Inventiones math. 10, 239–250 (1970)

Homeomorphisms between Homotopy Manifolds and Their Resolutions

MARSHALL M. COHEN* (Ithaca)

§ 1. Introduction

A *homotopy n-manifold without boundary* is a polyhedron M (i.e., a topological space along with a family of compatible triangulations by locally finite simplicial complexes) such that, for any triangulation in the piecewise linear (p.l.) structure of M, the link of each i-simplex $(0 \leqq i \leqq n)$ has the homotopy type of the sphere S^{n-i-1}. More generally, a *homotopy n-manifold* is a polyhedron such that the link of each i-simplex in any triangulation is homotopically an $(n-i-1)$ sphere or ball, and in which ∂M – the union of all simplexes with links which are homotopically balls – is itself a homotopy $(n-1)$-manifold without boundary. We note that if M is a homotopy manifold then ∂M is a well-defined subpolyhedron of M. Also, the question of whether a polyhedron M is a homotopy manifold is completely determined by a single triangulation of M (by Lemma LK 5 of [8]).

Our main purpose is to prove

Theorem 1. *Assume that M_1 and M_2 are connected homotopy n-manifolds where $n \geqq 6$ or where $n=5$ and $\partial M_1 = \partial M_2 = \emptyset$. Let f: $M_1 \to M_2$ be a proper p.l. mapping such that all point-inverses of f and of $(f|\partial M_1)$ are contractible. Let d be a fixed metric on M_2 and ε: $M_1 \to R^1$ a positive continuous function. Then there is a homeomorphism h: $M_1 \to M_2$ such that $d(h(x), f(x)) < \varepsilon(x)$ for all x in M_1.*

The statement that f is *proper* means that $f^{-1}(X)$ is compact, for every compact subset X of M_2. The hypothesis of Theorem 1 automatically implies (see the proof of (3.1)) that f is onto.

Theorem 1 can be used to add to our understanding of the relationships between homotopy manifolds, topological manifolds and p.l. manifolds which have recently come to light. These relationships are two-fold:

First, every homotopy n-manifold M is a topological manifold, provided that $n \neq 4$ or, if $n=5$, provided that ∂M is known to be a

* This work was supported by NSF Contract No. GP-9397.

topological manifold. This is demonstrated by Siebenmann in [10]. He omits the proof in the case where $\partial M \neq \emptyset$, but this is supplied by Glaser [4] in his recent work on homotopy manifolds. (It is unknown at present – March, 1970 – whether every triangulated topological manifold is, in fact, a homotopy manifold.)

Second, Sullivan [13] has constructed an elegant and essentially elementary obstruction theory connecting homotopy manifolds to p.l. manifolds. If M is a homotopy manifold whose boundary is a p.l. manifold then a *resolution of* M is defined[1] to be a pair (V, f) where V is a p.l. manifold and f: $(V, \partial V) \to (M, \partial M)$ is a p.l. surjection with compact point-inverses such that each $f^{-1}(x)$ is contractible and each $(f|\partial V)^{-1}(x)$ is collapsible. Given such an M, Sullivan proves that there is an element $z \in H^4(M, \partial M; \Theta_3)$ (where Θ_3 is the Kervaire-Milnor group [6] in the p.l. category) such that: $z = 0$ if and only if M has a resolution.

Theorem 1 and its proof show that Sullivan's obstruction is really the obstruction to building a p.l. manifold structure for the homotopy manifold M in a simple finite sequence of steps. In fact, since resolutions clearly satisfy the hypothesis of Theorem 1 on each component of M, we have

Theorem 2. *Suppose that M is a homotopy n-manifold where $n \geqq 6$ or where $n = 5$ and $\partial M = \emptyset$. Let d be a metric on M. Then, if (V, f) is any resolution of M and ε: $V \to R^1$ is a positive mapping, there is a homeomorphism h: $V \to M$ such that $d(h(x), f(x)) < \varepsilon(x)$ for all x.*

To prove Theorem 1, we shall use the results of [10] and [4] (basically the fact that the double suspension of a homotopy-manifold 3-cell or 4-cell is a ball) locally – i.e., to prove (3.1). We build the global picture from the Simplicial Factorization Lemma proved in § 2. This lemma, which sharpens a result of Homma ([5]; 2.2) and which depends heavily on [1], gives a very precise view of how a simplicial map is built. It leads (§ 3) to a homeomorphism in Theorem 1 which is constructed as the composite of a discrete sequence of moves – a "move" being a homeomorphism supported on a topological cone. Finally, in § 4, we explain the relationship between Sullivan's obstruction and the Simplicial Factorization Lemma.

I would like to thank Robert Connelly for several very enlightening comments during the evolution of this paper, and Dennis Sullivan for introducing me to the theory of resolutions.

Added in Proof. L. C. Siebenmann has announced [15] a generalization of Theorem 1 to cellular maps of topological manifolds.

[1] The definition here is equivalent to Sullivan's, though slightly different in form.

§ 2. The Simplicial Factorization Lemma

For the reader's convenience we reproduce here some of the definitions and concepts which are discussed (in greater detail) in [1]. We shall use these to prove two general lemmas about simplicial maps.

By a complex we mean a locally finite simplicial complex in some Euclidean space R^q. We write $K_0 < K$, or $K_0 \lhd K$, to indicate that K_0 is a subcomplex, or a full subcomplex, of K. We let $N(K_0, K)$ denote the stellar neighborhood of K_0 in K – i.e., the complex consisting of all simplexes which meet K_0, plus their faces. $C(K_0, K)$ consists of all simplexes which do not intersect K_0, and $\dot{N}(K_0, K) \equiv N(K_0, K) \cap C(K_0, K)$. If $K_0 \lhd K$ then every simplex of $N(K_0, K)$ is uniquely expressible as the join of a simplex in K_0 and a simplex in $\dot{N}(K_0, K)$. Finally K' denotes a first derived of K.

Small Greek letters represent (closed) simplexes. $\hat{\alpha}$ denotes the barycenter of the simplex α. If $\alpha, \beta < K$ then $\alpha \cdot \beta$ represents the smallest simplex of K containing both α and β or, if there is none, $\alpha \cdot \beta$ is the empty simplex. If $\alpha < K$, then $D(\alpha, K)$ is the well-known *dual cell* to α in K. Thus $D(\alpha, K) < K'$ and $D(\alpha, K) = \hat{\alpha} * \dot{D}(\alpha, K)$, where "$*$" denotes the simplicial join.

Suppose that $f\colon K \to L$ is a simplicial map. We may always (and do always) assume that K and L are joinable in some R^q and that first deriveds have been chosen so that $f\colon K' \to L'$ is simplicial. Then, if $\alpha < L$, we set

$$D(\alpha, f) = f^{-1} D(\alpha, L) = \{\hat{\sigma}_0 \dots \hat{\sigma}_q \mid \sigma_0 < \dots < \sigma_q;\ \alpha < f(\sigma_0)\} < K',$$

$$\dot{D}(\alpha, f) = f^{-1} \dot{D}(\alpha, L) = \{\hat{\sigma}_0 \dots \hat{\sigma}_q \mid \sigma_0 < \dots < \sigma_q;\ \alpha \lneqq f(\sigma_0)\},$$

$$C_f = \{\hat{\alpha}_0 \dots \hat{\alpha}_q\, \hat{\sigma}_{q+1} \dots \hat{\sigma}_p \mid \alpha_0 < \dots < \alpha_q < f(\sigma_{q+1});$$
$$\sigma_{q+1} < \dots < \sigma_p < K\} < (K' * L').$$

C_f is the simplicial mapping cylinder. It is the complex we called C_* on p. 233 of [1].

Suppose now that $\{J_i\}_{i \in S}$ is a family of non-empty full subcomplexes of J such that $J_i \cap N(J_j, J) = \emptyset$ if $i \neq j$. Let the 0-dimensional complex $\{x_i\}_{i \in S}$ be joinable with J. Then the simplicial map

$$g\colon J \to C\Big(\bigcup_i J_i, J\Big) \cup \bigcup_i \big(x_i * \dot{N}(J_i, J)\big)$$

which is defined by the conditions that

$$g \mid C\Big(\bigcup_i J_i, J\Big) = 1,$$
$$g(J_i) = x_i$$

is called a *simplicial squeeze of the family* $\{J_i\}$.

If $f\colon K\to L$ is simplicial and if $\alpha<L$, then, by (5.4) of [1], $D(\alpha,f)=N(f^{-1}\hat{\alpha}, D(\alpha,f))$ where $f^{-1}\hat{\alpha}$ is full and $\dot{D}(\alpha,f)=\dot{N}(f^{-1}\hat{\alpha}, D(\alpha,f))$. Thus a simple example of a simplicial squeeze is given by

$$F\colon [\dot{\alpha}'*D(\alpha,f)]\to[\dot{\alpha}'*\hat{\alpha}*\dot{D}(\alpha,f)]=[\alpha'*\dot{D}(\alpha,f)]$$

where $F|[\dot{\alpha}'*\dot{D}(\alpha,f)]=1$ and $F(f^{-1}\hat{\alpha})=\hat{\alpha}$. We prove that every simplicial map is the composition of squeeze maps built from examples of this type.

Lemma 2.1. *(The factorization lemma): If $f\colon K\to L$ is a simplicial map (where K, L are joinable complexes and $f\colon K'\to L'$ is also simplicial) and if*

$$P_0=K',$$
$$P_i=\bigcup\{(\dot{\alpha}^i)'*D(\alpha^i,f)\,|\,\alpha^i<L\}<C_f,\qquad 1\leqq i\leqq n=\dim L,$$
$$P_{n+1}=L'$$

then $f=f_n\ldots f_1 f_0$ where $f_i\colon P_i\to P_{i+1}$ is a well-defined simplicial map given by the conditions that

$$f_i|[\dot{\alpha}'*\dot{D}(\alpha,f)]=1,$$
$$f_i(f^{-1}\hat{\alpha})=\hat{\alpha}$$

for each i-simplex $\alpha<L$.

Remark. If α is an i-simplex of $L-f(K)$ then $f^{-1}\hat{\alpha}=D(\alpha,f)=\emptyset$ and $f_i\colon[\dot{\alpha}'*D(\alpha,f)]\to[\alpha'*\dot{D}(\alpha,f)]$ is just the inclusion $\dot{\alpha}'\subset\alpha'$.

Proof. First note, for each i-simplex $\alpha<L$, that $(f^{-1}\hat{\alpha})\lhd P_i$. For $f^{-1}\hat{\alpha}\lhd K'$ (since the pullback of a vertex is always full) and $K'\lhd C_f$. Thus $f^{-1}\hat{\alpha}\lhd C_f$ and, à fortiori, $f^{-1}\hat{\alpha}\lhd P_i$.

Now observe that $N(f^{-1}\hat{\alpha},P_i)=[\dot{\alpha}'*D(\alpha,f)]$ and

$$(f^{-1}\hat{\beta})\cap N(f^{-1}\hat{\alpha},P_i)=\emptyset\qquad\text{if }\beta=\beta^i\neq\alpha.$$

This is because

$$[\dot{\alpha}'*D(\alpha,f)]=N(f^{-1}\hat{\alpha},\dot{\alpha}'*D(\alpha,f))<N(f^{-1}\hat{\alpha},P_i)$$

and

$$[\dot{\alpha}'*D(\alpha,f)]\cap[\dot{\beta}'*D(\beta,f)]=[(\dot{\alpha}'\cap\dot{\beta}')*D(\alpha\cdot\beta,f)]<\dot{\alpha}'*\dot{D}(\alpha,f).$$

The above paragraphs show that f_i is a well-defined squeeze map for each i, and it is clear, if one looks at the vertices of an image simplex, that $f_i(P_i)\subset P_{i+1}$. Finally $f_n\ldots f_1 f_0\colon K'\to L'$ is a simplicial map which takes $f^{-1}\hat{\alpha}$ onto $\hat{\alpha}$ for every vertex $\hat{\alpha}$ of $f(K')$. Since a simplicial map is determined by what it does to vertices, this composition is precisely f. q.e.d.

Remark 1. In the topological category a map $f: K \to L$ can be realized by "sliding down" the rays of the topological mapping cylinder. The lemma above essentially says that a simplicial map can be realized in a finite sequence of steps down through the simplicial mapping cylinder. In fact this sequence corresponds to a simplicial collapse $C_f \searrow^{s} L'$ since

$$C_f = \bigcup \{\alpha' * D(\alpha, f) \mid \alpha < L\}$$

and

$$[\alpha' * D(\alpha, f)] = [\hat{\alpha} * \dot{\alpha}' * D(\alpha, f)] \searrow^{s} [\hat{\alpha} * \dot{\alpha}' * \dot{D}(\alpha, f)].$$

Remark 2. From the proofs of (3.3) and (5.5) of [1] one can see that each point inverse of the squeeze map $D(\alpha, f) \to \hat{\alpha} * \dot{D}(\alpha, f)$, other than $f^{-1}\hat{\alpha}$, is a convex cell. Thus point inverses of the map $[\dot{\alpha}' * D(\alpha, f)] \to [\dot{\alpha}' * \dot{D}(\alpha, f) * \hat{\alpha}]$ are either p.l. balls or are p.l. equivalent to $f^{-1}\hat{\alpha}$.

Lemma 2.2. *Suppose that $f: K \to L$ is a simplicial mapping and assume that α is an i-simplex of L and β is an $(i-1)$-face of α $(1 \leqq i \leqq n)$. Then $D(\alpha, f)$ is a regular neighborhood*[2] *of $f^{-1}(\hat{\alpha})$ in $\dot{D}(\beta, f)$ with boundary $\dot{D}(\alpha, f)$.*

Proof. It is shown in (5.4) of [1] that

$$D(\alpha, f) = N(f^{-1}\hat{\alpha}, \dot{D}(\beta, f)),$$
$$\dot{D}(\alpha, f) = \dot{N}(f^{-1}\hat{\alpha}, \dot{D}(\beta, f)).$$

From the Stellar Neighborhood Theorem (6.1 of [2]) it suffices to show that $\dot{D}(\alpha, f)$ is p.l. collared in $D(\alpha, f)$ and in $C(f^{-1}\hat{\alpha}, \dot{D}(\beta, f))$. This will be done (by 4.2 of [2]) once we show that, for each simplex $A < \dot{D}(\alpha, f)$, its links in $D(\alpha, f)$ and in $C(f^{-1}\hat{\alpha}, \dot{D}(\beta, f))$ are each p.l. equivalent (rel base) to the cone on $Lk(A, \dot{D}(\alpha, f))$.

Suppose $A = \hat{\sigma}_0 \ldots \hat{\sigma}_q < \dot{D}(\alpha, f)$. Thus $\alpha \lneqq f(\sigma_0)$. Denote

$$P = [\dot{D}(\sigma_0, \dot{\sigma}_1) * \cdots * \dot{D}(\sigma_{q-1}, \dot{\sigma}_q) * \dot{D}(\sigma_q, K)] < \dot{D}(\alpha, f).$$

Using (5.6) of [1] and its proof we see that

$$Lk(A, D(\alpha, f)) = D(\alpha, f \mid \dot{\sigma}_0) * P,$$
$$Lk(A, \dot{D}(\alpha, f)) = \dot{D}(\alpha, f \mid \dot{\sigma}_0) * P$$

where $D(\alpha, f \mid \dot{\sigma}_0)$ is a p.l. ball with boundary $\dot{D}(\alpha, f \mid \dot{\sigma}_0)$. Because $\dot{D}(\alpha, f \mid \dot{\sigma}_0) = D(\alpha, f \mid \dot{\sigma}_0) \cap \dot{D}(\alpha, f)$, it follows that there is a p.l. homeomorphism of pairs

$$\begin{aligned}[Lk(A, D(\alpha, f)), Lk(A, \dot{D}(\alpha, f))] &\cong [v * \dot{D}(\alpha, f \mid \dot{\sigma}_0) * P,\ \dot{D}(\alpha, f \mid \dot{\sigma}_0) * P] \\ &= [v * Lk(A, \dot{D}(\alpha, f)),\ Lk(A, \dot{D}(\alpha, f))].\end{aligned}$$

[2] We mean here a regular neighborhood in the sense of [2]; there is a triangulation of the polyhedron in which this is a first derived neighborhood.

To see that the link of A in the complement behaves the same way, notice that

$$\begin{aligned}&Lk(A, C(f^{-1}\hat{\alpha}, \dot{D}(\beta, f)))\\&=\{\hat{\tau}_1\ldots\hat{\tau}_p \mid \tau_p<\dot{\sigma}_0, \beta \lneqq f(\tau_1), (\hat{\tau}_1\ldots\hat{\tau}_p)\cap f^{-1}\hat{\alpha}=\emptyset\} * P\\&=C((f\mid\dot{\sigma}_0)^{-1}(\hat{\alpha}), \dot{D}(\beta, f\mid\dot{\sigma}_0)) * P.\end{aligned}$$

Let's denote the first factor in this join by C. Since (still using 5.6 of [1]), $\dot{D}(\beta, f\mid\dot{\sigma}_0)$ is a p.l. sphere and since the stellar neighborhood of $(f\mid\dot{\sigma}_0)^{-1}(\hat{\alpha})$ in this sphere is a p.l. ball, we see that C is the complement of a p.l. ball in a p.l. sphere – hence a p.l. ball. Its boundary is

$$\dot{N}((f\mid\dot{\sigma}_0)^{-1}(\hat{\alpha}), \dot{D}(\beta, f\mid\dot{\sigma}_0)) = C\cap\dot{D}(\alpha, f).$$

As in the previous case we deduce that $C*P$ is piecewise linearly the cone on $(C*P)\cap\dot{D}(\alpha, f)$.

§ 3. Proof of Theorem 1

Lemma 3.1. *Assume that M_1 and M_2 are connected homotopy n-manifolds where $n\geqq 6$ or where $n=5$ and $\partial M_1=\partial M_2=\emptyset$. Let $f\colon (M_1, \partial M_1)\to(M_2, \partial M_2)$ be a proper p.l. map such that all point-inverses of f and of $(f\mid\partial M_1)$ are contractible, and assume that M_1 and M_2 have been triangulated so that f is simplicial. Let α be a simplex of M_2. Then there is a homeomorphism*

$$h\colon [\dot{\alpha}' * D(\alpha, f)] \to [v * \dot{\alpha}' * \dot{D}(\alpha, f)]$$

such that $h\mid[\dot{\alpha}' * \dot{D}(\alpha, f)] = 1$.

Proof. We point out first of all that f is onto. To see this when $\partial M \neq \emptyset$ consider the map $F\colon 2M_1\to 2M_2$, where $2M_i$ is the double of M_i (i.e. $2M_i=M_i\times\{0,1\}$ with $(x,0)\equiv(x,1)$ for all $x\in\partial M_i$) and $F(x,j)=(f(x),j)$, $j=0,1$. One easily checks that F is also a proper p.l. map with contractible point inverses. Let $A=F(2M_1)$. Let $H^n(\ )$ denote Alexander cohomology with compact supports and Z_2 coefficients. Since M_1 is connected, $H^n(2M_1)\approx H_0(2M_1)\approx Z_2$. Hence, by the Vietoris-Begle mapping theorem ([12], p. 346) $H^n(A)\approx Z_2$. So, by duality, ([12], p. 342) $H_0(2M_2, 2M_2-A)\approx H^n(A)\approx Z_2$. Since M_2 is connected this implies that $2M_2=A=F(2M_1)$. Clearly then, $M_2=f(M_1)$. The proof when $\partial M=\emptyset$ is left to the reader.

Let $i=\dim\alpha$. We claim that $D(\alpha, f)$ is a homotopy $(n-i)$-manifold with boundary $\dot{D}(\alpha, f)\cup D(\alpha, f\mid\partial M_1)$. For if $A=\hat{\sigma}_0\ldots\hat{\sigma}_q$ is a simplex of $D(\alpha, f)$ then $Lk(A, D(\alpha, f))$ is a complex of the form $K*\dot{D}(\sigma_q, M_1)$ where K is a combinatorial ball or sphere according to whether A is in $\dot{D}(\alpha, f)$ or not; and where $\dot{D}(\sigma_q, M_1)$ – which is simplically isomorphic to $Lk(\sigma_q, M_1)'$ – has the homotopy type of a ball or sphere according to whether A is in ∂M_1 or not. The computations which justify these assertions are explicitly made in the proof of (5.6) of [1].

In fact $D(\alpha, f) = f^{-1} D(\alpha, M_2)$ is contractible and $\partial D(\alpha, f) = f^{-1}[\dot{D}(\alpha, M_2) \cup D(\alpha, \partial M_2)]$ is a homotopy sphere. This follows from [11] because point inverses are contractible, f is onto, and M_2 being a homotopy manifold, $[\dot{D}(\alpha, M_2) \cup D(\alpha, \partial M_2)]$ is homotopically a sphere.

For the rest of this proof we shall call a compact, contractible homotopy k-manifold whose boundary is homotopically a sphere a *homotopy k-cell*. We proceed to consider the various cases.

Case I. $n = 5$.

If $i \geqq 3$ then $D(\alpha, f)$ is a homotopy $(5-i)$-cell with boundary $\dot{D}(\alpha, f)$. By elementary results this is a ball. So $\dot{\alpha}' * D(\alpha, f)$ is just a ball with boundary $\dot{\alpha}' * \dot{D}(\alpha, f)$. Under these circumstances the lemma is trivial.

If $i = 2$, $\dot{\alpha}' * D(\alpha, f)$ is just the double suspension of a homotopy 3-cell. Siebenmann has proved [10] that such an object is a 5-ball.

If $i = 1$, we are asking whether the suspension of the homotopy 4-cell $D(\alpha, f)$ (which is *not* known to be a p.l. manifold) is homeomorphic to $v * (\dot{\alpha}') * \dot{D}(\alpha, f)$. By thinking of suspension as 2-point compactification, we will get an affirmative answer to this question once we prove the

Claim. *$D(\alpha, f) \times R^1$ is homeomorphic to $[v * \dot{D}(\alpha, f)] \times R^1$ by a homeomorphism which is 1 on $\dot{D}(\alpha, f) \times R^1$ and which is bounded in the R^1 direction.*

But notice that by Lemma 2.2, $D(\alpha, f)$ is a regular neighborhood of $f^{-1}(\hat{\alpha})$. Moreover $(\operatorname{Int} D(\alpha, f)) \times R^1$ is an open homotopy 5-manifold, hence by [10] a topological 5-manifold. Thus $D(\alpha, f)$ has the following properties:

A) $D(\alpha, f)$ is a mapping cylinder neighborhood of $f^{-1}\hat{\alpha}$ and $[D(\alpha, f) - f^{-1}(\hat{\alpha})]$ is p.l. homeomorphic to $\dot{D}(\alpha, f) \times [0, 1)$.

B) $\dot{D}(\alpha, f)$ is a p.l. manifold homotopy equivalent to S^3 and $D(\alpha, f)$ is contractible.

C) $(\operatorname{Int} D(\alpha, f)) \times R^1$ is homeomorphic to R^5 (by [7]) and so can be given a (new) p.l. structure under which it is p.l. equivalent to R^5.

Now Glaser has proved our claim (see [3]; 3.1 and 3.2) verbatim, except that wherever we use the symbol $D(\alpha, f)$ he puts in a p.l. 4-manifold, say F^4. However his proof only uses the fact that F^4 satisfies A) – C) above. Thus the claim follows by his argument.

Finally, if $i = 0$, we consider the 5-dimensional homotopy cell $D(\alpha, f)$, $\alpha = \alpha^0$. By Lemma 2.2, $D(\alpha, f)$ is a mapping cylinder neighborhood of $f^{-1}(\hat{\alpha})$. Hence $[D(\alpha, f)/f^{-1}(\hat{\alpha})]$ – the topological quotient space whose only non-degenerate element is $f^{-1}(\hat{\alpha})$ – is homeomorphic rel $\dot{D}(\alpha, f)$ to $v * \dot{D}(\alpha, f)$. On the other hand $\operatorname{Int}(D(\alpha, f))$ is an open 5-dimensional homotopy manifold simply connected at infinity; so by [10] and [7] it is topologically R^5. Since $f^{-1}(\alpha)$ has arbitrarily small mapping cylinder neighborhoods in $D(\alpha, f)$, each homeomorphic to R^5, $f^{-1}(\hat{\alpha})$ is cellular in $D(\alpha, f)$. Thus $[D(\alpha, f)/f^{-1}(\hat{\alpha})]$ is homeomorphic to $D(\alpha, f)$, rel $\dot{D}(\alpha, f)$. So $D(\alpha, f)$ is homeomorphic to $v * \dot{D}(\alpha, f)$, rel $\dot{D}(\alpha, f)$.

Case II. $n \geqq 6$ and $\alpha \nless \partial M_2$.

In this case $\dot{\alpha}' * D(\alpha, f)$ is a homotopy cell with boundary $\dot{\alpha}' * \dot{D}(\alpha, f)$. By [10] and [4] this is a topological manifold and by the topological Poincaré conjecture [9] it is a ball.

Case III. $n \geqq 6$ and $\alpha < \partial M_2$.

As in the previous paragraph, $\dot{\alpha}' * D(\alpha, f)$ is a topological ball, this time with boundary

$$\dot{\alpha}' * \partial D(\alpha, f) = \dot{\alpha}' * [\dot{D}(\alpha, f) \cup D(\alpha, f \mid \partial M_1)].$$

Similarly the homotopy cell $v * \dot{\alpha}' * \dot{D}(\alpha, f)$ is a ball and its boundary is

$$\begin{aligned} \dot{\alpha}' * \partial\big(v * \dot{D}(\alpha, f)\big) &= \dot{\alpha}' * [\dot{D}(\alpha, f) \cup v * \partial \dot{D}(\alpha, f)] \\ &= \dot{\alpha}' * [\dot{D}(\alpha, f) \cup v * \dot{D}(\alpha, f \mid \partial M_1)]. \end{aligned}$$

(We have used: $\partial \dot{D}(\alpha, f) = \dot{D}(\alpha, f \mid \partial M_1)$. Compare 5.6 of [1].)

Using Cases I and II, there is a homeomorphism

$$h_1\colon\ [\dot{\alpha}' * D(\alpha, f \mid \partial M_1)] \to [\dot{\alpha}' * v * \dot{D}(\alpha, f \mid \partial M_1)]$$

which is 1 on $\dot{\alpha}' * \dot{D}(\alpha, f \mid \partial M_1)$. This extends to

$$h_2\colon\ \partial\big(\dot{\alpha}' * D(\alpha, f)\big) \to \partial\big(\dot{\alpha}' * v * \dot{D}(\alpha, f)\big)$$

by simply setting $h_2 \mid [\dot{\alpha}' * \dot{D}(\alpha, f)] = 1$. Finally we extend h_2 "conewise" to a homeomorphism h of one ball onto the other such that $h \mid [\dot{\alpha}' * \dot{D}(\alpha, f)] = 1$, q.e.d.

Proof of Theorem 1. It suffices (by elementary arguments) to prove Theorem 1 when M_1 and M_2 are joinable Euclidean polyhedra. We are given the p.l. map $f\colon M_1 \to M_2$. Choose simplicial subdivisions, also called M_1 and M_2, so that f is simplicial and choose first deriveds so that $f\colon M_1' \to M_2'$ is also simplicial. Recall from the proof of Lemma 3.1 that f is onto.

By the Simplicial Factorization Lemma and the fact that f is onto, we may write $f = f_n \ldots f_0$ where $f_i\colon P_i \to P_{i+1}$ (same notation as (2.1)) and where for each simplex $\alpha^i < M_2$, f_i satisfies

$$\begin{aligned} &f_i[(\dot{\alpha}^i)' * D(\alpha^i, f)] = (\alpha^i)' * \dot{D}(\alpha^i, f), \\ &f_i \mid [(\dot{\alpha}^i)' * \dot{D}(\alpha^i, f)] = 1. \end{aligned}$$

Using Lemma (3.1) we may construct mappings h_i: $P_i \to P_{i+1}$ such that

$$h_i\,[(\dot{\alpha}^i)' * D(\alpha^i, f)] = (\alpha^i)' * \dot{D}(\alpha^i, f),$$

$$h_i|[(\dot{\alpha}^i)' * \dot{D}(\alpha^i, f)] = 1,$$

and

$$h_i|[(\dot{\alpha}^i)' * D(\alpha^i, f)] \quad \text{is a homeomorphism.}$$

Note that h_i is onto because every simplex in P_{i+1} lies in some $(\mathring{\beta}^{i+1})' * D(\beta^{i+1}, f)$. So for some $\alpha^i < \beta^{i+1}$ this simplex lies in

$$[(\alpha^i)' * D(\beta^{i+1}, f)] < [(\alpha^i)' * \dot{D}(\alpha^i, f)] < [\text{image } h_i].$$

Also h_i^{-1}: $P_{i+1} \to P_i$ is a well defined mapping because h_i is a homeomorphism on each complex of the form $(\dot{\alpha}^i)' * D(\alpha^i, f)$ and the image of this – $(\alpha^i)' * \dot{D}(\alpha^i, f)$ – meets any other such image only in $(\dot{\alpha}^i)' * \dot{D}(\alpha^i, f)$, where h_i is the identity. Setting $h = h_n \ldots h_0$: $M_1 \to M_2$ we arrive at the desired homeomorphism.

It remains to be shown that h can be made an ε-approximation to f by choosing the triangulations fine enough. In fact, it suffices to choose simplicial subdivisions of M_1 and M_2 under which f is simplicial and which have the following property:

For every $x \in M_1$ and for any chain of $(n+2)$ n-simplices $A_{-1}, A_0, A_1, \ldots, A_n$ of M_2 such that $f(x) \in A_{-1}$ and such that $A_i \cap A_{i+1} \neq \emptyset$ $(-1 \leqq i \leqq n-1)$, the diameter of $\bigcup A_i$ is less than $\varepsilon(x)$.

To see that such a choice of subdivisions is possible, start with arbitrary simplicial subdivisions such that f is simplicial. Order the simplices of M_1 as $\sigma_1, \sigma_2, \ldots$. Let $K_i = N(f(\sigma_i), M_2)$. Note that $\{K_i\}$ is a locally finite cover of M_2 by finite complexes, since f is proper and onto. Subdivide the complex K_i to K_{i*} where K_{i*} has mesh less than

$$\left(\frac{1}{n+2}\right) \cdot (\min\{\varepsilon(x) | x \in \sigma_i\})$$

and where no chain of length less than $n+3$ meets both $|f(\sigma_i)|$ and $|\dot{N}(f(\sigma_i))|$. Now let $\overline{M}_2$ be a subdivision of M_2 which contains a subdivision of each K_{i*} as subcomplex (this is possible by simple modifications of the results for compact polyhedra in Chapter 1 of [14]). Finally, since f: $M_1 \to \overline{M}_2$ is a proper p.l. map, choose subdivisions M_{1*} and M_{2*} such that f: $M_{1*} \to M_{2*}$ is simplicial. One easily checks that these subdivisions have the desired property. From now on we shall refer to these complexes M_{1*}, M_{2*} simply as M_1, M_2.

To see that such a choice of subdivision yields an ε-approximation consider the simplicial mapping p: $C_f \to M_2'$ (where C_f is the simplicial mapping cylinder defined in § 2) defined by

$$p(\hat{\alpha}_0 \ldots \hat{\alpha}_k\, \hat{\sigma}_{k+1} \ldots \hat{\sigma}_q) = \hat{\alpha}_0 \ldots \hat{\alpha}_k\, f(\hat{\sigma}_{k+1}) \ldots f(\hat{\sigma}_q).$$

Notice that $p|M'_1=f$ and $p|M'_2=1$. Notice also that

$$p[(\dot{\alpha}^i)'*D(\alpha^i,f)]=(\dot{\alpha}^i)'*D(\alpha^i,M_2)=(\alpha^i)'*\dot{D}(\alpha^i,M_2)$$
$$=p[(\alpha^i)'*\dot{D}(\alpha^i,f)]=p\,h_i[(\dot{\alpha}^i)'*D(\alpha^i,f)].$$

Thus, if $y\in(\dot{\alpha}^i)'*D(\alpha^i,f)$, both $p(y)$ and $p\,h_i(y)$ are in $(\dot{\alpha}^i)'*D(\alpha^i,M_2)=N(\hat{\alpha}^i,M'_2)$. So there are n-simplexes A_{i-1} and A_i of M'_2 which contain $p(y)$ and $p\,h_i(y)$ respectively and have non-empty intersection. In fact, if $h_i(y)\neq y$ then $p(y)$ and $p\,h_i(y)$ do not lie in $(\dot{\alpha}^i)'*\dot{D}(\alpha^i,M_2)=\dot{N}(\hat{\alpha}^i,M'_2)$, so *any* two n-simplices containing $p(y)$ and $p\,h_i(y)$, respectively, must have $\hat{\alpha}^i$ in their intersection. Now, if $x\in M_1$ and we apply the above observation inductively to the points $y=x$, $y=h_0(x)$, $y=h_1\,h_0(x)$, etc., we obtain a chain of simplexes $A_{-1},A_0,\ldots,A_n$ such that $f(x)=p(x)\in A_{-1}$ and $h(x)=h_n\ldots h_1\,h_0(x)\in A_n$. Since, by the choice of our subdivision, $\bigcup A_i$ has diameter less than $\varepsilon(x)$, this proves that $h(x)$ is an $\varepsilon(x)$ approximation to $f(x)$.

§ 4. Simplicial Factorization and the Resolution of Singularities

One means (long known to Sullivan but different from his treatment in [13]) of viewing Sullivan's resolution of singularities of a homotopy manifold is as follows. Suppose that M is a homotopy manifold (without boundary for convenience) for which we wish to build a resolution. Assume that M is triangulated and that M' is a first derived. Let M_0 be the dual cell complex, $\{D(\alpha,M)|\alpha<M\}$. *If there were* a resolution $f\colon V\to M$ with V a combinatorial manifold and f simplicial then $\{D(\alpha,f)|\alpha<M\}$ would be a cell complex formally isomorphic to M_0 and each "cell" $D(\alpha,f)$ would be a compact, contractible combinatorial manifold with homotopy sphere boundary (see [1], § 5, § 11). Moreover, by the Simplicial Factorization Lemma, f could be effected by successively squeezing (in order of increasing i) the spines $f^{-1}\hat{\alpha}^i$ of the "cells" $D(\alpha^i,f)$ to points and joining this map with the identity on $\dot{\alpha}'$.

Let us try to build a resolution by reversing this process – i.e., by successively "blowing up" (in order of decreasing i) the cone points of the dual cells $D(\alpha^i,M)$ in order to make the dual cells into contractible combinatorial manifolds.

The dual cells of dimension $\leqq 3$ are already contractible combinatorial manifolds (indeed balls) so they need not be touched. So set $M=M_0=M_1=M_2=M_3$. A 4-dimensional dual cell $D(\alpha,M)=\hat{\alpha}*\dot{D}(\alpha,M)$, $\alpha=\alpha^{n-4}$, is the cone on a homotopy 3-sphere which is itself a homotopy 3-manifold – hence a p.l. 3-manifold. Set $\dot{D}(\alpha,M)=\Sigma^3_\alpha$. Let us assume that Σ^3_α bounds a compact, contractible, combinatorial 4-manifold, Q^4_α. (We shall come back to this point later.) We change M_3 by cutting out $D(\alpha,M)$ and sewing in Q^4_α. (This amounts to "blowing $\hat{\alpha}$ up" to the spine of Q_α^4.) More precisely, we build the new space

$$(M-\operatorname{Int}[D(\alpha,M)*\dot{\alpha}'])\cup(Q_\alpha^4*\dot{\alpha}')$$

where the trivial identifications are made along $\Sigma_\alpha^3*\dot{\alpha}'$. Having done this for all 4-dimensional dual cells independently (which is possible since $\operatorname{Int}(N(\hat{\alpha}^{n-4},M'))\cap\operatorname{Int}(N(\hat{\beta}^{n-4},M'))=\emptyset$ if $\alpha^{n-4}\neq\beta^{n-4}$) we arrive at a new cell complex M_4.

Now M_4 contains the simplicial complex $(M^{n-5})'$ and it is still true that each barycenter $\hat{\alpha}^{n-5}$ has a stellar neighborhood of the form $N(\hat{\alpha},M_4)=\alpha'*\Sigma_\alpha^4$. Only now Σ_α^4 is not $\dot{D}(\alpha,M)$ but is the homotopy 4-sphere which resulted from excising the 4-dimensional cells of the cell complex $\dot{D}(\alpha,M)$ and sewing in 4-dimensional combinatorial manifolds. While it might seem ludicrous to think that Σ_α^4 is a combinatorial manifold just because its cells are such, this turns out to be true. {The 4-skeleton of M_4 is easily seen to have a p.l. manifold neighborhood. This allows one to find an open set in $\operatorname{Int}(\alpha'*\Sigma_\alpha^4)$, very close to Σ_α^4, which is a p.l. manifold and is p.l. homeomorphic to $\Sigma_\alpha^4\times R^{n-4}$. This implies that Σ_α^4 is a p.l. manifold.}[3] Then, since $\Theta_4=0$, there is a contractible, combinatorial 5-manifold Q_α^5 with $\Sigma_\alpha^4=\partial Q_\alpha^5$. So for each $(n-5)$-simplex α we cut out $(\alpha'*\Sigma_\alpha^4)$ and sew in $(\dot{\alpha}'*Q_\alpha^5)$ to form M_5.

We proceed in this fashion, inductively building M_j such that the j-skeleton in the "cell" structure of M_j has a p.l. manifold neighborhood. For $j\geqq 5$ no trouble occurs because the homotopy spheres Σ_α^j are real p.l. spheres by the Poincaré conjecture. In fact the construction gives $M_5=M_6=\cdots=M_n$ and $M_n\equiv V$ is a p.l. manifold because its n-skeleton has a p.l. manifold neighborhood. Clearly the process gives a p.l. surjection $f_k\colon M_{k+1}\to M_k$ with compact, contractible point inverses, and $f=f_n\ldots f_0\colon V\to M$ is a resolution of M.

The only hole in the preceding sketch is that the spheres Σ_α^3 might not bound contractible combinatorial manifolds. This leads to an obstruction theory as follows. Orient all the dual cells arbitrarily. This then gives a block dissection of $M_3=M$ and a chain complex $C(M)$ generated by the dual cells. Let $c\colon C_4(M)\to\Theta_3$ be given by

$$c(D(\alpha,M))=[\Sigma_\alpha^3]$$

where Σ_α^3 is the homotopy 3-sphere $\dot{D}(\alpha,M)$ with induced orientation and $[\Sigma_\alpha^3]$ is the h-cobordism class of Σ_α^3 in Θ_3. If c is the 0-cochain, each Σ_α^3 bounds a contractible combinatorial manifold and we are done. In general, c is conceivably not 0. But c is always a cocycle and this is a typical obstruction theory. If $\langle c\rangle=0\in H^4(M;\Theta_3)$ then the 3-dimensional cells in M (which are balls and were previously left untouched) may be

[3] This is not the simplest proof for Σ_α^4, but it is simplest for Σ_α^i ($i\geqq 4$), later in the induction.

replaced by new compact contractible combinatorial manifolds, so that in the new complex M_3^*, the cochain occurring is indeed 0.

Finally, suppose that $f: V \to M$ is a resolution and that $f: V \to M$ and $f: V' \to M'$ are simplicial. Then the cochain $\bar{c}: C_4(M) \to \Theta_3$ given by $\bar{c}(D(\alpha^{n-4}, M)) = [\dot{D}(\alpha^{n-4}, f)]$ is the 0 cochain since $\dot{D}(\alpha^{n-4}, f)$ bounds the contractible combinatorial manifold $D(\alpha^{n-4}, f)$. One can show that $c - \bar{c}$ is a coboundary because $\dot{D}(\alpha^{n-4}, M)$ and $\dot{D}(\alpha^{n-4}, f)$ have essentially the same 2-skeleton. Hence, if there is a resolution, $\langle c \rangle = 0$. Thus the vanishing of this cocycle is necessary and sufficient for the existence of a resolution.

References

1. Cohen, M. M.: Simplicial structures and transverse cellularity. Ann. of Math. **85**, 218–245 (1967).
2. – A general theory of relative regular neighborhoods. Trans. A.M.S. **136**, 189–229 (1969).
3. Glaser, L. C.: On the double suspension of certain homotopy 3-spheres. Ann. of Math. **85**, 494–507 (1967).
4. – To appear.
5. Homma, T.: P.l. approximations of embeddings of manifolds. Florida State U., 1965. (Mimeographed.)
6. Kervaire, M., Milnor, J.: Groups of homotopy spheres. Ann. of Math. **77**, 504–537 (1963).
7. Luft, E.: On contractible open topological manifolds. Inventiones Math. **4**, 192–201 (1967).
8. Morton, H. R.: Joins of polyhedra. To appear in Topology.
9. Newman, M. H. A.: The engulfing theorem for topological manifolds. Ann. of Math. **84**, 555–571 (1966).
10. Siebenmann, L. C.: Are nontriangulable manifolds triangulable? Proceedings, 1969 Georgia Topology Conference. To appear.
11. Smale, S.: A Vietoris mapping theorem for homotopy. Proc. A. M. S. **8**, 604–610 (1957).
12. Spanier, E. H.: Algebraic topology. New York, N.Y.: McGraw Hill Book Co. 1966.
13. Sullivan, D.: Lecture at the 1969 Georgia Topology Conference (paper to appear).
14. Zeeman, E. C.: Seminar on combinatorial topology. Institut des Hautes Etudes Scientifiques, 1963. (Mimeographed.)
15. Siebenmann, L. C.: Approximating cellular maps by homeomorphisms. Notices A. M. S., **17**, 532 (1970).

Marshall M. Cohen
Dept. of Mathematics
White Hall
Cornell University
Ithaca, New York 14850, USA

(Received January 16, 1970)