-Singer condition on $\Sigma \times [-1, 0]$,

$$D = \pi^* \sigma \circ \left(\frac{\partial}{\partial t} + \pi^* \hat{D} \right)$$

where $\hat{\mathbf{E}} = \mathbf{E} \mid \Sigma$, $\sigma\colon \hat{\mathbf{E}} \to \hat{\mathbf{E}}$ is a bundle automorphism, $\hat{D}\colon \Gamma(\hat{\mathbf{E}}) \to \Gamma(\hat{\mathbf{E}})$ is a self-adjoint operator, and π^* stands for the pullback induced by the projection $\pi\colon \Sigma \times [-1, 0] \to \Sigma$.

Now the complex linearity condition on D implies that both σ and $\hat{D}$ are complex linear, and in addition,

(4.1)
- (i) $\sigma^* = \sigma$, i.e., σ is Hermitian skew-adjoint,
- (ii) $\hat{D}$ is a first-order, elliptic, Hermitian, self-adjoint operator on $\Gamma(\hat{\mathbb{E}})$, and
- (iii) $\sigma \hat{D} = -\hat{D} \sigma$.

Note that, from (2.4) and (2.5), $-\sigma^2 = \sigma\sigma^*$ is in general a positive self-adjoint operator. With our applications in mind, we will assume throughout that

$$\sigma^2 = -\mathrm{Id}\,. \tag{4.2}$$

The above condition, (4.2), can be viewed as introducing a new complex structure $(\mathbf{E}, J)$ on $\mathbf{E}$ with $J = -\sigma$ as the multiplication by the imaginary element $\sqrt{-1}$. Denote the original complex structure on $\mathbf{E}$ by $(\mathbf{E}, i)$. Then since the two complex multiplications commute with each other, $Ji = iJ$, there is a decomposition of $\Gamma(\hat{\mathbf{E}})$ into (± 1)-eigenspaces $\Gamma(\hat{\mathbf{E}})_\pm$ of Ji:

$$\Gamma(\hat{\mathbf{E}}) = \Gamma(\hat{\mathbf{E}})_+ \oplus \Gamma(\hat{\mathbf{E}})_- \quad \text{where } \Gamma(\hat{\mathbf{E}})_\pm = \{s : Jis = \pm s\} = \{s : \sigma s = \pm is\}.$$

Note that Ji is self-adjoint, $(Ji)^* = Ji$, and so the above decomposition is also an orthogonal decomposition.

Let $\Pi_\pm\colon \Gamma(\mathbf{E}) \to \Gamma(\mathbf{E})_\mp$ denote the orthogonal projection onto $\Gamma(\mathbf{E})_\mp$. Then $\Pi_\pm v = 1/\sqrt{2}(v \pm Jv)$ and anticommutes with $\hat{D}$, $\hat{D} \circ \Pi_\pm = \Pi_\mp \hat{D}$. It follows that

$\hat{D}$ induces two complex operators

$$\hat{D}_{\pm} = (\hat{D} \mid \Gamma(\mathbf{E})_{\pm}) \colon \Gamma(\mathbf{E})_{\pm} \to \Gamma(\mathbf{E})_{\mp}$$

on these eigensubspaces and can be expressed as the sum of these two operators. In matrix form, we have

$$\hat{D} = \begin{pmatrix} 0 & \hat{D}_- \\ \hat{D}_+ & 0 \end{pmatrix} : \begin{matrix} \Gamma(\mathbf{E})_+ \\ \oplus \\ \Gamma(\mathbf{E})_- \end{matrix} \longrightarrow \begin{matrix} \Gamma(\mathbf{E})_+ \\ \oplus \\ \Gamma(\mathbf{E})_- \end{matrix} \tag{4.3}$$

Both operators $\hat{D}_{\pm}$ are first-order elliptic over Σ, and the formal adjoint of one is equal to the other, $(\hat{D}_{\pm})^* = \hat{D}_{\mp}$.

In the situation of a smooth family of operators $\{\hat{D}(u) : u \in B\}$ over a parameter space B, the above decomposition leads to two continuous families $\{\hat{D}_{\pm}(u) : u \in B\}$ of operators. Following Quillen [21], we consider the *determinant line bundle* $\det\{\hat{D}_{\pm}(u) \mid u \in B\}$ over B whose fiber at $u \in B$ can be identified with $\det(\ker D_{\pm}(u)) \otimes \det(\operatorname{coker} D_{\pm}(u))^*$. Since $\ker D_{\pm}(u) = \operatorname{coker} D_{\mp}(u)$, the two complex line bundles $\det\{\hat{D}_{+}(u) \mid u \in B\}$ and $\det\{\hat{D}_{-}(u) \mid u \in B\}$ are dual to each other, and therefore we can concentrate on just one, say, $\det\{\hat{D}_{-}(u) \mid u \in B\}$. To justify this claim, we consider a small K-band of eigenmodes

$$\mathcal{H}(u, K) = \operatorname{span}\{\phi : \hat{D}(u)\phi = \mu\phi,\ |\mu| \leq K\}.$$

This last complex vector space is finite-dimensional and invariant under σ. Hence, as before, there is an orthogonal decomposition

$$\mathcal{H}(u, K) = \mathcal{H}(u, K)_+ \oplus \mathcal{H}(u, K)_-$$

where $\mathcal{H}(u, K)_{\pm}$ can be identified with the corresponding band of eigenmodes for the operators $\hat{D}(u)_{\pm}$. Around a neighborhood of u, the fiber $\det\, \hat{D}_{\pm}(u)$ of the determinant line bundle is $\bigwedge^{\max} \mathcal{H}_{\pm}(u, K)^* \otimes \bigwedge^{\max} \mathcal{H}_{\mp}(u, K)$ for some K. From this, it is easy to see that the line bundles $\det\, \hat{D}_{\pm}$ are locally isomorphic, and by the usual partition-of-unity argument, we can patch up the local isomorphisms together to obtain a global isomorphism $\det\, \hat{D}_+ \simeq \det\, \hat{D}_-$.

The space $\mathcal{H}(u, K)$ is a symplectic vector space with respect to the pairing $\{\alpha, \beta\} = (\alpha, \sigma(u)\beta)$. In Part II, we considered various Lagrangian subspaces $\mathcal{L}(u, K)$, $L(u) \oplus [P_{\pm}(u) \cap \mathcal{H}(u, K)]$ of this symplectic space. For instance, $\mathcal{L}(u, K)$ is constructed by first considering the solutions $D(\psi) = 0$ of D and then projecting these solutions ψ onto $\Pi[K](\psi \mid \Sigma)$ in $\mathcal{H}(u, K)$ via the orthogonal projection $\Pi[K] \colon \Gamma(E) \to \mathcal{H}(u, K)$. In the present setting, $\mathcal{H}(u, K)$ has a Hermitian structure $(\mathcal{H}(u, K), i, \langle\cdot,\cdot\rangle_J)$ defined by

$$\langle \alpha, \beta \rangle_J = (\alpha, \beta)_\Sigma + \sqrt{-1}(\alpha, J\beta)_\Sigma\,. \tag{4.4}$$

With respect to this new structure, $\mathcal{L}(u, K)$ and $L(u) \oplus [P_{\pm}(u) \cap \mathcal{H}(u, K)]$ are "complex" Lagrangian subspaces. To avoid repetition, we will explain this property only for $\mathcal{L}(u, K)$ as in the following:

PROPOSITION 4.1 (i) *$\mathcal{L}(u,K)$ is a complex subspace of $(\mathcal{H}(u,K),i)$.*

(ii) *$\mathcal{L}(u,K)$ is a totally real subspace with respect to the Hermitian pairing $\langle\cdot,\cdot\rangle_J$, i.e., $\langle\alpha,\beta\rangle_J \in \mathbb{R}$ for all $\alpha,\beta \in \mathcal{L}(u,K)$.*

(iii) $\dim_{\mathbb{C}} \mathcal{L}(u,K) = \frac{1}{2}\dim_{\mathbb{C}} \mathcal{H}(u,K)$.

(iv) $\mathcal{L}(u,K) \cap \mathcal{H}(u,K)_+ = 0$.

(v) *There exists a unique unitary isomorphism $\varphi\colon \mathcal{H}(u,K)_+ \to \mathcal{H}(u,K)_-$ such that $L(u,K)$ coincides with the graph Δ_φ of φ,*

$$\Delta_\varphi = \{v \oplus \varphi(v) \in \mathcal{H}(u,K)_+ \oplus \mathcal{H}(u,K)_-\}\,.$$

PROOF: Assertion (i) follows from the definition of $\mathcal{L}(u,K)$ and the fact that D and $\hat{D}$ have complex eigensections for a given eigenvalue. As for (ii), the imaginary component $\operatorname{Im}\langle\cdot,\cdot\rangle_J$ of $\langle\cdot,\cdot\rangle_J$ is related to the symplectic pairing $\{\cdot,\cdot\}$ by the formula

$$\{\alpha,\beta\} = (\alpha,\sigma(u)\beta) = -\operatorname{Re}\langle\alpha,J\beta\rangle_J = -\operatorname{Im}\langle\alpha,\beta\rangle_J\,.$$

Thus the vanishing of $\operatorname{Im}\langle\alpha,\beta\rangle_J$ is a reformulation of the isotropy property of $\mathcal{L}(u,K)$. In the same manner, (iii) is a reformulation of the real dimension count, $\dim_{\mathbb{C}} \mathcal{L}(u,K) = \frac{1}{2}\dim_{\mathbb{R}} \mathcal{H}(u,K)$.

To prove (iv), we write an element $v = \mathcal{H}(u,K)$ as the sum $v_1 \oplus v_2$ of its two orthogonal components $v_1 \in \mathcal{H}(u,K)_+$ and $v_2 \in \mathcal{H}(u,K)_-$. Suppose $v = v_1 \oplus v_2$ is an element in $\mathcal{L}(u,K)$; then from (i) the multiple $iv = iv_1 \oplus iv_2$ is also in $\mathcal{L}(u,K)$. On the other hand, from the isotropy property of $\mathcal{L}(u,K)$, we have

$$\begin{aligned} 0 = \{v,iv\} &= (v_1 \oplus v_2,\ \sigma(iv_1 \oplus iv_2)) \\ &= (v_1 \oplus v_2,\ -v_1 \oplus v_2) \\ &= -\|v_1\|^2 + \|v_2\|^2 \end{aligned}$$

and so $\|v_1\|^2 = \|v_2\|^2 = \frac{1}{2}\|v\|^2$. Hence, neither of the components v_1 or v_2 can be zero unless $v = 0$. This proves $\mathcal{L}(u,K) \cap \mathcal{H}(u,K)_+ = 0$.

Let $\alpha_\pm$ denote the composite

$$\mathcal{L}(u,K) \hookrightarrow \mathcal{H}(u,K) \xrightarrow{\Pi_\pm} \mathcal{H}(u,K)_\pm$$

of the inclusion with the projection $\Pi_\pm = (1/\sqrt{2})(1 \pm J)$. From (iv), these mappings $\alpha_\pm$ are one-to-one. On the other hand, from the formula

$$\dim_{\mathbb{C}} \mathcal{H}(u,K) = \dim_{\mathbb{C}} \mathcal{H}(u,K)_+ + \dim \mathcal{H}(u,K)_-$$

and from (iii), we have

$$\dim_{\mathbb{C}} \mathcal{L}(u,K) = \dim_{\mathbb{C}} \mathcal{H}(u,K)_\pm\,,$$

and hence $\alpha_\pm$ are isomorphisms by dimension count. Let

$$\varphi\colon \mathcal{H}(u,K)_+ \to \mathcal{H}(u,K)_-$$

denote the composite isomorphism $\alpha_- \circ \alpha_+^{-1}$. Then

$$\mathcal{L}(u,K) = \{\alpha_+(v) \oplus \alpha_-(v) : v \in \mathcal{L}(u,K)\}$$

$$
\begin{aligned}
&= \{v \oplus \varphi(v) : v \in \mathcal{H}(u,K)_+\} \\
&= \Delta_\varphi\,.
\end{aligned}
$$

Before showing that φ is unitary, let us clarify the Hermitian structures on $\mathcal{H}(u,K)_\pm$. First, whenever a complex or Hermitian structure is not explicitly mentioned, the convention is to take the structure inherited from $(\Gamma(\mathbf{E}), i, \langle\cdot,\cdot\rangle)$. Now there are new complex structures $(\mathcal{H}_\pm(u,K), J)$ specified by J. From the definition of $\mathcal{H}_\pm(u,K)$ as eigenspaces, we have

$$
\begin{aligned}
(\mathcal{H}(u,K)_-, J) &= (\mathcal{H}(u,K)_-, i) = \mathcal{H}(u,K)_- \\
(\mathcal{H}(u,K)_+, J) &= (\mathcal{H}(u,K)_+, -i) = \overline{\mathcal{H}(u,K)_+} \\
(\mathcal{H}(u,K), J) &= \mathcal{H}(u,K)_- \oplus \overline{\mathcal{H}(u,K)_+}
\end{aligned}
\tag{4.5}
$$

where $\overline{\mathcal{H}(u,K)_+}$ stands for the conjugate complex structure. As for the Hermitian pairings on these spaces, we have

$$
\begin{aligned}
\langle\cdot,\cdot\rangle_J \mid \mathcal{H}(u,K)_+ \times \mathcal{H}(u,K)_+ &= \overline{\langle\cdot,\cdot\rangle}\,, \\
\langle\cdot,\cdot\rangle_J \mid \mathcal{H}(u,K)_- \times \mathcal{H}(u,K)_- &= \langle\cdot,\cdot\rangle.
\end{aligned}
\tag{4.6}
$$

Returning to the proof of (v), we observe that

$$
\begin{aligned}
0 &= \{\alpha \oplus \varphi(\alpha), \beta \oplus \varphi(\beta)\} = \mathrm{Re}\langle \alpha \oplus \varphi(\alpha), \sigma(\beta \oplus \varphi(\beta))\rangle \\
&= \mathrm{Re}\langle \alpha \oplus \varphi(\alpha), i\beta \oplus -i\varphi(\beta)\rangle = \mathrm{Re}\,[\langle \alpha, i\beta\rangle + \langle \varphi(\alpha), -i\varphi(\beta)\rangle] \\
&= \mathrm{Re}\,[-i\langle \alpha,\beta\rangle + i\langle \varphi(\alpha),\varphi(\beta)\rangle]
\end{aligned}
$$

and so $\mathrm{Im}\langle\varphi(\alpha),\varphi(\beta)\rangle = \mathrm{Im}\langle\alpha,\beta\rangle$. Replacing α by $i\alpha$, the last identity yields $\mathrm{Re}\langle\varphi(\alpha),\,\varphi(\beta)\rangle = \mathrm{Re}\langle\alpha,\beta\rangle$. Thus, $\langle\varphi(\alpha),\varphi(\beta)\rangle = \langle(\alpha,\beta)\rangle$ and completes the proof of (v). □

REMARK 4.2 In the literature, a complex subspace L in a Hermitian vector space $(V, i, \langle\cdot,\cdot\rangle)$ is known as a *complex Lagrangian* if L is totally real and is maximal with respect to this property. Thus, in Proposition 4.1(i)–(iii), we have shown that $\mathcal{L}(u,K)$ is a complex Lagrangian in $(\mathcal{H}(u,K), i, \langle\cdot,\cdot\rangle)$. Furthermore, denote by $\mathrm{Lag}_{\mathbb{C}}(V)$ the space of all complex Lagrangians in V. Then, as in [24, p. 14], $\mathrm{Lag}_{\mathbb{C}}(V)$ is isomorphic to the group of unitary transformations $U(V_+, V_-)$ where $V_\pm$ is the (± 1)-eigenspace of the operator $J\colon V \to V$ and J is given by the formula $\mathrm{Im}\langle\alpha,\beta\rangle = \mathrm{Re}\langle\alpha, J\beta\rangle$. In Proposition 4.1(v), we establish this relation explicitly for the complex Lagrangian subspace $\mathcal{L}(u,K)$ in $\mathrm{Lag}_{\mathbb{C}}(u,K)$. All these arguments remain valid for the Lagrangians $L(u) \oplus [P_\pm(u,K) \cap \mathcal{H}(u,K)]$.

In Theorem C and D of Part II, we studied the spectral flow of a family of operators $\{D(u) : a \le u \le b\}$ on either a closed manifold X with a splitting $X = X_1 \cup X_2$, $X_1 \cap X_2 = \Sigma$, or on a compact manifold X with boundary $\partial X = \Sigma$. There the method is to choose some smooth families of Lagrangians $\{\mathcal{L}(u) : a \le u \le b\}$ in $\mathcal{H}(u,K)$ that connects up the subspace $\mathcal{L}(a) \oplus [P_\pm(a) \cap \mathcal{H}(a,K)]$ and $\mathcal{L}(b) \oplus [P_\pm(b) \cap \mathcal{H}(b,K)]$ at the endpoints $u = a, b$. Then the spectral flow can be expressed in terms of the sum of Maslov indices and other relevant terms. In the present Hermitian setting, since $\mathcal{L}(a) \oplus [P_\pm(a) \cap \mathcal{H}(a;K)]$ and $\mathcal{L}(b) \oplus [P_\pm(b) \cap \mathcal{H}(b;K)]$ are complex Lagrangians, we can connect them up by a family of smoothly varying complex Lagrangians $\mathcal{L}(u)$ in $\mathcal{H}(u;K)$. By Remark 4.2, these Lagrangians in turn give rise to a smooth family of unitary transformations

$$\varphi(u) : \mathcal{H}(u,K)_+ \to \mathcal{H}(u,K)_- .$$

Taking the highest exterior power, we have

$$\det \varphi(u) = \bigwedge^{\max} \varphi(u) : \bigwedge^{\max} \mathcal{H}(u,K)_+ \to \bigwedge^{\max} \mathcal{H}(u,K)_- ,$$

or in other words, an element

$$\det\{\varphi(u)\} \quad \text{in } \left[\bigwedge^{\max} \mathcal{H}(u,K)_+\right]^* \otimes \left[\bigwedge^{\max} \mathcal{H}(u,K)_-\right].$$

Since the latter is the fiber of the determinant line bundle $\det\{\hat{D}_+(u)\}$, we obtain by the above procedure a smooth section $\det\{\varphi(u)\}$ of this line bundle. In fact, because $\varphi(u)$ is unitary, $\|\det \varphi(u)\| = 1$ and so $\det\{\varphi(u)\}$ lies in the unit circle bundle of $\det\{\hat{D}_+(u)\}$, denoted by $\det^1\{\hat{D}_+(u)\}$.

The above definition of $\det \varphi(u)$ can be compared with another construction: Given $\mathcal{L}(u)$ in $\mathrm{Lag}_{\mathbb{C}}[\mathcal{H}(u,K)]$, we choose a real orthonormal basis $e_1, \dots, e_n$ with the orientation $\bigwedge_{j=1}^n e_j$ induced from the complex structure $(\mathcal{L}(u), i)$. Since $\mathcal{L}(u)$ is totally real, a real, oriented basis $\{e_1, \dots, e_n\}$ is also a J-complex basis of $\mathcal{H}(u,K)$. Thus, $\bigwedge_{j=1}^n e_j$ gives an element in $\bigwedge^{\max}[\mathcal{H}(u,K)J]$ and is of norm 1 because $\{e_1, \dots, e_n\}$ is an orthonormal basis.

PROPOSITION 4.3 *There exists a natural isomorphism between*

$$\bigwedge^{\max} [\mathcal{H}(u,K), J] \quad \textit{and} \quad \left[\bigwedge^{\max} \mathcal{H}(u,K)_+\right]^* \otimes \left[\bigwedge^{\max} \mathcal{H}(u,K)_-\right],$$

and under this isomorphism the element $(i)^n \bigwedge_{j=1}^n e_j$ *is sent to* $\det \varphi(u)$.

PROOF: As is well-known, the adjoint $\mathrm{Ad}\langle\cdot,\cdot\rangle$ of a nonsingular Hermitian pairing $\langle\cdot,\cdot\rangle$ gives rise to an isomorphism

$$\begin{aligned} \overline{\mathcal{H}(u,K)} \simeq \mathrm{Hom}(\mathcal{H}(u,K), \mathbb{C}) &= \mathcal{H}(u,K)^* \\ \alpha \longmapsto (v \mapsto \langle v, \alpha\rangle) \end{aligned}$$

between the complex conjugate $\overline{\mathcal{H}(u,K)}$ of $\mathcal{H}(u,K)$ and its dual $\mathcal{H}(u,K)^*$. Combining this isomorphism with (4.5), we have

$$\begin{aligned}
&\bigwedge^{\max}[\mathcal{H}(u,K)_+]^* \otimes \bigwedge^{\max}\mathcal{H}(u,K)_- \\
&\quad\cong \bigwedge^{\max}\overline{\mathcal{H}(u,K)_+} \otimes \bigwedge^{\max}\mathcal{H}(u,K)_- \\
&\quad\cong \bigwedge^{\max}[\mathcal{H}(u,K)_+, J] \otimes \bigwedge^{\max}[\mathcal{H}(u,K)_-, J] \\
&\quad\cong \bigwedge^{\max}[\mathcal{H}(u,K), J].
\end{aligned} \tag{4.7}$$

Note that $\mathcal{H}(u,K)$ can be thought of as a complex vector space $(\mathcal{H}(u,K), i)$ with a representation of the cyclic group $\mathbb{Z}/4\langle\sigma\rangle$ of order 4 and $\sigma^2 = -\mathrm{Id}$. From representation theory it follows that $(\mathcal{H}(u,K), (\cdot,\cdot))$ has a real orthonormal basis $\{\phi_j, \sigma\phi_j, i\phi_j, i\sigma\phi_j : j = 1,\dots,n\}$ and as complex vector spaces

$$\begin{aligned}
\mathcal{H}(u,K) \quad &= \mathrm{span}_{\mathbb{C}}\{\phi_j, \sigma\phi_j : j = 1,\dots,n\}\,, \\
\mathcal{H}(u,K)_+ &= \mathrm{span}_{\mathbb{C}}\left\{\left(\frac{1-i\sigma}{\sqrt{2}}\right)\phi_j : j = 1,\dots,n\right\}, \\
\mathcal{H}(u,K)_- &= \mathrm{span}_{\mathbb{C}}\left\{\left(\frac{1+i\sigma}{\sqrt{2}}\right)\phi_j : j = 1,\dots,n\right\}.
\end{aligned}$$

In particular,

$$\left\{\left(\frac{1-i\sigma}{\sqrt{2}}\right)\phi_j, \left(\frac{1+i\sigma}{\sqrt{2}}\right)\phi_j : j = 1,\dots,n\right\}$$

can be regarded as a basis system for $\mathcal{H}(u,K)$. On the other hand, as a J-complex vector space, $\mathcal{H}(u,K)$ is spanned by $\{\phi_j, i\phi_j : j = 1,\dots,n\}$. The above two basis systems are related by

$$\sqrt{2}\,\phi_j = \left(\frac{1-i\sigma}{\sqrt{2}}\right)\phi_j + \left(\frac{1+i\sigma}{\sqrt{2}}\right)\phi_j$$

and

$$\sqrt{2}\,i\phi_j = i\left(\frac{1-i\sigma}{\sqrt{2}}\right)\phi_j + i\left(\frac{1+i\sigma}{\sqrt{2}}\right)\phi_j\,,$$

and so

$$\phi_j \wedge i\phi_j = \left(\frac{1-i\sigma}{\sqrt{2}}\right)\phi_j \wedge i\left(\frac{1+i\sigma}{\sqrt{2}}\right)\phi_j\,.$$

In particular, under the identification in (2.8), the exterior power

$$\bigwedge_{j=1}^{n}\left(\frac{1-i\sigma}{\sqrt{2}}\right)\phi_j \wedge \left(\frac{1+i\sigma}{\sqrt{2}}\right)\phi_j$$

is sent to $(i)^n \cdot \bigwedge_{j=1}^{n}(\phi_j \wedge i\phi_j)$.

Let $\mathcal{L}(u)$ be a complex Lagrangian given by the graph Δ_φ of a unitary transform $\varphi\colon \mathcal{H}(u,K)_+ \to \mathcal{H}(u,K)_-$. By choosing the above basis ϕ_j suitably, we may assume without loss of generality that φ is of diagonal form, i.e.,

$$\varphi\left[\left(\frac{1-i\sigma}{\sqrt{2}}\right)\phi_j\right] = e^{i\theta_j}\left(\frac{1+i\sigma}{\sqrt{2}}\right)\phi_j\,.$$

In addition, because the determinant has obvious multiplicative property, we may work one diagonal block at a time and assume that $n=1$. Then

$$\begin{aligned}\det\varphi &= \left(\frac{1-i\sigma}{\sqrt{2}}\right)\phi_1 \wedge \varphi\left[\left(\frac{1-i\sigma}{\sqrt{2}}\right)\phi_1\right] \\ &= e^{i\theta_1}\left(\frac{1-i\sigma}{\sqrt{2}}\right)\phi_1 \wedge \left(\frac{1+i\sigma}{\sqrt{2}}\right)\phi_1\end{aligned} \tag{4.8}$$

in $\mathcal{H}(u,K)_+^* \otimes \mathcal{H}(u,K)_-$.

As for $\mathcal{L}(u)$, it has the following real, oriented basis:

$$\begin{aligned} e_1 &= \left(\frac{1-i\sigma}{\sqrt{2}}\right)\phi_1 + e^{i\theta_1}\left(\frac{1+i\sigma}{\sqrt{2}}\right)\phi_1\,,\\ ie_1 &= i\left(\frac{1-i\sigma}{\sqrt{2}}\right)\phi_1 + e^{i\theta_1} i\left(\frac{1+i\sigma}{\sqrt{2}}\right)\phi_1\,.\end{aligned}$$

A straightforward calculation shows that

$$e_1 \wedge ie_1 = (\cos\theta + J\sin\theta)\phi_1 \wedge i\phi_1\,. \tag{4.9}$$

The proof of the proposition follows by comparing (4.8) and (4.9). □

REMARK 4.4 By [10, section 5], the Maslov index $\mathrm{Mas}_{\mathbb{R}}\{\mathcal{L}_1(u),\mathcal{L}_2(u)\}$ for a pair of (real) Lagrangian paths $(\mathcal{L}_1(u),\mathcal{L}_2(u))$ in $\mathrm{Lag}_{\mathbb{R}}\{\mathcal{H}(u,K)\}$ can be computed by choosing smoothly varying basis systems $\{e_1(u),\dots,e_n(u)\}$ and $\{e_1'(u),\dots,e_n'(u)\}$ of $\mathcal{L}_1(u)$ and $\mathcal{L}_2(u)$. Since for real Lagrangians there is no preferred choice of orientation, we form the determinant square "$\det^2$" $\mathcal{L}_1(u)$ and "$\det^2$" $\mathcal{L}_2(u)$ as two continuous sections of the line bundle

$$S^1\left[\left(\bigwedge_{\mathbb{C}}^n \mathcal{H}(u,K)\right)^{\otimes 2}\right].$$

The Maslov index $\mathrm{Mas}[(\mathcal{L}_1(u),\mathcal{L}_2(u)\colon a\le u\le b]$ is then the intersection number of these two sections in the cylinder $[a,b]\times S^1[(\bigwedge_{\mathbb{C}}^n \mathcal{H}(u,K))^{\otimes 2}]$.

In the present Hermitian setting, both $\mathcal{L}_1(u)$ and $\mathcal{L}_2(u)$ are complex Lagrangians in $\mathrm{Lag}_{\mathbb{C}}\{\mathcal{H}(u,K)\}$; for the definition of the Maslov index $\mathrm{Mas}_{\mathbb{C}}\{\mathcal{L}_1(u),\mathcal{L}_2(u)\}$, we consider the sections $\det\varphi_1(u)$ and $\det\varphi_2(u)$ in the determinant line bundle $\det\hat{D}_+(u)$ and count the intersection number

$$\mathrm{Mas}_{\mathbb{C}}\{\mathcal{L}_1(u),\mathcal{L}_2(u)\} = \#\det{}_1\varphi_1 \cap \det\varphi_2\,.$$

Note that if we forget about the complex structures on $\mathcal{L}_1(u)$ and $\mathcal{L}_2(u)$, then $\mathrm{Mas}_{\mathbb{C}}\{\mathcal{L}_1(u),\mathcal{L}_2(u)\}$ is one half of $\mathrm{Mas}_{\mathbb{R}}\{\mathcal{L}_1(u),\mathcal{L}_2(u)\}$, the real Maslov index defined before. This agrees with the intuitive notion that when we count the complex dimension $\dim_{\mathbb{C}} \mathcal{L}_1(u) \cap \mathcal{L}_2(u)$, it is one half of the real dimension.

In the same manner, the spectral flow of a family of self-adjoint, Hermitian operators equals one half of its counterpart. Since in the real setting we have demonstrated in [10] various formulae relating spectral flows with Maslov indices, dividing them by $\frac{1}{2}$ we have the corresponding formulae for Hermitian operators. In short, these formulae remain valid when we replace $\mathrm{Mas}\{\mathcal{L}_1(u), \mathcal{L}_2(u)\}$ by $\mathrm{Mas}_{\mathbb{C}}\{\mathcal{L}_1(u), \mathcal{L}_2(u)\}$.

5 B^{ev}-Operator, de Rham Operator, and the Cauchy-Riemann Operator

From now on we will focus on the situation where X is an oriented, Riemannian 3-manifold with boundary a Riemann surface, $\partial X = \Sigma$, and D is the B^{ev}-operator coupled with an $\mathrm{SU}(2)$ (or $\mathrm{U}(1)$) connection. First introduced by Atiyah-Singer-Patodi in [2, p. 63], the B^{ev}-operator is the tangential component of the Sign_+ operator; it has more recently played an important role in the theory of Floer homology (e.g., [14, 23, 25]). In a product neighborhood $\Sigma \times [-1, 0]$ of $\Sigma = \Sigma \times 0$, the B^{ev}-operator is of Atiyah-Patodi-Singer type and its tangent component, as we will see, is related to the Cauchy-Riemann operators and also the de Rham operators.

Let $\mathbb{E}$ denote a Hermitian vector bundle on X, and A a unitary connection on $\mathbb{E}$. Then, associated to A there is the differential operator

$$d_A : \Omega^*(X, \mathbb{E}) \to \Omega^{*+1}(X, \mathbb{E})$$

obtained by taking the exterior derivatives on forms with coefficients in $\mathbb{E}$. Combining d_A with the star operator $* : \Omega^*(X; \mathbb{E}) \to \Omega^{3-*}(X; \mathbb{E})$ on X, $* = *_X$, we have

$$B^{\mathrm{ev}} = (-1)^p(*d_A - d_A*) \colon \Omega^{\mathrm{ev}}(X; \mathbb{E}) \to \Omega^{\mathrm{ev}}(X; \mathbb{E})$$

where $\Omega^{\mathrm{ev}}(X; \mathbb{E}) = \Omega^2(X; \mathbb{E}) \oplus \Omega^0(X; \mathbb{E})$ and $p = 1$ or 0 depending on whether we are operating on 2- or 0-forms.

Over a collar neighborhood $\Sigma \times [-1, 0]$ of $\Sigma = \Sigma \times 0$, we assume that A is the pullback $A = \pi^*\hat{A}$ of a unitary connection $\hat{A}$ on $\hat{\mathbb{E}} = \mathbb{E} \mid \Sigma$. Here $\pi \colon \Sigma \times [-1, 0] \to \Sigma$ is the projection onto the first factor. With a product metric on $\Sigma \times [-1, 0]$, the operator B^{ev} restricted to the subspace $\Omega^{\mathrm{ev}}(\Sigma \times [-1, 0])$ takes the form $\pi^*\sigma \circ (\partial/\partial t + \pi^*\hat{D}_{\hat{A}})$ (cf. [25, p. 282] and [17, section 3]). To write down explicit formulae for σ and $\hat{D}_{\hat{A}}$, we represent an $\mathbb{E}$-valued, even-degree form $\omega \mid \Sigma \times [-1, 0]$ as a sum

$$\omega \mid \Sigma \times [-1, 0] = (dt \wedge (*P) + R, Q) \tag{5.1}$$

where P, Q, and R are, respectively, 1-, 0-, and 2-forms on Σ with values in $\mathbb{E}$, and $* = *_\Sigma$ is the star operator on Σ. (Note the $*$ in this formulation differs from the conventions of [25, p. 282] and [17, section 3].) By viewing $(P, Q, R) \in$

$(\Omega^1 \oplus \Omega^0 \oplus \Omega^2)(\Sigma; \mathbb{E})$ as a column vector, the operator σ and $\hat{D}_{\hat{A}}$ has the following block matrix representation:

$$\sigma = \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & -* \\ 0 & * & 0 \end{bmatrix}, \qquad \hat{D}_{\hat{A}} = \begin{bmatrix} 0 & *d_{\hat{A}} & d_{\hat{A}}* \\ -*d_{\hat{A}} & 0 & 0 \\ -d_{\hat{A}}* & 0 & 0 \end{bmatrix}. \tag{5.2}$$

It is not difficult to check that

$$\begin{aligned} &B^{\mathrm{ev}}(dt \wedge *P + Q + R) \\ &\quad = \left[-*\left(\frac{\partial R}{\partial t}\right) - dt \wedge d_{\hat{A}}(*R) \right] + \left[*(d_{\hat{A}}(*P)) - dt \wedge \frac{\partial P}{\partial t} - d_{\hat{A}}P \right] \\ &\qquad + \left[*\frac{\partial Q}{\partial t} - dt \wedge *(d_{\hat{A}}Q) \right] \\ &\quad = dt \wedge * \left[(*d_{\hat{A}}*)R + *\frac{\partial P}{\partial t} - d_{\hat{A}}Q \right] + \left[-*\frac{\partial R}{\partial t} + (*d_{\hat{A}} * P) \right] \\ &\qquad + \left[-d_{\hat{A}}P + *\frac{\partial Q}{\partial t} \right]. \end{aligned}$$

In matrix form, this last formula[1] reads

$$B^{\mathrm{ev}} = \begin{bmatrix} *\frac{\partial}{\partial t} & -d_{\hat{A}} & *d_{\hat{A}}* \\ *d_{\hat{A}}* & 0 & -*\frac{\partial}{\partial t} \\ -d_{\hat{A}} & *\frac{\partial}{\partial t} & 0 \end{bmatrix}$$

and explains (5.2). Furthermore, we have the following:

PROPOSITION 5.1 *Let B^{ev}, $\hat{D}_{\hat{A}}$, and σ be defined as above. Then*

(i) *B^{ev} is a complex linear, first-order, elliptic, and self-adjoint operator on $(\Omega^0 \oplus \Omega^2)(X, \mathbb{E})$;*
(ii) *$B^{\mathrm{ev}} \mid \Sigma \times [-1, 0] = \pi^* \circ \left(\frac{\partial}{\partial t} + \pi^* \hat{D}_{\hat{A}}\right)$;*
(iii) *σ is Hermitian skew-adjoint, $\sigma = \sigma^*$, $\sigma^2 = -\mathrm{Id}$;*
(iv) *$\hat{D}_{\hat{A}}$ is a first-order, elliptic, Hermitian, self-adjoint operator on $(\Omega^1 \oplus \Omega^0 \oplus \Omega^2)(\Sigma; \hat{\mathbb{E}})$ $\hat{\mathbb{E}} = \mathbb{E} \mid \Sigma$.*

In short, the operator B^{ev} and its decomposition $\pi^*\sigma \circ (\partial/\partial t + \pi^* \hat{D}_{\hat{A}})$ fits into the framework discussed in Section 4. In particular, we can decompose the operator $\hat{D}_{\hat{A}}$ into a sum of two operators:

$$\hat{D}_{\hat{A}} = \begin{pmatrix} 0 & \hat{D}_- \\ \hat{D}_+ & 0 \end{pmatrix}, \quad \hat{D}_{\hat{A}} : \left(\Omega^*(\Sigma; \hat{\mathbb{E}})\right)_{\pm} \to \left(\Omega^*(\Sigma; \hat{\mathbb{E}})\right)_{\mp}. \tag{5.3}$$

[1] In [6, 17, 25] different authors use different conventions in presenting the B^{ev} operator. For instance, Booss and Wojciechowski [6] present the operator in the form $\sigma\left(\frac{\partial}{\partial t} - \pi^* \hat{D}\right)$ and also use $(\Omega^1 \oplus \Omega^0 \oplus \Omega^0)(\Sigma; \hat{\mathbb{E}})$, both of which agree with the presentation of Yoshida in [25]. On the other hand, Kirk and Klassen [17] use (as here) $\pi^*\sigma \circ \left(\frac{\partial}{\partial t} + \pi^* \hat{D}_{\hat{A}}\right)$ because of the application of the index theorem with $\frac{\partial}{\partial t}$ as the outward-pointing normal.

Here $\Omega^*(\Sigma;\hat{\mathbb{E}})_\pm$ denotes the subspace of $\hat{\mathbb{E}}$-valued forms (P,Q,R) in $(\Omega^1\oplus\Omega^0\oplus\Omega^2)(\Sigma;\hat{\mathbb{E}})$ satisfying

$$\sigma\begin{pmatrix}P\\Q\\R\end{pmatrix}=\pm i\begin{pmatrix}P\\Q\\R\end{pmatrix}\quad\text{or}\quad *P=\pm iP\,,\quad *Q=\pm iR\,. \tag{5.4}$$

The above operators $\hat{D}_\pm$ can be compared with the Cauchy-Riemann operator $\bar\partial$. First of all, the Hodge star operator $*$ can be thought of as an automorphism of the cotangent bundle $T^*\Sigma$ with $*^2=-\mathrm{Id}$, or in other words, an almost complex structure on $T^*\Sigma$. Let

$$(T^*\Sigma)\underset{\mathbb{R}}{\otimes}\mathbb{C}$$

denote the complexification of the cotangent bundle. Then the $(\pm i)$-eigenbundles of $*$ provide us with an orthogonal decomposition

$$(T^*\Sigma)\underset{\mathbb{R}}{\otimes}\mathbb{C}=T^{0,1}\Sigma\oplus T^{1,0}\Sigma \tag{5.5}$$

where

$$T^{0,1}\Sigma=\{(v,p):p\in\Sigma\,,\ v\in T_p^*\Sigma\otimes\mathbb{C},\ *v=+iv\}\,,$$
$$T^{1,0}\Sigma=\{(v,p):p\in\Sigma\,,\ v\in T_p^*\Sigma\otimes\mathbb{C},\ *v=-iv\}\,.$$

As is well-known, the Riemannian structure on Σ specifies a unique holomorphic structure within the same conformal class. With respect to this complex analytic structure, $T^{0,1}\Sigma$ and $T^{1,0}\Sigma$ are, respectively, the holomorphic and antiholomorphic cotangent bundle. This justifies the notation $T^{0,1}\Sigma$ and $T^{1,0}\Sigma$ and the Cauchy-Riemann operator $\bar\partial$.

Analogous to (5.5), there is an orthogonal decomposition of the vector bundle

$$T^*\Sigma\underset{\mathbb{R}}{\otimes}\hat{\mathbb{E}}=\left(T^{0,1}\Sigma\underset{\mathbb{C}}{\otimes}\mathbb{E}\right)\oplus\left(T^{1,0}\Sigma\underset{\mathbb{C}}{\otimes}\mathbb{E}\right)$$

and a corresponding splitting on the space of C^∞-sections

$$\begin{aligned}\Omega^1(\Sigma;\hat{\mathbb{E}})&=\Gamma\left(T^*\Sigma\underset{\mathbb{R}}{\otimes}\hat{\mathbb{E}}\right)\\&=\Gamma\left(T^{0,1}\Sigma\underset{\mathbb{C}}{\otimes}\hat{\mathbb{E}}\right)\oplus\Gamma\left(T^{1,0}\Sigma\underset{\mathbb{C}}{\otimes}\hat{\mathbb{E}}\right)\\&=\Omega^{0,1}(\Sigma;\hat{\mathbb{E}})\oplus\Omega^{1,0}(\Sigma;\hat{\mathbb{E}})\end{aligned}$$

where

$$\Omega^{0,1}(\Sigma;\hat{\mathbb{E}})=\{P\in\Omega^1(\Sigma;\hat{\mathbb{E}}):*_\Sigma P=iP\}$$
$$\Omega^{1,0}(\Sigma;\hat{\mathbb{E}})=\{P\in\Omega^1(\Sigma;\hat{\mathbb{E}}):*_\Sigma P=-iP\}\,.$$

Comparing the above formula with (5.4), it is clear that $\Omega^{0,1}(\Sigma;\hat{\mathbb{E}})$ and $\Omega^{1,0}(\Sigma;\hat{\mathbb{E}})$ are the $(\pm i)$-eigenspace of σ on $\Omega^1(\Sigma;\hat{\mathbb{E}})$, $\Omega^{0,1}(\Sigma;\hat{\mathbb{E}}) = \Omega^1(\Sigma;\hat{\mathbb{E}})_+$ and $\Omega^{1,0}(\Sigma;\hat{\mathbb{E}}) = \Omega^1(\Sigma;\hat{\mathbb{E}})_-$, respectively. In the literature (e.g., [15, p. 25], the Cauchy-Riemann operators ∂ and $\bar{\partial}$ are defined by the formulae

$$\bar{\partial} = \left(\frac{1-i*}{\sqrt{2}}\right) d_A : \Omega^0(\Sigma;\hat{\mathbb{E}}) \to \Omega^{0,1}(\Sigma;\hat{\mathbb{E}})\,,$$

$$\partial = \left(\frac{1+i*}{\sqrt{2}}\right) d_A : \Omega^0(\Sigma;\hat{\mathbb{E}}) \to \Omega^{1,0}(\Sigma;\hat{\mathbb{E}})\,,$$

i.e., the composite of d_A with the orthogonal projection

$$\left(\frac{1\mp i*}{\sqrt{2}}\right) : \Omega^1(\Sigma;\hat{\mathbb{E}}) \to \Omega(\Sigma;\hat{\mathbb{E}})_\pm\,.$$

In fact, in view of the following proposition, it will be more convenient to multiply these operators by a factor of $(\mp i)$:

PROPOSITION 5.2 (i) *Let $\mathcal{P}_\pm$ denote the operator*

$$(\mp i)\left(\frac{1\pm i*}{\sqrt{2}}\right) d_A : \Omega^0(\Sigma;\hat{\mathbb{E}}) \to \Omega^1(\Sigma;\hat{\mathbb{E}})_\mp\,.$$

Then its adjoint is equal to $-\sqrt{2}\, * d_{\hat{A}} : \Omega^1(\Sigma;\hat{\mathbb{E}})_\mp \to \Omega^0(\Sigma;\hat{\mathbb{E}})$.

(ii) *There exist natural isomorphisms between*

$$\Big[\Omega^1(\Sigma;\hat{\mathbb{E}}) \oplus \Omega^2(\Sigma;\hat{\mathbb{E}}) \oplus \Omega^0\ (\Sigma;\hat{\mathbb{E}})\Big]_\pm$$

and $\Omega^1(\Sigma;\hat{\mathbb{E}})_\pm \oplus \Omega^0(\Sigma;\hat{\mathbb{E}})$. *Under these isomorphisms, the operators $\hat{D}_\pm$ coincide with*

$$\begin{pmatrix} 0 & \mathcal{P}_\pm \\ \mathcal{P}^*_\mp & 0 \end{pmatrix} : \begin{matrix} \Omega^1(\Sigma;\hat{\mathbb{E}})_\pm \\ \oplus \\ \Omega^0(\Sigma;\hat{\mathbb{E}}) \end{matrix} \longrightarrow \begin{matrix} \Omega^1(\Sigma;\hat{\mathbb{E}})_\mp \\ \oplus \\ \Omega^0(\Sigma;\hat{\mathbb{E}})\,. \end{matrix}$$

PROOF: Note the adjoint of $(\mp i)(\frac{1\pm i*}{\sqrt{2}})d_A$ is the operator

$$(\pm i)(d_{\hat{A}})^*\left(\frac{1\pm i*_\Sigma}{\sqrt{2}}\right)$$

acting on $\Omega^1(\Sigma;\hat{\mathbb{E}})_\mp$. Since $*$ acts as $\mp i$ on $\Omega^1(\Sigma;\hat{\mathbb{E}})_\mp$, we have

$$\begin{aligned} \pm i(-*d_{\hat{A}}*)(1\pm i*)\frac{1}{\sqrt{2}} = \pm i(-*d_{\hat{A}}*)(2)\frac{1}{\sqrt{2}} &= \pm i(-*d_{\hat{A}})(\mp i)\sqrt{2} \\ &= -\sqrt{2}\, * d_{\hat{A}} \end{aligned}$$

as claimed.

From (5.4), it is easy to see that the $[\Omega^1(\Sigma;\hat{\mathbb{E}}) \oplus \Omega^0(\Sigma;\hat{\mathbb{E}}) \oplus \Omega^2(\Sigma;\hat{\mathbb{E}})]_\pm$ consist of elements of the form $(P, Q, \mp i * Q)$ where $P \in \Omega^1(\Sigma;\hat{\mathbb{E}})_\pm$, $Q \in \Omega^0(\Sigma;\hat{\mathbb{E}})$. In particular, the assignment $(P, Q) \mapsto (P, (1/\sqrt{2})\, Q, (1/\sqrt{2})(\mp i) * Q)$ gives rise to an isomorphism of $\Omega^1(\Sigma;\hat{\mathbb{E}})_\pm \oplus \Omega^0(\Sigma;\hat{\mathbb{E}})$ onto $[\Omega^1(\Sigma;\hat{\mathbb{E}}) \oplus \Omega^0(\Sigma;\hat{\mathbb{E}}) \oplus \Omega^2(\Sigma;\hat{\mathbb{E}})]_\pm$. On the other hand, we have

$$\hat{D}\begin{pmatrix} P \\ Q \\ \mp i *_\Sigma Q \end{pmatrix} = \begin{pmatrix} \mp i(1 \pm i*_\Sigma)(d_{\hat{A}}Q) \\ - *_\Sigma d_{\hat{A}}P \\ (\pm i*_\Sigma)(- *_\Sigma d_{\hat{A}}P) \end{pmatrix}$$

and so, under the above isomorphisms, $\hat{D}$ sends (P,Q) to $(\frac{1}{\sqrt{2}}(\mp i)(1 \pm i*_\Sigma)d_{\hat{A}}Q, -\sqrt{2}(*_\Sigma d_{\hat{A}}P)) = (\mathcal{P}_\pm Q, \mathcal{P}^*_\mp P)$. This proves (3.2). □

From (5.2) we see that $\det D_+$ and $\det D_-$ are the same as the determinant line bundles associated to the Cauchy-Riemann operators $(\mathcal{P}_\pm, \mathcal{P}^*_\pm)$. In particular, this identification can be used to study the spectral flow of operators $\{D_{A(u)} : a \le u \le b\}$ induced by a smoothly varying family of connections $A(u)$. For our application it is more convenient to use the de Rham operator

$$d_{\hat{A}} + \delta_{\hat{A}} \colon \Omega^1(\Sigma,\hat{\mathbb{E}}) \to \Omega^0(\Sigma,\hat{\mathbb{E}}) \oplus \Omega^2(\Sigma,\hat{\mathbb{E}})$$

where $\delta_{\hat{A}} = -*_\Sigma d_{\hat{A}} *_\Sigma$. When $\hat{A}$ is a flat connection, it is well-known by Hodge theory that $\ker(d_{\hat{A}} + \delta_{\hat{A}})$ is isomorphic to the cohomology $H^1(\Sigma;\hat{\mathbb{E}})$ of Σ associated to the flat coefficient system $(\hat{E},\hat{A})$, while $\operatorname{coker}(d_{\hat{A}} + \delta_{\hat{A}})$ is the sum $(H^0 \oplus H^1)(\Sigma,\hat{E})$. Moreover, if $\hat{A}$ can be extended to a flat connection A on X, then there are Lagrangian subspaces in $\ker(d_{\hat{A}} + \delta_{\hat{A}})$ and $\operatorname{coker}(d_{\hat{A}} + \delta_{\hat{A}})$ defined by

$$\begin{aligned} L &= \operatorname{Im}(H^1(X,E) \to H^1(\Sigma,\hat{\mathbb{E}})) \\ &\quad \oplus \operatorname{Im}((H^0 \oplus H^2)(X,E) \to (H^0 \oplus H^2)(\Sigma,\hat{\mathbb{E}}))\,, \end{aligned} \tag{5.6}$$

the image of the natural induced homomorphism.

The operator $(d_A + \delta_A)$ is clearly complex linear with respect to the complex structure induced from $\mathbb{E}$. However, the relevant complex structures are given by the star operator $*\,(= -J)$ on the domain $\Omega^1(\Sigma,\hat{\mathbb{E}})$ and $\left(\begin{smallmatrix} 0 & * \\ -* & 0 \end{smallmatrix}\right)(= J)$ on the range $(\Omega^0 \oplus \Omega^2)(\Sigma,\hat{\mathbb{E}})$. It is not difficult to check that $(d_A + \delta_A)$ is also complex linear with respect to these structures (cf. [24, p. 16]).

The complex structure $*\,(= -J)$ on $\Omega^1(\Sigma,\hat{\mathbb{E}})$ is no stranger to us, since we have already discussed the splitting of $\Omega^1(\Sigma,\hat{\mathbb{E}})$ into the sum $\Omega^1(\Sigma,\hat{\mathbb{E}})_+ \oplus \Omega^1(\Sigma,\hat{\mathbb{E}})_-$ of its $(\pm i)$-eigenspaces $\Omega^1(\Sigma,\hat{E})_\pm$. As for the complex structure $\left(\begin{smallmatrix} 0 & * \\ -* & 0 \end{smallmatrix}\right)$ on $(\Omega^0 \oplus \Omega^2)(\Sigma,\hat{\mathbb{E}})$, it coincides with the J-complex structure discussed in Section 4. Accordingly, there is a splitting of $(\Omega^0 \oplus \Omega^2)(\Sigma,\hat{\mathbb{E}})$ into $(\pm i)$-eigenspaces:

$$(\Omega^0 \oplus \Omega^2)_\pm(\Sigma,\hat{\mathbb{E}}) = \left\{ \left(\frac{Q}{\sqrt{2}}, \mp i * \frac{Q}{\sqrt{2}}\right) : Q \in \Omega^0(\Sigma,\hat{\mathbb{E}}) \right\}.$$

As in (5.2), these subspaces are naturally isomorphic to $\Omega^0(\Sigma, \hat{\mathbb{E}})$

$$\Omega^0(\Sigma, \hat{\mathbb{E}}) \simeq (\Omega^0 \oplus \Omega^2)(\Sigma, \hat{\mathbb{E}})_\pm \,,$$
$$Q \longmapsto \left(\frac{Q}{\sqrt{2}}, \mp \frac{i * Q}{\sqrt{2}}\right)$$

which in turn are the images of $\mathcal{P}^*_+$ and $\mathcal{P}^*_-$.

With respect to the above $(\pm i)$-eigenspaces, the operator $(d_{\hat{A}} + \delta_{\hat{A}})$ can be written as a sum of two operators $(d_{\hat{A}} + \delta_{\hat{A}})_\pm \colon \Omega^1(\Sigma, \hat{\mathbb{E}})_\pm \to (\Omega^0 \oplus \Omega^2)(\Sigma, \hat{\mathbb{E}})_\pm$. Comparing with the Cauchy-Riemann operators $\mathcal{P}^*_\pm$, we have the following:

PROPOSITION 5.3 $(d_{\hat{A}} + \delta_{\hat{A}})_\pm = (\pm i)\mathcal{P}^*_\pm$.

The proof of Proposition 5.3 follows from the well-known relation between de Rham and Cauchy-Riemann operators. Explicitly, for a 1-form P with $*P = \pm iP$, we have

$$\begin{aligned}(d_{\hat{A}} + \delta_{\hat{A}})P &= (d_A - *d_{\hat{A}}*)P \\ &= d_A P \mp i * d_{\hat{A}} P \\ &= (\mp i * d_{\hat{A}} P, \pm i * (\mp i * d_{\hat{A}} P)) \,.\end{aligned}$$

Hence $(d_{\hat{A}} + \delta_{\hat{A}})_\pm \colon \Omega^1(\Sigma, \hat{\mathbb{E}})_\pm \to \Omega^0(\Sigma, \hat{\mathbb{E}})$ sends P to $\sqrt{2}(\mp i) * d_{\hat{A}} P$ under the above identification. Comparing this with $\mathcal{P}^*_\pm = (-\sqrt{2} \, *_\Sigma \, d_{\hat{A}})$, we see the factor of $(\pm i)$ between these two operators in (5.3).

For a smooth family of connections $\{\hat{A}(u) : u \in B\}$, we have a corresponding smooth family of first-order, elliptic operators $d_{\hat{A}(u)} + \delta_{\hat{A}(u)}$, $u \in B$. Using the method of Quillen discussed in Section 2, we can construct the determinant line bundle $\det\{d_{\hat{A}(u)} + \delta_{\hat{A}(u)} : u \in B\}$ over B. More precisely, let $\mathcal{H}(u,K) = \mathcal{H}^1(u,K) \oplus \mathcal{H}^0(u,K) \oplus \mathcal{H}^2(u,K)$ be the subspace in $(\Omega^1 \oplus \Omega^0 \oplus \Omega^2)(\Sigma, \hat{\mathbb{E}})$ of eigenforms with respect to the Laplace operator $\delta^*_{\hat{A}(u)} d_{\hat{A}(u)} + d_{\hat{A}(u)} \delta^*_{\hat{A}(u)}$ with eigenvalues in the band $[0, K]$. Choose K to be an excluded value in the neighborhood of u; then these subspaces, $\mathcal{H}^1(u,K)$, $\mathcal{H}^0(u,K)$, and $\mathcal{H}^2(u,K)$, vary smoothly and we can form the exterior product

$$\bigwedge^{\max} [\mathcal{H}^1(u,K)]^* \otimes \bigwedge^{\max} [(\mathcal{H}^0 \oplus \mathcal{H}^2)(u,K)] \,,$$

which serves as the fiber of the determinant line bundle in the neighborhood. Here, as before, $\mathcal{H}^1(u,K)$ and $(\mathcal{H}^0 \oplus \mathcal{H}^2)(u,K)$ are Hermitian vector spaces with complex structures defined, respectively, by $*$ and $\begin{pmatrix} 0 & * \\ -* & 0 \end{pmatrix}$.

By comparing these complex structures, it follows that

$$
\begin{aligned}
&\bigwedge^{\max}[\mathcal{H}^1(u,K),*]^*\otimes\bigwedge^{\max}\Big[(\mathcal{H}^0\oplus\mathcal{H}^2)(u,K),\left(\begin{smallmatrix}0&*\\ *&0\end{smallmatrix}\right)\Big]\\
&\quad\cong\bigwedge^{\max}[\mathcal{H}^1(u,K),-*]\otimes\bigwedge^{\max}[(\mathcal{H}^0\oplus\mathcal{H}^2)(u,K),J]\\
&\quad\cong\bigwedge^{\max}[(\mathcal{H}^1\oplus\mathcal{H}^0\oplus\mathcal{H}^2)(u,K),J]\\
&\quad\cong\bigwedge^{\max}[\mathcal{H}(u,K),J]\,.
\end{aligned}
\tag{5.7}
$$

In conclusion, for the one-parameter family $\{B^{\mathrm{ev}}\otimes A(u)\,:\,0\le u\le 1\}$ of B^{ev}-operators, we can compute its spectral flow by choosing, as in theorem C of Part II, families of complex Lagrangians in $(\mathcal{H}(u,K),J)$. In view of (5.7), these complex Lagrangians give rise to sections of the determinant line bundles $\det\{d_{A(u)}+\delta_{A(u)}\,:\,0\le u\le 1\}$. Finally, as in Remark 4.4, the above spectral flows are related to the complex Maslov index of these Lagrangians, which in turn are related to the intersection number of the determinant sections.

6 Geometry of the Representation Spaces

Let M^3 be a closed, connected, oriented, Riemannian 3-manifold with cyclically finite fundamental groups $\pi_1(M)$. As explained in Section 1, there are two different generalizations of Casson's invariant in this situation: one by Boyer-Nicas $\lambda_{\mathrm{BN}}(M)$ and the other by Walker $\lambda_{\mathrm{W}}(M)$ (cf. [7, 24].

PROPOSITION 6.1 *The difference $\lambda_{\mathrm{W}}(M)-\lambda_{\mathrm{BN}}(M)$ of these two extensions of Casson's invariant is equal to $\sum_{[\rho]}I(\rho)$, where $I(\rho)$ is the correction term $I(\rho)$ of Walker—one $I(\rho)$ for each equivalence class $[\rho]$ of reducible $\mathrm{SU}(2)$ representations ρ of $\pi_1(M)$.*

PROOF: Consider a Heegaard decomposition of M into a union of two handle bodies W_1 and W_2 along a splitting surface Σ of genus g, $M=W_1\cup W_2$, $W_2\cap W_2=\partial W_1=\partial W_2=\Sigma$. Following the notation in [24], we let

$$R=\mathrm{Hom}(\pi_1(\Sigma),\mathrm{SU}(2))/\mathrm{SU}(2) \tag{6.1}$$

denote the space of equivalence classes $[\rho]$ of $\mathrm{SU}(2)$-representations of $\pi_1(\Sigma)$. Let Q_j, $j=1,2$, denote the subspace in R consisting of representations that can be factored through $\pi_1(W_j)$. From van Kampen's theorem,

$$Q_j=\mathrm{Hom}(\pi_1(Wj),\mathrm{SU}(2))/\mathrm{SU}(2)$$

and the intersection $Q_1\cap Q_2$ coincides with the space

$$\mathrm{Hom}(\pi_1(M),\mathrm{SU}(2))/\mathrm{SU}(2)$$

of $\mathrm{SU}(2)$-representations of $\pi_1(M)$.

As is well-known, the above representation spaces R, Q_1, and Q_2 are singular algebraic varieties with singular strata given by reducible representations. Note that a reducible $\mathrm{SU}(2)$-representation $\rho : \pi_1(\Sigma) \to \mathrm{SU}(2)$ is equal to the sum $\sigma \oplus \sigma^{-1}$ of a $U(1)$-representation $\sigma : \pi_1(\Sigma) \to U(1)$ and its conjugate σ^{-1}. (From now on, we will also use σ to denote any $U(1)$-representation of $\pi_1(\Sigma)$. This unfortunately conflicts with the notation σ used before, but from the context it will not result in any confusion.) Let $\widetilde{S} = \mathrm{Hom}(\pi_1(\Sigma), U(1))$ denote the space of $U(1)$-representations of $\pi_1(\Sigma)$. Then the Weyl group of $U(1)$ induces a $\mathbb{Z}/2$-action $\tau : \widetilde{S} \to \widetilde{S}$ on $\widetilde{S}$ by sending σ to σ^{-1}. The quotient space $S = \widetilde{S}/(\mathbb{Z}/2)$ can be identified with the subspace of reducible $\mathrm{SU}(2)$-representations in R. Similarly, let $\widetilde{T}_j = \mathrm{Hom}(\pi_1(W_j), U(1))$, $j = 1, 2$, and let T_j denote the quotient space $\widetilde{T}_j/(\mathbb{Z}/2)$ of $\widetilde{T}_j$ under the induced Weyl group action. Then $T_j = Q_j \cap S$, $j = 1, 2$, coincides with the space of reducible $\mathrm{SU}(2)$-representations of $\pi_1(\Sigma)$ that can be factored through $\pi_1(W_j)$, and their intersection $T_1 \cap T_2$ is the subspace of reducible $\mathrm{SU}(2)$-representations of $\pi_1(M)$. Since M, by definition, is a rational homology sphere, there are at most finitely many $U(1)$-representations of $\pi_1(M)$ and so $|T_1 \cap T_2| < \infty$. In addition, the condition of cyclic finiteness implies that at each of these finite intersection points $\rho \in T_1 \cap T_2$, the subspaces Q_1 and Q_2 intersect each other transversely in the Zariski tangent cone of ρ. Thus, keeping a neighborhood of S fixed, we can perturb the subspaces $Q_j^- = Q_j - Q_j \cap S$ of Q_j into a transverse position with respect to each other by a motion compactly supported in $R - S$. By definition,

$$\lambda_{\mathrm{BN}}(M) = \sum_{[\rho] \in Q_1^- \cap Q_2^-} \mathrm{Sign}(\rho)\,,$$

$$\lambda_{\mathrm{W}}(M) = \sum_{[\rho]} I(\rho) + \sum_{[\rho] \in Q_1^- \cap Q_2^-} \mathrm{Sign}(\rho)\,,$$

and so

$$\lambda_{\mathrm{W}}(M) - \lambda_{\mathrm{BN}}(M) = \sum_{[\rho]} I(\rho)\,.$$

□

In view of (6.1), Theorem A of Section 1 is a consequence of the following:

THEOREM 6.2 *Let ρ be the sum $\sigma \oplus \sigma^{-1}$ of a $U(1)$-representation $\sigma : \pi_1(M) \to U(1)$ and its complex conjugate σ^{-1}. Then*

$$I(\rho) = -\frac{\rho(M, \sigma^2)}{2}$$

where $\rho(M, \sigma^2)$ stands for the rho-invariant of M associated to the representation σ^2.

To see this implication, consider the abelian covering $f : M_{\mathrm{ab}} \to M$ associated to the homomorphism $\pi_1(M) \to H_1(M, Z) \to H_1(M, Z)/\{\text{elements of order } 2\}$.

We fix a metric on M and use the induced metric on M_{ab}. Now, following Atiyah-Patodi-Singer [3], the rho-invariant of the covering f

$$\rho(f : M_{\mathrm{ab}} \to M) = (-\rho(M_{\mathrm{ab}}) + |A| \cdot \rho(M))$$

can be computed as minus the sum over the nontrivial characters χ of

$$A = H_1(M, Z)/\{\text{elements of order } 2\}$$

of the twisted rho-invariants $\rho(M, \chi)$ of M. This is the same as the sum over the squares of the characters of $\pi_1(M)$ since such a square factors through A. Since the rho-invariant for χ^2 is the same as for $\bar{\chi}^2$ and since it vanishes for trivial characters, this sum is minus twice the sum over the the equivalence classes under $\rho \equiv \bar{\rho}$ of the nontrivial squares of characters of $\pi_1(M)$. But by Theorem 6.2 this last is the sum $\sum_{[\rho]} I(\rho)$, which by Theorem 6.2 equals the difference $\lambda_{\mathrm{W}}(M) - \lambda_{\mathrm{BN}}(M)$. This proves Theorem A of Section 1.

The proof of Theorem 6.2 will occupy the rest of this section and Section 7.

Let us first recall the definition of $I(\rho)$ in [24, (2.1)]. For definiteness, we concentrate at a single point $[\rho_0]$ in $T_1 \cap T_2$ where $\rho_0 = \sigma_0 \oplus \sigma_0^{-1}$ and $\sigma_0 \in \widetilde{T}_1 \cap \widetilde{T}_2$. Since $\widetilde{T}_j$ is pathwise connected, we can choose a path $\alpha_j = \{\alpha_j(t) : 0 \le t \le 1\}$ in $\widetilde{T}_j$ starting from the trivial representation $\alpha_j(0) = \mathrm{Id}$ and terminating at $\alpha_j(1) = \sigma_0$. Under the above involution τ, such a path α_j is sent to a similar path $\beta_j = \{\beta_j(t) : 0 \le t \le 1\}$ where $\beta_j(t) = \alpha_j(t)^{-1}$, which goes from Id to the point σ_0^{-1}. Note the composite paths $\alpha_2^{-1} * \alpha_1$ and $\beta_2^{-1} * \beta_1$ form two loops in $\widetilde{S}$ with the same base point Id. Therefore, composing $\alpha_2^{-1} * \alpha_1$ with $\beta_2^{-1} * \beta_1$, we obtain a figure-8 loop $\beta_2^{-1} * \beta_1 * \alpha_2^{-1} * \alpha_1$ in $\widetilde{S}$.

It is not difficult to see that the homomorphism $\tau : H_1(\widetilde{S}; \mathbb{Z}) \to H_1(\widetilde{S}; \mathbb{Z})$ induced by the $\mathbb{Z}/2$-action is the same as multiplication by -1. Since the figure-8 loop consists of two subloops $\beta_2^{-1} * \beta_1$ and $\alpha_2^{-1} * \alpha_1$ that are interchanged by the involution, it is homologous to zero in $H_1(\widetilde{S}; \mathbb{Z})$ and hence is the boundary of a (possibly singular) surface E in $\widetilde{S}$, $E \subseteq \widetilde{S}$, $\partial E = \beta_2^{-1} * \beta_1 * \alpha_2^{-1} * \alpha_1$. Associated to a point ρ in $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SU}(2))$, there is the adjoint representation $\mathrm{Ad}\,\rho : \pi_1(\Sigma) \to \mathrm{Aut}(\mathrm{su}(2))$ of $\pi_1(\Sigma)$ on the Lie algebra $\mathrm{su}(2)$ of $\mathrm{SU}(2)$. In the case of a reducible representation $\rho = \sigma \oplus \sigma^{-1}$, this adjoint representation $\mathrm{Ad}\,\rho$ decomposition into the sum of $h \oplus h^{\perp}$ where h is the trivial representation given by action on the Lie algebra of $U(1) = \{(e^{i\theta}, e^{-i\theta})\}$ and $h^{\perp}$, the orthogonal complement of h in $\mathrm{su}(2)$. In fact, the latter is a real two-dimensional representation that can be identified with σ^2 (or σ^{-2}) after forgetting the complex structure.

Following [24, (1.13)], we have, for each $\rho \in \widetilde{S}$, the de Rham operator of Σ with coefficients in $h_{\rho}^{\perp}$:

$$D \otimes h^{\perp} = (d + d^*) \otimes h^{\perp} : \Omega^1(h^{\perp}) \to \Omega^0(h^{\perp}) \oplus \Omega^2(h^{\perp})\,. \tag{6.2}$$

Using the star operator, $* = *_\Sigma$, as the complex multiplication, we obtain a family of Hermitian operators and hence a determinant line bundle $\det(D \otimes h^\perp)$ (cf. Section 5). Associated to this complex line bundle there is a natural connection and a first Chern form $\omega = c_1(\det(D \otimes h^\perp))$, and integration $\int_E \omega$ of ω over the surface provides us with one of the terms in the following formula for $I(\rho_0)$:

$$I(\rho_0) = \begin{cases} \frac{1}{2}(c_1(\det(D \otimes h^\perp) \mid E; \Phi) - \int_E \omega), & \sigma_0^2 \neq \mathrm{Id}, \\ \frac{1}{4}(c_1(\det(D \otimes h^\perp) \mid E; \Phi) - \int_E \omega), & \sigma_0^2 = \mathrm{Id}. \end{cases} \tag{6.3}$$

Walker shows that for the natural connection, the first Chern form ω equals -8 times the symplectic form on $\widetilde{S}$. The notation $c_1(\det(D \otimes h^\perp) \mid E; \Phi)$ stands for the relative first Chern class of $\det(D \otimes h^\perp)$ on E with respect to a trivialization Φ on ∂E to be specified.

First of all, over a point $\rho = \sigma \oplus \sigma^{-1}$ in $\widetilde{S}$, the fiber of the line bundle $\det(D \otimes h^\perp)$ is given by

$$\bigwedge^{\max} H^1(\Sigma, \sigma^2) \otimes \bigwedge^{\max} H^0(\Sigma, \sigma^2)^* \otimes \bigwedge^{\max} H^2(\Sigma, \sigma^2)^* \tag{6.4}$$

[24, (1.14)]. In the case $\sigma^2 \neq \mathrm{Id}$, the terms $H^0(\Sigma, \sigma^2)$ and $H^2(\Sigma, \sigma^2)$ are equal to zero and formula (6.4) reduces to $\bigwedge^{\max} H^1(\Sigma, \sigma^2)$. Suppose this point ρ lies in $\widetilde{T}_j$. Then, as it factors through $\pi_1(W_j)$, we have cohomology $H^1(W_j, \sigma^2)$ of W_j with coefficients in σ^2. The image of $H^1(W_j, \sigma^2)$ under the induced homomorphism

$$H^1(W_j, \sigma^2) \longrightarrow H^1(\Sigma, \sigma^2)$$

gives us a complex Lagrangian subspace $\eta_{j,\rho}$ in $H^1(\Sigma, \sigma^2)$, the Hermitian vector space. As we vary ρ along the arcs α_1, α_2^{-1}, β_1, and β_2^{-1} of ∂E, these complex Lagrangians give rise to smooth sections

$$\det{}^1(\eta_1)\big|_{\alpha_1}, \quad \det{}^1(\eta_2)\big|_{\alpha_2^{-1}}, \quad \det{}^1(\eta_1)\big|_{\beta_1}, \quad \det{}^1(\eta_2)\big|_{\beta_2^{-1}},$$

over the corresponding arcs α_1, α_2^{-1}, β_1, and β_2^{-1}.

At the points where $\sigma^2 = \mathrm{Id}$, there are jumps in the dimension of $H^1(\Sigma, \sigma^2)$, because $H^0(\Sigma, \sigma^2)$ and $H^2(\Sigma, \sigma^2)$ are nontrivial. In [24, (1.13)] it has been shown that the above four sections have continuous limits and therefore can be extended to ∂E as if there were no jumps. We will give a more detailed discussion in Section 7. It is quite similar to the case $\sigma = \mathrm{Id}$, where again there are jumps and zero- and two-dimensional cohomologies $H^0(\Sigma, \mathbb{C}) = \mathbb{C}$ and $H^2(\Sigma, \mathbb{C}) = \mathbb{C}$.

At the corner points [Id], $[\sigma_0]$, and $[\sigma_0^{-1}]$, the above sections of $\det(D \otimes h^\perp)$ from different handle bodies do not necessarily agree. Indeed, because of the cyclically finite assumption, $\{H^1(W_1, \sigma^2), H^1(W_2, \sigma^2)\}$, the pairs of Lagrangians, are transverse to each other at these points. Now, given an ordered pair of transverse complex Lagrangians L_1 and L_2 in a symplectic space V, there exist two well-defined deformations $P_\pm(L_1, L_2)$ bringing L_1 to L_2 through complex Lagrangians $P_\pm(L_1, L_2)_t$, $0 \leq t \leq 1$. More precisely, choose an oriented basis $e_1, \ldots, e_n$ of L_1 and $f_1, \ldots, f_n$ of L_2 such that $\langle e_i, f_j \rangle = \delta_{ij}$. Define $P_\pm(L_1, L_2)_t$ to be the (oriented) span of $(1-t)e_i \pm t f_i$. As t varies, we obtain two paths of Lagrangians $P_{\pm}(L_0, L_1)$ connecting L_0 and L_1. Applying this construction to $\eta_{1,\rho}$ $(= L_0)$ and $\eta_{2,\rho}$ $(= L_1)$ at [Id] and $[\sigma_0 \oplus \sigma_0^{-1}]$, we can "round off" the sections $\det^1(\eta_1)$ and $\det^1(\eta_2)$ to produce a nonzero section

(6.5)
$$\det{}^1(\eta_1)\big|_{\alpha_1} * \det{}^1\left(P_{\pm}(\eta_{1,\rho_0}, \eta_{2,\rho_0})\right) * -\det{}^1(\eta_2)\big|_{\alpha_2^{-1}} * \det{}^1 P_{\pm}(\eta_{1,\mathrm{Id}}, \eta_{2,\mathrm{Id}})$$

of $\det(D\otimes h^{\perp})\big|_{\alpha_1 * \alpha_2^{-1}}$. Repeating the same procedure for $\beta_1 * \beta_2^{-1}$ and combining with the above, we obtain a trivialization $\Phi_{\pm}$ of $\det(D\otimes h^{\perp})$ over ∂E. With respect to these trivializations $\Phi_{\pm}$, the relative Chern numbers $c_1(\det(D\otimes h^{\perp}) \mid E, \Phi_+)$ and $c_1(\det(D\otimes h^{\perp}) \mid E, \Phi_-)$ are defined, and the term $c_1(\det(D\otimes h^{\perp}) \mid E, \Phi)$ is the average of these two numbers

$$(6.6)\quad c_1(\det(D\otimes h^{\perp}) \mid E, \Phi) = \frac{1}{2}\{c_1(\det(D\otimes h^{\perp}) \mid E, \Phi_+) + c_1(\det(D\otimes h^{\perp}) \mid E, \Phi_-)\}$$

This completes the definition of $I(\rho_0)$.

As the first step in computing $I(\rho_0)$, we will reformulate $I(\rho_0)$ so that it becomes more symmetrical and also reduce our computation to one over $E/(\mathbb{Z}/2)$ in S instead of E in $\tilde{S}$.

First of all, in identifying the representation $h^{\perp}$ with σ_0^2, we have made a choice between σ_0^2 and σ_0^{-2}. It will be convenient to consider the complexification $h^{\perp}\otimes \mathbb{C} = \sigma_0^2 \oplus \sigma_0^{-2}$ where both σ_0^2 and σ_0^{-2} have equal footing. Second, in the above definition of $I(\rho_0)$, we have been using the de Rham operator $D\otimes h^{\perp}\colon \Omega^1(h^{\perp}) \to \Omega^0(h^{\perp}) \oplus \Omega^2(h^{\perp})$ with the complex structure given by the star operation. It will be convenient to replace $D\otimes h^{\perp}$ by the Cauchy-Riemann operator $\partial \otimes h^{\perp}\otimes\mathbb{C}$

$$(6.7)\qquad \partial\otimes h^{\perp}\otimes\mathbb{C} : \Omega^{1,0}(\Sigma, h^{\perp}\otimes\mathbb{C}) \to \Omega^{1,1}(\Sigma, h^{\perp}\otimes\mathbb{C})$$

with coefficients in $h^{\perp}\otimes\mathbb{C}$.

From Section 5 the determinant line bundle $\det(\partial\otimes h^{\perp}\otimes\mathbb{C})$ is the same as $\det(D\otimes h^{\perp})$ and so are the relative Chern classes and Chern forms, i.e., $c_1(\det(\partial\otimes h^{\perp}\otimes\mathbb{C}), \Phi) = c_1(\det(D\otimes h^{\perp}), \Phi)$ and $c_1(\det(\partial\otimes h^{\perp}\otimes\mathbb{C})) = \omega$. Also under these identifications the rotations over Id, σ, and $\bar{\sigma}$ connecting the paths of Lagrangians are the same. In other words, if we define $I(\rho_0, h^{\perp}\otimes\mathbb{C})$ by

$$(6.8)\quad I(\rho_0, h^{\perp}\otimes\mathbb{C}) = \begin{cases} \frac{1}{2}[c_1(\det(\partial\otimes h^{\perp}\otimes\mathbb{C}) \mid E, \Phi) - \int_E \omega]\,, & \sigma_0^2 \neq \mathrm{Id}\,, \\ \frac{1}{4}[c_1(\det(\partial\otimes h^{\perp}\otimes\mathbb{C}) \mid E, \Phi) - \int_E \omega]\,, & \sigma_0^2 = \mathrm{Id}\,, \end{cases}$$

then $I(\rho_0, h^{\perp}\otimes\mathbb{C}) = I(\rho_0)$.

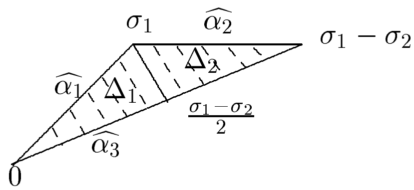

FIGURE 6.1. $\Delta \subset \mathrm{Hom}(\pi_1\Sigma, \mathbb{R})$

Next we give an explicit description of the singular chain E in $\widetilde{S}$. Note $\widetilde{S}$ and its subspace $\widetilde{T}_j$ can be interpreted as cohomology with $U(1)$-coefficients

(6.9)
$$\begin{aligned}\widetilde{S} &= \mathrm{Hom}(\pi_1(\Sigma), U(1)) = H^1(\Sigma, U(1)),\\ \widetilde{T}_j &= \mathrm{Hom}(\pi_1(T_j), U(1)) = H^1(T_j, U(1)).\end{aligned}$$

From the exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \xrightarrow{\exp} U(1) \to 0$, we have a commutative diagram with exact rows:

(6.10)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(\Sigma,\mathbb{Z}) & \longrightarrow & H^1(\Sigma,\mathbb{R}) & \xrightarrow{\exp} & H^1(\Sigma,U(1)) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H^1(W_j,\mathbb{Z}) & \longrightarrow & H^1(W_j,\mathbb{R}) & \xrightarrow{\exp} & H^1(W_j,U(1)) & \longrightarrow & 0\,. \end{array}$$

Hence, if we let $\hat{S} = H^1(\Sigma, \mathbb{R})$ and $\hat{T}_j = H^1(W_j, \mathbb{R})$, then (6.10) gives us universal covering spaces $\hat{S} \xrightarrow{\exp} \widetilde{S}$ and $\hat{T}_j \xrightarrow{\exp} \widetilde{T}_j$ with $H^1(\Sigma, \mathbb{Z})$ and $H^1(W_j, \mathbb{Z})$ as the covering transformation groups.

Given $\sigma_0 \in \widetilde{T}_1 \cap \widetilde{T}_2$, we choose two points $\sigma_1 \in \hat{T}_1$ and $\sigma_2 \in \hat{T}_2$ in its preimage, $\exp(\sigma_1) = \exp(\sigma_2) = \sigma_0$, and form the triangle Δ spanned by the three vertices 0, σ_1, and σ_2 in $\hat{S}$

(6.11) $$\Delta = \{t_1\sigma_1 + t_2(\sigma_1 - \sigma_2) : 0 \le t_1 + t_2 \le 1,\ t_1 \ge 0,\ t_2 \ge 0\}.$$

Let $F' = \exp(\Delta)$ denote the projection of Δ in $\widetilde{S}$, let F'' denote the surface in $\widetilde{S}$ obtained by applying the involution τ to F', and let $\bar{F}$ denote the quotient of $F' \cup F''$ in S.

Since $\exp(\sigma_1) = \exp(\sigma_2) = [\sigma_0]$, the difference $\sigma_1 - \sigma_2$ is an element in $H^1(\Sigma, \mathbb{Z})$. Hence, under the projection $\exp : \hat{S} \to \widetilde{S}$, the two vertices 0 and $\sigma_1 - \sigma_2$ of Δ are sent to the trivial representation Id and the third vertex σ_1 is sent to σ_0. The three edges of Δ are given by

(6.12)
$$\begin{aligned}\hat{\alpha}_1(t) &= t\sigma_1, & 0 \le t \le 1,\\ \hat{\alpha}_2(t) &= t\sigma_1 + (1-t)(\sigma_1 - \sigma_2), & 0 \le t \le 1,\\ \hat{\alpha}_3(t) &= (1-t)(\sigma_1 - \sigma_2), & 0 \le t \le 1,\end{aligned}$$

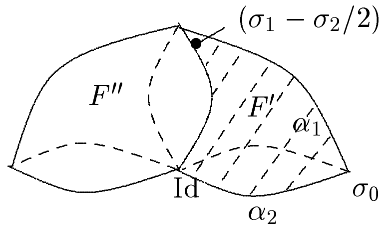

FIGURE 6.2. $F' \cup F'' \subset \widetilde{S}$

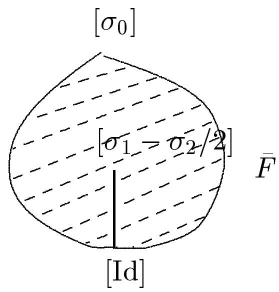

FIGURE 6.3. $\bar{F} \subset S$

respectively (see Figure 6.1). Under the projection the first edge $\hat{\alpha}_1(t)$ is sent to $\alpha_1 = \{\exp \hat{\alpha}_1(t) : 0 \leq t \leq 1\}$ in $\widetilde{T}_1$ connecting Id to σ_0, while the second edge $\hat{\alpha}_2(t)$ is sent to $\alpha_2 = \{\exp(\hat{\alpha}_2(t)) : 0 \leq t \leq 1\}$ in $\widetilde{T}_2$ connecting Id to σ_0. Thus, combining α_1 and α_2, we obtain the loop $\alpha_2^{-1} * \alpha_1$, which forms part of the boundary of F' in $\widetilde{S}$. As for the third edge $\hat{\alpha}_3$, it is mapped onto a loop $\alpha_3 = \{\exp \hat{\alpha}_3(t) : 0 \leq t \leq 1\}$ from Id to Id in $\widetilde{S}$ (see Figure 6.2). Since $-\hat{\alpha}_3(t) = \hat{\alpha}_3(1-t)$ modulo $H^1(\Sigma, \mathbb{Z})$, this loop is invariant under τ and is mapped onto the intersection of the boundary of F' and F''. Thus, $F' \cup F''$ has boundary $\tau(\alpha_2^{-1} * \alpha_1) * (\alpha_2^{-1} * \alpha_1) = \beta_2^{-1} * \beta_1 * \alpha_2^{-1} * \alpha_1$ and can serve as the singular chain E in (6.8).

Let τ' denote the rotation through 90° in $\hat{S}$ about the midpoint $(\sigma_1 - \sigma_2)/2$. That is, $\tau'((\sigma_1 - \sigma_2)/2 + v) = (\sigma_1 - \sigma_2)/2 - v$. This covers the mapping τ on $\tilde{S}$. Over $\hat{S} \times \Sigma$ we have the natural "tautological" line bundle L. Moreover, there is a natural unitary connection on L that descends to $\tilde{S}$. Here L over $\rho \times \Sigma$ is the flat, complex, unitary line bundle with holonomy given by ρ. More explicitly, the connection and bundle is as in [24] (recalled below) or generally, as in [5]. The involution τ of $\tilde{S}$ lifts to the conjugate linear mapping of L to itself. The pieces F' and F'' are the images of Δ and $\tau'(\Delta)$, respectively. It is natural to introduce the sum of complex line bundles with unitary connection $E = L \oplus (\tau')^*(L)$. Since τ' is of order 2, this complex plane bundle has a natural involution τ'' interchanging the two summands that covers the map τ'. Note that τ'' is complex linear and has square the identity. Over the midpoint $m = (\sigma_1 - \sigma_2)/2$, the map τ'' is the permutation $(v, w) \to (v, w)$.

Let $\diamond$ denote the union of Δ and $\tau'(\Delta)$. Thus τ' acts on $\diamond$ in $\hat{S}$ with a single fixed point at $m = (\sigma_1 - \sigma_2)/2$. Choose a small positive number δ such that the closed ball $B(\delta, m)$ of radius δ about the midpoint $m = (\sigma_1 - \sigma_2)/2$ of $\diamond$ is in the interior of $\diamond$. Note that τ' preserves such balls. Choose a smooth mapping $f : \diamond \to \diamond$ such that $f(B(\delta/2, m)) = m$ and f maps the region $\diamond - B(\delta/2, m)$ diffeomorphically onto $\diamond - \{m\}$. Here the ball is of radius $\delta/2$. This f is chosen to commute with the action of τ'. We may and will arrange that f maps the region $\diamond - B(\delta, m)$ by the identity. Note that $\bar{F}$ is the image of $\diamond$ in S.

Observe that the midpoint $m = (\sigma_1 - \sigma_2)/2$ of the edge $\hat{\alpha}_3$ is a "half" lattice point in $\frac{1}{2}H^1(\Sigma, \mathbb{Z})$, and so it is mapped to the point $\exp[(\sigma_1 - \sigma_2)/2 \cdot 2\pi] = \pm\mathrm{Id}$ in $\widetilde{S}$. Since this last point and also the base point Id of the loop α_3 are fixed under the involution τ, under the quotient map α_3 is folded into a single edge $\bar{\alpha}_3$ in S. After folding up the edge $\bar{\alpha}_3$, the cell F' becomes a two-dimensional singular disk $\bar{F}$ in S. Because of the singularities, $\bar{F}$ is far from an embedding and should be regarded as the image of a continuous map $\phi : I \times I \to S$ with the following properties (see Figure 6.3 for a picture of $\bar{F}$):

(i) ϕ is a degree-1 mapping from the square $I \times I = \{(s, t) : 0 \leq s \leq 1,\ 0 \leq t \leq 1$ onto $\bar{F}$.
(ii) $\phi(0, t) = [\alpha_1(t) \oplus \alpha_1(t)^{-1}]$, $\phi(1, t) = [\alpha_2(t) \oplus \alpha_2(t)^{-1}]$, $\phi(s, 0) = [\mathrm{Id}]$, and $\phi(s, 1) = [\rho_0]$.
(iii) ϕ maps the line segment $\frac{1}{2} \times [0, \frac{1}{2}]$ onto $\bar{\alpha}_3$.
(iv) ϕ restricted to $(I \times I - \frac{1}{2} \times [0, \frac{1}{2}])$ factors through $F' \to S$.

The smooth mapping $f : \diamond \to \diamond$ is equivariant and so covers a continuous mapping $\bar{f} : \bar{F} \to \bar{F}$. This (under the identification ϕ) becomes a continuous mapping $f' : I \times I \to I \times I$ that preserves the distance from the point m. This map is the identity off of a δ-neighborhood of the midpoint $[m]$, maps $I \times I$ minus a closed $\delta/2$-neighborhood homeomorphically to $I \times I - [m]$, and maps this closed $\delta/2$ ball neighborhood to the point $[m]$.

Recall that the projective unitary group $\mathrm{PSU}(2)$ is the quotient of the unitary group $U(2)$ by its center. Equivalently, $\mathrm{PSU}(2)$ is the quotient of the special unitary group $\mathrm{SU}(2)$ by its center $\pm\mathrm{Id}$. The adjoint action $U(2) \to \mathrm{Aut}(u(2))$ factors through the action of $\mathrm{PSU}(2)$ since the center of $U(2)$ acts trivially. A principal $\mathrm{PSU}(2)$-bundle is called reducible if its structure group can be reduced to the image of the normalizer $N = (\mathbb{Z}/2 \rhd U(1) \times U(1))$ of $U(1) \times U(1)$ in $U(2)$.

PROPOSITION 6.3 *Over the product manifold $\Sigma \times I \times I$ there exists a principal $\mathrm{PSU}(2)$-bundle $\mathbb{E}$ together with a reducible $\mathrm{PSU}(2)$-connection C such that, restricted to the subspace $\Sigma \times s \times t$, the connection $C(s, t) = C \mid \Sigma \times s \times t$ is flat and the holonomy representation in $\mathrm{PSU}(2)$ associated to $C(s, t)$ is $\phi(f'(s, t))$.*

PROOF: We construct a special unitary 2-plane bundle $\mathbb{E}'$ with unitary connection over the subset $\diamond - B(\frac{3}{8}\delta, m)$ of $\hat{S}$ together with a connection-preserving bundle map covering τ' that over the fixed point m lies in the center of $\mathrm{SU}(2)$. The associated principal $\mathrm{SU}(2)$ bundle then passes down to give the desired principal $\mathrm{PSU}(2)$-bundle $\mathbb{E}$ and connection over $I \times I - B(\frac{3}{8}\delta, [m])$. Then the missing closed ball $B(\frac{3}{8}\delta, m)$ is filled in.

As is well-known, there is a one-to-one correspondence between $U(1)$-connections on the trivial line bundle $\Sigma \times \mathbb{C} \to \Sigma$ and Lie-algebra-valued 1-forms in $\Omega^1(\Sigma, \mathbb{R})$, $\mathbb{R} = \mathrm{Lie}\, U(1)$. Under this identification the subspace $\mathbb{H}^1(\Sigma, \mathbb{R})$ of harmonic 1-forms in $\Omega^1(\Sigma, \mathbb{R})$ consists of flat $U(1)$-connections on Σ. As $\diamond$ is embedded in $\mathbb{H}^1(\Sigma, \mathbb{R})$ $(= H^1(\Sigma, \mathbb{R}))$, this procedure leads us to a $U(1)$-connection B on $L := \{\Sigma \times \diamond \times \mathbb{C} \to \Sigma \times \diamond\}$ such that the restriction of (L, B) to the subspace $\Sigma \times p$, $p \in \diamond$, is flat and has holonomy the representation p in $\tilde{S}$.

Now consider the direct sum $E = L \oplus (\tau')^*(L)$ over $\diamond$. The natural involution τ'' covering τ' that interchanges the two summands is a free action off of the midpoint m and, moreover, preserves the connection $B \oplus (\tau')^*(B)$. Hence, the pullback $E' = f^*(E)$ has over $\diamond$ the structure of a $\mathrm{SU}(2)$-bundle with connection and the pullback $f^*(\tau'')$ acts as a bundle and connection-preserving involution. Over the region $\diamond - B(\frac{1}{4}\delta, m)$ the action of $f^*(\tau')$ is free; we define $\mathbb{E}' \mid (\diamond - B(\frac{3}{8}\delta, m))/(\mathbb{Z}/2))$ as the quotient of this free $\mathbb{Z}/2$-action and C the associated $\mathrm{SU}(2)$-connection. We take $\mathbb{E}$ over this piece to be the associated $\mathrm{PSU}(2)$ bundle with connection.

By definition, the bundle $E' \mid B(\delta/2, m)$ is the trivial product

$$L' \oplus L' \times B(\delta/2, m) \to \Sigma \times B(\Delta/2, m)$$

as a bundle with connection. Here L' denotes the complex line bundle with flat connection given by $L \mid \{m\}$. Also, the induced map $f^*(\tau'')$ maps a point $(v, w, m+p)$ to $(w, v, m-p)$. That is, the first two are merely interchanged while the point $m+p$ of the disk is rotated by 180° around m.

We parameterize the quotient $B(\delta/2, m)/(\mathbb{Z}/2)$ by the radius r and the angle θ which ranges $0 \le \theta \le \pi$ with 0 and π identified. (Recall that τ' is the 180° rotation about the midpoint m.) Choose a smooth, decreasing, real function $g(r)$ for $0 \le r \le \delta/2$ such that g equals 1 for $0 \le r \le \delta/8$ and g equals 0 for $\delta/4 \le r \le \delta/2$. Then we may introduce the product bundle with connection over the product: $[0, \delta/2] \times [0, \pi] \times \Sigma$ given by $[0, \delta/2] \times [0, \pi] \times L' \oplus L'$. We may make the following identifications along $([0, \delta] \times \pi \times L' \oplus L')$ and $([0, \delta] \times 0 \times L' \oplus L')$ by $(r, \pi, (v, v)) \to (r, 0, i \cdot \exp(\frac{1}{2}\pi g(r)) \cdot (v, v))$ and $(r, \pi, (v, -v)) \to (r, 0, i \cdot \exp(-\frac{1}{2}\pi g(r)) \cdot (v, -v)$.

Again, these are bundle-preserving maps over $r \times \theta \times \Sigma$ for $r > 0$. We may modify the connection in a neighborhood of the "edge" $[\delta/8, \delta/4] \times [0, \pi]$ so that they become connection-preserving. Moreover, over the piece $\delta/4 \le r \le \delta/2$ the identifications are via $(r, \pi, (v, v)) \to (r, 0, (v, v))$ and $(r, \pi, (v, -v)) \to$

$(r, 0, (-v, v))$; that is, $(v, w) \to (w, v)$. Thus, as bundles with connections we have a natural identification of the pullback of this constructed 2-plane bundle over $B(3\delta/8, [m]) - B(\delta/2, [m])$ with the restriction of $\mathbb{E}'$ to the corresponding piece.

We define the identifications over $0 \times [0, \pi] \times L' \oplus L'$ by $(0, \theta, (z, w)) \to (0, 0, (z, w))$ for any θ and (z, w). The corresponding identifications of $[0, \delta/2] \times [0, \pi/2]$ gives back the disk $B(\delta/2, [m])$. The induced identifications over the point $[m]$ are by $(v, w) \to i \cdot (v, w)$, so on the fibers we must identify by multiplication by i. Hence, we only get a principal $\mathrm{PSU}(2)$-bundle after this identification. We define $\mathbb{E} \mid B(\delta/2, m) \times \Sigma$ as this quotient. It matches the previous definition over $(B(\delta, m) - B(\delta/2, m)) \times \Sigma$.

The statement in Proposition 6.3 about the connection $C(s, t)$ and holonomy of $\mathbb{E}$ is true by definition.

Pulled back to $\diamond$, we can unwind the transformation in $B(\delta/2, m)$ that makes the identification multiplication by i instead of the permutation of the two factors. This is done by rotating the first factor (v, v) to match the second $(v, -v)$. Since the exponentials cancel and the two i's give -1, this deformation reproduces the bundle E. That is, we have a parameter family C_u of unitary connections on E such that C_0 is the original connection on E, and the principal $\mathrm{PSU}(2)$ associated to (E, C_1) is the pullback of the $\mathrm{PSU}(2)$-bundle with connection $\mathbb{E}$; moreover, the connections are constant (independently of u) off the disk $B(\delta/2, m)$ about the midpoint $m = (\sigma_1 - \sigma_2)/2$.

In particular, the integral $\int \omega$ of the first Chern class ω of the determinant bundle associated to the adjoint bundle $\mathrm{Ad}(\mathbb{E})$ coupled to the $\bar\partial$ operator over the singular disk will be equal to $\frac{1}{2}$ times the integral $\int \omega$ of the first Chern class of the determinant line bundle of the natural bundle $\mathrm{Ad}(L \oplus (\tau')^*(L))$ with connection coupled to the $\bar\partial$ over the union $F' \cup F''$. That is, so far as the integral terms in Walker's correction is concerned, we may pass down to the quotient space $\bar F$ and use the bundle and connection $\mathrm{Ad}(\mathbb{P}, C)$ instead. Similarly, the obstructions to extending the determinant sections are the same since the bundles with connection are all identical in a neighborhood of the boundary of $\diamond$. The point is that under the involution $(v, w) \to (w, v)$ over the boundary of $\diamond$, the Lagrangians are preserved.

To conclude this section, we explain how the above singular chain $\psi : I \times I \to E \subset S$ can be used to evaluate $I(\rho_0, h^\perp \otimes \mathbb{C})$. From the above proof of (6.3), it is not difficult to see that the $\mathrm{PSU}(2)$-connection C over $\Sigma \times I \times I$ is reducible. Hence, the adjoint bundle $\mathrm{Ad}\ \mathbf{P}$ and the associated connection are decomposed into the sum $\mathbf{H} \oplus \mathbf{H}^\perp$ of a real 1-dimensional bundle $\mathbf{H}$ and its orthogonal complement $\mathbf{H}^\perp$, endowed with connections. After forming the complexification of $\mathbf{H}^\perp$, we obtain an $\mathrm{SU}(2)$-bundle, $\mathbb{E} = \mathbf{H}^\perp \otimes \mathbb{C}$, with connection A such that the restriction $A(s, t)$ of A to $\Sigma \times s \times t$, $(s, t) \in I \times I$, is flat and has $(h^\perp \times \mathbb{C}) \circ \phi(f'(s, t))$ as its holonomy representation.

By coupling the Cauchy-Riemann operator ∂ with the connection $A(s, t)$, we obtain a family of operators $\{\partial \otimes A(s, t) : 0 \le s \le 1, 0 \le t \le 1\}$ and hence a determinant line bundle $\det(\partial \times h^\perp \times \mathbb{C})$ with a Bismut-Freed connection [5] and Chern form $\omega = c_1[\det(\partial \otimes h^\perp \otimes \mathbb{C})]$. Using these data, we can, as before, form the relative Chern number $c_1(\det(\partial \otimes h^\perp \otimes \mathbb{C}) \mid \bar{E}, \Phi_\pm)$ and the integral $\int_{\bar{E}} \omega$. Here, because of a lack of good notation, we use the symbol $\bar{E}$ to denote the domain of the singular chain $\phi : I \times I \to S$. As explained, above this integral is $\frac{1}{2}$ times the integral in Walker's correction.

Since $\bar{\alpha}_3$ is a contractible subspace in $\bar{E}$, we can extend the trivialization $\Phi_\pm$ from $\partial\bar{E} = \alpha_2^{-1} * \alpha_1$ to $\bar{\alpha}_3$. Lifting this last trivialization from $\bar{\alpha}_3$ to α_3, we have trivializations $\Phi'_\pm$ and $\Phi''_\pm$ along $\partial E'$ and $\partial E''$ and hence well-defined relative Chern numbers $c_1(\det(\partial \otimes h^\perp \otimes \mathbb{C}) \mid E', \Phi'_\pm)$ and $c_1(\det(\partial \otimes h^\perp \otimes \mathbb{C}) \mid E'', \Phi''_\pm)$. By additivity and functorial properties of the first Chern class, it is easy to see

$$
\begin{aligned}
c_1(\det(\partial \otimes h^\perp \otimes \mathbb{C}) \mid E' \cup E'', \Phi'_\pm \cup \Phi''_\pm) &= 2c_1(\det(\partial \otimes h^\perp \otimes \mathbb{C}) \mid E', \Phi_\pm) \\
&= 2c_1(\det(\partial \otimes h^\perp \otimes \mathbb{C}) \mid \bar{E}, \Phi_\pm)\,.
\end{aligned}
$$

Thus, as we replace E by $\bar{E}$, the formula in (6.8) reads as

(6.13)

$$
I(\rho_0, h^\perp \otimes \mathbb{C}) = \begin{cases} c_1(\det(\partial \times h^\perp \otimes \mathbb{C}) \mid \bar{E}, \Phi) - \int_{\bar{E}} \omega\,, & \sigma_0^2 \neq \mathrm{Id}\,, \\ \frac{1}{2}\{c_1(\det(\partial \otimes h^\perp \otimes \mathbb{C}) \mid \bar{E}, \Phi) - \int_{\bar{E}} \omega\}\,, & \sigma_0^2 = \mathrm{Id}\,. \end{cases}
$$

□

7 Walker's Correction Term and Spectral Flow

At the end of Section 6, we constructed over $\Sigma \times I \times I$ an $\mathrm{SU}(2)$-bundle $\mathbb{E}$ with connection A whose restriction $A(s,t)$ to $\Sigma \times s \times t$ is flat with holonomy $(h^\perp \otimes \mathbb{C}) \circ \phi(f(s,t)) = \sigma(f(s,t))^2 \oplus \sigma(f(s,t))^{-2}$. Along $\Sigma \times 0 \times I$ and $\Sigma \times 1 \times I$, the representations $\sigma(0,t)$ $(= \alpha_1(t))$ and $\sigma(1,t)$ $(= \alpha_2(t))$ can be extended to representations of $\pi_1(W_1)$ and $\pi_1(W_2)$, respectively. It follows that the $\mathrm{SU}(2)$-bundles $\mathbb{E} \mid \Sigma \times 0 \times I$ and $\mathbb{E} : \Sigma \times 1 \times I$ and connections $\{A(0,t) : 0 \le t \le 1\}$ and $\{A(1,t) : 0 \le t \le 1\}$ can be extended to corresponding bundles $\mathbb{E}_1$ and $\mathbb{E}_2$ and connections A_1 and A_2 over $W_1 \times I$ and $W_2 \times I$.

Consider the 4-manifold $M \times I$ as obtained from $W_1 \times I$, $\Sigma \times I \times I$, and $W_2 \times I$ by gluing the relevant boundary components together, i.e., $\partial W_1 \times I = \Sigma \times 0 \times I$ and $\partial W_2 \times I = \Sigma \times 1 \times I$. Then over $M \times I$ there is the $\mathrm{SU}(2)$-bundle $\mathbb{E}_0$ with connection A_0 given by gluing $(\mathbb{E}_1, A_1)$, $(\mathbb{E}_2, A_2)$, and $(\mathbb{E}, A)$ along these boundary components.

As in [2, p. 63], the signature operator $\mathrm{Sign} : \Omega \to \Omega$ of a 4-manifold is a sum $\mathrm{Sign}_+ \oplus \mathrm{Sign}_-$ of two operators

$$
\mathrm{Sign}_+ : \Omega_+^{\mathrm{ev}} \to \Omega_-^{\mathrm{ev}}\,, \qquad \mathrm{Sign}_- : \Omega_+^{\mathrm{odd}} \to \Omega_-^{\mathrm{odd}}\,.
$$

Coupling Sign_+ with the above bundle $\mathbb{E}_0$ and connection A_0 data, we obtain a first-order, elliptic operator

$$
\mathrm{Sign}_+ \otimes \mathbb{E}_0 : \Omega_+^{\mathrm{ev}}(M \times I, \mathbb{E}_0) \to \Omega_-^{\mathrm{ev}}(M \times I, \mathbb{E}_0)
$$

over $M \times I$. By adding collars to $\Sigma \times I \times 0$ and $\Sigma \times I \times 1$, we may assume that the bundle $\mathbb{E}$ and connection A are productlike in these collar neighborhoods. It follows that the bundle $\mathbb{E}_0$, the connection A_0, and the operator $\mathrm{Sign} \otimes \mathbb{E}_0$ are also productlike. By imposing the L_2-boundary condition as in [2, (3.22)], we have a well-defined index $\mathrm{Ind}(M \times I)$ of $\mathrm{Sign}_+ \otimes \mathbb{E}_0$, which can in turn be computed by the formula

$$\mathrm{Ind}(M \times I) = \int \alpha(\mathrm{Sign}_+ \otimes \mathbb{E}_0) - \frac{\eta(M \times \partial I, B^{\mathrm{ev}} \otimes \mathbb{E}_0) + h}{2} \tag{7.1}$$

where $\eta(M \times \partial I, B^{\mathrm{ev}} \otimes \mathbb{E}_0)$ is the eta-invariant, h is the zero-mode correction, and $\alpha(\mathrm{Sign}_+ \otimes \mathbb{E}_0)$ is a Chern form given by the symbol of the operator.

Along each slice $M \times t$, the tangential component of $\mathrm{Sign}_+ \otimes \mathbb{E}_0$ gives us the operator $B^{\mathrm{ev}} \otimes (\mathbb{E}_0 \mid M \times t)$:

$$\begin{gathered}(\Omega^2 \oplus \Omega^0)(M \times t, \mathbb{E}_0 \mid M \times t) \to (\Omega^2 \oplus \Omega^0)(M \times t, \mathbb{E}_0 \mid M \times t)\,, \\ (a, b) \longmapsto (d_{A_0 * \mid M \times t} a + * d_{A_0 \mid M \times t} b,\ - * d^*_{A_0 \mid M \times t} b)\,.\end{gathered}$$

Denote by $D(t)$ these operators $B^{\mathrm{ev}} \otimes (\mathbb{E}_0 \mid M \times t)$; then as t varies we obtain a family $\{D(t) : 0 \leq t \leq 1\}$ of self-adjoint elliptic operators and hence a well-defined $(-\varepsilon / + \varepsilon)$-spectral flow. As explained in section 8 of [12], this last spectral flow can be identified with $\mathrm{Ind}(M \times I)$, and so

$$\begin{aligned}&(-\varepsilon / + \varepsilon) - \text{spectral flow}\{D(t) : 0 \leq t \leq 1\} = \\ &\qquad\qquad \int \alpha(\mathrm{Sign}_+ \otimes \mathbb{E}_0) - \frac{\eta(M \times \partial I, B^{\mathrm{ev}} \otimes \mathbb{E}_0) + h}{2}\,.\end{aligned}$$

Here $\varepsilon > 0$ is chosen so that both $D(0)$ and $D(1)$ have at most zero eigenvalues in the range $[-\varepsilon, +\varepsilon]$. The $-\varepsilon/+\varepsilon$–spectral flow measures the number (with signs and multiplicities) of eigenvalues of $\{D(t) : 0 \leq t \leq 1\}$ that cross the line in $R \times [0,1]$ from $[-\varepsilon, 0]$ to $[+\varepsilon, 1]$.

In a similar fashion, we may take the $(+\varepsilon/+\varepsilon)$–spectral flow utilizing the line from $[+\varepsilon, 0]$ to $[+\varepsilon, 1]$, and the $(-\varepsilon/-\varepsilon)$–spectral flow utilizing the line from $[-\varepsilon, 0]$ to $[-\varepsilon, 1]$.

On the other hand, there is also the average spectral flow,

$$\mathcal{A} - \text{spectral flow}\{D(t) : 0 \leq t \leq 1\}\,,$$

given by taking the average of $(+\varepsilon)$- and $(-\varepsilon)$-spectral flows. Using that, we have (as explained in section 8 of [12],

$$\begin{aligned}&\mathcal{A} - \text{spectral flow}\{D(t) : 0 \leq t \leq 1\} = \\ &\qquad\qquad \int \alpha(\mathrm{Sign}_+ \otimes \mathbb{E}_0) - \frac{\eta(M \times \partial I, B^{\mathrm{ev}} \otimes \mathbb{E}_0)}{2}\,.\end{aligned} \tag{7.2}$$

This averaging procedure cancels out the zero-mode terms in the previous formula.

The idea underlying our proof of Theorem 6.2 is to compare term by term the above formula (7.2) with (6.13).

PROPOSITION 7.1 *The integral* $\int_{M\times I}\alpha(\mathrm{Sign}_+\otimes\mathbb{E}_0)$ *in* (7.2) *equals* $2\int_{\bar{E}}\omega$ *in* (6.13)*, and the eta-invariant* $\eta(M\times\partial I, B^{\mathrm{ev}}\otimes\mathbb{E}_0)/2$ *equals the rho-invariant* $\rho(M,\sigma_0^2)$.

PROOF: By [2, (4.19)], the integral in (7.2) is given by

$$\frac{1}{2}\int\limits_{M\times I}[L(M\times I)\,\mathrm{ch}(\mathbb{E}_0)-e]$$

where $L(M\times I)$ stands for the unstable L-polynomial and e the Euler characteristic form. In the present setting, $\int e$ equals zero since $M\times I$ is a metric product. On the other hand, since $c_1(\mathbb{E},C)$ vanishes by $\mathbb{E}$ an $\mathrm{SU}(2)$ bundle,

$$\begin{aligned}L(M\times I)&=\prod\left(\frac{x_i}{\tanh(x_i/2)}\right)=4[1+\cdots]\,,\\ \mathrm{ch}(\mathbb{E}_0)&=[2+\mathrm{ch}_{(2)}(\mathbb{E}_0)+\cdots]\,,\end{aligned}$$

and so

$$\int\limits_{M\times I}[L(M\times I)\,\mathrm{ch}(\mathbb{E}_0)-e]=4\int\limits_{M\times I}\mathrm{ch}_{(2)}(\mathbb{E}_0)\,.$$

Since $M\times I$ is the union of $W_1\times I$, $W_2\times I$, and $\Sigma\times I\times I$, we can calculate $\int_{M\times I}\mathrm{ch}_{(2)}(\mathbb{E}_0)$ by performing the integral over these regions separately and then sum up the answers.

Over each slice $W_j\times t$, $j=1,2$, we have a flat connection $A_j(t)$ for the bundle $\mathbb{E}_j\mid W_j\times t$. In particular, the curvature 2-form is the product of dt with a Lie-algebra-valued 1-form. Thus its square has trace zero. It follows that the Chern form $\mathrm{ch}_{(2)}(\mathbb{E}_0)\mid W_j\times I$ is zero and so

$$\int\limits_{M\times I}\alpha(\mathrm{Sign}_+\otimes\mathbb{E}_0)=2\int\limits_{\Sigma\times I\times I}\mathrm{ch}_{(2)}(\mathbb{E}).$$

On the other hand, the integral $\int_{\bar{E}}\omega$ has the integrand $c_1(\det\partial\otimes\mathbb{E}_0)$, which by the family index theorem is $\int_\Sigma\mathrm{ch}(\mathbb{E}_0)td(\Sigma)$. Because $td(\Sigma)=(1+\cdots)$ and $\mathbb{E}$ is a $\mathrm{SU}(2)$-bundle, we have

$$\int_{\bar{E}}\omega=\int\limits_{I\times I}\int\limits_{\Sigma}\mathrm{ch}(\mathbb{E}_0)td(\Sigma)=\int\limits_{\Sigma\times I\times I}\mathrm{ch}_{(2)}(\mathbb{E}_0)\,. \tag{7.3}$$

From the definition, the rho-invariant $\rho(M,\sigma_0^2)$ equals the difference of two eta-invariants, $\eta(M,\sigma_0^2)-\eta(M,\mathbb{C})$, i.e., the reduced eta-invariant. Hence,

$$\begin{aligned}\frac{\eta(M\times\partial I,B^{\mathrm{ev}}\otimes\mathbb{E}_0)}{2}&=\frac{\eta(M\times 1,\sigma_0^2\oplus\sigma_0^{-2})-\eta(M\times 0,\mathbb{C}\oplus\mathbb{C})}{2}\\ &=\frac{\rho(M,\sigma_0^2)+\rho(M,\sigma_0^{-2})}{2}\\ &=\rho(M,\sigma_0^2)\,.\end{aligned}$$

This proves (7.1). ☐

As a consequence of (7.1), to prove Theorem 6.2, it suffices to prove that

$$\text{(7.4)} \quad \mathcal{A} - \text{spectral flow}\{D(t) : 0 \leq t \leq 1\} = 2c_1(\det(\partial \otimes h^{\perp} \otimes \mathbb{C}) \mid \bar{E}, \Phi) \,.$$

Once this is established, we will have

$$\begin{aligned} -\rho(M, \sigma_0^2) &= -\frac{\eta(M \times \partial I, B^{\text{ev}} \otimes \mathbb{E}_0)}{2} \\ &= \mathcal{A} - \text{spectral flow}\{D(t) : 0 \leq t \leq 1\} - \int\limits_{M \times I} \alpha(\text{Sign}_+ \otimes \mathbb{E}_0) \\ &= 2c_1(\det \partial \otimes h^{\perp} \otimes \mathbb{C} \mid \bar{E}, \Phi) - 2\int\limits_{\bar{E}} \omega \\ &= 2I(\rho_0). \end{aligned}$$

If $\sigma_0^2 = \mathrm{Id}$, the last equality is to be replaced by $I(\rho_o)$. Since $\rho(M, \sigma_0^2) = 0$ in this case, we again get $I(\rho_0) = 0 = -\frac{1}{2}\rho(M, \sigma_0^2)$.

To prove (7.4), we use the method in [12] to compute the average spectral flow. Without loss of generality, we may assume in a collar neighborhood of $0 \times I$ in $I \times I$ the map $\phi : I \times I \to S$ is constant along the s-directions, i.e., $\phi(s,t) = \phi(s',t)$. It follows that the connection $A(s,t) = A \mid \Sigma \times s \times t$ is also constant along the s-directions, $A(s,t) = A(s',t)$, and the operator $D(t)$ can be written in the form $D(t) = \sigma(\partial/\partial s + \hat{D}(s,t))$. Here σ and $\hat{D}(s,t)$ are given as in (5.2) by the following:

For (a,b,c) in $\Omega^1 \oplus \Omega^0 \oplus \Omega^2$:

$$\sigma(a,b,c) = (*a, - * c, *b) \,,$$

$$\hat{D}(s,t)(a,b,c) = (*d_{A(s,t)}b + *d_{A(s,t)} * c, - * d_{A(s,t)}a, - * d_{A(s,t)}a) \,.$$

From the above formula, the tangential component $\hat{D}(s,t)$ of the operator $D(t)$ is constant along the s-directions in a neighborhood of $\Sigma \times 0 \times I$. Hence, as we split $M \times I$ into two pieces $W_1 \times I$ and $\Sigma \times I \times I \cup W_2 \times I$, the family $\{D(t) : 0 \leq t \leq 1\}$ satisfies the Atiyah-Patodi-Singer condition in [12]. In particular, we can apply theorem C of [12] after a family of restricted Lagrangian pairs have been chosen along $\Sigma \times 0 \times I$ meeting special restraints at the endpoints such that they in addition are "specially complementary."

If $\hat{D}$ is a self-adjoint operator with pure point spectrum extending from $-\infty$ to $+\infty$ acting on a complex Hilbert space $\mathbb{H}$, we use the symplectic form given by $\{v,w\} = -\Im\langle v,w\rangle$. A Lagrangian subspace is then by definition the closure of a complete orthonormal basis of this complex Hilbert space with Hermitian inner product. We say that two Lagrangians (L_1, L_2) are a *special complementary pair* if conditions A and B below hold:

A. L_1 intersects the span of the positive eigenvectors of D in a subspace of finite codimension.
B. L_2 intersects the span of the negative eigenvectors of D in a subspace of finite codimension.

These are *very* strong conditions.

For a continuous-parameter family of specially complementary pair

$$\{(L_1(t), L_2(t)) : 0 \leq t \leq 1\}$$

of continuously varying self-adjoint operators $\hat{D}(t)$ obtained by partitioning the interval into a finite number of pieces, we can reduce the spectral decomposition results to that treated in our paper [12]. That is, it is a mild restatement of the results of [12] to consider such families with the appropriate boundary condition specified there. In particular, we have no ambiguity about the definition of Maslov indices of such simple families. (The work of [10] on the finite-dimensional case is sufficient for this purpose.)

These chosen Lagrangians in our problem serve a dual role. They impose boundary conditions so that we can define for the manifolds with boundary $W_1 \times I$ and $\Sigma \times I \times I \cup W_2 \times I$ a family of self-adjoint elliptic operators for B^{ev} coupled to the connection. Also, the Maslov index of these varying Lagrangians appears in the formula for the spectral flow. The condition of special complementarity translates into the fact that the associated boundary value problem is of Atiyah-Patodi-Singer type up to an operator of finite rank. In particular, we see that these operators are Fredholm.

The spectral flow decomposition theorem was formulated in [12] in great generality, so it could be applied to many situations (for example, Dirac operators coupled to bundles). As such, the spectral flows appearing in the theorems of [12] are those that arise only after the manifold is stretched along the product neighborhood of the separating manifold (here Σ). After a tube of length r with r large is inserted, we may by [11] be assured that the resultant manifold $M(r)$ and operators at the ends $D(M(r), 0)$ and $D((M(r), 1)$ have no eigenvalues equal to $\pm(1/r^2)$. (The rough idea is that all "small" eigenvalues are proved in Part 1 to be exponentially small for r large.) Consequently, for r large the $(+r^{-2}/+r^{-2})$–spectral flow of $D(M(r), t)$ is well-defined. It is this spectral flow that [12] computes.

In the special case considered here, the operators at the ends $t = 0$ and $t = 1$ are completely topological. Their zero modes are identifiable with certain elements of the cohomology of M with trivial (at $t = 0$) or twisted (by σ^2 at $t = 1$) coefficients. In particular, the space of zero modes of the operators $D(M(r), 0)$ and $D(M(r), 1)$ is of constant dimension and varies smoothly with r as r varies. Therefore, stretching r gives a smooth-parameter family of self-adjoint elliptic operators with the same dimensional, smoothly varying subspaces of zero modes at the ends. Consequently, the $(+r^{-2}, +r^{-2})$–spectral flow for r large appearing in the theorems of [12] are in our special case the *same* as the $(+\varepsilon/+\varepsilon)$–spectral flow computed *without* stretching. In the same way we may use the averaged spectral flow of the unstretched operators. This will be done without further comment in what follows.

Before choosing these Lagrangians, we observe that, because $A(s,t)$ is flat with holonomy $\sigma(s,t)^2 \oplus \sigma(s,t)^{-2}$, the operator $\hat{D}(s,t)$ admits the following cohomological interpretation:

(7.5) The null space $\ker \hat{D}(s,t)$ of $\hat{D}(s,t)$ is canonically isomorphic to the total sum $H^*\left(\Sigma, \sigma(s,t)^2 \oplus \sigma(s,t)^{-2}\right)$ of cohomologies of Σ with twisted coefficients $\sigma(s,t)^2 \oplus\ \sigma(s,t)^{-2}$. The Hermitian structure on $\ker \hat{D}(s,t)$ is induced by the cup product.

(7.6) Along the two boundary components $\Sigma \times 0 \times I$ and $\Sigma \times 1 \times I$, the connections $A(0,t)$ and $A(1,t)$ extend to flat connections on $W_1 \times t$ and $W_2 \times t$, respectively. Let $L_1(0,t)$ and $L_2(1,t)$ denote subspaces in $\ker \hat{D}(0,t)$ and $\ker \hat{D}(1,t)$ given by the extended L^2-solutions from $W_1 \times t$ and $W_2 \times t$. Then, under the isomorphism in (7.5) above, $L_1(0,t)$ and $L_2(1,t)$ coincide, respectively, with the images of the induced homomorphisms

$$H^*(W_1 \times t, \sigma(0,t)^2 \oplus \sigma(0,t)^{-2}) \to H^*(\Sigma \times 0 \times t, \sigma(0,t)^2 \oplus \sigma(0,t)^{-2})\,,$$
$$H^*(W_2 \times t, \sigma(1,t)^2 \oplus \sigma(1,t)^{-2}) \to H^*(\Sigma \times 1 \times t, \sigma(1,t)^2 \oplus \sigma(1,t)^{-2})\,.$$

(7.7) The L^2-solution spaces of the operators $D(t) \mid W_1 \times t$ and $D(t) \mid W_2 \times t$ are isomorphic to the images of the natural homomorphisms:

$$H^*_{\mathrm{comp}}(W_1 \times t, \sigma(0,t)^2 \oplus \sigma(0,t)^{-2}) \to H^*(W_1 \times t, \sigma(0,t)^2 \oplus \sigma(0,t)^{-2})\,,$$
$$H_{\mathrm{comp}}(W_2 \times t, \sigma(1,t)^2 \oplus \sigma(1,t)^{-2}) \to H^*(W_2 \times t, \sigma(1,t)^2 \oplus \sigma(1,t)^{2})\,,$$

from cohomology with compact support to singular cohomology. The image of the cohomology with compact support is the image of the relative cohomology in the absolute cohomology [2]. Since the W_j are handle bodies, we have $H^1(W_j, \partial W_j, \cdot) = H_2(W_j, \cdot) = 0$, $H^0(W_j, \partial W_j, \cdot) = H^3(W_j, \cdot) = 0$, and $H^2(W_j, \cdot) = 0$ for any local coefficient system $\cdot$ on W_j. In view of these vanishing results, we see that the L^2-solution spaces of the operators on these pieces $W_j \times t$ vanish.

In particular, if we impose on a subinterval, say $I' = [t_0, t_1]$, of $[0,1]$ the boundary conditions $L(D(t) \mid W_1 \times I', +) \oplus L_1(0,t)$, $L(D(t) \mid W_2 \times I', -) \oplus L_2(1,t)$, where $L(D(t), \pm)$ denotes the L^2-closure of the space of eigensolutions to $D(t) \mid \partial W_1, W_2$ with eigenvalues positive, respectively, negative, then these families of self-adjoint operators have no L^2-solutions; moreover, their zero-mode spaces are isomorphic to $L_1(0,t)$ and $L_2(1,t)$, respectively. In particular, for subintervals where these vary smoothly and are of constant dimension, we have vanishing spectral flow.

Let R denote the subspace in $I \times I$ consisting of points (s,t) where $\sigma(s,t)^2 \neq \mathrm{Id}$. Given a point (s,t) in R, we have, by item (1) above,

$$\mathcal{H}(s,t) = \ker \hat{D}(s,t) = H^1(\Sigma \times s \times t, \sigma(s,t)^2 \oplus \sigma(s,t)^{-2}),$$

because in this case

$$H^0(\Sigma \times s \times t, \sigma(s,t)^2 \oplus \sigma(s,t)^{-2}) = H^2(\Sigma \times s \times t, \sigma(s,t)^2 \oplus \sigma(s,t)^{-2}) = 0.$$

Since $\mathcal{H}(s,t)$ is of constant dimension $2(2g-2)$, it is not difficult to see that $\{\mathcal{H}(s,t) : (s,t) \in R\}$ forms a Hermitian vector bundle over R. In a similar manner, the subspaces $P_\pm(s,t)$ in the spectral decomposition $L^2(\mathbb{E}_0 \mid \Sigma \times s \times t) = P_\pm(s,t) \oplus \mathcal{H}(s,t) \oplus P_-(s,t)$ also form smoothly varying, infinite-dimensional vector bundles over R. Along the the boundary line $\{(0,t) : (0,t) \in R\}$, where we split $M \times I$, the above Hermitian vector bundle has, by (7.6) above, a Lagrangian subbundle:

Supply closing right braces in (7.8).

$$\begin{aligned} \mathcal{B}_1(t) &= \mathrm{Im}\{H^1(W_1 \times 0 \times t, \sigma(0,t)^2 \oplus \sigma(0,t)^{-2}) \\ &\qquad \to H^1(\Sigma \times 0 \times t, \sigma(0,t)^2 \oplus \sigma(0,t)^{-2})\} \\ \mathcal{B}_2(t) &= \mathrm{Im}\{H^1(W_2 \times 1 \times t, \sigma(1,t)^2 \oplus \sigma(1,t)^{-2}) \\ &\qquad \to H^1(\Sigma \times 1 \times t, \sigma(1,t)^2 \oplus \sigma(1,t)^{-2})\}. \end{aligned} \tag{7.8}$$

Thus the sum $\mathcal{B}_1(t) \oplus P_+(0,t)$ and $\mathcal{B}_1(t) \oplus P_-(1,t)$ form two restricted Lagrangians that can be used as smoothly varying boundary conditions for our operators over the subintervals of $\{(0,t) : (0,t) \in R\}$. Here M is decomposed into W_1 and $\Sigma \times I \cup W_2$. Unfortunately, the endpoint $(0,0)$ is outside this good region, so we must explicitly address this problem to make progress.

We have the corresponding restricted Lagrangian sum $\mathcal{B}_2(t) \oplus P_+(1,t)$ and $\mathcal{B}_2(t) \oplus P_-(1,t)$ forming two restricted Lagrangians that can be used as smoothly varying boundary conditions for our operators over the subintervals of $\{(1,t) : (1,t) \in R\}$. Here M is cut open along ∂W to get pieces, $W_1 \cup \Sigma \times I$ and W_2.

On the other hand, for a point (s,t) outside of R, i.e., $\sigma(s,t)^2 = \mathrm{Id}$, the Hermitian vector space $\ker \hat{D}(s,t)$, by (7.5), is isomorphic to $H^0(\Sigma \times s \times t, \mathbb{C} \oplus \mathbb{C}) \oplus H^1(\Sigma \times s \times t, \mathbb{C} \oplus \mathbb{C}) \oplus H^2(\Sigma \times s \times t, \mathbb{C} \oplus \mathbb{C})$. Since the latter has dimension $2(2g+2)$, it presents a jump from $\{\mathcal{H}(s,t) : (s,t) \in R\}$. For example, this occurs along a neighborhood of the bottom line $I \times 0 = \{(s,0) : 0 \leq s \leq 1\}$. Along the two edges $0 \times I$ and $1 \times I$ and away from the neighborhoods of $(0,0)$ and $(1,0)$, there may also be isolated points $(0,t^*)$ and $(1,t^*)$ outside of R. They correspond to the half-lattice points in $\frac{1}{2} \cdot H^1(\Sigma, \mathbb{Z})$, which happen to lie on the lines $\hat{\alpha}_1$ and $\hat{\alpha}_2$. In the case when the genus of Σ is bigger than 1, we can remove these isolated jumps because the dimension of $H^1(W_j, \mathbb{R})$ is at least 2. By a general position argument, we can perturb $\hat{\alpha}_1(t)$ and $\hat{\alpha}_2(t)$, keeping the endpoints fixed so that they stay inside $H^1(W_1,\mathbb{R})$ and $\sigma_1 + H^1(W_2,\mathbb{R})$ but avoid points in $\frac{1}{2}H^1(\Sigma,\mathbb{Z})$. Because $\hat{\alpha}_3(t)$ is unchanged, the construction of the region S and the singular chain $\phi : I \times I \to S$ remains the same. Since λ_{W} and λ_{BN} are independent of the choice of Heegaard decomposition, we can always assume that the genus of the splitting surface Σ is at least 2.

We will assume for the moment that the point σ_0 is not of order 2. This means that our problems with jumping Lagrangians only occurs at the point Id. We will treat the remaining order 2 cases later.

Because of the jump phenomenon near $(0,0)$, the above family will not meet the conditions necessary to apply theorem C of Part II. We will have to modify the Lagrangian subspaces $\mathcal{B}_1(t)$ in (7.8).

We may arrange that there exists $\varepsilon_1 > 0$ such that

1. $\sigma \mid I \times [0,\varepsilon_1]$ is the constant trivial representation and the bundle with connection over this piece $(\times\Sigma)$ is the trivial bundle with connection,
2. $\sigma \mid I \times [1-\varepsilon_1]$ is the constant representation σ_0 and the bundle with connection over this piece $(\times\Sigma)$ is the pullback from $\sigma_0 \times \Sigma$ as a bundle with connection,
3. $\sigma \mid [0,\varepsilon_1] \times I$ is the pullback by the projection to $\sigma \mid 0 \times I$ and similarly for connections, and
4. $\sigma \mid [1-\varepsilon_1, 1] \times I$ is the pullback by the projection to $\sigma \mid 1 \times I$ and similarly for connections.

We may and do arrange that at the times $t = \varepsilon_1,\ 1-\varepsilon_1$, the representations move quickly away from the constant representations. Of course, we have only a piecewise-smooth-parameter family, constant on $[0,\varepsilon_1]$ and on $[1-\varepsilon_1, 1]$ while smooth on $[\varepsilon_1, 1-\varepsilon_1]$, having nonzero right and left first derivatives at $t = \varepsilon_1, 1-\varepsilon_1$, along all of $M \times \varepsilon_1$ and $M \times (1-\varepsilon_1)$, respectively.

Now choose $K > 0$ so that the only eigenvalues for the operator $\hat{D}(0,0)$ inside the band $[-K,K]$ are the zero eigenvalue. Since $\hat{D}(s,t) = \hat{D}(0,0)$ for $0 \le s \le 1$, $0 \le t \le \varepsilon_1$, the same holds for the operator $\hat{D}(s,t)$ over the collar neighborhood of $0 \times I$. By upper semicontinuity there exists ε, $1 > \varepsilon > \varepsilon_1$, such that in addition K is not an eigenvalue of $\hat{D}(s,t)$ for $0 \le s \le \varepsilon$, $0 \le t \le 1$. Let $\mathcal{H}(s,t;K)$ denote the span of the eigensections ψ, $\hat{D}(s,t)\psi = \lambda\psi$ with $|\lambda| < K$. Then by the spectral projection theorem, the family of vector spaces $\{\mathcal{H}(s,t;K) : 0 \le s \le 1,\ 0 \le t \le \varepsilon\}$ form a smooth Hermitian vector bundle over $[0,1]\times[0,\varepsilon]$. By the choice of K, over $[0,1]\times[0,\varepsilon_1]$ the fiber $\mathcal{H}(s,t;K)$ can be identified with $\ker \hat{D}(0,0) = \oplus H^*(\Sigma;\mathbb{C}\oplus\mathbb{C})$. On the other hand, over $[0,1]\times(\varepsilon_1,\varepsilon]$ the fiber $\mathcal{H}(s,t;K)$ contains $\ker \hat{D}(s,t) = H^*(\Sigma, \sigma(s,t)^2 \oplus \sigma(s,t)^{-2})$ as a subspace of codimension 8 $(= 2[(2g+2)-(2g-2)])$. That is, we have bifurcation at $t = \varepsilon_1$. We will be more explicit below.

Let $P_+(s,t;K)$ and $P_-(s,t;K)$ denote, respectively, the subspaces in $P_+(s,t)$ and $P_-(s,t)$ spanned by $(\pm)$-eigensections of $\hat{D}(s,t)$ with eigenvalues lying outside of the band $[-K,K]$. Then, again by the spectral projection theorem, these

Hilbert spaces $P_\pm(s,t;K)$ vary smoothly over the region $\{(s,t) : 0 \le s \le 1,\ 0 \le t \le \varepsilon,\}$. Moreover, $P_\pm(s,t;K) = P_\pm(0,0;K)$, for (s,t) in $[0,1] \times [0,\varepsilon_1]$.

Furthermore, there are the following decompositions over (s,t) in $M \times [0,\varepsilon]$:

$$\begin{aligned} P_\pm(s,t) &= P_\pm(s,t;K) \oplus \mathcal{H}_\pm(s,t;K)\,, \\ \mathcal{H}(s,t;K) &= \mathcal{H}_+(s,t;K) \oplus \mathcal{H}(s,t) \oplus \mathcal{H}_-(s,t;K)\,, \end{aligned} \tag{7.9}$$

where $\mathcal{H}_\pm(s,t;K) = \mathcal{H}(s,t) \cap P_\pm(s,t;K)$. These last spaces $\mathcal{H}_\pm(s,t;K)$ are of dimension 4 and vary smoothly over the region $0 \le s \le 1$, $\varepsilon_1 < t \le \varepsilon$. (Recall that σ skew-commutes with $\hat{D}$ so to each eigenvector ϕ of eigenvalue λ we have $\sigma\phi$ of eigenvalue $-\lambda$).

Recall that in the symplectic spaces $\mathcal{H}(j-1,t;K)$, $j = 1,2$, $\varepsilon_1 < t \le \varepsilon$, there are the Lagrangian subspaces $\mathcal{B}_j(t)$ given by the extended L^2-solutions of $D(t) \mid W_j \times t$. Their sum with $\mathcal{H}_\pm(j-1,t;K)$ give us the Lagrangian subspaces:

$$\begin{aligned} \mathcal{B}'_1(t) &= \mathcal{B}_1(t) \oplus \mathcal{H}_+(0,t;K)\,, \\ \mathcal{B}'_2(t) &= \mathcal{B}_2(t) \oplus \mathcal{H}_-(1,t;K)\,. \end{aligned} \tag{7.10}$$

in $\mathcal{H}(j-1,t;K)$, $j = 1,2$. In (1.13) of [24], Walker proves that as $t \to \varepsilon_1$, the subspaces $\mathcal{B}_j(t)$, $\mathcal{H}_\pm(j-1,t;K)$, and $\mathcal{B}'_j(t)$ converge to the following subspaces:

$$\mathcal{B}_j(\varepsilon_1 + 0) = \lim_{t\to\varepsilon_1} \mathcal{B}_j(t)\,, \tag{7.11a}$$

$$\mathcal{H}_\pm(j-1,\varepsilon_1+0;K) = \lim_{t\to\varepsilon_1} \mathcal{H}_\pm(j-1,t;K)\,, \tag{7.11b}$$

$$\mathcal{B}'_j(\varepsilon_1 + 0) = \lim_{t\to\varepsilon_1} \mathcal{B}'_j(t)\,, \tag{7.11c}$$

in $\mathcal{H}(j-1,\varepsilon_1;K)$. We now give an explicit description of these subspaces, using some standard facts from perturbation theory. As we will soon see, there is a jump of Lagrangian subspaces $\mathcal{B}_j(t)$ at $t = \varepsilon_1$ and so $\mathcal{B}_j(\varepsilon_1 + 0) \ne \mathcal{B}_j(\varepsilon_1)$.

First, over $[\varepsilon_1,\varepsilon]$ our operators depend analytically on t. Hence, by the curve selection lemma of Kato [16, theorem 2.6], there exist two sets of $2(2g+2)$ real analytic functions $\{\lambda_j(k,t) : k = 1,\dots 2(2g+2),\ \varepsilon_1 \le t \le \varepsilon\}$ such that they form the sets of eigenvalues of $\hat{D}(j-1,t)$ in the band $[-K,K]$. At $t = \varepsilon_1$, all these eigenvalues vanish, and so we have to analyze the phenomenon of bifurcation at $t = \varepsilon_1$. By perturbation theory, the rate of bifurcation, i.e., the set of the derivatives $\lambda'_j(t,\varepsilon_1+0) = \frac{d}{dt}\lambda_j(k;t)|_{t=\varepsilon_1+0}$, coincides with the set of eigenvalues of the operator

$$T_j = \pi \circ \left(\frac{d}{dt}\, \hat{D}(j-1,t)\Big|_{t=\varepsilon_1+0} \right) \tag{7.12}$$

where π is the orthogonal projection of the Hilbert space onto $\mathcal{H}(j-1,\varepsilon_1) = \ker \hat{D}(j-1,\varepsilon_1)$. For all practical purposes, we can regard T_j as an operator acting on the finite-dimensional vector space $\mathcal{H}(j-1,\varepsilon_1)$, which can be identified with the sum of harmonic 1-forms, 0-forms, and 2-forms on Σ with coefficients in $\mathbb{C} \oplus$

$\mathbb{C}$. Note that in the present setting the operator $\hat{D}(j-1,0)$ and its derivatives commute. Hence we can apply the treatment of [16, theorem 2.6] to deduce that the eigenvalues have a linear rate of decay.

Let ω_j denote the tangent vector $\omega_j = \frac{d}{dt}\alpha_j(t)|_{t=\varepsilon_j+0}$ of the path $\alpha_j(t) \subset R(\Sigma)$ at $t=\varepsilon_j$. We have arranged in advance that $\omega_j \neq 0$ along all of $[0,1]\times\varepsilon_1$. Then, as in Section 6, ω_j can be regarded as a cohomology class in $H^1(\Sigma, h^\perp)$, which can in turn be identified with a harmonic 1-form $\Omega^1(\Sigma, h^\perp)$. In the same manner, the complexification $\omega_j \otimes \mathbb{C}$ gives rise to a harmonic 1-form $v_j = \omega_j \otimes \mathbb{C}$ in $\Omega^1(\Sigma, h^\perp \otimes \mathbb{C}) = \Omega^1(\Sigma, \mathbb{C}\oplus\mathbb{C})$. We have prearranged that $v_j \neq 0$. Recall that

$$\hat{D}(s,t)(a,b,c) = \left(d_{A(s,t)}b - *d_{A(s,t)}c, -d^*_{A(s,t)}a, -*d_{A(s,t)}a\right)$$

where along $(j-1)\times[\varepsilon_1,\varepsilon]$ the operator $d_{A(j-1,t)}$ is given by $d + t(v_j\wedge)$. Here $v_j\wedge$ stands for the endomorphism on $\oplus\Omega^*(\Sigma, h^\perp\otimes\mathbb{C})$ given by the wedge product with v_j.

Putting the above information into (7.12), we can view T_j as the block matrix

$$T_j = \begin{bmatrix} 0 & *(v_j\wedge) & (v_j\wedge)* \\ -*(v_j\wedge) & 0 & 0 \\ -(v_j\wedge)* & 0 & 0 \end{bmatrix} \tag{7.13}$$

acting on the space of column vectors with entries in harmonic 1-forms, 0-forms, and 2-forms. From (7.13) it is not difficult to deduce that the eigenvalues of T_j are 0, $+1$, and -1. The 0-eigenspace is generated as a $(h^\perp\otimes\mathbb{C})$–module by the column vectors

$$\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \tag{7.14}$$

where x is a harmonic 1-form in $\Omega^1(\Sigma, h^\perp\otimes\mathbb{C})$ with the property $x\wedge v_j = x\wedge *v_j = 0$. Hence, it has dimension $2(2g-2)$ and coincides with the limit space $\lim_{t\to\varepsilon_1+}\mathcal{H}(j-1,t)$ of the nearby zero modes. In particular, it contains $\mathcal{B}_j(\varepsilon_1+0)$ in (7.11a) as a Lagrangian subspace. On the other hand, there are the column vectors

(7.15)

$$a_j = \begin{bmatrix} v_j \\ 0 \\ -i*1 \end{bmatrix}, \qquad b_j = \begin{bmatrix} *v_j \\ -i1 \\ 0 \end{bmatrix}, \qquad c_j = \begin{bmatrix} *v_j \\ i1 \\ 0 \end{bmatrix}, \qquad d_j = \begin{bmatrix} -v_j \\ 0 \\ -i*1 \end{bmatrix},$$

where a_j and b_j are the generators of the $(+1)$-eigenspace, and c_j and d_j are the generators of the (-1)-eigenspace. In view of the bifurcation, these (±1)-eigenspaces of T_j are the respective limit spaces $\mathcal{H}_\pm(j-1,\varepsilon_j+0;K)$ of the nearby $(\pm)$-eigenmodes.

The extended L^2-solution space $\mathcal{B}_j(\varepsilon_1)$ can also be described in terms of the eigenvectors in (7.14) and (7.15). First of all, it contains all the column vectors in

(7.14) with x in the image of the induced homomorphism $H^1(W_j \times \varepsilon_1, \mathbb{C} \oplus \mathbb{C}) \to H^1(\Sigma \times \varepsilon_1, \mathbb{C} \oplus \mathbb{C})$. Since this homomorphism is the limit of the nearby maps $H^1(W_j \times \varepsilon_1, h^\perp \otimes \mathbb{C}) \to H^1(\Sigma \times \varepsilon_1, h^\perp \otimes \mathbb{C})$, its image coincides with $\mathcal{B}_j(\varepsilon_1 + 0)$. A dimension count shows that the rest of $\mathcal{B}_j(\varepsilon_1)$ is generated by the following:

$$\begin{bmatrix} v_j \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}(a_j - d_j)\,, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2i}(-b_j + c_j)\,. \tag{7.16}$$

Comparing the two Lagrangians $\mathcal{B}_j(\varepsilon_1)$ and $\mathcal{B}'_j(\varepsilon_1 + 0)$ in $\mathcal{H}(j-1, t; K)$, we see that both of them contain $\mathcal{B}_j(\varepsilon_1 + 0)$ and become different on its orthogonal complements:

$$\mathcal{B}'_j(\varepsilon_1) = \begin{cases} \mathcal{B}_j(\varepsilon_1 + 0) \oplus (\mathbb{C} \oplus \mathbb{C})[a_j] \oplus (\mathbb{C} \oplus \mathbb{C})[b_j] & \text{for } j = 1\,, \\ \mathcal{B}_j(\varepsilon_1 + 0) \oplus (\mathbb{C} \oplus \mathbb{C})[c_j] \oplus (\mathbb{C} \oplus \mathbb{C})[d_j] & \text{for } j = 2\,, \end{cases}$$
$$\mathcal{B}_j(\varepsilon_1) = \mathcal{B}_j(\varepsilon_1 + 0) \oplus (\mathbb{C} \oplus \mathbb{C})[a_j - d_j] \oplus (\mathbb{C} \oplus \mathbb{C})[b_j - c_j]\,.$$

To define a continuous Lagrangian boundary condition suitable for applying theorem C of [12], we have to connect the following two orthogonal complements:

$$\begin{aligned} U'_j(t) &= (\mathbb{C} \oplus \mathbb{C})[a_j] \oplus (\mathbb{C} \oplus \mathbb{C})[b_j]\,, & 0 \le t \le \varepsilon_1\,, \\ U_j(t) &= (\mathbb{C} \oplus \mathbb{C})[a_j - d_j] \oplus (\mathbb{C} \oplus \mathbb{C})[b_j - c_j]\,, & 0 \le t \le \varepsilon_1\,, \end{aligned}$$

by families of Lagrangians in $\mathcal{H}(j-1, t)$, $0 < t \le \varepsilon_1$. (Here, as before, we identify $\mathcal{H}(j-1, t)$ with $\mathcal{H}(j-1, \varepsilon_1)$, and so $U'_j(t)$ and $U_j(t)$ are Lagrangians in the same symplectic space.) As in Section 5, there is the complex structure on $\mathcal{H}(j-1, t)$ given by

$$J = \begin{bmatrix} -* & 0 & 0 \\ 0 & 0 & * \\ 0 & -* & 0 \end{bmatrix}.$$

Note $c_j = -Ja_j$ and $d_j = Jb_j$, and so $U_j(t)$ is generated by $(a_j + Jb_j)$ and $(b_j + Ja_j)$ while $U'_j(t)$ is generated by $x_j = (a_j + b_j)$ and $y_j = (a_j - b_j)$. With respect to the generators $\{x_j, y_j, Jx_j, Jy_j\}$, there is the unitary transformation $u_j(t)$ on $\mathcal{H}(j-1, t)$ given by

$$\begin{cases} x_j \longmapsto \cos\left(\frac{\varepsilon_1 - t}{2\varepsilon_1}\right) x_j + \sin\left(\frac{\varepsilon_1 - t}{2\varepsilon_1}\right) Jx_j \\ y_j \longmapsto \cos\left(\frac{\varepsilon_1 - t}{2\varepsilon_1}\right) y_j - \sin\left(\frac{\varepsilon_1 - t}{2\varepsilon_1}\right) Jy_j \\ z \longmapsto z \quad \text{for } z \in \mathcal{B}(j-1, \varepsilon_1 + 0). \end{cases} \tag{7.17}$$

Note at $t = \varepsilon_1$ the map $u(\varepsilon_1)$ is the identity while at $t = \varepsilon_1/2$,

$$u_j(\varepsilon_1/2)x_j = \frac{1}{\sqrt{2}}(1 + J)x_j = \frac{1}{\sqrt{2}}\{(a_j + Jb_j) + (b_j + Ja_j)\}\,,$$
$$u_j(\varepsilon_1/2)y_j = \frac{1}{\sqrt{2}}(1 - J)y_j = \frac{1}{\sqrt{2}}\{(a_j + Jb_j) - (b_j + Ja_j)\}\,.$$

Applying $u_j(t)$ on $\mathcal{B}'_j(t)$, we obtain a smooth family $\{u_j(t)\cdot\mathcal{B}'_j(t) : \varepsilon_1/2 \le t \le \varepsilon_1\}$ connecting up $\mathcal{B}'_j(\varepsilon_1)$ at $t=\varepsilon_1$ to $\mathcal{B}_j(\varepsilon_1)$ at $t=\varepsilon_1/2$.

PROPOSITION 7.2 *Let $\mathcal{B}_j(t)$, $P_\pm(j-1,t)$, and $u_j(t)\cdot\mathcal{B}'_j(t)$ be defined as above. Then the formulae*

$$\mathcal{C}_j(t)=\begin{cases}\mathcal{B}_j(t)\oplus P_\pm(j-1,t) & \text{for } \varepsilon_1\le t\le 1\,,\\ u_j(t)\cdot\mathcal{B}'_j(t)\oplus P_\pm(j-1,t) & \text{for } \varepsilon_1/2\le t\le\varepsilon_1\,,\\ \mathcal{B}_j(t)\oplus P_\pm(j-1,t) & \text{for } 0\le t\le \varepsilon_1/2\,,\end{cases}\tag{7.18}$$

$j=1,2$, *define two continuous families of restricted Lagrangians in* $(L^2(\mathbb{E}\mid \Sigma\times(j-1))\times t)$. *Using them as boundary conditions for the operators $D\mid W_j\times t$, $j=1,2$, we obtain two continuous families of self-adjoint elliptic operators*

$$D(t,\mathcal{C}_j(t)) : L^2_1(\mathbb{E}_j\mid W_j\times t,\mathcal{C}_j(t))\to L^2(\mathbb{E}_j\mid W_j\times t)\,.$$

Moreover, this family satisfies the endpoint conditions necessary to apply theorem C *of* [12].

From the definition, the isotropic subspaces $\mathcal{B}_j(t)$, $P_\pm(j-1,t)$, and $u_j(t)\cdot\mathcal{B}'_j(t)$ vary smoothly over their domains of definition; it is enough to check that the various terms in (7.17) match up at $t=\varepsilon_1$ and $t=\varepsilon_1/2$. Since $\mathcal{B}_j(t)\oplus P_\pm(j-1,t)=\mathcal{B}_j(t)\oplus\mathcal{H}_\pm(j-1,t;K)\oplus P_\pm(j-1,t;K)=\mathcal{B}'_j(t)\oplus P_\pm(j-1,t;K)$, we have the sum $\mathcal{B}'_j(\varepsilon_1)\oplus P_\pm(j-1,\varepsilon_1;K)$ at $t=\varepsilon_1$. This last sum coincides with $u_j(\varepsilon_1)\cdot\mathcal{B}'_j(\varepsilon_1)\oplus P_\pm(j-1,\varepsilon_1)$ because $u_j(\varepsilon_1)=\mathrm{Id}$. Similarly, the Lagrangians agree at $t=\varepsilon_1/2$ because $u_j(\varepsilon_1/2)\cdot\mathcal{B}'_j(\varepsilon_1/2)=\mathcal{B}_j(\varepsilon_1/2)$.

Note the signs in $P_\pm(j-1,t)$ are $+$ for $j=1$ and $-$ for $j=2$. Since the $\mathcal{C}_j(t)$ coincide with $P_\pm(j-1,t)$ except for some finite-dimensional subspaces, it follows that they are restricted Lagrangians and the operators $D(t,\mathcal{C}_j(t))$ are self-adjoint.

We now define two continuous families of special complementary pairs of Lagrangians $\mathcal{C}_1(s,t)$ and $\mathcal{C}_2(s,t)$ over $I\times I$ in each of the symplectic spaces $L^2(\mathbb{E}\mid\Sigma\times(s,t))$ as follows: $\mathcal{C}_j(s,t)$ over $I\times([0,\varepsilon_1]\cup[1-\varepsilon_1,1])$ is the pullback of $\mathcal{C}(j-1,t)$ over $(j-1)\times([0,\varepsilon_1]\cup[1-\varepsilon_1])$ for $j=1,2$. $\mathcal{C}_1(s,t)$ over $[0,\varepsilon_1]\times I$ is the pullback of $\mathcal{C}(0,t)$ over $0\times I$. $\mathcal{C}_1(s,t)$ over $[1-\varepsilon_1]\times I$ is the pullback of $\mathcal{C}(1,t)$ over $I\times I$. The rest of the restricted Lagrangians are chosen as any smooth extension of these given Lagrangians $\mathcal{C}_j(s,t)$ that enjoy the special complementarity property: The $\mathcal{C}_j(s,t)$ intersect the closure of the span of the positive and negative eigenspaces of $\hat{D}(s,t)$ in a subspace of finite codimension for $j=1,2$, respectively. This is easily done.

These Lagrangians can be utilized as appropriate boundary conditions. They differ only by finite rank operators from the Atiyah-Patodi-Singer boundary conditions [3], and thus the operators with boundary conditions $(D(t)\mid(W_1\times t)\cup(\Sigma\times[0,s])\times t,\mathcal{C}_1(s,t))$, $(D(t)\mid((\Sigma\times[s,1])\cup W_2)\times t),\mathcal{C}_2(s,t))$ are self-adjoint elliptic operators with well-defined spectrum and excellent properties. For example, the associated spaces of extended L^2-solutions by Section 3 vary continuously and are Lagrangians in the Hilbert spaces $L^2(\mathbb{E} \mid \Sigma \times (s,t))$.

From (7.17) it is clear that $\mathcal{C}_j(t)$, $j = 1,2$, satisfy the endpoint condition of [12], and so they can be used as the Lagrangian families in the calculation of the average spectral flow of $\{(D(t) \mid (W_1 \times t), \mathcal{C}_1(t)) : 0 \leq t \leq 1\}$ and $\{(D(t) \mid (W_2 \times t), \mathcal{C}_2(t)) : 0 \leq t \leq 1\}$.

We seek to apply our spectral flow decomposition theorem to the family $D(t)$ over M when we split M along the surface $\Sigma = \Sigma \times 0$. This splits M into W_1 and $I \cup W_2$. An ingredient in these calculations is the following:

PROPOSITION 7.3 (a) *The average spectral flow of* $\{D(t)|(W_j \times t), \mathcal{C}_j(t)) : 0 \leq t \leq 1\}$ *equals zero.*

(b) *The average spectral flow of* $\{D(t)|([0,1] \cup W_2) \times t, \mathcal{C}_2(t) : 0 \leq t \leq 1\}$ *equals zero.*

PROOF: By the additivity of spectral flows, we can break up the calculation of $\mathcal{A}\text{–spectral flow}\{D(t) \mid W_j \times t, \mathcal{C}_j(t) : 0 \leq t \leq 1\}$ into calculations over the intervals $[0, \varepsilon_1/2]$, $[\varepsilon_1/2, \varepsilon_1]$, and $[\varepsilon_1, 1]$. By (7.7) there are no L^2-solutions for $D(t) \mid W_j \times t$, and so the extended L^2-solution spaces are mapped isomorphically onto $\mathcal{B}_j(t)$. Since over $[0, \varepsilon_1/2]$ and $[\varepsilon_1, 1]$ the boundary condition is $\mathcal{B}_j(t) \oplus P_+(t)$, it follows that $\ker D_j(t)(W_j \times t, \mathcal{C}_j(t))$ has the same dimension as $\mathcal{B}_j(t)$. Since these spaces have constant dimensions over the intervals $[0, \varepsilon_1/2]$ and $[\varepsilon_1, 1]$ and vary smoothly, the corresponding average spectral flows equal zero for $j = 1, 2$.

It remains to calculate the $\mathcal{A}$–spectral flow of $\{D_j(t)(W_j \times t, \mathcal{C}_j(t)) : \varepsilon_1/2 \leq t \leq \varepsilon_1\}$. Since over $[\varepsilon_1/2, \varepsilon_1]$ the operators $D_j(t) \mid W_j \times t$ are constant, by theorem D of [12], these spectral flows are equal to the following averaged Maslov indices:

(7.19a) $\quad \mathcal{A} - \operatorname{Mas}\{(u_1(t) \cdot \mathcal{B}_1'(t), \mathcal{B}_1(t)) : \varepsilon_1/2 \leq t \leq \varepsilon_1\} \quad \text{for } j = 1,$

(7.19b) $\quad \mathcal{A} - \operatorname{Mas}\{(\mathcal{B}_2(t), u_2(t) \cdot \mathcal{B}_2'(t)) : \varepsilon_1/2 \leq t \leq \varepsilon_1\} \quad \text{for } j = 2.$

Recall that over the subspace $\mathcal{B}(j-1, \varepsilon_1 + 0)$ the transformation $u_j(t)$ is the identity, and so the corresponding average Maslov indices are equal to zero. Thus, by the symplectic additivity of the Maslov index, the terms in (7.19a) and (7.19b) are, respectively, $\mathcal{A} - \operatorname{Mas}\{(u_1(t) \cdot U_1'(t), U_1(t)) : \varepsilon_1/2 \leq t \leq \varepsilon_1\}$ and $\mathcal{A} - \operatorname{Mas}\{(U_2(t), u_2(t) \cdot U_2'(t)) : \varepsilon_1/2 \leq t \leq \varepsilon_1\}$. From (7.17) the Lagrangians $U_j'(t)$ are the sum of two components of the same dimension; moreover, the transformation $u_j(t)$ is a clockwise rotation on one and counterclockwise on the other. It follows that the average Maslov indices are equal to zero. This proves the first assertion of Proposition 7.3.

As for the second statement, by the homotopy invariance of spectral flow for fixed endpoints, we may conclude that the families $\{(D(t)|(\Sigma \times [s,1] \cup W_2) \times t, \mathcal{C}_2(s,t)) : \varepsilon_1 \leq t \leq 1\}$ have the same spectral flow for all s since the endpoint operators are independent of s. As for the region $0 \le t \le \varepsilon_1$, the operators $(D(t) \mid (\Sigma \times [s,1] \cup W_2) \times t, \mathcal{C}_2(s,t))$ are again independent of s so the spectral flow is the same as for $s = 1$. Hence, this case follows from the above. □

With the family of restricted Lagrangians given as above, we consider the Sobolev spaces of sections of $\mathbb{H}^\perp \otimes \mathbb{C}$ over $\Sigma \times [s,1] \times t \cup W_2 \times t$ with $\mathcal{C}_2(s,t)$ as the boundary condition. Over these spaces there are the continuous family of self-adjoint elliptic operators $D(s,t;\mathcal{C}_2(s,t))$ induced by $D(t)$. In particular, along $\Sigma \times 0 \times I$ we have the family of restricted Lagrangians $\mathcal{C}_2(0,t) = \mathcal{C}(0,t)$ and operators $D(0,t;\mathcal{C}(0,t))$, which can be compared with $\mathcal{C}_1(0,t)$ and $D(t)(W_1 \times t, \mathcal{C}_1(0,t))$ defined before over W_1.

PROPOSITION 7.4 *The average spectral flow of* $\{D(t) : 0 \le t \le 1\}$ *over* M *equals the average Maslov index* $\{(\mathcal{C}_2(0,t), \mathcal{C}_1(0,t)) : 0 \le t \le 1\}$.

PROOF: We split $M \times I$ along $\Sigma \times 0 \times I$ into two pieces $W_1 \times I$ and $\Sigma \times I \times I \cup W_2 \times I$. On $L^2(\mathbb{E} \mid \Sigma \times 0 \times I)$, there are two families of restricted Lagrangians $\{(\mathcal{C}_1(t), \mathcal{C}'_2(t))\}$. As explained above, the average spectral flow is independent of the stretching of the manifold M along Σ. Hence, by section 7 of [12], the average spectral flow of $\{D(t) : 0 \le t \le 1\}$ equals

$$\begin{aligned}&\mathcal{A}\text{−spectral flow}\{D(t) \mid W_1, \mathcal{C}_1(0,t) : 0 \le t \le 1\}\\ &\quad + \mathcal{A}\text{−spectral flow}\{D_2(t) \mid (((I \times I) \cup W_2), \mathcal{C}_2(0,t)) : 0 \le t \le 1\}\\ &\quad + \mathcal{A}\text{–Mas}\{(\mathcal{C}_1(0,t), \mathcal{C}_2(0,t)) : 0 \le t \le 1\}\,.\end{aligned}$$

□

As explained above, averaged spectral flow is independent of r in our case.

PROPOSITION 7.5 *The average Maslov index of* $\{(\mathcal{C}_1(0,t), \mathcal{C}_2(0,t)) : 0 \le t \le 1\}$ *equals* $2c_1(\det(\partial \otimes h^\perp \otimes \mathbb{C}) \mid E, \Phi)$.

Recall that by our choice, our restricted Lagrangians have the special feature that $\mathcal{C}_1(s,t)$ intersects the closure of the span of the positive eigenvectors of $\hat{D}(s,t)$ in a subspace of finite codimension, while $\mathcal{C}_2(s,t)$ intersects the closure of the span of the negative eigenvectors of $\hat{D}(s,t)$ in a subspace of finite codimension. Thus, the consideration of Maslov index for these Lagrangians is almost the same as that for finite-dimensional Lagrangians considered in [10].

As in (2.28), associated to the restricted Lagrangians $\{\mathcal{C}(s,t) : 0 \le s \le 1,\ 0 \le t \le 1\}$ and $\{\mathcal{C}_1(t) : 0 \le t \le 1\}$, there are the corresponding sections $\{\det \mathcal{C}(s,t) : 0 \le s \le 1,\ 0 \le t \le 1\}$, $\{\det \mathcal{C}_1(t,0) : 0 \le t \le 1\}$, and $\{\det \mathcal{C}_2(t,1) : 0 \le t \le 1\}$ of the determinant line bundles $\{\det \hat{D}(s,t) : 0 \le s \le 1,\ 0 \le t \le 1\}$ and $\{\det \hat{D}(j-1,t) : 0 \le t \le 1\}$. The average Maslov index of $\{(\mathcal{C}_1(t,0), \mathcal{C}_2(t,0)) : 0 \le t \le 1\}$ can be expressed in terms of these sections.

More precisely, at the two endpoints $t = 1, 0$, we can connect the Lagrangians $(\mathcal{C}_2(0,1), \mathcal{C}_1(0,1))$ and $(\mathcal{C}_1(0,0), \mathcal{C}_2(0,0))$ by clockwise- and counterclockwise-rotating Lagrangians $v_{++}(1), v_{--}(1)$, and $v_{++}(2), v_{--}(2)$, respectively, on the transversal parts of these Lagrangians (cf. (6.5) or section 13 of [10]). Accordingly, the sections $\{\det \mathcal{C}_2(0,t) : 0 \le t \le 1\}$ and $\{\det \mathcal{C}_1(0, 1-t) : 0 \le t \le 1\}$ can be connected to form two loops of sections f_{++} and f_{--} in $\det \hat{D}$ (following the notation of [10] composing from left to right),

$$f_{\pm\pm} = [\mathcal{C}_2(0,t)] * v_{\pm\pm}(1) * [\mathcal{C}_1(0, 1-t)] * v_{\pm\pm}(0)\,. \tag{7.20}$$

Let $\hat{\mu} : \pi_1(\mathrm{Lag}_{\mathbb{C}}) \to \mathbb{Z}$ be defined as in [10]. Then, by (13.1) of [10], the average $(\hat{\mu}(f_{++}) + \hat{\mu}(f_{--}))/2$ is the same as the averaged Maslov index,

$$\mathcal{A}\text{–}\mathrm{Mas}\{(\mathcal{C}_1(0,t), \mathcal{C}_2(0,t)) : 0 \le t \le 1\}\,.$$

(Note the reversal in the order of 1 and 2 in (7.20)).

Since everything is constant in a neighborhood of the segments $I \times 0$ and $I \times 1$, we can view these Lagrangians $v_{\pm\pm}(0)$ and $v_{\pm\pm}(1)$ as defined over the corner points $(0,1)$, $(0,0)$, and $[0,s] \times I$. Combining them with $\{\mathcal{C}_2(s,t) : 0 \le t \le 1\}$ and $\{\mathcal{C}_1(0, 1-t) : 0 \le t \le 1\}$, we have two loops of Lagrangians around the boundary of the square $[0,s] \times I$:

$$\begin{aligned} f'(s)_{\pm\pm} = {} & [\mathcal{C}_2(s',0) : 0 \le s' \le s] * [\mathcal{C}_2(s,t)] * [\mathcal{C}_2(s-s',1) : 0 \le s' \le s] \\ & * v_{\pm\pm}(1) * [\mathcal{C}_1(0,1-t)] * v_{\pm\pm}(0)\,. \end{aligned} \tag{7.21}$$

Here the parameters s and t run from 0 to 1 in an increasing direction.

By the deformation invariance of the mapping μ, the obstruction to extending the section $f'_{\pm\pm}$ of the determinant bundle across the disk $[0,s] \times I$ is equal to the obstruction for $f_{\pm\pm}$. Thus the averaged obstruction for $f'(s)_{\pm\pm}$ is independent of s and equals that for $f'(0)_{\pm\pm}$, which is $f_{\pm\pm}$.

The determinant sections $\Psi_\pm$ associated to these two loops of Lagrangians give two trivializations of the determinant line bundle $\det \hat{D}$ over the boundary of $I \times I$. From this it is not difficult to see in view of the above that

$$\hat{\mu}(f'(1)_{\pm\pm}) = c_1[\det \hat{D} \mid I \times I, \Psi_\pm]$$

and

$$\frac{\hat{\mu}(f_{++}) + \hat{\mu}(f_{--})}{2} = c_1[\det \hat{D} \mid I \times I, \Psi]\,.$$

Here $c_1[\det \hat{D} \mid I \times I, \Psi]$ is the average of $c_1[\det \hat{D} \mid I \times I, \Psi_\pm]$, the two relative Chern numbers, as in (6.6).

We can now compare $c_1[\det \hat{D} \mid I \times I, \Psi]$ with $c_1[\det \partial \otimes h^\perp \mid \mathbb{E}, \Phi]$. In the line bundle $\det \hat{D}$, we have been using the B^{ev}-operator and in $\det \partial \otimes h^\perp$ the Cauchy-Riemann operator ∂. From the discussion in Section 5, there is a canonical isomorphism $\det \hat{D} \cong [\det \partial \otimes h^\perp]^{\otimes 2}$. Under this isomorphism we claim that the trivialization Ψ becomes $\Phi^{\otimes 2}$. Recall in (6.3) the determinant section $(\det^1 \eta_j)$ is defined by the complex Lagrangians $\mathrm{Im}[H^1(W_j, \sigma^2) \to H^1(\Sigma, \sigma^2)]$. Comparing this with $\mathcal{B}_j(t) = \mathrm{Im}[H^1(W_j, \sigma^2 \oplus \sigma^{-2}) \to H^1(\Sigma, \sigma^2 \oplus \sigma^{-2})]$, it is easy to see that $(\det^1 \eta_j)^{\otimes 2}$ is the same as $[\det \mathcal{C}_j(t)]$ for t outside of $[\varepsilon_1/2, \varepsilon_1]$. Inside $[\varepsilon_1/2, \varepsilon_2]$ the Lagrangian $\mathcal{C}_j(t)$ is obtained from $\mathcal{B}_j(0)$ by composing it with a unitary transformation $u_j(t)$ in (7.18). Since the latter has determinant 1 (cf. (7.17)), this deformation does not affect the determinant section and so $[\det \mathcal{C}_j(t)] = (\det^1 \eta_j)^{\otimes 2}$ for all t. This proves our claim and hence Proposition 7.5.

PROOF OF THEOREM 6.2: As explained before, the proof follows from the formula (7.4). In the case with σ_0^2 not of order 2, this last formula follows from combining Proposition 7.4 and 7.5.

In the case with σ_0^2 of order 2, we must confront the jump in zero modes at the points $(s, 1)$ in $I \times I$ where the representation is constantly σ_0^2 in addition to the points $(s, 0)$, where as before σ_0^2 is trivial. This is achieved in precisely the same manner as we employed for the jumps at Id for the points $(s, 0)$. We need but use the method of choice of Lagrangian pairs at the end $(s, 1)$ as we employed at the end $(s, 0)$. The result is again Theorem 6.2 in this special case. □

8 Canonical Perturbations

We return to the setting discussed at the beginning of Section 6. M is a rational homology 3-sphere with a Heegaard decomposition $M = W_1 \cup W_2$, $\Sigma = W_1 \cap W_2 = \partial W_1 = \partial W_2$. However, we drop the assumption of the fundamental group $\pi_1(M)$ being cyclically finite. Therefore, near a reducible representation $\rho : \pi_1(M) \to \mathrm{SU}(2)$ of $\pi_1(M)$, the Lagrangian subspaces Q_1 and Q_2 need not be in a transverse position. In this setting we will define two "canonical" perturbations, Q_{2R} and Q_{2L}, of Q_2 in the Zariski tangent cone of ρ that intersect Q_1 transversely. Roughly speaking, there is a complex structure on such a neighborhood of ρ, and with respect to this complex structure Q_{2R} is the result of right-handed rotation while Q_{2L} is the result of a left-handed one. (The appropriate complex structure is chosen so as to be compatible with the underlying symplectic structure.)

After these perturbations, we will have subspaces $(Q_1 \cap Q_{2R}) \cap (R - S)$ and $(Q_1 \cap Q_{2L}) \cap (R - S)$ compactly supported in the irreducibles $(R - S)$. Therefore, as in the cyclically finite case, we can perform further perturbations on Q_{2R} and Q_{2L} away from a neighborhood of the reducibles S to get stratified Lagrangian subspaces $Q_{2R}^{\#}$ and $Q_{2L}^{\#}$ such that their smooth strata $\bar{Q}_{2R}^{\#} = Q_{2R}^{\#} - S$ and $\bar{Q}_{2L}^{\#} = Q_{2L}^{\#} - S$ intersect $\bar{Q}_1 = Q_1 - S$ transversely and compactly. Once this is achieved, we define natural extensions $\lambda(M)_R$ and $\lambda(M)_L$ of the Boyer-Nicas invariant to all rational homology spheres by the formulae

$$(8.1) \qquad \lambda(M)_R = \sum_{P \in \bar{Q}_1 \cap \bar{Q}_{2R}^{\#}} \mathrm{Sign}(P)\,, \qquad \lambda(M)_L = \sum_{P \in \bar{Q}_1 \cap \bar{Q}_{2L}^{\#}} \mathrm{Sign}(P)\,.$$

In the case when $\pi_1(M)$ is cyclically finite, the canonical perturbations Q_{2r} and Q_{2L} are isotopic to Q_2 through a family of Lagrangians always transverse to Q_1 near the reducibles. From this we have the following:

PROPOSITION 8.1 *For a rational homology 3-sphere M whose fundamental group $\pi_1(M)$ is cyclically finite,*

$$\lambda_{\mathrm{BN}}(M) = \lambda(M)_R = \lambda(M)_L$$

Another feature of these canonical perturbations Q_{2R} and Q_{2L} is that they are complex Lagrangians at reducible representations ρ of $\pi_1(M)$ in the same sense as discussed in Section 6. Note that this "Walker" complex structure is different from the one compatible with the symplectic structure. It uses a choice of ordering of the splitting of ρ as a direct sum $\rho = \rho_1 \oplus \rho_2$ of one-dimensional representations. It is well-defined only on the space of representations of the fundamental group into $U(1)$, not its image as reducible $\mathrm{SU}(2)$-representations. In this quotient the order is forgotten. Following the treatment there, we obtain two well-defined Walker correction terms from these perturbations, $I(\rho, Q_1, Q_{2R})$ and $I(\rho, Q_1, Q_{2L})$.

PROPOSITION 8.2 *For a rational homology 3-sphere M, we have*

$$\begin{aligned}\lambda_W(M) &= \lambda(M)_R + \sum_\rho I(\rho, Q_1, Q_{2R}) \\ &= \lambda(M)_L + \sum_\rho I(\rho, Q_1, Q_{2L})\end{aligned}$$

where the terms in the above sum go through all the reducible $\mathrm{SU}(2)$*-representations ρ of $\pi_1(M)$.*

As a generalization of Theorem B in the introduction, we can express the above Walker corrections $I(\rho, Q_1, Q_{2R})$ and $I(\rho, Q_1, Q_{2L})$ in terms of known invariants of M.

THEOREM 8.3 *Let M be a rational homology 3-sphere and let $[\rho] = [\sigma \oplus \sigma^{-1}]$, $\sigma : \pi_1(M) \to U(1)$, be a reducible* $\mathrm{SU}(2)$*-representation of $\pi_1(M)$. Then the Walker correction terms $I(\rho, Q_1, Q_{2R})$ and $I(\rho, Q_1, Q_{2L})$ associated to the right-handed and left-handed perturbations at ρ, respectively, satisfy the following equalities*:

(a) $I(\rho, Q_1, Q_{2R}) - I(\rho, Q_1, Q_{2L}) = \dim_{\mathbb{C}} H^1(M, \sigma^2)$,
(b) $I(\rho, Q_1, Q_{2R}) + I(\rho, Q_1, Q_{2L}) = -(1/2)\rho(M, \sigma^2)$.

Here $\rho(M, \sigma^2)$ stands for the rho-invariant of M associated to the representation σ^2 and $H^1(M, \sigma^2)$ for the cohomology of M with coefficients in σ^2.

Putting the above in Proposition 8.2 results in the following generalization of Theorem A of the introduction.

THEOREM 8.4 *Let M be a rational homology 3-sphere. Then*

(a) $\lambda(M)_R - \lambda(M)_L = \sum_{\sigma^2} \dim_{\mathbb{C}} H^1(M, \sigma^2)$ *and*

(b) $\frac{1}{2}[\lambda(M)_R + \lambda(M)_L] = \lambda_W(M) - \frac{1}{2}\operatorname{Def}(M_{\text{ab}} \to M)$ *where the sum goes over all squares* $[\sigma^2]$ *of characters* $\sigma : \pi_1(M) \to U(1)$ *and the signature defect* $\operatorname{Def}(M_{\text{ab}} \to M)$ *is as in Theorem* A.

We begin by defining canonical perturbations in the situation of a symplectic vector space $(V, \{\cdot,\cdot\})$ and then go on to explain how this simple idea extends to our setting.

Suppose in addition to a symplectic structure $\{\cdot,\cdot\}$ the vector space V is endowed with a metric, i.e., a real, symmetric, positive definite bilinear pairing $(\cdot,\cdot) : V \times V \to \mathbb{R}$. Then by linear algebra there is a unique complex structure (V, J),

$$J : V \to V\,, \quad J^2 = -\mathrm{Id}\,,$$

and Hermitian pairing

$$\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$$

such that the real and imaginary parts are

$$\Re\langle\alpha,\beta\rangle = (\alpha,\beta)\,, \qquad \Im\langle\alpha,\beta\rangle = -\{\alpha,\beta\}\,. \tag{8.2}$$

A typical example of the above is the space $\Gamma(\mathbb{E})$ of C^∞-sections of $\mathbb{E}$ with the pairings $\langle f, g\rangle$ and $\{f, g\}$ already mentioned in (1.6).

Let L be a Lagrangian subspace in V. Then a canonical right-handed perturbation of L is given by

$$\phi_t(L)_R = \{e^{-Jt} \cdot L : 0 \le t < \varepsilon\}\,, \tag{8.3}$$

and a left-handed perturbation is given by

$$\phi_t(L)_L = \{e^{+Jt} \cdot L : 0 \le t < \varepsilon\} \tag{8.4}$$

with ε sufficiently small and positive. Our convention is that a "right" rotation should rotate the second Lagrangian clockwise with respect to the first.

Note that $e^{Jt} = I + Jt + (Jt)^2/2! + \cdots$ represents a family of unitary and so symplectic automorphisms of V. Hence, as t varies, we obtain two families $\{\phi_t(L)_R\}$ and $\{\phi_t(L)_L\}$ of Lagrangians in V connecting up L to, respectively, $e^{-J\varepsilon} \cdot L$ and $e^{+J\varepsilon} \cdot L$.

Let (L_1, L_2) be a pair of Lagrangian subspaces in V that *need not* be transverse to each other. The following proposition, whose simple proof can be found in [10, lemma 2.1], allows us to perturb L_2 using either the right-handed or the left-handed perturbation to a transverse position with respect to L_1.

PROPOSITION 8.5 *Let* L_1 *and* L_2 *be two Lagrangian subspaces in* V. *Then there exists an* $\varepsilon > 0$ *such that* $(L_1 \cap e^{Jt} \cdot L_2) = \{0\}$ *for all* t *in the half-open intervals* $0 < t \le \varepsilon$ *and* $-\varepsilon \le t < 0$.

REMARK 8.6 It is clear from the definition that both $\{\phi_t(L)_R\}$ and $\{\phi_t(L)_L\}$ depend on the complex structure, which in turn depends on the choice of metric $(\cdot,\cdot)$. However, as is well-known, the space of metrics is convex, i.e., given two metrics $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_1$ on V, there exists the smooth family of interpolating metrics

$\{(\cdot,\cdot)_s = (1-s)(\cdot,\cdot)_0 + s(\cdot,\cdot)_1 : 0 \le s \le 1\}$ connecting them. Accordingly, there are associated smooth families of complex structures J_s, right-handed perturbations $\phi_{t,s}(L)_R$, and left-handed perturbations $\phi_{t,s}(L)_L$ connecting the corresponding data. In all this the underlying symplectic structure $\{\cdot,\cdot\}$ is regarded as fixed. As is evident these various perturbations are *isotopic* as symplectic motions of Lagrangians starting from the identity. Moreover, the result of any of these motions is to render L_1 and L_2 transverse; i.e., $\{\phi_{t,s}(L_2)_R\}$ and $\{\phi_{t,s}(L_2)_L\}$ are transverse with respect to L_1 for all t and s with $0 < |t| \le \varepsilon$ and $0 \le s \le 1$, for ε sufficiently small. In this strong symplectic sense, we may regard these families of perturbations as equivalent "canonical" ways of making L_1 and L_2 transverse. Thus, in our application of Proposition 8.5, we fix one complex structure and proceed with the discussion for this single case.

Another feature of the motion $e^{Jt} : V \to V$ is that it is the result of the Hamiltonian flow associated to the functions

$$-\frac{r^2}{2} : V \to V\,, \qquad \alpha \to -\frac{1}{2}(\alpha,\alpha)\,. \tag{8.5}$$

This can easily be seen when $V = \mathbb{C}$ with the standard symplectic structure $\omega = dx \wedge dy$ and $J = i$. In this case, the rotational motion $z \to e^{it}\cdot z$ is generated by the vector field $i(x+iy) = (-y,x) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. Clearly,

$$\Big(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\Big)\omega = -y\,dy - x\,dx = d\Big(\frac{-r^2}{2}\Big)\,.$$

In general, the assertion follows from considering $V \cong \mathbb{C}^n$ as a direct sum of n copies of the above example.

Next we extend the definition of the above perturbations to the setting of a symplectic reduction. Let (G,V) be a complex unitary representation of a Lie group G on a Hermitian vector space $(V,\langle\cdot,\cdot\rangle,J)$. Associated to this situation, there is a moment map $\mu : V \to \mathrm{Lie}(G)^*$ from V to the dual of the Lie algebra. More explicitly, for a in the Lie algebra of G, let $\exp(ta)$ denote the one parameter group generated in G. By definition we have

$$(\mu(v))(a) = \lim_{t\to 0}\left[\frac{\{v,\exp(ta)v\}}{t}\right]$$

where v is any vector in V. This mapping is G-equivariant, where G acts by the conjugation (i.e., adjoint action) on its Lie algebra $\mathrm{Lie}(G)$. By definition, the symplectic reduction $\mu^{-1}(0)/G$ is obtained by taking the zero set $\mu^{-1}(0)$ of the moment map and then forming the quotient space $\mu^{-1}(0)/G$ of $\mu^{-1}(0)$ modulo the action of G.

Let L be a Lagrangian subspace of V that is invariant under the action of G. Then, from the isotropy property ($\{v,w\} = 0$ for v,w in L) of Lagrangian subspaces, L is contained in $\mu^{-1}(0)$ and, by passing to the quotient L/G, is a (stratified) Lagrangian subspace in the symplectic reduction $\mu^{-1}(0)/G$. To perturb L/G, we observe that the function $-r^2/2 : V \to \mathbb{R}$ in (8.5) is invariant under G. Hence, the associated Hamiltonian flow and symplectic automorphism $\alpha \to e^{Jt}\alpha$ are G-equivariant. As a result, throughout the deformation, $\{\phi_t(L)_R\}$ and $\{\phi_t(L)_L\}$ remain G-invariant and descend to "canonical" right-handed and left-handed perturbations $\{\phi_t(L/G)_R\}$ and $\{\phi_t(L/G)_L\}$ of L/G in $\mu^{-1}(0)/G$.

Now given a pair of G-invariant Lagrangians (L_1, L_2) in V, the perturbations $\{\phi_t(L_2)_R\}$ and $\{\phi_t(L_2)_L\}$ in Proposition 8.5 allow us to deform L_2 into linear transverse subspaces with respect to L_1, i.e., $L_1 \cap \{\phi_t(L_2)_R\} = \{0\}$ and $L_1 \cap \{\phi_t(L_2)_L\} = \{0\}$ for $0 < t \le \varepsilon$. Here ε is sufficiently small and positive. Passing to the quotient, we see that $(L_1/G) \cap \{\phi_t(L_2/G)_R\} = \{0\}$ and $(L_1/G) \cap \{\phi_t(L_2/G)_L\} = \{0\}$. In other words, $\{\phi_t(L_2/G)_R\}$ and $\{\phi_t(L_2/G)_L\}$ for $0 < t \le \varepsilon$ have no intersection with L_1/G except at the cone point $\{0\} \in \mu^{-1}(0)/G$.

In fact, in our application, we have to make sure that the desired perturbations take place only in a small prescribed δ-neighborhood of $\{0\}$ in $\mu^{-1}(0)/G$ and remain at the original position outside a slightly larger δ'-neighborhood, $0 < \delta < \delta'$; that is, the motion is the identity outside this larger neighborhood. For this, we choose a smooth, nonnegative, decreasing function

$$g(r) : \mathbb{R}^+ \to \mathbb{R}^+$$

such that

$$g(r) = 1 \quad \text{for } 0 \le r \le \delta\,, \tag{8.6a}$$

$$g(r) = 0 \quad \text{for } \delta' \le r < \infty\,, \tag{8.6b}$$

$$0 \le g(r) \le 1 \quad \text{for } \delta \le r \le \delta'\,. \tag{8.6c}$$

Then we consider the Hamiltonian flow associated to the function

$$\alpha \to \frac{-g(r)\cdot r^2}{2}$$

with $r^2 = (\alpha, \alpha)$ and the symplectic automorphism Ψ_t generated by this flow. Clearly, this new automorphism is the same as ϕ_t for $0 \le |\alpha| \le \delta$ and is the identity outside $\delta' \le |\alpha|$. Moreover, the above function is G-invariant and so passes down to the symplectic quotient $\mu^{-1}(0)/G$ with the same properties.

Let $\rho : \pi_1(M) \to \mathrm{SU}(2)$ be a reducible $\mathrm{SU}(2)$-representation of $\pi_1(M)$. Then as in Section 6, we write ρ as a sum $\rho = \sigma \oplus \sigma^{-1}$ where $\sigma : \pi_1(M) \to U(1)$. This involves a choice of ordering. That is, following Walker and Section 6, we are considering the twofold covering of the reducible $\mathrm{SU}(2)$-representation consisting of the representations $\sigma : \pi_1(M) \to U(1)$. A neighborhood of this reducible point $[\rho]$ can be explicitly evaluated in terms of a symplectic reduction $\mu^{-1}(0)/G$. The symplectic space in question is the Zariski tangent space of R at $[\rho]$:

$$\begin{aligned} H^1(\Sigma, \operatorname{Ad}\rho) &= H^1_{\bar\partial}(\Sigma, \operatorname{Ad}\rho \otimes \mathbb{C}) \\ &= H^1_{\bar\partial}(\Sigma, \sigma^2 \oplus \mathbb{C} \oplus \sigma^{-2}) \\ &= V \oplus H^1_{\bar\partial}(\Sigma, \mathbb{C}) \oplus \bar V \end{aligned}$$

with $V \equiv H^1_{\bar\partial}(\Sigma, \sigma^2)$ and $\bar V \equiv H^1_{\bar\partial}(\Sigma, \sigma^{-2})$. Here the symplectic structure on $H^1(\Sigma, \operatorname{Ad}\rho)$ comes from the combination of the cup product pairing on cohomology with the natural Ad-invariant inner product $[\cdot,\cdot]$ on the coefficients $\operatorname{Ad}\rho = \mathrm{su}(2)$:

$$\begin{aligned} \{\cdot,\cdot\} : H^1(\Sigma, \operatorname{Ad}\rho) \otimes H^1(\Sigma, \operatorname{Ad}\rho) &\xrightarrow{\cap} H^2(\Sigma, \operatorname{Ad}\rho \otimes \operatorname{Ad}\rho) \\ &\xrightarrow{[\cdot,\cdot]} H^2(\Sigma, \mathbb{R}) \xrightarrow{\cong} \mathbb{R}\,. \end{aligned}$$

Note that this is independent of the choice of σ. However, the complex structure and so the $\bar\partial$-operator come into the choice of σ.

In the definition of the above $\bar\partial$-cohomology, we have fixed a Riemannian (hence holomorphic) structure on Σ. With respect to this Riemannian metric and complex structure, there is a unique Hermitian pairing $\langle\cdot,\cdot\rangle$ on $H^1(\Sigma, \operatorname{Ad}\rho)$ that is compatible with the skew-symmetric pairing as in (8.2). In terms of the identification (8), this Hermitian pairing is a sum of the natural Hermitian pairings on the three summands given by $\int \alpha \wedge \bar\beta$ where the forms α and β with values in the local systems are paired using the natural complex unitary metrics on these three coefficient systems, σ^2, $\mathbb{C}$, and σ^{-2}.

The stabilizer of the reducible representation $[\rho] = [\sigma \oplus \sigma^{-1}]$ is the circle subgroup

$$U(1) = \{\operatorname{diag}[e^{i\theta}, e^{-i\theta}] : 0 \le \theta \le 2\pi\}$$

of $\mathrm{SU}(2)$ if σ is not of order 2. Here $U(1)$ operates on $\operatorname{Ad}\rho$ as a subgroup of $\mathrm{SU}(2)$ and hence on the Zariski tangent space $H^1(\Sigma, \operatorname{Ad}\rho)$ preserving the Hermitian structure. With respect to this last action, the decomposition of (8) is an eigenspace decomposition $V \oplus H^1_{\bar\partial}(\Sigma, \mathbb{C}) \oplus \bar V$ with weights $(2, 0, -2)$. Geometrically the middle term $H^1_{\bar\partial}(\Sigma, \mathbb{C})$ represents the tangent space $(TS)_\rho$ of the reducible stratum S at $[\rho]$, while $V \oplus \bar V$ stands for the Zariski normal bundle to S inside of R.

If σ is a point of order 2, then $\operatorname{Ad}\rho$ is trivial and hence the stabilizer is all of $\mathrm{SU}(2)$. In this case Q_1 and Q_2 are already transverse; see below.

For σ not of order 2, from a straightforward computation the moment map

$$\mu : V \oplus H^1_{\bar\partial}(\Sigma, \mathbb{C}) \oplus \bar V \longrightarrow \mathbb{R}$$

of the above $U(1)$-action is given by

$$\mu(a, b, c) = 2\|a\|^2 - 2\|c\|^2\,, \quad a \in V\,,\ b \in H^1_{\bar\partial}(\Sigma, \mathbb{C})\,,\ c \in \bar V\,. \tag{8.7}$$

Using (8.7) we have the following description of $\mu^{-1}(0)/U(1)$, the symplectic reduction in this case:

(8.8)
$$\begin{aligned}\mu^{-1}(0)/U(1) &= \{(a,b,c) : a \in V,\ b \in H^1_{\bar\partial}(\Sigma,\mathbb{C}),\ c \in \bar V,\ \|a\|^2 = \|c\|^2\}/S^1 \\ &= H^1_{\bar\partial}(\Sigma,\mathbb{C}) \times \text{cone on } \{(a,c) : \|a\|^2 = \|c\|^2 = 1\}/S^1 \\ &= H^1(\Sigma,\mathbb{R}) \times \text{cone on } (S(V) \times S(\bar V)/S^1)\,.\end{aligned}$$

Here $S(V)$ and $S(\bar V)$ are the unit spheres of the Hermitian vector spaces V and $\bar V$.

Next we consider the situation when $[\rho] = [\sigma \oplus \sigma^{-1}]$ is a point in the intersection $Q_1 \cap Q_2$ of Q_1 and Q_2. The Zariski tangent spaces $(TQ_1)_\rho$ and $(TQ_2)_\rho$ at this point $[\rho]$ are naturally isomorphic to the cohomology $H^1(W_1, \operatorname{Ad}\rho)$, $H^1(W_2, \operatorname{Ad}\rho)$, which in turn are decomposed into the following orthogonal sums:

$$\begin{aligned}H^1(W_1, \operatorname{Ad}\rho) &= H^1(W_1,\mathbb{R}) \oplus H^1(W_1, \operatorname{Ad}^\perp \rho)\,, \\ H^1(W_2, \operatorname{Ad}\rho) &= H^1(W_2,\mathbb{R}) \oplus H^1(W_2, \operatorname{Ad}^\perp \rho)\,,\end{aligned} \tag{8.9}$$

where $\operatorname{Ad}^\perp = \sigma^2 \oplus \sigma^{-2}$. Using the Mayer-Vietoris sequence in cohomology and the fact that M is a rational homology sphere, it is easy to deduce

$$H^1(W_1,\mathbb{R}) \oplus H^1(W_2,\mathbb{R}) = H^1(\Sigma,\mathbb{R})\,. \tag{8.10}$$

Geometrically the summands $H^1(W_1,\mathbb{R})$ and $H^1(W_2,\mathbb{R})$ are the tangent spaces $T(Q_1 \cap S)_\rho$ and $T(Q_2 \cap S)_\rho$ in the reducibles S at $[\rho]$, and the equality (8.10) indicates that the subspaces $Q_1 \cap S$ and $Q_2 \cap S$ intersect each other transversely at $[\rho] \in S$. The true neighborhood of Q_1 and Q_2 at $[\rho]$ is the above Zariski tangent bundle divided out by the action of the stabilizer $U(1)$.

Let $L_j = H^1(W_j, \operatorname{Ad}^\perp \rho)$, $j = 1, 2$, be the second component in the decomposition (8.10). Then as explained in Section 6 (cf. [24, p. 15]), L_j is a complex Lagrangian subspace of $V \oplus \bar V$. In other words, there is a *unitary* mapping $\chi(L_j) : V \to \bar V$ between the Hermitian vector spaces V and $\bar V$, and L_j is the graph of this map,

$$L_j = \text{graph of } \chi(L_j) = \{(a, \chi(L_j)(a) : a \in V\}\,. \tag{8.11}$$

Because the graph is always isomorphic to the domain space, we have $L_j \cong V$ and, by passing to the quotient,

$$L_j/S^1 \cong \text{cone on } S(V)/S^1\,. \tag{8.12}$$

Thus a neighborhood of Q_j at $[\rho]$ is modeled on the inclusion of $H^1(W_j,\mathbb{R}) \times$ cone on $S(V)/S^1$ into $H^1(\Sigma,\mathbb{R}) \times$ cone on $(S(V) \times S(\bar V)/S^1)$.

In the present situation of a singular cone, two subspaces such as Q_1 and Q_2 are said to be transverse at $[\rho]$ if they are transverse in the Zariski tangent space. Geometrically this means that in a δ-neighborhood of $[\rho]$, Q_1 and Q_2 intersect only at the cone point. As discussed before, $Q_1 \cap S$ and $Q_2 \cap S$ are transverse in the tangent direction of the stratum S; we can therefore concentrate on their normal cones L_1/S^1 and L_2/S^1. Note that on applying the canonical Hamiltonian perturbation $\Psi_t : V \oplus \bar{V} \to V \oplus \bar{V}$, we deform an S^1-invariant, complex Lagrangian L to another via a family of such complex Lagrangians. In fact, the procedure can be thought of as replacing the graph of the unitary map $\chi(L)$ from V to $\bar{V}$,

$$L = \text{graph of } \chi(L) = \{(a, \chi(L)(a)) : a \in V\}\,,$$

by the family of graphs of $e^{it} \cdot \chi(L)$ on a δ-ball neighborhood, $B_\delta(0)$ of 0.

$$\phi_t(L) \cap B_\delta(0) = \{(a, e^{it}\chi(L)(a)) : a \in V\} \cap B_\delta(0)\,. \tag{8.13}$$

Thus given Q_1 and Q_2, which may or may not be transverse at $[\rho]$, we can perturb Q_2 to transverse position with respect to Q_1 by either the canonical right-handed perturbation $Q_{2R} = H^1(W_2, \mathbb{R}) \times \phi_t(L_2/S^1)_R$ or the left-handed one $Q_{2L} = H^1(W_2, \mathbb{R}) \times \phi_t(L_2/S^1)_L$.

In addition, given the result of two right-handed perturbations Q_{2R} and Q'_{2R} (or left-handed Q_{2L} and Q'_{2L}), they can be deformed from one to another by Hamiltonian motions $(Q_{2R})_s$ in R that are the identity on R and are transverse to Q_1 throughout the deformation at all reducibles in $Q_1 \cap Q_2 \cap R$. In other words, $(Q_{2R})_s$ is an admissible family of deformations in the sense of Walker (cf. [24]). As a consequence, $\lambda(M)_R$ defined using such a Q_{2R} and $\lambda(M)_L$ defined using such a Q_{2L} are well-defined independently of the choices of handle bodies W_1 and W_2 and Riemannian metrics on Σ.

In the case where the Lagrangians Q_1 and Q_2 are already transverse at the reducibles, then the aforementioned Q_{2R} and Q_{2L} leave all the transverse data unchanged. That is, the above symplectic isotopy from Q_2 to Q_{2R} or Q'_{2R} is then always transverse to Q_1. Hence, $\lambda(M)_R$ and $\lambda(M)_L$ agree with $\lambda_{\mathrm{BN}}(M)$ in the cyclically finite case. This proves Propositions 8.1 and 8.2.

The proofs of Theorems 8.3 and 8.4 follow by the same argument used for their counterparts of Theorems A and B. The average correction

$$\frac{1}{2}(I(\rho, Q_1, Q_{2L}) + I(\rho, Q_1, Q_{2R}))$$

at a reductible $[\rho]$ in $Q_1 \cap Q_2 \cap R$ is computed without essential changes, as in that discussion. This correction is by definition expressed as an integral term plus an average Maslov index. The average Maslov index is then, using the results of Part 2, re-expressed as an average spectral flow on $M \times [0, 1]$. There are no zero-mode correction terms in our formulae for this average. The integral plus average spectral flow is then expressed as $-\frac{1}{2}$ times an η-invariant as in Section 7. This proves one-half of our Theorem 8.3. Addition also gives one-half of Theorem 8.4. It was an essential feature of our analysis in [12] that we dealt with the problems of zero modes occurring at the ends of the families of self-adjoint elliptic operators. There the formulae were given in terms of averaged spectral flows, and with this one caveat all proceeds in the present proof as before.

The other half of Theorem 8.3 and 8.4 concerns the difference of these invariants. It is apparent that the positive rotation e^{+Jt}, $-\varepsilon < t < +\varepsilon$, with t increasing effects (near each reducible) a rotation that carries the "right" Lagrangian into the "left" Lagrangian. The result of this motion is to introduce precisely $\sum_{\sigma^2} \dim_{\mathbb{C}} H^1(M, \sigma^2)$ additional points (each with positive sign) into the intersection. This demonstrates part (b) of Theorem 8.4.

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