

# Inversive Localization in Noetherian Rings

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## 1. Introduction

Localization is a most useful tool in commutative algebra: With every prime ideal  $\mathfrak{p}$  of a commutative ring  $R$  one associates a local ring  $R_{\mathfrak{p}}$  whose residue class field is  $Q(R/\mathfrak{p})$ , the field of fractions of the integral domain  $R/\mathfrak{p}$ , together with a homomorphism  $\lambda: R \rightarrow R_{\mathfrak{p}}$  such that the accompanying diagram commutes. The kernel of  $\lambda$  is the  $C$ -component of 0, where  $C$  is the

$$\begin{array}{ccc} R & \xrightarrow{\lambda} & R_{\mathfrak{p}} \\ \downarrow & & \downarrow \\ R/\mathfrak{p} & \longrightarrow & Q(R/\mathfrak{p}) \end{array}$$

complement of  $\mathfrak{p}$  in  $R$ , as is well known and easily verified. When  $R$  is Noetherian,  $\ker \lambda$  may also be characterized as the intersection of all the primary components of 0 associated with prime ideals contained in  $\mathfrak{p}$  (cf. [12], Chapter IV).

Now the process of forming fractions has been generalized to non-commutative rings in a number of ways. The simplest is Ore's method (see e.g. [2]), but this is of limited applicability. Another method, much studied lately, depends on forming injective hulls; it is usually expressed within the framework of torsion theories (cf. [6], [7], [11]). It leads to a generalized quotient ring which reduces to Ore's construction whenever the latter is applicable, and like the latter it is not left-right symmetric. We shall call this the *injective* method. A second way of generalizing Ore's method is to invert matrices rather than elements. This allows one to obtain an explicit form for the quotient ring produced; unlike the injective method it leads to actual inverses and is left-right symmetric, but in general it is more difficult to determine the kernel of the canonical mapping. This may be called the *inversive* method; it is described in [3]. If instead of inverting matrices, we merely make them right invertible, we obtain an intermediate method, which may be called *semi-inversive*.

It is natural to try to apply these methods to obtain a non-commutative localization process. For a Noetherian (semi)prime ring, Ore's method can always be applied to yield a (semi)simple Artinian quotient ring (Goldie's theorem), but this is no longer so when we try to localize at a prime ideal of a Noetherian ring. A number of ways of performing such a localization have

been proposed; in particular, Goldie [4] has given a construction for prime ideals in Noetherian rings which exploits the  $p$ -adic topology, giving rise to a "topological localization", while Lambek and Michler in [8], [9] construct a quotient ring by means of a torsion theory. This "injective localization" has the advantage that the kernel of the canonical mapping can be more easily determined, but the resulting ring need not be a local ring.

The object of this note is to show how the inversive method may be applied to obtain a localization at a semiprime ideal  $\mathfrak{n}$  of a Noetherian ring  $R$ . The ring thus obtained is always a semilocal ring whose residue class ring is the classical quotient ring of  $R/\mathfrak{n}$  (Section 4). This is followed by some illustrative examples in Section 5. We begin with a general discussion of the torsion theory (generally non-hereditary) that can be associated with a multiplicative set of matrices (Section 2), and in Section 3 obtain a general theorem on the relation between localizing and forming factor rings, from which our main results follow almost immediately.

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## 2. Multiplicative Matrix Sets and Their Torsion Theories

Throughout, all rings are associative with a unit element which is preserved by homomorphisms, inherited by subrings, and acts unittally on modules.

Let  $R$  be a ring, then a set  $\Sigma$  of square matrices over  $R$  is said to be *multiplicative* if  $1 \in \Sigma$  and, for any  $A, B \in \Sigma$  and any matrix  $C$  of the right size,  $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \Sigma$ . Thus  $\Sigma$  contains square matrices of all orders; we write  $\Sigma_n$  for the set of  $n \times n$  matrices in  $\Sigma$ . If  $\Sigma$  is multiplicative and moreover admits all elementary (row and column) transformations, it is called *admissible*. A homomorphism  $f: R \rightarrow S$  is said to be  $\Sigma$ -*inverting* if each matrix of  $\Sigma$  is mapped to an invertible matrix by  $f$ . It is easily seen (cf. [3]) that for any set  $\Sigma$  of square matrices over  $R$  there exists a ring  $R_\Sigma$  and a  $\Sigma$ -inverting homomorphism

$$(1) \quad \lambda: R \rightarrow R_\Sigma$$

which is universal in the sense that every  $\Sigma$ -inverting homomorphism  $f: R \rightarrow S$  can be factored uniquely by  $\lambda$ , i.e., there exists a unique homomorphism

$$\begin{array}{ccc} R & \xrightarrow{\lambda} & R_\Sigma \\ & \searrow & \downarrow \\ & & S \end{array}$$

$f': R_{\Sigma} \rightarrow S$  such that the accompanying triangle commutes. We observe that  $\lambda$  is always an epimorphism in the category of rings.

The ring  $R_{\Sigma}$  is called the *universal  $\Sigma$ -inverting ring*. If  $\Sigma$  is multiplicative, the elements of  $R$  may be obtained as the components of the solutions of the matrix equations

$$(2) \quad uA + a = 0,$$

where  $a$  is a row over  $R$  and  $A \in \Sigma$ .

The advantage of inverting matrices rather than elements is that the solutions of the equations (2) actually *form* (and not merely generate) the inverting ring. As is well known, in the case of elements this is so only for the denominator sets (sets satisfying the Ore condition). We recall that a *right denominator set* in a ring  $R$  is a subset  $S$  which includes 1, is multiplicatively closed, and is such that

- (i) for all  $a \in R, s \in S, aS \cap sR \neq \emptyset$ ,
- (ii) for all  $a \in R, s \in S$ , if  $sa = 0$ , then  $at = 0$  for some  $t \in S$ .

Let  $S$  be a right denominator set and let  $\Sigma$  be the multiplicative set of matrices generated by  $S$ ; thus  $\Sigma$  consists of all upper triangular matrices with elements of  $S$  on the main diagonal. Then we can form the ring of fractions  $R_S$  and the universal  $\Sigma$ -inverting ring  $R_{\Sigma}$ ; we claim that these rings are isomorphic. More precisely, there is an isomorphism  $\omega: R_S \rightarrow R_{\Sigma}$  such that the triangle shown

$$\begin{array}{ccc} & R & \\ \lambda' \swarrow & & \searrow \lambda \\ R_S & \longrightarrow & R_{\Sigma} \end{array}$$

commutes, where  $\lambda, \lambda'$  are the canonical mappings. This follows easily from the universal properties of these mappings; thus  $\lambda': R \rightarrow R_S$  is  $\Sigma$ -inverting and  $\lambda: R \rightarrow R_{\Sigma}$  is  $S$ -inverting.

We also recall that  $\ker \lambda'$  is the left  $S$ -component of 0 (cf. [2]):

$$(3) \quad \ker \lambda' = \{x \in R \mid xs = 0 \text{ for some } s \in S\}.$$

Given any set  $\Sigma$  of square matrices over  $R$ , let  $\bar{\Sigma}$  be the set of matrices of  $R$  that are inverted by the canonical mapping  $\lambda: R \rightarrow R_{\Sigma}$ . This set  $\bar{\Sigma}$  is called the *saturation* of  $\Sigma$ , and if  $\Sigma = \bar{\Sigma}$ , we call  $\Sigma$  *saturated*. From the universal properties one sees that, for any  $\Sigma$ ,

$$(4) \quad R_{\Sigma} \cong R_{\bar{\Sigma}}.$$

It is clear that the set  $\Sigma$  is admissible (see [3], p. 249 for the multiplicative case); hence the multiplicative closure of  $\Sigma$  and the admissible closure of  $\Sigma$

both have the same saturation as  $\Sigma$  itself, and by (4) all have isomorphic universal inverting rings.

We now come to torsion theories. As is well known, with any right denominator set a torsion theory may be associated (cf. [11]). Instead of a denominator set we can also start with a multiplicative matrix set, and obtain a torsion theory as before:

**PROPOSITION 2.1.** *Let  $R$  be a ring and  $\Sigma$  a multiplicative set of matrices over  $R$ . For any right  $R$ -module  $M$  denote by  $t(M)$  the set of elements of  $M$  occurring as component in a row  $u$  such that  $uA = 0$  for some  $A \in \Sigma$ . Then the correspondence  $M \mapsto t(M)$  is an idempotent radical. Further, both  $\Sigma$  and its admissible closure give rise to the same radical.*

The first assertion means that  $t(M)$  is a submodule of  $M$  and the assignment  $M \mapsto t(M)$  is an idempotent subfunctor of the identity such that  $t(M/t(M)) = 0$  (cf. [11]).

**Proof:** Let  $x, y \in t(M)$ , say  $x = u_1, y = v_1$ , where  $uA = 0, vB = 0$  and  $A, B \in \Sigma$ . Denote the first row of  $B$  by  $b_1$  and write  $v'$  for the row  $v$  with its first element removed, then

$$(u, v_1 - u_1, v') \left( \begin{array}{c|c} A & b_1 \\ \hline O & B \end{array} \right) = 0;$$

hence  $x - y \in t(M)$ , and a similar argument applies if  $x, y$  occur in places other than the first. Secondly, if  $uA = 0$ , then, for any  $c \in R$ ,

$$(u, uc) \left( \begin{array}{cc} A & -cA \\ O & A \end{array} \right) = 0,$$

and this shows that if  $u_1 \in t(M)$ , then  $u_1 c \in t(M)$ , so that  $t(M)$  is a submodule. Clearly any homomorphism  $M \rightarrow N$  maps  $t(M)$  into  $t(N)$ , and the correspondence  $M \mapsto t(M)$  is easily seen to be a functor. It is also clear that

$$t(t(M)) = t(M).$$

Next we show that  $t(M)$  is unchanged if we replace  $\Sigma$  by its 'admissible closure', i.e., the set of all matrices  $PAQ$ , where  $A \in \Sigma$  and  $P, Q$  are products of elementary matrices. For, given  $c \in R$ , the pair of elements  $u_1, u_2 + u_1 c$  lies in  $t(M)$  if and only if  $u_1$  and  $u_2$  lie in  $t(M)$ ; thus  $u \cdot A = 0$  is equivalent to  $uP^{-1} \cdot PA = 0$ , where  $P$  is an elementary matrix, and by induction this holds for a product of elementary matrices. Of course right multiplication of

$A$  by  $Q$  presents no difficulty. Thus in what follows we may take  $\Sigma$  to be admissible.

It remains to prove the radical property:  $t(M/t(M)) = 0$ . If

$$uA \equiv 0 \pmod{t(M)},$$

where  $A \in \Sigma$ , write  $uA = v'$  and let  $v = (v', v'')$  be a row including all components of  $v'$  such that  $vB = 0$  for some  $B \in \Sigma$ . We can take  $v$  in this special form because  $\Sigma$  is admissible. Then

$$(v, u, v'') \left( \begin{array}{c|cc} B & -I & \\ \hline & A & 0 \\ 0 & 0 & I \end{array} \right) = 0,$$

and this shows that  $u \equiv 0 \pmod{t(M)}$ . Hence  $t(M/t(M)) = 0$ , and the proof is complete.

When we take  $M = R$ ,  $t(R)$  is just the 'left  $\Sigma$ -component of 0', i.e., the set of elements of  $R$  occurring as component in some row  $u$  such that  $uA = 0$  for some  $A \in \Sigma$ . We have seen in Proposition 2.1 that this is a right ideal, and it is clearly also a left ideal, so we obtain the

**COROLLARY.** *The left  $\Sigma$ -component of 0 is a two-sided ideal of  $R$ .*

By the symmetry of the construction the same holds for the right  $\Sigma$ -component of 0.

The functor  $t(M)$  can now be used to define a torsion theory, as described e.g. in [11]; we refer to this as the  $\Sigma$ -torsion theory. In general this need not be hereditary, i.e., a submodule of a  $\Sigma$ -torsion module need not be a  $\Sigma$ -torsion module. However, let us assume that  $\Sigma$  is such that the  $\Sigma$ -torsion theory is hereditary. Then for any  $x \in t(M)$  there exist  $u_2, \dots, u_n \in R$  and  $A \in \Sigma_n$  such that  $(x, xu_2, \dots, xu_n)A = 0$ , since this expresses the fact that  $xR$  is a  $\Sigma$ -torsion module containing  $x$ . More generally, let us define a *unimodular row* over  $R$  as a row  $u$  of elements of  $R$  such that for a suitable column  $u'$  over  $R$  we have  $uu' = 1$ . Then we have

**PROPOSITION 2.2.** *Let  $R$  be a ring and  $\Sigma$  a multiplicative matrix set such that the  $\Sigma$ -torsion theory is hereditary. Then, for any right  $R$ -module  $M$ , the  $\Sigma$ -torsion submodule  $t(M)$  consists of all  $x \in M$  such that*

$$(5) \quad xuA = 0$$

*for some  $A \in \Sigma$  and some unimodular row  $u$  over  $R$ .*

Proof: Let  $x \in t(M)$ ; then  $xR$  is a  $\Sigma$ -torsion module containing  $x$  and this means that (5) holds for some  $A \in \Sigma$  and a row  $u$  one of whose components is 1; thus it is certainly unimodular. Conversely, if (5) holds with a unimodular row  $u$ , then  $xu_i \in t(M)$  and if  $uu' = 1$ , then  $t(M)$  also contains  $xuu' = x$ .

The hereditary torsion theories so obtained form a special class; they are the theories obtainable by flat epimorphisms. Given a ring homomorphism  $f: R \rightarrow S$ , we shall call a row  $u$  over  $R$  *f-unimodular* if its image under  $f$  is unimodular in  $S$ . A homomorphism  $f: R \rightarrow S$  is a left flat epimorphism (i.e.,  $f$  is a ring epimorphism such that  ${}_R S$  is flat) if and only if for each  $a \in \ker f$  there exists an *f-unimodular* row  $u$  such that  $a \cdot u = 0$  and for each  $b \in S$  there is an *f-unimodular* row  $u$  such that  $b \cdot u' \in \text{im } f$  (see [11], p. 78).

Now consider a ring  $R$  and a family  $\Pi$  of rows over  $R$  such that, for any right  $R$ -module  $M$ , the set  $t(M)$  of elements annihilated by some row of  $\Pi$  forms a submodule and that  $t$  is an idempotent radical subfunctor of the identity. Such a  $t$  then defines a torsion theory, this time necessarily hereditary, and the quotient  $Q(M)$  of any module  $M$  may be constructed as follows: form  $M_0 = M/t(M)$  and define  $Q(M)$  by the equation

$$(6) \quad Q(M)/M_0 = t(I(M_0)/M_0),$$

where  $I(M)$  is the injective hull of  $M$  (cf. [7]). Regarded as  $R$ -module,  $Q(M)$  is closed; we recall that an  $R$ -module  $N$  is *closed* if  $t(N) = 0$  and  $t(I(N)/N) = 0$  (such modules are called 'torsion free divisible' by Lambek).

In particular,  $Q(R)$  is the ring obtained by first dividing out by  $t(R)$ , so as to get  $R_0 = R/t(R)$ , and then taking the set of those elements of  $I(R_0)$  that are mapped to a row over  $R_0$  by right multiplication by a row of  $\Pi$ .

If we apply this method to a non-hereditary torsion theory, we can again define  $Q(M)$  by (6), but there will be no guarantee now that  $Q(M)$  is closed (clearly this will be so provided that every essential extension of a torsion free module is torsion free). It is not known whether the torsion theory associated with every multiplicative matrix set has this property, but even if it does not, we can form the module  $M \bigotimes R_\Sigma$ . Before comparing this with  $Q(M)$  (when this is closed), we introduce an intermediate notion, the semi-inversive theory.

For any set  $\Sigma$  of matrices over  $R$  we can form the right  $\Sigma$ -inverting ring  $R_\Sigma$ , with canonical homomorphism  $\mu: R \rightarrow R_\Sigma$ . This is the universal ring over  $R$  in which every matrix of  $\Sigma$  has a right inverse; it is obtained by taking a presentation of  $R$  and for every matrix  $A$  of order  $n$  adjoining  $n^2$  indeterminates  $a'_{ij}$ , written as an  $n \times n$  matrix  $A' = (a'_{ij})$  with defining relations (in matrix form)  $AA' = I$ . Since every matrix of  $\Sigma$  has a unique inverse in  $R_\Sigma$ , the canonical mapping  $\lambda: R \rightarrow R_\Sigma$  can be factored uniquely by  $\mu$ . The passage from  $M$  to  $M \bigotimes R_\Sigma$ , for some set  $\Sigma$  of matrices, is a form of localization which we shall call the *semi-inversive* method.

Here we have confined ourselves to square matrices throughout, but the semi-inversive method can more generally be applied to sets of rectangular matrices; indeed Proposition 2.1 can be generalized to such sets, but we shall not pursue the matter here.

Let us compare the ring  $R_\Sigma$  with the ring  $Q(R)$  obtained by the injective method (in case this has closed quotients). Any right  $R_\Sigma$ -module  $P$  may be regarded as a right  $R$ -module, by pullback along  $\mu$ . We claim that this module  $P_R$  is closed in the  $\Sigma$ -torsion theory: if  $u$  is a row in  $P$  such that  $uA = 0$  for some  $A \in \Sigma$ , then  $u = uAA' = 0$ ; hence  $P_R$  is torsion free. Now let  $u$  be a row in  $I(P_R)$  such that  $v = uA$  is a row in  $P$ , for some  $A \in \Sigma$ . Then  $u = uAA' = vA'$  is a row in  $P$ ; hence  $\iota(I(P)/P) = 0$ , and  $P$  is indeed closed. In particular, this applies to any module of the form  $M \bigotimes R_\Sigma$ , where  $M$  is a right  $R$ -module. By the universal property of  $Q(M)$ , this leads to the commutative triangle shown:

$$\begin{array}{ccc} & M & \\ \swarrow & & \searrow \\ Q(M) & \longrightarrow & M \bigotimes R_\Sigma \end{array}$$

In particular, for  $M = R$  we obtain a canonical homomorphism

$$(7) \quad Q(R) \rightarrow R_\Sigma.$$

By interpreting the elements of  $Q(R)$  and  $R_\Sigma$  as left multiplications we see that (7) is in fact a ring homomorphism. If we combine the above triangle with the canonical homomorphism from  $M \bigotimes Q(R)$  to  $Q(M)$  (see [11], p. 72), we obtain the commutative diagram shown:

$$\begin{array}{ccc} M & \longrightarrow & Q(M) \\ \downarrow & & \downarrow \\ M \bigotimes Q(R) & \longrightarrow & M \bigotimes R_\Sigma \end{array}$$

When (7) is an isomorphism, the bottom arrow is an isomorphism for all  $R$ -modules  $M$ . Conversely, if this arrow is an isomorphism for all  $M$ , then by taking  $M = R$  we see that the mapping (7) is an isomorphism.

### 3. Inverse Localization at a Factor Ring

Let  $R$  be a ring and  $\mathfrak{a}$  an ideal in  $R$ ; our object in this section is to lift a localization of  $R/\mathfrak{a}$  to one of  $R$ . We begin with some general properties of multiplicative matrix sets. We recall that the set of all matrices inverted under a homomorphism is a saturated, hence admissible, set. To ensure that only

square matrices occur we shall assume our rings to be weakly finite; by definition, a ring is *weakly finite* if every invertible matrix over it is square.

Let  $f: R \rightarrow S$  be any homomorphism and let  $\Sigma_f$  be the set of all square matrices over  $R$  mapped to invertible matrices by  $f$ . We shall write  $R_f$  instead of  $R_{\Sigma_f}$  and call this the *localization associated with the homomorphism  $f$* . Thus with every homomorphism  $f: R \rightarrow S$  there is associated a ring  $R_f$  and a commutative triangle as shown:

$$\begin{array}{ccc} & R & \\ \lambda \swarrow & & \searrow f \\ R_f & \xrightarrow{f} & S \end{array}$$

**THEOREM 3.1.** *Let  $f: R \rightarrow S$  be a homomorphism, and  $R_f$  the associated localization; then any square matrix over  $R_f$  mapped to an invertible matrix over  $S$  is already invertible over  $R_f$ .*

**Proof:** Any  $x \in R_f$  occurs as a component  $x = u_1$ , say, in an equation

$$(8) \quad Au = a,$$

where  $A \in \Sigma$  and  $a$  is a column over  $R$ . Write

$$A = (a_1, \dots, a_n), \quad A_1 = (a, a_2, \dots, a_n);$$

then  $xf$  is invertible in  $S$  if and only if  $A_1f$  is invertible over  $S$  (by Cramer's rule; see [3], Proposition 7.1.3), and this is so if and only if  $A_1 \in \Sigma$ ; but then  $x$  is invertible in  $R_f$ . This proves the result for  $1 \times 1$  matrices. In the general case, of an  $n \times n$  matrix, consider the corresponding triangle of matrix rings: Any matrix over  $R_n$  that becomes invertible over  $S_n$  lies in  $\Sigma$  and so becomes invertible over  $(R_f)_n$ ; hence any element of  $(R_f)_n$  whose image is invertible in  $S_n$  is already invertible in  $(R_f)_n$ .

We recall that the *rational closure* of  $R$  in  $S$  (under the mapping  $f$ ) is the set of all solutions of equations (8) with  $A \in \Sigma_f$ ; in particular, if this is the whole of  $S$ , the latter is said to be *matrix-rational* over  $R$ . Below we denote the Jacobson radical of a ring  $R$  by  $J(R)$ .

**COROLLARY.** *Let  $f: R \rightarrow S$  be a homomorphism such that  $S$  is matrix-rational over  $R$ ; then  $f: R_f \rightarrow S$  is surjective and  $J(R_f) = f^{-1}(J(S))$ .*

**Proof:** The surjectivity is clear, and it implies that  $J(R_f) \subseteq f^{-1}(J(S))$ . Conversely, let  $af \in J(S)$ ; then, for any  $x \in R_f$ ,  $1 - ax$  has an invertible image in  $S$  and so is invertible in  $R_f$ , and hence  $a \in J(R_f)$ .



As a further consequence we note

$$(9) \quad R_f/J(R_f) \cong S/J(S) .$$

To describe the form of the localization we recall the definition of a 'local ring': Let  $R$  be a ring and  $J = J(R)$  its Jacobson radical. If  $R/J$  is a (skew) field,  $R$  is called a *local ring*; if  $R/J$  is simple Artinian (and hence, by Wedderburn's theorem, a total matrix ring over a field),  $R$  is called a *matrix local ring*. The previous case is the special case of  $1 \times 1$  matrices, also called a *scalar local ring* for emphasis. If  $R/J$  is Artinian (and hence a direct product of a finite number of total matrix rings over fields),  $R$  is called a *semilocal ring*. In each case  $R/J$  is called the *residue class ring*.

In particular, if  $S$  is a field, a simple or semisimple Artinian ring, then  $R_f$  is a local, matrix local or semilocal ring, respectively, with residue class ring  $S$ .

The next result shows that localizing and going to factor rings are commutative operations.

**THEOREM 3.2.** *Let  $R$  be a ring,  $\mathfrak{a}$  an ideal and  $\Sigma$  a set of square matrices over  $R/\mathfrak{a}$ . Write  $S = (R/\mathfrak{a})_\Sigma$ , let  $f: R \rightarrow S$  be the canonical mapping and  $f_f: R_f \rightarrow S$  the induced mapping from the localization. Then  $\ker f = (\mathfrak{a}\lambda)$ , the ideal of  $R_f$  generated by  $\mathfrak{a}\lambda$ , and*

$$S = (R/\mathfrak{a})_\Sigma \cong R_f/(\mathfrak{a}\lambda) .$$

**Proof:** Since  $f_f: R_f \rightarrow S$  annihilates  $\mathfrak{a}\lambda$ , it induces a mapping  $g: R_f/(\mathfrak{a}\lambda) \rightarrow S$ .

$$\begin{array}{ccc} R & \xrightarrow{\lambda} & R_f \\ \downarrow & & \downarrow \\ R/\mathfrak{a} & \longrightarrow & R_f/(\mathfrak{a}\lambda) \\ & \searrow g & \nearrow h \\ & S & \end{array}$$

On the other hand, denote by  $\Sigma'$  the inverse image of  $\Sigma$  in  $R$ . Over  $S$  the elements of  $\Sigma$  are invertible; hence  $\Sigma'$  becomes invertible over  $R_f$ . Thus  $R_f/(\mathfrak{a}\lambda)$  is an  $R$ -ring in which  $\mathfrak{a}$  is mapped to 0, and regarding it as an  $(R/\mathfrak{a})$ -ring we see that the elements of  $\Sigma$  become invertible over it. Therefore there is a mapping  $h: S \rightarrow R_f/(\mathfrak{a}\lambda)$ , and it is easily verified that  $g, h$  are inverse to each other.

#### 4. Application to Noetherian Rings

In a Noetherian ring, the results of the last section lead to a rather explicit form of localization if we use Goldie's theorem. We recall that an element  $c$  of a ring  $R$  is called *left regular* if  $cx = 0$  implies  $x = 0$ ; *right regular* elements

are defined similarly and a *regular* element is one which is left and right regular. A matrix set  $\Sigma$  is called (*left, right*) *regular* if for all  $n$  each matrix  $A \in \Sigma_n$  is (*left, right*) regular in the total matrix ring  $R_n$ .

In a right Noetherian semiprime ring, any left regular element is right regular (because it generates a large right ideal, cf. e.g. [5], p. 174), so there is no need to distinguish between left regular and regular in this case. This result carries over to matrices, because the matrix ring  $R_n$  is semiprime right Noetherian whenever  $R$  itself is.

Let  $R$  be a right Noetherian semiprime ring and  $\mathfrak{n}$  a semiprime ideal in  $R$ ; then  $R/\mathfrak{n}$  is a right Noetherian semiprime ring. A matrix  $A$  over  $R$  is said to be *left  $\mathfrak{n}$ -regular* if for any column  $u$  over  $R$  (of the right length)

$$Au \equiv 0 \pmod{\mathfrak{n}} \text{ implies } u \equiv 0 \pmod{\mathfrak{n}}.$$

*Right  $\mathfrak{n}$ -regularity* is defined similarly, and a matrix satisfying both is called  *$\mathfrak{n}$ -regular*. By what has been said, every left  $\mathfrak{n}$ -regular matrix over  $R$  is  $\mathfrak{n}$ -regular. We denote by  $\Gamma = \Gamma_{\mathfrak{n}}$  the set of all  $\mathfrak{n}$ -regular matrices over  $R$ . This set is admissible, for clearly  $1 \in \Gamma$  and if  $A, B \in \Gamma$ , and

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \equiv 0 \pmod{\mathfrak{n}},$$

then  $Au + Cv \equiv 0$  and  $Bv \equiv 0$ ; hence  $v \equiv 0$  and so  $Au \equiv 0$ , i.e.,  $u \equiv 0 \pmod{\mathfrak{n}}$ . Further, the elements of  $\Gamma$  admit elementary transformations, so  $\Gamma$  is in fact admissible.

We shall write  $Q_{cl}(R)$  for the quotient ring of  $R$  by the set of all regular elements, whenever the latter form a (*left or*) right denominator set (this ring is often called the 'classical' quotient ring). We also recall Goldie's theorem, in the following form:

*Let  $R$  be a right Noetherian (semi)prime ring; then the set  $S$  of regular elements in  $R$  is a right denominator set, and the quotient ring  $R_S$  is right Artinian (semi)simple.*

If we now apply Theorem 3.2 to a right Noetherian ring  $R$ , taking  $\mathfrak{a} = \mathfrak{n}$  to be a semiprime ideal of  $R$  and  $\Sigma = \Gamma_{\mathfrak{n}}$  the set of all  $\mathfrak{n}$ -regular matrices, we find that  $S = Q_{cl}(R/\mathfrak{n})$ , while  $R_f = R_{\Gamma}$  is the universal  $\Gamma$ -inverting ring because  $\Gamma$  is the precise set of matrices inverted over  $R_f$ . By Goldie's theorem,  $S$  is Artinian semisimple, and by (9),  $S = S/J(S) \cong R_f/J(R_f)$ . Hence  $R_f$  is a semilocal ring and we obtain our main result:

**THEOREM 4.1.** *Let  $R$  be a right Noetherian ring and  $\mathfrak{n}$  a semiprime ideal in  $R$ . Denote by  $\Gamma = \Gamma_{\mathfrak{n}}$  the set of all  $\mathfrak{n}$ -regular matrices over  $R$ . Then the universal  $\Gamma_{\mathfrak{n}}$ -inverting ring  $R_{\Gamma}$  is a semilocal ring with residue class ring  $Q_{cl}(R/\mathfrak{n})$ . In particular, when  $\mathfrak{n}$  is prime,  $R_{\Gamma}$  is a matrix local ring.*

The local rings constructed here are in general very much smaller than the quotient rings constructed by Lambek and Michler; as we saw in Section 2, their quotient ring has the universal right  $\Gamma$ -inverting ring  $R_\Gamma$  as a homomorphic image, so that we always have a homomorphism  $Q(R) \rightarrow R_\Gamma$ . But there is an important case where the two constructions coincide: We first note the following result from [8] (Proposition 5.5):

**LEMMA.** *Let  $R$  be a right Noetherian ring and  $\mathfrak{p}$  a prime ideal in  $R$ . Denote by  $C = C(\mathfrak{p})$  the set of  $\mathfrak{p}$ -regular elements of  $R$ , and by  $Q(R)$  the quotient ring with respect to the torsion theory defined by  $C$ . Then the following assertions are equivalent:*

- (a) *the elements of  $C$  map to units of  $Q(R)$ ,*
- (b) *the elements of  $C$  map to right-invertible elements of  $Q(R)$ ,*
- (c)  *$C$  is a right denominator set in  $R$ .*

**THEOREM 4.2.** *Let  $R$  be a right Noetherian ring and  $\mathfrak{p}$  a prime ideal of  $R$ , and denote by  $C = C(\mathfrak{p})$  the set of  $\mathfrak{p}$ -regular elements in  $R$ . Further, let  $Q(R)$  be the ring of right quotients at  $\mathfrak{p}$ , as in [8], page 381, and let  $R_\Gamma$  be the inversive localization at  $\mathfrak{p}$  constructed as in Theorem 4.1. Then there is a natural homomorphism*

$$(10) \quad Q(R) \rightarrow R_\Gamma,$$

*and this is an isomorphism if and only if  $C$  is a right denominator set in  $R$ .*

**Proof:** The homomorphism (10) is obtained by combining the mapping (7) with the natural mapping  $R_\Gamma \rightarrow R_\Gamma$ . If (10) is an isomorphism, then the elements of  $C$  are inverted in  $Q(R)$ , so  $C$  is a right denominator set, by the lemma. Conversely, when this is the case, the elements of  $C$  are inverted in  $Q(R)$ , and (10) is then an isomorphism, by the universal property of  $R_\Gamma$ .

## 5. Examples

An obvious shortcoming of the method of inversive localization is that the kernel of the natural mapping  $\lambda : R \rightarrow R_\Gamma$  is not explicitly determined. Taking  $\Gamma$  to be the set of  $\mathfrak{p}$ -regular matrices over  $R$ , for some prime ideal  $\mathfrak{p}$ , let us denote by  $l(\Gamma)$  and  $r(\Gamma)$  the left and right  $\Gamma$ -components of 0, respectively. By Corollary to Proposition 2.1, these sets are two-sided ideals, and it is clear that

$$(11) \quad l(\Gamma) + r(\Gamma) \subseteq \ker \lambda \subseteq \mathfrak{p},$$

but it is not known when equality holds. All we can say is that (i) the relation (11) provides bounds for  $\ker \lambda$ , and (ii)  $\ker \lambda$  represents the smallest ideal mapped to 0 by any  $\Gamma$ -inverting mapping.

As an illustration consider the following ring (in symbolic matrix notation) and semiprime ideal:

$$R = \begin{pmatrix} R_1 & M \\ 0 & R_2 \end{pmatrix}, \quad \mathfrak{n} = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix},$$

where  $R_1, R_2$  are Noetherian prime rings and  $M$  is an  $(R_1, R_2)$ -bimodule. Here  $\ker \lambda = l(\Gamma) + r(\Gamma)$ , as is easily checked. If moreover,  $l(\Gamma) + r(\Gamma) = M$ , then  $R_\Gamma = Q(R/\mathfrak{n}) = Q(R_1) \oplus Q(R_2)$ .

Our second example is taken from [8]: Let  $D$  be a commutative discrete valuation ring with maximal ideal  $\mathfrak{m}$  and field of fractions  $F$ . We put

$$R = \begin{pmatrix} D & \mathfrak{m} \\ D & D \end{pmatrix}, \quad \mathfrak{p} = \begin{pmatrix} \mathfrak{m} & \mathfrak{m} \\ D & D \end{pmatrix}.$$

Clearly,  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \Gamma$ ; hence  $\begin{pmatrix} 0 & 0 \\ D & D \end{pmatrix} \in r(\Gamma)$ , and the two-sided ideal generated is  $\mathfrak{p}$  itself. Thus  $r(\Gamma) = \mathfrak{p}$ , even though  $R$  is  $\mathfrak{p}$ -torsion free (cf. [8]). Of course, here  $R_\Gamma = Q(R/\mathfrak{p}) = D/\mathfrak{m}$ . We observe that  $R$  is itself prime, with quotient ring  $Q(R) = F_2$ .

Thirdly we quote an example from [1] of a Noetherian ring in which some non-zero elements are mapped to 0 by every inversive localization. We denote by  $C_n$  the cyclic group of order  $n$  and consider the ring  $R = \text{End}(C_4 \oplus C_2)$ . This ring is finite (it has 32 elements); hence it is Noetherian and moreover semilocal. In matrix form we may represent it as

$$R = \begin{pmatrix} A & M \\ N & B \end{pmatrix},$$

where  $A = \text{End}(C_4)$ ,  $B = \text{End}(C_2)$ ,  $M$  is an  $(A, B)$ -bimodule and  $N$  a  $(B, A)$ -bimodule, both isomorphic to  $C_2$  as additive groups. Further,

$$MN \subseteq 2A, \quad NM = 0.$$

This ring has two prime ideals:

$$\mathfrak{p}_1 = \begin{pmatrix} 2A & M \\ N & B \end{pmatrix}, \quad \mathfrak{p}_2 = \begin{pmatrix} A & M \\ N & 0 \end{pmatrix}.$$

Let  $\Gamma_i$  be the set of  $\mathfrak{p}_i$ -regular matrices ( $i = 1, 2$ ), then it is easily verified that  $l(\Gamma_i) = r(\Gamma_i) = \mathfrak{p}_i$ ; hence every element in  $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \begin{pmatrix} 2A & M \\ N & 0 \end{pmatrix}$  is mapped to 0 in the localization (10).

Thus although itself semilocal,  $R$  is not embedded in the localization constructed in Theorem 4.1.

Finally here is an example in which  $l(\Gamma) + r(\Gamma) \neq \ker \lambda$ . Let  $R$  be an algebra over a commutative field, on 8 generators  $a_{ij}$ ,  $b_{ij}$ ,  $i, j = 1, 2$ , with defining relations (in matrix form)

$$AB = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, \quad A = (a_{ij}), \quad B = (b_{ij}).$$

This is essentially Malcev's example of an integral domain not embeddable in a field (cf. [10]). If  $\Gamma$  is the multiplicative set generated by  $a_{22}$  and  $b_{11}$ , then  $\Gamma$  consists of triangular matrices, and since  $R$  is an integral domain, it follows that  $l(\Gamma) = r(\Gamma) = 0$ , but it may be verified (as in [10]) that  $c = a_{11}b_{12} + a_{12}b_{22} \in \ker \lambda$ . By adjoining another element  $t$  with the relation  $ta_{22} = 0$ , we obtain an example in which  $l(\Gamma) \neq r(\Gamma) = 0$ . Of course,  $R$  is not Noetherian in this example, but it is not even known whether, for a Noetherian ring  $R$  and a multiplicative set  $\Sigma$  of matrices, the localization  $R_\Sigma$  is necessarily Noetherian.

To get a better idea of the size of  $\ker \lambda$  one may have to use something like Proposition 7.1.3 of [3] (Cramer's rule), but it is far from clear whether this would provide a usable criterion.

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