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DETERMINATION OF THE COBORDISM RING

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The cobordism ring was first defined by R. Thom [15], and is sometimes known as the Thom algebra. Consider the set of closed oriented manifolds of dimension k (here, and throughout this paper, all manifolds are supposed differentiable, of what class it does not matter), and if V is an oriented manifold, denote by $-V$ the same manifold with the opposite orientation. Introduce the relation $V \sim W$ (pronounced: V is cobordant with W) if there is a compact oriented manifold M with oriented boundary $\partial_0(M) = V + (-W)$, where $+$ denotes disjoint union. It is easy to see that $\sim$ is an equivalence relation, compatible with $+$ and $-$, so that the equivalence classes form an abelian group, Ω_k, the cobordism group in dimension k. Since, if V is closed, $\partial_0(M \times V) = \partial_0 M \times V$, topological product is compatible with $\sim$, and induces a product $\Omega_k \times \Omega_l \to \Omega_{k+l}$, with respect to which the direct sum $\Omega = \sum_k \Omega_k$ becomes an associative and skew-commutative ring. In an informal way we can consider the boundary operator ∂_0 as a nilpotent endomorphism of the set of all compact oriented manifolds and define $\Omega = \operatorname{Ker} \partial_0 / \operatorname{Im} \partial_0$; hence the name 'intrinsic homology' adopted by Rohlin for cobordism.

If orientation is not required in the above, we obtain an equivalence relation $V \sim_2 W$ (pronounced: V is cobordant with W mod 2) for non-oriented manifolds, and a new cobordism ring $\mathfrak{N} = \sum_k \mathfrak{N}_k$. We will denote by $r : \Omega \to \mathfrak{N}$ the natural map obtained by ignoring orientation. $\Omega, \mathfrak{N}$ are rings in the ordinary algebraic sense, and r is a homomorphism between them, and the problem with which we are concerned is to give a purely algebraic description of them.

Now the structure of $\mathfrak{N}$ was already completely determined in [15]: $\mathfrak{N}$ is a ring of polynomials mod 2, with one generator x_i in each dimension i not of the form $2^j - 1$. A necessary and sufficient condition that two manifolds be cobordant mod 2 is that they have the same Stiefel numbers, which are defined as follows. Let w^i be the i^{th} Stiefel class mod 2 of the manifold M_k, so $w^i \in H^i(M_k, Z_2)$. (For a definition of the Stiefel classes of a manifold see [5] or [13].) Form any homogeneous polynomial of degree k in the w^i, $f(w^1, \cdots, w^k) \in H^k(M_k, Z_2)$ and evaluate it on the fundamental cycle (mod 2) of M_k. It is frequently convenient to regard the w^i as the elementary symmetric functions of k (or even more) indeterminates t_j; then let $s_k(w^1, \cdots, w^k)$ be that polynomial in them equal identically to $\sum_j t_j^k$. The corresponding Stiefel number has the property that M_k can be chosen as a k-dimensional generator of $\mathfrak{N}$ if and only if $s_k[M_k] = 1$. In particular, Thom showed that real projective spaces $P_{2n}(R)$ can be taken as generators in even dimensions, and Dold in [2] gave generators for the odd dimensions which I shall define and use in § 3.

Much work has been done on the structure of Ω. An early theorem [9], [14] asserts that Stiefel numbers and Pontrjagin numbers are invariants of cobordism class. Pontrjagin numbers are defined as Stiefel numbers, but using Pontrjagin classes $p^i \in H^{4i}(M_k, Z)$, (as defined in [4]), evaluated with integral coefficients, and only definable if $k \equiv 0 \pmod 4$. The main results on Ω are also contained in [15], viz.,

(i) The Ω_k are finitely generated abelian groups.

(ii) $\Omega \otimes Q$ (Q denotes the field of rational numbers) is a polynomial algebra, whose generators may be taken as the complex projective spaces $P_{2n}(C)$.

(iii) Two manifolds determine the same element of Ω mod torsion if and only if they have the same Pontrjagin numbers.

These results have been extended by Milnor [7], [16] who has shown, using a spectral sequence due to Adams [1],

(iv) Ω has no odd torsion.

(v) The torsion free part of Ω is a polynomial ring, and a manifold M_{4k} qualifies as generator if and only if the Pontrjagin number

$$\begin{aligned} s_k(p^1, \cdots, p^k)[M_{4k}] &= \pm q \qquad \text{if } 2k+1 \text{ is a power of the prime } q \\ &= \pm 1 \qquad \text{if } 2k+1 \text{ is not a prime power.} \end{aligned}$$

Thus all that remains to be evaluated is the 2-torsion of Ω. The most notable worker on this is Rohlin, whose work is based on a study of $r : \Omega \to \mathfrak{N}$; in particular he finds in [10] the exact sequence

$$\Omega \xrightarrow{2} \Omega \xrightarrow{r} \mathfrak{N}$$

(where 2 denotes the homomorphism $x \to 2x$) for which an alternative proof has since been given by Dold [3]. Unfortunately, Rohlin's papers contain a mere outline of proofs, and his evaluation of the 2-torsion of Ω in [11] is incorrect, as this paper will show.

The object of this paper is to prove the following results:

(vi) Ω contains no elements of order 4.

(vii) Two manifolds are cobordant if and only if they have the same Stiefel and Pontrjagin numbers.

These were conjectured by Thom. The proof will occupy the greater part of this paper; the last two paragraphs make various deductions, as my proof shows rather more than these results, and in particular yields a complete algebraic description of Ω. In fact, all properties of Ω are as simple as they could possibly be (if my results should strike the reader as complicated, let him try and work out cobordism theory for the spinor group). The main ideas of this paper were announced in [17].

I should like to express my gratitude to J. F. Adams for suggesting this problem to me, and for simplifying several of my proofs, and to E. C. Zeeman for unfailing encouragement of my work.

1. The main construction

Let M_n be a closed manifold, whose first Stiefel class w^1 is the restriction mod 2 of a class with (simple) integer coefficients, which must correspond to a map $f: M_n \to K(Z, 1) = S^1$ (for definition and properties of Eilenberg-MacLane spaces $K(\pi, n)$ see [12]). Let u generate $H^1(S^1, Z_2)$, then $f^*(u) = w^1$. But there is a (1–1) correspondence between Z_2-bundles over a space X and $H^1(X, Z_2)$, each corresponding to homotopy classes of maps of X into $K(Z_2, 1)$. Hence the bundle corresponding to w^1, with group Z_2, and which I may describe as the orientation bundle, is induced by f from the double covering of S^1.

We may now approximate to f by a differentiable map, and then apply Theorem I.5 of [15]; we find a map $g: M_n \to S^1$, homotopic to f, and t-regular at 0 (we regard S^1 as the interval [0, 1] with 0, 1 identified). The map g is t-regular at 0, and so in some neighbourhood. We choose δ so that $(-\delta, \delta)$ is contained in the interior of some such neighbourhood.

Since g is t-regular at 0, $g^{-1}(0) = V_{n-1}$ is a differentiable submanifold of M_n. Its normal bundle is induced from that of 0 in S^1, so is trivial. The orientation bundle of $M - V$ is induced from a bundle over (0, 1) so is trivial. Hence the manifold $g^{-1}[0, \delta)$ with boundary V is orientable, and its orientation induces one of V. We may note that if we do the same for $(1 - \delta, 1]$ we induce the same orientation of V.

Finally, since g induces $w^1(M)$, and $V = g^{-1}(0)$, the homology class of V in M is dual (in M) to $w^1(M)$, working mod 2.

LEMMA 1. *An oriented manifold V_{n-1} can be obtained by the above construction from some M_n if and only if $2V \sim 0$.*

PROOF. If V can be so obtained, cut M along V to obtain a manifold M' with boundary. $M - V$ is orientable, so M' is. Since the normal bundle of V in M was trivial, the boundary of M' consists of two disjoint copies of V. The orientation of $M - V$ induces orientations of M' and of V, and by a remark above, $\partial_0 M' = 2V$. Hence $2V \sim 0$.

Conversely, suppose $2V \sim 0$, and let M' be an oriented manifold with boundary $\partial_0 M' = 2V$. Let M be obtained by identifying the two copies of V in M'; clearly we can give M a differentiable structure using that on M'. Let ρ be a differentiable metric on M, inducting ρ' on M', normalised so that the distance apart of the two copies of V in M', say V_1, V_2, is $\geqq 1$. Define $f' : M' \to [0, 1]$ by

If $\rho'(x, V_1) < 1/2$, $\quad f'(x) = \rho'(x, V_1)$.
If $\rho'(x, V_2) < 1/2$, $\quad f'(x) = 1 - \rho'(x, V_2)$.
Otherwise, $\quad f'(x) = 1/2$.

Let f' induce $f : M \to S^1$ on identifying V_1, V_2 and $0, 1$. Then f is differentiable in a neighbourhood of V, and $f^{-1}(0) = V$. The double covering of S^1 induces the orientation bundle of M, since the corresponding statement is true for f', both bundles being trivial, and remains so for f, as the local orientation of M induced from M' changes across V. Thus $w^1(M) = f^*(u)$. Since u is the restriction of an integer class, so is w^1. We now see that the above construction leads from M to V.

2. Definition and first properties of $\mathfrak{W}$ and ∂_1

Let M_n, V_{n-1} be as above, let $i : V \to M$ be the inclusion map, and let ξ, η, ζ be the tangent bundles of M and V and the normal bundle of V in M. Then $i^*\xi = \eta \oplus \zeta$ and so, by the Whitney product theorem for the total Whitney classes, $i^*w(\xi) = w(\eta)w(\zeta)$. But ζ is trivial, so $w(\zeta) = 1$, and $i^*w(\xi) = w(\eta)$, or $i^*w(M) = w(V)$.

If X is a topological space, $x \in H_r(X, Z_2)$ and $y \in H^r(X, Z_2)$, let $[y, x]$ denote their Kronecker product.

Let $\omega = (a, b, \cdots, c)$ be a partition of $n-1$, and write $w^\omega = w^a w^b \cdots w^c$; thus $[w^\omega(V), V]$ denotes a typical Stiefel number of V.

LEMMA 2. $[w^\omega(V), V] = [w^\omega(M)w^1(M), M]$.

PROOF.
$$\begin{aligned}[w^\omega(V), V] &= [i^*w^\omega(M), V] \\ &= [w^\omega(M), i_* V] \\ &= [w^\omega(M), w^1(M) \frown M] ,\end{aligned}$$

where $\frown$ denotes the cap product of cohomology and homology, and $i_* V = w^1(M) \frown M$ since the class $i_*(V)$ is dual to $w^1(M)$; and

$$[w^\omega(M), w^1(M) \frown M] = [w^\omega(M)w^1(M), M] .$$

It follows from the lemma that since Stiefel numbers determine cobordism class mod 2, the class, $\{M\}$, of M determines $\{V\}$. We shall write $\{V\} = \partial_1\{M\}$. We also define $\mathfrak{W}$ as the subset of $\mathfrak{N}$ formed by classes containing a manifold such as M. Then $r(\Omega) \subset \mathfrak{W} \subset \mathfrak{N}$, and the image of

$\partial_1 : \mathfrak{W} \longrightarrow \mathfrak{N}$ is contained in $r(\Omega)$, since V is orientable, so ∂_1 induces maps from $\mathfrak{W}$ to $r(\Omega)$ and $\mathfrak{W}$, which we may also denote by ∂_1. We note that clearly $\partial_1 r : \Omega \longrightarrow \mathfrak{N}$ is zero.

We shall prove later (in § 8) that $\{M\}$ determines the class in Ω of V, which will enable us to define $\partial_3 : \mathfrak{W} \longrightarrow \Omega$.

LEMMA 3. *$\mathfrak{W}$ is a subalgebra of $\mathfrak{N}$, and ∂_1 is a derivation of $\mathfrak{W}$.*

PROOF. The set of manifolds with first Stiefel class $w^1(M)$ the restriction of an integer class u is closed under addition. To prove the first part of the lemma, we remark that it is also closed under multiplication. For the tangent bundle of the product $M \times M'$ is the direct sum of the bundles induced from the tangent bundles of M, M' by the projections on these factors. By the Whitney product theorem,

$$w^1(M \times M') = w^1(M) \otimes 1 + 1 \otimes w^1(M') ,$$

the restriction of the integer class $u \otimes 1 + 1 \otimes u'$.

Now let $\mathfrak{B}$ be the polynomial algebra on generators w^i, and $\Delta : \mathfrak{B} \rightarrow \mathfrak{B} \otimes \mathfrak{B}$ the homomorphism defined by $\Delta(w^i) = \sum_{j+k=i} w^j \otimes w^k$. If $X \in \mathfrak{B}$, and M_k is a manifold, $X(M)$ denotes the appropriate polynomial in the Stiefel classes of M, and $X[M]$ denotes $[X^k(M), M]$, where X^k is the k-dimensional part of X. Since $H^*(M \times M', Z_2) = H^*(M, Z_2) \otimes H^*(M', Z_2)$ we may define

$$X \otimes Y(M, M') = X(M) \otimes Y(M') ,$$
$$X \otimes Y[M, M'] = X[M] \cdot Y[M'] .$$

Since, by Whitney's product theorem,

$$w(M \times M') = (w(M) \otimes 1)(1 \otimes w(M')) = w(M) \otimes w(M') ,$$

for any element X of $\mathfrak{B}$,

$$X(M \times M') = \Delta(X)(M, M') . \tag{1}$$

In our present notation, lemma 2 can be written as

$$X[\partial_1 M] = w^1 X[M] , \tag{2}$$

and using (1) and (2) we can compute as follows

$$\begin{aligned} X[\partial_1(M \times M')] &= w^1 X[M \times M'] \\ &= \Delta(w^1 X)[M, M'] \\ &= (w^1 \otimes 1)\Delta X[M, M'] + (1 \otimes w^1)\Delta X[M, M'] \\ &= \Delta X[\partial_1 M, M'] + \Delta X[M, \partial_1 M'] \\ &= X[\partial_1 M \times M'] + X[M \times \partial_1 M'] . \end{aligned}$$

Thus all Stiefel numbers X agree on $\partial_1(M\times M')$ and $\partial_1 M\times M'+M\times\partial_1 M'$, whence these two manifolds are cobordant mod 2. Hence ∂_1 is a derivation. (For this streamlined version of my proof I am indebted to J. F. Adams.)

3. The Dold manifolds $P(m, n)$ and $Q(m, n)$

In this paragraph we introduce the study of the manifolds $P(m, n)$ (first used by Dold [2]) and $Q(m, n)$. Our main aim is Lemma 6 in the next section, affirming indecomposability in $\mathfrak{N}$ of certain elements of $\mathfrak{W}$, and we refer the reader to the next section for motivation.

Let S^m denote the subset $\sum_0^m x_i^2 = 1$ of R^{m+1}, and $P_n(C)$ the complex projective space with homogeneous coordinates $(z_0, \cdots, z_n)$. The Dold manifold $P(m, n)$ is the orbit space of the action $(x, z) \to (-x, \bar{z})$ of Z_2 on $S^m \times P_n(C)$, where $\bar{z}$ denotes complex conjugates. The projection $(x, z) \to x$ induces a fibre map α: $P(m, n) \to P_m(R)$ with fibre $P_n(C)$. Let T reflect S^m in the plane $x_m = 0$. Then $(x, z) \to (Tx, z)$ is compatible with the above action of Z_2, and induces an autohomeomorphism A of the orbit space. In the case when m is odd and n even, $P(m, n)$ is orientable, and A reverses the orientation. All these remarks are due to Dold [2].

We define $Q(m, n)$ as the manifold formed from $P(m, n) \times [0, 1]$ by identifying $(p, 0)$ to $(Ap, 1)$ for each $p \in P(m, n)$. The projection $(x, z, t) \to t$ induces a fibre map $\beta: Q(m, n) \to S^1$, with fibre $P(m, n)$. The projection $(x, z, t) \to (x, t)$ induces another fibre map $\gamma: Q(m, n) \to Q(m, 0)$ with fibre $P_n(C)$, and group Z_2. Finally a classifying map $Q(m, 0) \to P_{m+1}(R)$ for γ is covered by a bundle map $\theta: Q(m, n) \to P(m + 1, n)$, which may be defined by

$$(x, z, t) \to (x_0, \cdots, x_{m-1}, x_m \cos \pi t, x_m \sin \pi t, z) ,$$

from which it follows that θ has degree 1.

LEMMA 4. *$H^*(Q(m, n), Z_2)$ has three generators x, c, d in dimensions 1, 1, 2 respectively, which are bound by the sole relations*

$$x^2 = 0 , \quad c^{m+1} = c^m x , \quad d^{n+1} = 0 .$$

PROOF. The mod 2 cohomology of $P(m, n)$ was determined by Dold, who showed that it was the ring with two generators c, d and relations $c^{m+1} = d^{n+1} = 0$. A acts trivially on this, so the fibre map β gives rise to a spectral sequence in which for reasons of dimension, all differentials are zero. Thus if x is induced by β from the generator u of $H^1(S^1)$, and c, d induce the classes of the same names in $H^*(P(m, n))$, $H^*(Q(m, n))$ has the additive base $\{c^r d^s t^\varepsilon: 0 \leqq r \leqq m, 0 \leqq s \leqq n, 0 \leqq \varepsilon \leqq 1\}$. (The spectral sequence argument could be circumvented by finding an explicit cell decomposition.) Also, we have $x^2 = \beta^*(u^2) = 0$.

We define d to be induced by θ from the class d in $H^*(P(m+1, n))$; this class does indeed induce d on the submanifold $P(m, n)$. Then since $d^{n+1} = 0$ in $P(m+1, n)$, the same holds in $Q(m, n)$.

It remains only to define c and compute c^{m+1}. In the case $n = 0$, we define c as induced by θ from the class c in $P_{m+1}(R)$. Now θ has degree 1, and so maps the top dimensional class c^{m+1} of $P_{m+1}(R)$ onto that, $c^m x$, of $Q(m, 0)$. But as $\theta^*(c) = c$, $\theta^*(c^{m+1}) = c^{m+1}$, so $c^{m+1} = c^m x$ in $Q(m, 0)$. The result in the general case now follows by defining c to be induced by γ, and since x also is, the equation $c^{m+1} = c^m x$ is preserved.

We note finally that these relations define an algebra with the correct additive base, which would thus be disturbed if more relations were required to hold.

LEMMA 5. *The Stiefel class of $Q(m, n)$ for $m > 0$ is*

$$(1 + c + x)(1 + c)^{m-1}(1 + c + d)^{n+1} .$$

PROOF. We use the results and methods of [2] and will proceed by induction on n.

Induction Basis: $n = 0$. $Q(m, 0)$ has a submanifold $P_{m-1}(R) \times S^1$ of codimension 1. This has unit intersection numbers with the cycles $P_1(R) \times 0$ and $P_0(R) \times S^1$, so its dual cohomology class is $c + x$. Hence by Thom's definition, the Whitney class of the normal bundle is $1 + c + x$. But

$$w(P_{m-1}(R) \times S^1) = (1 + c)^m .$$

Let j be the inclusion map of $P_{m-1}(R) \times S^1$ in $Q(m, 0)$. Then

$$j^*(w(Q(m, 0))) = (1 + c + x)(1 + c)^m ,$$

so that if we define

$$D = w(Q(m, 0)) - (1 + c + x)(1 + c)^m ,$$

we have $j^*D = 0$, so D is a multiple of c^m.

$Q(m, 0)$ also has a submanifold $P(m, 0) = P_m(R)$ of codimension 1, and with trivial normal bundle. Since $w(P_m(R)) = (1 + c)^{m+1}$, if i denotes the inclusion of $P_m(R)$ in $Q(m, 0)$,

$$i^*(w(Q(m, 0))) = (1 + c)^{m+1} = i^*((1 + c + x)(1 + c)^m)$$

so $i^*D = 0$, so D is also a multiple of x, thus of $c^m x$. Hence we know all the Stiefel classes except that in the top dimension, which is given by the Euler class; now using Lemma 4, we see that the Euler number of $Q(m, 0)$ is 0, so $w^{m+1} = 0$, as stated.

Induction Step: assume the result for $n - 1$. This is quite similar to the above: we prove the difference

$$D = w\big(Q(m, n)\big) - (1 + c + x)(1 + c)^{m-1}(1 + c + d)^{n+1}$$

divisible by x, by considering the submanifold $P(m, n) \times 0$; by c^m, by considering the submanifold $P(m-1, n) \times S^1$, and by d^n, by considering the submanifold $Q(m, n-1)$ and finally check the top class in exactly the same way. The only steps needing comment are the computations of the Whitney classes of the normal bundles. In fact, the first is trivial, and the second is not essentially different from the case $n = 0$. The normal bundle of $Q(m, n-1)$ in $Q(m, n)$ is induced by θ from the normal bundle of $P(m+1, n-1)$ in $P(m+1, n)$, and the Whitney class of the latter was shown by Dold to be $1 + c + d$.

4. The polynomial algebra $\mathfrak{W}''$

In the case when m is odd and n even, as already remarked, $P(m, n)$ is orientable, and A reverses the orientation. Also, $w^1\big(Q(m, n)\big) = x$, which is induced (by β) from the class u on S^1. It is now clear that in this case the passage from $Q(m, n)$ to $P(m, n)$ is precisely the construction of Lemma 1.

Dold defined odd dimensional generators of $\mathfrak{N}$ as follows. Let $2k - 1$ be an odd number with k not a power of 2. We write $k = 2^{r-1}(2s + 1)$ with $s \neq 0$. Then X_{2k-1} is the class in $\mathfrak{N}$ of

$$V_{2k-1} = P(2^r - 1, 2^r s) .$$

We now define, under the same conditions, X_{2k} as the class in $\mathfrak{N}$ of

$$M_{2k} = Q(2^r - 1, 2^r s) .$$

LEMMA 6. *X_{2k} is an indecomposable element of $\mathfrak{N}$.*

PROOF. It is sufficient, by a result of Thom mentioned in the introduction to this paper, to prove $s_{2k}[M_{2k}] = 1$.

Write formally $1 + c + d = (1 + \mu)(1 + \nu)$. Then since

$$w\big(Q(m, n)\big) = (1 + c + x)(1 + c)^{m-1}(1 + \mu)^{n+1}(1 + \nu)^{n+1} ,$$

the indeterminates t_j, of which the w^i are elementary symmetric functions, may be taken as $c + x$, c ($m - 1$ times), μ, ν ($n + 1$ times each). Recall that m is odd, n even and $0 < n$, and $2k = m + 2n + 1$.

$$\begin{aligned} s_{2k}\big(Q(m, n)\big) &= (c + x)^{m+2n+1} + (m - 1)c^{m+2n+1} + (n + 1)(\mu^{m+2n+1} + \nu^{m+2n+1}) \\ &= \mu^{m+2n+1} + \nu^{m+2n+1} . \end{aligned}$$

Now by induction on r, using the inductive definition of binomial coefficients and the relation

$$s_{r+1} \equiv (u + v)s_r + (uv)s_{r-1} \pmod 2 ,$$

where $s_r = u^r + v^r$, we find

$$s_r \equiv \sum_{0\leq s<r/2} \{s, r-2s-1\}(u+v)^{r-2s}(uv)^s \pmod 2 ,$$

So

$$\begin{aligned} s_{2k}(Q(m,n)) &= \sum_s \{s, m+2n-2s\}\, c^{m+2n-2s+1}d^s \\ &= \{n, m\}\, c^{m+1}d^n , \end{aligned}$$

$$s_{2k}[M_{2k}] = \{n, m\} = \{2^r s, 2^r - 1\} \equiv 1 \pmod 2 .$$

Let X_{2^j} denote the class in $\mathfrak{N}$ of $P_{2^j}(R)$. We note that since we have defined one X_i in each dimension not of the form $2^j - 1$, and they are all indecomposable, by Thom's result, $\mathfrak{N}$ is the polynomial algebra with them as generators.

LEMMA 7. $P_n(C) \sim_2 (P_n(R))^2$.

PROOF. The cohomology and characteristic classes (mod 2) of $P_n(C)$ are isomorphic to those of $P_n(R)$ on doubling all dimensions. Hence the nonzero Stiefel numbers of $P_n(C)$ are obtainable from those of $P_n(R)$ by doubling all dimensions. Now for any manifold M, the Stiefel numbers of M^2 can be computed in terms of those of M by the methods of Lemma 3. We remark that

$$w^k(M^2) = \sum_{i+j=k} w^i(M) \otimes w^j(M)$$

and that in computing Stiefel numbers all terms with $i \neq j$ cancel out by symmetry. Thus we see that the nonzero Stiefel numbers of M^2 are obtainable from those of M by doubling all dimensions. The lemma now follows. It was first announced by Rohlin [11]. The author feels that it ought to be proved by constructing a manifold with appropriate boundary, but has been unable to find one.

Now X_{2k-1} (since it represents an orientable class) and X_{2k} (by construction) belong to $\mathfrak{W}$. Also, by Lemma 7, $X_{2^j}^2$ belongs to $\mathfrak{W}$. Hence $\mathfrak{W}$ contains the whole polynomial algebra $\mathfrak{W}''$ generated by these elements. On this algebra, the derivation ∂_1 is determined by its values on the generators:

$$\left.\begin{aligned} \partial_1 X_{2k-1} &= 0 \\ \partial_1 X_{2k} &= X_{2k-1} \\ \partial_1 X_{2^j}^2 &= 0 \end{aligned}\right\} (k \text{ not a power of } 2)$$

The next two paragraphs will apply Thom theory to prove that $\mathfrak{W}'' = \mathfrak{W}$, and when this is known we shall be able to assemble our results and prove the main theorems. We define $\mathfrak{W}'$ as the set of classes in $\mathfrak{N}$ such that all Stiefel numbers with $(w^1)^2$ as a factor vanish on them. Since if w^1 is induced from an integer class, $(w^1)^2 = 0$, we have $\mathfrak{W}'' \subset \mathfrak{W} \subset \mathfrak{W}'$. However Thom's theory will enable us to prove that there are no more elements in $\mathfrak{W}'$ than in $\mathfrak{W}''$. (*Note*: Since the first draft of this paper was written, a simpler proof that $\mathfrak{W}'' = \mathfrak{W}$ has been found by M. F. Atiyah.)

5. Thom theory

We must now go rather more deeply into the methods of [15]. Thom's work depends on the following construction: consider the classifying space $B(\mathrm{O}_n)$ of the orthogonal group in n variables, O_n (for definition see [5] or [13]). Over it there is a canonical O_n-bundle. Let $A(\mathrm{O}_n)$ be the associated bundle with fibre B^n (the n-ball in R^n defined by $\sum_1^n x_i^2 \leqq 1$), and $M(\mathrm{O}_n)$ be the space obtained from $A(\mathrm{O}_n)$ by identifying the boundary to a point. If a manifold M_k is contained in S^{n+k}, we find the map $f: M_k \rightarrow B(\mathrm{O}_n)$ inducing the normal bundle of M_k in S^{n+k}, extend to a map $f_1: N \rightarrow A(\mathrm{O}_n)$ of a small tubular neighbourhood of M_k (we regard $B(\mathrm{O}_n)$ as embedded in $A(\mathrm{O}_n)$ by the zero cross-section) sending the boundary of N to that of $A(\mathrm{O}_n)$ and thus to a point in $M(\mathrm{O}_n)$, and finally define $f_2: S^{n+k} \rightarrow M(\mathrm{O}_n)$ by mapping the rest of S^{n+k} to that point. Thom showed that this set up a (1–1) correspondence between cobordism classes of manifolds M_k and homotopy classes of maps f_2, provided $k < n$, i.e., set up an isomorphism $\mathfrak{N}_k \cong \pi_{n+k}(M(\mathrm{O}_n))(k < n)$. By an analogous procedure with the group of rotations, he also defined $M(\mathrm{SO}_n)$ and proved $\Omega_k \cong \pi_{n+k}(M(\mathrm{SO}_n))$ if $k < n$.

Thom computed $\mathfrak{N}_k$ by showing that $H^*(M(\mathrm{O}_n))$ was, up to dimension $2n$, a free module over the Steenrod algebra $\mathcal{A}_2$ (see [6]). (Note that all cohomology is here supposed to have coefficient group Z_2.) Choose a free $\mathcal{A}_2$-basis $\{a_i\}$, and for each element of it a characteristic map $g_i: M(\mathrm{O}_n) \rightarrow K(Z_2, \dim a_i)$ (see [12]); then the product map $g: M(\mathrm{O}_n) \rightarrow \prod_i K(Z_2, \dim a_i)$ induces an isomorphism of cohomology up to dimension $2n$, hence also of integer homology, since for these spaces and in these dimensions this consists only of torsion of order 2. The two spaces are both simply-connected if $1 < n$, and so by a result of J. H. C. Whitehead [19], g induces an isomorphism of homotopy up to dimension $2n - 1$. Thus the known homotopy of the latter space gives that of $M(\mathrm{O}_n)$.

We wish to extend these results, so must investigate more precisely. Let i denote the inclusion map of $B(\mathrm{O}_n)$ into $M(\mathrm{O}_n)$; then i^* can be described as follows. $H^*(B(\mathrm{O}_n))$ is the polynomial algebra in $w^1, \cdots, w^n$. i^* is monomorphic and its image is the ideal generated by w^n. We follow the practice of Thom in regarding the w^i as the elementary symmetric functions of $t_1, \cdots, t_n$. Then $H^*(B(\mathrm{O}_n))$ is the algebra of symmetric functions of the t_j. Let now

$$S = t_1^{a_1} \cdots t_n^{a_n}$$

by any monomial, and consider the set of all distinct monomials formed by permuting the variables in S; let $s(a_1, \cdots, a_n)$ be their sum. Clearly the $s(a_1, \cdots, a_n)$ with $a_1 \geqq \cdots \geqq a_n$ form an additive base for $H^*(B(\mathrm{O}_n))$. In writing such expressions out in the future we shall omit any a_i which happen to be zero.

We now introduce a partial ordering on monomials S. We say that t_i is dyadic for S if a_i has the form $2^j - 1$, and call the submonomial of S made up with all other t_i its nondyadic factor. Denote by $u(S)$ the number of these variables, and by $v(S)$ the degree of this factor. Then we define $S < T$ if $u(S) < u(T)$ or if $u(S) = u(T)$ and $v(S) > v(T)$. For any $h \leqq n$ we form the classes

$$S^h_\omega = s(a_1 + 1, \cdots, a_r + 1, 1, \cdots, 1) = w^n s(a_1, \cdots, a_r)$$

where $\omega = (a_1, \cdots, a_r)$ runs through all partitions of h into integers not of the form $2^j - 1$. Then for any $m < n$, the classes $\mathrm{Sq}^I S^h_\omega$ for $h \leqq m$, S^h_ω varying as above, and Sq^I running through a base of $(\mathcal{A}_2)_{m-h}$ are linearly independent. For the highest terms in

$$\mathrm{Sq}^I(t_1^{a_1+1} \cdots t_r^{a_r+1} t_{r+1} \cdots t_n)$$

are formed by

$$t_1^{a_1+1} \cdots t_r^{a_r+1} \mathrm{Sq}^I(t_{r+1} \cdots t_n)$$

since if t_j is dyadic for S, it is so also for $\mathrm{Sq}^I S$. But Thom shows that for $m \leqq n$, the $\mathrm{Sq}^I(t_{r+1} \cdots t_n)$ are linearly independent. The result now follows from the further remark (easily verified) that there are the correct number of elements $\mathrm{Sq}^I S^h_\omega$ for a base of $H^{m+n}(M(\mathrm{O}_n))$.

From this proof we may draw the following conclusion. If each S^h_ω is replaced by T^h_ω, where all terms $T^h_\omega - S^h_\omega$ are less than S^h_ω, then $\mathrm{Sq}^I T^h_\omega$ has the same top terms as $\mathrm{Sq}^I S^h_\omega$, and so the T^h_ω also form a free base of the $\mathcal{A}_2$-module $H^*(M(\mathrm{O}_n))$ up to dimension $2n$, by the same argument.

6. Proof of Theorem 1

Note that $w^n S \to w^{n+1} S$ induces an isomorphism of $H^{m+n}(M(\mathrm{O}_n))$ onto $H^{m+n+1}(M(\mathrm{O}_{n+1}))$ for $m \leqq n$, compatible with the operation of $\mathcal{A}_2$. Thus $w^n S \to S$ induces an isomorphism of $H^{m+n}(M(\mathrm{O}_n))$ onto $\mathfrak{B}_m$, ($\mathfrak{B}$ still denoting the algebra of all w^i) which we can (and do) use to introduce operations of $\mathcal{A}_2$ on $\mathfrak{B}$ independent of n. The above result may now be paraphrased by the statement: $\mathfrak{B}$ is a free $\mathcal{A}_2$-module (and we know a free base of it).

Let $\omega = (a_1, \cdots, a_r)$ be a partition; we write $s(\omega) = s(a_1, \cdots, a_r)$, and recall that these form a base for $\mathfrak{B}$. Let $\mathfrak{C}$ be the graded dual of $\mathfrak{B}$, and let the dual base to $\{s(\omega)\}$ be $\{\sigma(\omega)\}$. Define products in $\mathfrak{C}$ by $\sigma(\varphi)\sigma(\psi) = \sigma(\varphi\psi)$, where $\varphi\psi$ is obtained by the juxtaposition of the partitions φ and

ψ. We may verify that this is dual to the diagonal homomorphism Δ in $\mathfrak{B}$ of Lemma 3. $\mathfrak{C}$ is clearly a polynomial algebra, with generators the $\sigma(i)$. Denote multiplication by $(w^1)^2 = s(2)$ in $\mathfrak{B}$ by δ_2, and the dual homomorphism in $\mathfrak{C}$ by ∂_2.

LEMMA 8. *∂_2 is the derivation of $\mathfrak{C}$ with*

$$\partial_2\sigma(1) = 0 \,, \quad \partial_2\sigma(2) = 1 \,, \quad \partial_2\sigma(i) = \sigma(i-2) \qquad (\textit{for } 2 < i) \,.$$

PROOF. An analogous argument to that of Lemma 3, using the primitivity of $(w^1)^2$ for Δ, could be used to prove ∂_2 a derivation. We shall proceed otherwise. Let ω be the partition in which i occurs λ_i times as a part, and write

$$\omega = (1^{\lambda_1}2^{\lambda_2} \cdots r^{\lambda_r}) \,.$$

Then

$$\begin{aligned}\delta_2 s(1^{\lambda_1}2^{\lambda_2} \cdots r^{\lambda_r}) &= (\lambda_2 + 1)s(1^{\lambda_1}2^{\lambda_2+1} \cdots r^{\lambda_r}) \\ &\quad + \textstyle\sum_{i \geqq 1} (\lambda_{i+2} + 1)s\big(1^{\lambda_1} \cdots i^{\lambda_i - 1}(i+1)^{\lambda_{i+1}}(i+2)^{\lambda_{i+2}+1} \cdots r^{\lambda_r}\big)\end{aligned}$$

and so, taking the dual,

$$\begin{aligned}\partial_2\sigma(1^{\lambda_1}2^{\lambda_2} \cdots r^{\lambda_r}) &= \lambda_2\sigma(1^{\lambda_1}2^{\lambda_2-1} \cdots r^{\lambda_r}) \\ &\quad + \textstyle\sum_{i \geqq 1} \lambda_{i+2}\sigma\big(1^{\lambda_1} \cdots i^{\lambda_i+1}(i+1)^{\lambda_{i+1}}(i+2)^{\lambda_{i+2}-1} \cdots r^{\lambda_r}\big) \,,\end{aligned}$$

and recalling the definition of the product in $\mathfrak{C}$, we see that this shows ∂_2 to be as stated.

We may now determine Ker ∂_2 by means of the following lemma.

LEMMA 9. Ker ∂_2 *contains an element τ_i of each degree i not a power of* 2, *which is a sum of $\sigma(i)$ and decomposable elements of $\mathfrak{C}$.*

PROOF. We may take

$$\tau_{2i+1} = \sigma(2i+1) + \sigma(2)\sigma(2i-1) + \cdots + \sigma(2i)\sigma(1)$$

even if $i = 0$. For even degrees $2i$, we write $\upsilon = \sigma(2i-2k)\sigma(4k-2i-2)$. Then $\partial_2^k(\upsilon) = 0$, and so

$$\begin{aligned}\partial_2\big(\sigma^2(2i-2k)\sigma(4k-2i)\big) &= \sigma(2i-2k)\upsilon \\ &= \partial_2\big(\sigma(2i-2k+2)\upsilon + \sigma(2i-2k+4)\partial_2\upsilon + \cdots + \sigma(2i)\partial_2^{k-1}\upsilon\big)\end{aligned}$$

giving an equation of the form $\partial_2\chi = 0$, where the coefficient of $\sigma(2i)$ in χ is

$$\partial_2^{k-1}\upsilon = \partial_2^{k-1}\big(\sigma(2i-2k)\sigma(4k-2i-2)\big) = \{i-k, 2k-i-1\}$$

by Leibniz' theorem. If $i = 2^r(2s+1)$, where $r \geqq 0$, $s \geqq 1$, we choose $k = 2^{r+1}s$; then the coefficient is 1, and we define τ_{2i} to be the corresponding χ. (For more about this method of proof see [18].)

Now δ_2, as multiplication by $(w^1)^2$ in the polynomial ring $\mathfrak{B}$, is a monomorphism, thus ∂_2 is an epimorphism. Ker ∂_2 contains the τ_i for i not a power of 2, also $\sigma(1)$ and $\sigma^2(2^j)$ for $0 < j$, and hence, being itself a subalgebra, contains the polynomial subalgebra that they generate, which has one generator in each dimension except 2. If V is a graded vector space, let V_n denote its component of degree n, and $d_n(V) = \dim V_n$. Since ∂_2 is onto, $d_n(\text{Ker } \partial_2) = d_n(C) - d_{n-2}(C) = d_n(Q)$, where Q is a polynomial algebra with one generator in each dimension except 2. Hence the above is the whole of Ker ∂_2.

Let the $\sigma(\omega)$ be ordered in the same way as the $s(\omega)$. Since the product in $\mathfrak{C}$ is defined by juxtaposition, the ordering is compatible with the product. Now $\tau_i = \sigma(i) +$ greater terms, for this is clear for odd i, and holds for even i since τ_{2i} is a polynomial in the $\sigma(2k)$. Hence the monomials in the above generators for Ker ∂_2 are equal to the corresponding monomials in the $\sigma(i)$ for i not a power of 2, and $\sigma(1)$ and $\sigma^2(2^j)$ for $0 < j$; added to greater terms. I restate this as

LEMMA 10. *Let m be a monomial in the $\sigma(i)$ in which each $\sigma(2^j)$ for $j \geqq 1$ occurs to an even power. Then there is an element of* Ker ∂_2, *which consists of m, plus other monomials greater than m.*

LEMMA 11. *Let x be an element of* Im δ_2, *and $s(\omega)$ one of the greatest terms in it. Then some 2^j with $j \geqq 1$ occurs an odd number of times as a part in ω.*

PROOF. We shall obtain Lemma 11 from Lemma 10 by dualising. Proceed by assuming that each 2^j with $j \geqq 1$ occurs an even number of times as a part in ω; we shall establish a contradiction. By Lemma 10, there is an element χ, say, of Ker ∂_2, which consists of $\sigma(\omega)$, plus other greater terms. Then since the $s(\omega)$ and $\sigma(\omega)$ form dual bases, the inner product $[x, \chi]=1$. But $x = \delta_2 y$, say, and

$$[x, \chi] = [\delta_2 y, \chi] = [y, \partial_2 \chi] = 0 ,$$

giving the required contradiction.

LEMMA 12. *Let ω be a partition in which some 2^j with $j \geqq 1$ occurs an odd number of times as a part. Then there is an element of* Im δ_2 *equal to $s(\omega)$, plus smaller terms.*

PROOF. This follows from the previous lemma by a dimensional argument. For dimensional reasons, $d_n(\text{Im } \delta_2)$ is the same as the number of partitions of n of the above form. If we now consider the associated graded algebra $G(\mathfrak{B})$ to $\mathfrak{B}$, $d_n(G(\text{Im } \delta_2)) = d_n(\text{Im } \delta_2)$. But $G_n(\text{Im } \delta_2)$ is contained in the vector space spanned by the images of the $s(\omega)$ with ω of the type in question, which has the same dimension as its own. Hence it is the whole space, from which the result follows.

Now $\delta_2\mathfrak{B} = (w^1)^2\mathfrak{B}$ is an $\mathcal{A}_2$-submodule of $\mathfrak{B}$, for in $H^*(B(\mathrm{O}_n))$,

$$\mathrm{Sq}(w^1)^2 = (w^1)^2 + (w^1)^4$$
$$\mathrm{Sq}^i(x(w^1)^2) = (w^1)^2\mathrm{Sq}^i x + (w^1)^4\mathrm{Sq}^{i-2}x$$

so the ideal generated by $(w^1)^2$ in this is a submodule, as is $H^*(M(\mathrm{O}_n))$, hence so is their intersection $\delta_2 H^*(M(\mathrm{O}_n))$, as required.

THEOREM 1. *$(w^1)^2\mathfrak{B}$ is a free direct summand of the free $\mathcal{A}_2$-module $\mathfrak{B}$.*

PROOF. We know an $\mathcal{A}_2$-base of $\mathfrak{B}$, and that each element may be modified by smaller terms. Lemma 12 may be interpreted as saying that many elements of this base, suitably modified, fall in $(w^1)^2\mathfrak{B}$, and so generate a free $\mathcal{A}_2$-submodule of it, a direct summand of $\mathfrak{B}$. We prove the theorem by showing the submodule to be the whole of $(w^1)^2\mathfrak{B}$, by a simple computation of dimensions, which may safely be left to the reader.

With the proof of this theorem, the hard toiling in this paper is finished, and we can proceed in a more relaxed mood.

7. Determination of $\mathfrak{W}$ and of Ker ∂_1/Im ∂_1

Let $\mathcal{A}_2^+$ denote the set of elements of positive dimension in $\mathcal{A}_2$. Then by Thom's result, $\mathfrak{N}$ may be regarded as the dual vector space to $\mathfrak{B}/\mathcal{A}_2^+\mathfrak{B}$. We defined $\mathfrak{W}'$ as the annihilator of $(w^1)^2\mathfrak{B}$, and thus as dual to $\mathfrak{B}/\mathcal{A}_2^+\mathfrak{B}+(w^1)^2\mathfrak{B}$, so with the same number of linearly independent elements in each dimension as a free $\mathcal{A}_2$-base for $\mathfrak{B}/(w^1)^2\mathfrak{B}$, since, by Theorem 1, this is a free $\mathcal{A}_2$-module. Now such a base is determined by nondyadic partitions, in which a power of 2 occurs an even number of times as a part, and these again are in (1–1) correspondence with monomials in symbols x_i, one in each dimension i not of the form $2^j - 1$, and not 2; i.e., with a vector space base for $\mathfrak{W}''$. Hence as promised, since $\mathfrak{W}'' \subset \mathfrak{W} \subset \mathfrak{W}'$, we can deduce $\mathfrak{W}'' = \mathfrak{W} = \mathfrak{W}'$. Moreover, we have in § 4 a complete algebraic description of the action of ∂_1.

LEMMA 13.

(i) Ker ∂_1/Im ∂_1 *is the polynomial algebra on the* X_{2k}^2.

(ii) *Every element of it is uniquely representable as a polynomial* mod 2 *in the* $P_{2n}(C)$.

(iii) *The dual vector space is the space of Pontrjagin numbers, reduced* mod 2.

PROOF. The ∂_1-module $\mathfrak{W}$ is the tensor product of algebras

(a) polynomial on X_{2k}, X_{2k-1} with $\partial_1 X_{2k} = X_{2k-1}$, $\partial_1 X_{2k-1} = 0$, for k not a power of 2, and

(b) polynomial on X_{2j}^2, with $\partial_1 X_{2j}^2 = 0$.

The homology of (a) with respect to the differential operator ∂_1 is the polynomial algebra on X_{2k}^2; that of (b) is (b) itself. Since the coefficient ring is a field, Z_2, we take the tensor product of these for the total homology. This proves (i).

Pontrjagin numbers reduced mod 2 are Stiefel numbers, so give linear functions on Ker ∂_1. An element of Im ∂_1 is representable by an orientable manifold whose class in Ω has order 2, so Pontrjagin numbers vanish on it. Hence they determine linear functionals on Ker ∂_1/Im ∂_1. Products of the $P_{2n}(C)$ are oriented, so determine elements of Ker ∂_1/Im ∂_1. Now it follows from a computation in [5] that the determinant of values of a base for the Pontrjagin numbers on the products of the $P_{2n}(C)$ is odd, so these functionals on, and elements of Ker ∂_1/Im ∂_1 are linearly independent. Since there are the right number of each, by (i), the lemma follows.

8. Proof of main theorems

First, we recall that the natural restriction $r : \Omega \to \mathfrak{N}$ lies in Rohlin's exact sequence

$$(1) \qquad \Omega \xrightarrow{2} \Omega \xrightarrow{r} \mathfrak{N} .$$

THEOREM 2. *Ω contains no elements of order* 4.

PROOF. Let c be an element of Ω_m, of maximal order 2^x: suppose if possible $1 < x$. Then as $\partial_1 r = 0$, rc is in Ker ∂_1. But, c being a torsion element of Ω, all Pontrjagin numbers vanish on it, so by Lemma 13, rc determines the zero element of Ker ∂_1/Im ∂_1, and so is in Im ∂_1. Hence, by Lemma 1, there is a class c' in Ω, with $2c' = 0$ and $rc' = rc$. Thus since $1 < x$, $c - c'$ has order 2^x, and $r(c - c') = 0$. By (1), for some d, $c - c' = 2d$. But then d has order 2^{x+1}, contrary to the maximality of x.

COROLLARY 1. *Two manifolds are cobordant if and only if they have the same Stiefel and Pontrjagin numbers.*

PROOF. The necessity of the condition is known. Suppose it satisfied. Since the manifolds have the same Pontrjagin numbers, their difference gives a torsion element, c, of Ω. Since they also have the same Stiefel numbers, $rc = 0$, so by (1) there is an element d of Ω such that $2d = c$. d is a torsion element of Ω, which has no odd torsion by a result of Milnor, hence by Theorem 2, $2d = 0$, i.e., $c = 0$, as asserted.

Now recall the construction of § 1, leading from a manifold M_n to an orientable manifold V_{n-1} representing the first Stiefel class of M.

THEOREM 3. *The class of M determines the class in Ω of V. Write*

$[V] = \partial_3\{M\}$. *Then there is an exact triangle*

$$\begin{array}{ccc} \Omega & \xrightarrow{2} & \Omega \\ & {\scriptstyle \partial_3}\nwarrow \quad \swarrow{\scriptstyle r} & \\ & \mathfrak{W} & \end{array} .$$

PROOF. By Lemma 2, the Stiefel numbers of M determine those of V, and since by Lemma 1, $2V \sim 0$, all Pontrjagin numbers of V vanish. From the corollary to Theorem 2, we deduce that the class of V in Ω is now determined. The exactness of $(2, r)$ is (1) above, and of $(\partial_3, 2)$ is Lemma 1. Also, $\partial_3 r = 0$, so it only remains to show $\operatorname{Ker} \partial_3 \subset \operatorname{Im} r$. We shall give two alternative proofs of this.

Certainly, $\operatorname{Im} \partial_1 \subset \operatorname{Im} r$, since V in § 1 is orientable. Also each coset of $\operatorname{Im} \partial_1$ in $\operatorname{Ker} \partial_1$ is represented (by Lemma 13) as a polynomial in the $P_{2n}(C)$, hence as an orientable manifold. Hence $\operatorname{Ker} \partial_1 \subset \operatorname{Im} r$. But $\operatorname{Ker} \partial_3 = \operatorname{Ker} \partial_1$, as since all Pontrjagin numbers of V vanish, it determines the zero class in Ω if and only if it does so in $\mathfrak{N}$.

Alternatively, suppose M determines $V \sim 0$. Let M' be obtained (as in Lemma 1) by cutting M along V, and let N be the oriented manifold with boundary V. Form N' from M' and two copies of N by identifying the copies of V on the boundary of M' to those on the boundaries of the copies of N. Clearly, N' is orientable, and we show $M \sim_2 N'$. For define W by identifying the two copies of $N \times 1$ in $N' \times I$, and straightening the corners at $V \times 1$ (see [8]). Then we see $\partial_0 W = M \times 1 + N' \times 0$, thus $M \sim_2 N'$.

Combined with all we already know, Theorem 3 is essentially a structure theorem, determining the algebraic structure of Ω; we shall exploit this in the next paragraph.

Recent results of M. F. Atiyah already referred to in § 4 yield a fairly simple direct proof of Theorem 3, and even extend it to the following, where ∂_2' is not unrelated to the derivation ∂_2 of § 6:

$$\begin{array}{ccc} \Omega \oplus \mathfrak{N} & \xrightarrow{(2,\,0)} & \Omega \\ & {\scriptstyle (\partial_3,\, \partial_2')}\nwarrow \quad \swarrow{\scriptstyle r} & \\ & \mathfrak{N} & \end{array} .$$

9. Further results

We first give some all but trivial propositions, and then prove algebraic descriptions of the algebra Ω, and the $\mathcal{A}_2$-module $H^*(M(\mathrm{SO}_n))$ in stable dimensions.

PROPOSITION 1. *Ω is commutative.*

PROOF. We know that Ω is skew-commutative. This implies commutativity of a product in which at least one factor is either of order 2 or in even dimension. But every element of Ω is of one of these two forms.

PROPOSITION 2. *The product of an orientable and a nonorientable class in $\mathfrak{N}$ is nonorientable. (The proof of this result given in [11] is incorrect.)*

PROOF. Let the classes be x, y respectively. It is clear that if y is not in $\mathfrak{W}$, the product is not even in $\mathfrak{W}$, and if it is,

$$\partial_1 x = 0\ , \quad \partial_1 y \neq 0\ , \qquad \text{so } \partial_1(xy) = x\partial_1 y \neq 0\ .$$

Thus xy is not in $\operatorname{Ker} \partial_1 = \operatorname{Im} r$.

PROPOSITION 3. *The square of any manifold is cobordant with an orientable manifold.*

PROOF. Since we are working in $\mathfrak{N}$, and so mod 2, the square of any polynomial in the generators is equal to the same polynomial in the squares of the generators. Now $(P_{2n}(R))^2 \sim_2 P_{2n}(C)$, which is orientable, and V_{2n-1} is already orientable, whence the result. (Although this proof is simple, and uses only previously known results, we should much like to see a direct geometrical proof.)

PROPOSITION 4. *Any class in $\mathfrak{N}$, on which all Stiefel numbers with w^1 as factor vanish, contains orientable manifolds.*

PROOF. We note that the corresponding result for $(w^1)^2$ is that $\mathfrak{W}'=\mathfrak{W}$. If c is a class of the type above, it is in $\mathfrak{W}'=\mathfrak{W}$, and by Lemma 2, $\partial_1 c=0$. Hence by Theorem 3, $c \in \operatorname{Im} r$.

We now proceed to the long-announced description of Ω.

LEMMA 14. *Generators h_{4k} can be chosen for the torsion-free part of Ω, such that $r(h_{4k}) = X_{2k}^2$.*

PROOF. If $P_{2k}(C)$ is expressed in terms of generators of the torsion-free part of Ω (the quotient of Ω by the ideal of torsion elements) the generator of dimension $4k$ has an odd coefficient, by results of Milnor, as in Lemma 13. Hence we may choose new generators H_{4k} such that $r(H_{4k}) = r[P_{2k}(C)] = \{P_{2k}(R)\}^2$. Now $\{P_{2k}(R)\}$ equals X_{2k}, plus decomposable elements of $\mathfrak{N}$. Hence $\{P_{2k}(R)\}^2$ equals X_{2k}^2, plus a sum of products of squares of elements of $\mathfrak{N}$, which can all (Proposition 3) be represented by orientable manifolds. Subtracting the resulting decomposable classes from H_{4k} (which, we note, leaves it a generator) we obtain the required class h_{4k}.

Note: When we speak of a generator for the torsion-free part of Ω, we refer throughout to classes in Ω (not taken mod torsion).

For each partition $\omega = (a_1, \cdots, a_r)$ with unequal parts a_i none of which is a power of 2, define an element of Ω by

$$g_\omega = \partial_3(X_{2a_1}, \cdots, X_{2a_r}) ,$$

and note that

$$r(g_\omega) = \partial_1(X_{2a_1} \cdot \cdots \cdot X_{2a_r}) = \sum_i X_{2a_1} \cdots X_{2a_i - 1} \cdots X_{2a_r} .$$

THEOREM 4. *The g_ω and h_{4k} form an irredundant set of generators for Ω.*

PROOF. The torsion-free part of Ω is a polynomial ring: the h_{4k} are generators of it by definition and are irredundant. Let x be a torsion element of Ω. Then $x \in \operatorname{Ker} 2 = \operatorname{Im} \partial_3$, say $x = \partial_3 y$, $y \in \mathfrak{W}$. y is a sum of monomials $(\prod X_{2a}^2)(\prod X_{2b-1})X_{2a_1} \cdots X_{2a_r}$, so

$$\begin{aligned} rx = \partial_1 y &= \sum (\prod X_{2a}^2)(\prod X_{2b-1})\partial_1(X_{2a_1} \cdots X_{2a_r}) \\ &= r[\sum (\prod h_{4a})(\prod g_b) g_\omega] \end{aligned}$$

where $a_1, \cdots, a_r$ may be taken distinct (letting the first bracket contain all the squared factors) and then none can be powers of 2, since $y \in \mathfrak{W}$, so they form a partition ω of the type above; and where b denotes the partition with b as sole part.

Since the torsion element $x - \sum (\prod h_{4a})(\prod g_b)g_\omega$ of Ω restricts mod 2 to zero, it must already be zero. Hence we have indeed a set of generators.

If any g_ω is a redundant generator, let it be expressed in terms of the other generators by a polynomial, which may clearly be supposed homogeneous, $g_\omega = P(g_\varphi, h_{4j})$. Now restrict the whole equation mod 2, and express in terms of the generators X_i of $\mathfrak{N}$. Equate coefficients of the leading term $X_{2a_1-1}X_{2a_2} \cdots X_{2a_r}$ of g_ω. Now each term in each g_φ has one X_i with odd suffix as a factor, so no term on the right hand side with two g_φ as factors can equal the above term on the left. But, nor can any term with an h_{4j} as a factor, for these will have repeated factors. Hence the corresponding term on the right is a single g_φ. But no other rg_φ except rg_ω contains the above term, hence the above equation is impossible, as required.

COROLLARY. *Orientable Dold manifolds will not suffice to generate the torsion of Ω.*

(There are not nearly enough of them to go round.)

By similar arguments we may deduce the relations between the generators. First, of course, we have

(1) $$2g_\omega = 0 .$$

Consider the uniqueness of the expression in Theorem 4 of a torsion element x in terms of the generators. The element y was indeterminate by an element of $\operatorname{Ker} \partial_3 = \operatorname{Im} r$, rz, say. Using the expression for z by the generators, we find that the new expression for x is deducible from the old one by relations of the following type (which express $\partial_3 r \partial_3 = 0$). Let $\omega = (a_1, \cdots, a_r)$ be a partition into unequal parts, none powers of 2, and with more than two parts, and let ω_i be formed by omitting a_i. Then

$$(2) \qquad \sum_i g_{\omega_i} g_{a_i} = 0 .$$

Finally, every product of two generators can be put in the standard form, and this leads to some new relations

$$(3) \qquad g_\varphi g_\psi = \sum (\prod h_{4j} \prod g_b \cdot g_\omega)$$

such as

$$(3') \qquad g_\omega^2 = g_{a_1}{}^2 h_{4a_2} \cdots h_{4a_r} + \cdots + h_{4a_1} \cdots h_{4a_{r-1}} g_{a_r}^2 .$$

The proof of the independence of these relations is not difficult, and we leave it to the reader.

We now investigate the $\mathcal{A}_2$-module $H^*(M(\mathrm{SO}_n))$ in stable dimensions. This is not free, but is the next simplest possibility. Let $\mathfrak{M}$ be an $\mathcal{A}_2$-module with one generator z and one relation $\mathrm{Sq}^1 z = 0$, this agrees with the stable part of $H^*(Z, n, Z_2)$.

THEOREM 5. *In stable dimensions, $H^*(M(\mathrm{SO}_n))$ is a direct sum of a free $\mathcal{A}_2$-module and modules of type $\mathfrak{M}$.*

PROOF. As when we were considering the modules $H^*(M(\mathrm{O}_n))$ in stable dimensions, we can restrict consideration to a single module, which in the other case was $\mathfrak{B}$, and is seen here to be $\mathfrak{B}/w^1\mathfrak{B}$.

$\mathfrak{N}$ is dual to $\mathfrak{B}/\mathcal{A}_2^+\mathfrak{B}$, and of finite type. By Proposition 4, $r(\Omega)$ is the annihilator of $w^1\mathfrak{B}$, hence dual to $\mathfrak{B}/\mathcal{A}_2^+\mathfrak{B} + w^1\mathfrak{B}$. Let $\mathfrak{B}'$ be the algebra $\mathfrak{B}/w^1\mathfrak{B}$; then $r(\Omega) = \operatorname{Ker} \partial_3 = \operatorname{Ker} \partial_1$ is dual to $\mathfrak{B}'/\mathcal{A}_2^+\mathfrak{B}'$, i.e., to a vector space with base a minimal set of generators of the $\mathcal{A}_2$-module $\mathfrak{B}'$.

Among these generators are the mod 2 restrictions of products of Pontrjagin classes (the linear independence of restrictions of Pontrjagin numbers, and so of products of Pontrjagin classes modulo decomposable elements, was shown in Lemma 13), dual to a base of $\operatorname{Ker} \partial_1/\operatorname{Im} \partial_1$, and all satisfying $\mathrm{Sq}^1 z = 0$, as the restrictions of integral classes. $\mathfrak{B}'$ is thus a quotient of a direct sum of a free module, with generators dual to a base of $\operatorname{Im} \partial_1$, and modules of type $\mathfrak{M}$, with generators dual to a base of $\operatorname{Ker} \partial_1/\operatorname{Im} \partial_1$. A further computation of dimensions shows $\mathfrak{B}'$ as big as this module, hence isomorphic to it.

It seems that a direct proof of this theorem would be extremely difficult.

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