Rings of Fractions

P. M. Cohn

The American Mathematical Monthly, Vol. 78, No. 6 (Jun. - Jul., 1971), 596-615.

Stable URL:
http://links.jstor.org/sici?sici=0002-9890%28197106%2F07%2978%3A6%3C596%3AROF%3E2.0.CO%3B2-O

7. J. H. Curtiss, Harmonic interpolation in Fejér points with Faber polynomials as a basis, Math. Z., 86 (1964) 75–92.
8. ———, Transfinite diameter and harmonic polynomial interpolation, J. Analyse Math., 22 (1969) 371–389.
9. Georg Faber, Über polynomische Entwicklungen, Math. Ann., 57 (1903) 398–408.
10. ———, Über Tschebyscheffsche Polynome, J. Reine Angew. Math., 150 (1920) 79–106.
11. Helmut Grunsky, Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen, Math. Z., 45 (1939) 29–61.
12. Einar Hille, Analytic Function Theory, Ginn, Boston, 1962.
13. J. A. Jenkins, Some area theorems and a special coefficient theorem, Ill. J. Math., 8 (1964) 80–99.
14. T. Kövari and Ch. Pommerenke, On Faber polynomials and Faber expansions, Math. Z., 99 (1967) 193–206.
15. Ch. Pommerenke, Über die Faberschen Polynome schlichter Funktionen, Mathematische Z., 85 (1964) 197–208.
16. ———, Konform Abbildung und Fekete-Punkte, Math. Z., 89 (1965) 422–438.
17. ———, Über die Verteilung der Fekete-Punkte, Math. Ann., 168 (1967) 111–127.
18. P. C. Rosenbloom and S. E. Warschawski, Approximation by polynomials, Lectures on Functions of a Complex Variable, ed. Wilfred Kaplan et al., the University of Michigan Press, 1955.
19. Menahem Schiffer, Faber polynomials in the theory of univalent functions, Bull. Am. Math. Soc., 54 (1948) 503–517.
20. I. Schur, On Faber Polynomials, Amer. J. Math., 67 (1945) 33–41.
21. W. E. Sewell, Degree of Approximation by Polynomials in the Complex Domain, Princeton University Press, 1942.
22. V. I. Smirnov and N. A. Lebedev, Functions of a Complex Variable-Constructive Theory, the M.I.T. Press, 1968.
23. P. K. Suetin, The basic properties of Faber polynomials (Russian), Uspehi Mat. Nauk, No. 4 (118), 19 (1964) 125–154.
24. Gabor Szegö, Orthogonal Polynomials, Amer. Math. Soc., 1939.
25. Antoni Zygmund, Trigonometric Series, Cambridge University Press, 1959.

RINGS OF FRACTIONS

P. M. COHN, Bedford College, London

Introduction. A well-known (and easily proved) theorem states that each integral domain can be embedded in a field [37]. This was generalized to certain noncommutative rings in a brilliant paper by Ore [33] in 1930; his results are essentially as simple as in the commutative case, and the proofs, though longer, are no harder. Beyond this, very little is known, so little that it can be set down in quite a brief article. I thought this was worth doing because some interesting

Professor Cohn did his Cambridge Ph.D. under Philip Hall, and he has held positions at the Univ. de Nancy, Manchester University, Queen Mary College London, and (presently) Bedford College, London. He has spent leaves-of-absence at Yale Univ., Univ. of Chicago, and Rutgers. He has published extensively in universal algebra and in many branches of algebra; in 1965–67 he was the Secretary of the London Mathematical Society. His books are *Lie Groups* (Cambridge Univ. Press 1957), *Linear Equations* (Routledge and Kegan Paul 1958), *Solid Geometry* (Routledge and Kegan Paul 1961), and *Universal Algebra* (Harper and Row 1965). *Editor.*

problems on the embedding of rings in skew fields remain, and because it gives me the chance to mention some recent work which makes it seem that these embedding problems are not quite as hard as they appear at first sight.

The central problem, finding *fields* of fractions, is really part of the problem of constructing *rings* of fractions (i.e., inverting certain elements), which also has other important applications. We shall therefore arrange the discussion so as to include this more general case.

Conventions. Every ring has a unit-element, denoted by 1, which is preserved by homomorphisms, inherited by subrings and acts as the identity operator on modules. The same conventions apply to semigroups. Of course a ring may very well consist of 0 alone; this is the case precisely when $1=0$. We use 0 to denote both the zero element and the set consisting of the zero element; the context will always make clear which is intended.

In any ring R the set of nonzero elements is denoted by R^*. A ring R, not necessarily commutative, such that R^* is a group under multiplication will be called a *field*. In the current literature this is often called a *skew field* or *division ring*; we shall occasionally use the prefix 'skew' for emphasis. A ring R such that R^* is a semigroup under multiplication is said to be *entire*, and a commutative entire ring is called an *integral domain*. Note that in a field $1\neq 0$; the same is true in entire rings, by our convention about semigroups.

An element u in a ring is *invertible* or a *unit* if it has an *inverse* u^{-1} satisfying $uu^{-1}=u^{-1}u=1$; of course the inverse is unique if it exists at all. In an entire ring, if $uv=1$, then $u(vu-1)=(uv-1)u=0$, hence $vu=1$ and v is the inverse of u. Thus all one-sided inverses are two-sided in this case. An element u is called a *zero-divisor* if $u\neq 0$ and if for some $v\neq 0$, either $uv=0$ or $vu=0$. A *nonzero-divisor* is a nonzero element which is not a zero-divisor. Thus by our convention 0 is neither a zero-divisor nor a nonzero-divisor.

If R is any ring, a *field of fractions* of R is a field containing R as a subring and generated, as a field, by R.

Outline. In Section 1 we review the commutative case; Section 2 introduces the obvious but rather useful notion of a 'universal S-inverting ring' and also gives Malcev's example of an entire ring not embeddable in a field. The Ore construction occupies Section 3, with applications in Section 4, including the theorems of Goldie and Posner. The remaining sections describe methods of constructing fields of fractions which go beyond Ore's theorem. In Section 5 we examine generalized inverses, and the relation to the Johnson-Utumi 'ring of quotients'; this turns out to be not very close in the case of chief interest to us, that of entire rings. The topological methods in Section 6 essentially generalize the notion of a decimal fraction. The final Section 7 reports briefly on the author's recent method of embedding rings in fields by inverting matrices rather than elements.

1. The commutative case. Let R be a commutative ring. The need for fractions arises when we try to enlarge R so as to ensure that equations of the form

$$xb = a \tag{1}$$

can be solved. At this stage our instinct should tell us to beware of the case $b=0$, but we shall leave this question aside for the moment. If we denote the solution of (1) by ab^{-1} or a/b, we see immediately that $a/b=ac/bc$, or more generally,

$$a/b = a'/b' \quad \text{if and only if } ab' = ba', \tag{2}$$

under suitable restrictions to exclude division by 0. Further, if the solutions are to form a ring containing R, they must add and multiply according to the rules

$$\frac{a}{b}+\frac{a'}{b'}=\frac{ab'+a'b}{bb'}, \qquad \frac{a}{b}\cdot\frac{a'}{b'}=\frac{aa'}{bb'}\,. \tag{3}$$

We learn at an early stage of our algebra course that if R is an integral domain, then we can find a field containing R as subring by taking all fractions a/b with $b\neq 0$ and combining them according to the rules (3), bearing in mind the cancellation rule (2).

To generalize this construction we observe that to form the new denominators in (3), we need only *multiply* the denominators b and b' together, not add them; this suggests taking a subsemigroup S of R as our stock of denominators. Now we shall in general no longer obtain a field; we may not even get a ring containing R as subring, but by following essentially the same construction we get a ring R_S say, with a homomorphism

$$\lambda: R \to R_S$$

which maps the elements of S to invertible elements, and it will be a simple matter to find out when λ is injective.

Thus we are given a subsemigroup S of a commutative ring R and we define a relation on the product set $R\times S$ by the rule:

$$(a, s)\sim(a', s') \text{ if and only if } as't=a'st \text{ for some } t\in S. \tag{4}$$

This reduces to (2) when S consists of nonzero-divisors; in general the form (4) is necessary to make sure that '$\sim$' is really an equivalence relation. Let us only check transitivity (reflexivity and symmetry are obvious): If $(a, s)\sim(a', s')$ and $(a', s')\sim(a'', s'')$, then $as't=a'st$ and $a's''t'=a''s't'$ for some $t, t'\in S$; hence

$$as''\cdot s'tt' = a'ss''tt' = a''s'stt' = a''s\cdot s'tt'.$$

Since $s'tt'\in S$, this shows that $(a, s)\sim(a'', s'')$ and the transitivity is proved. We denote the equivalence class containing (a, s) by a/s and define addition and multiplication by the formulae (3); of course we must verify that the definitions really only depend on the classes of a/s, a'/s' and not on the representatives of these classes used in the formulae. This is a routine verification, as is the proof that the set of classes a/s with these operations forms a ring, denoted by R_S say, with zero $0/1$ and unit-element $1/1$. The mapping

$$\lambda: a \mapsto a/1 \tag{5}$$

of R into R_S is clearly a homomorphism; it maps each element of S to an invertible element of R_S, because $(s/1)(1/s) = s/s = 1/1$.

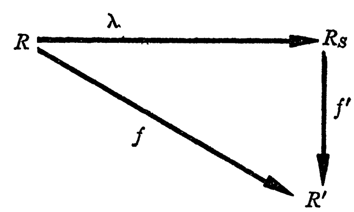
Let us say that a homomorphism $f: R \to R'$ is *S-inverting* if each element of S is mapped by f to an invertible element of R'. For example the mapping (5) is S-inverting, but we can say more than this. Let $f: R \to R'$ be any S-inverting homomorphism and define a mapping $f_1: R \times S \to R'$ by the rule

$$(a, s)^{f_1} = a^f(s^f)^{-1}.$$

This makes sense because f is S-inverting, by hypothesis. We observe now that f_1 takes the same value on pairs that are equivalent according to (4): If $as't = a'st$, then $a^f s'^f t^f = a'^f s^f t^f$ and hence $a^f(s^f)^{-1} = a'^f(s'^f)^{-1}$. This means that we obtain a well-defined mapping f' of R_S into R' by putting $(a/s)^{f'} = (a, s)^{f_1}$. This mapping f' has the property that for any $a \in R$,

$$(a/1)^{f'} = a^f, \tag{6}$$

an equation which may also be expressed by saying that the accompanying diagram commutes, i.e., $\lambda f' = f$. Moreover, f' is uniquely determined by (6), because



that equation determines its value on the elements $a/1$, and its value on $1/s$ must be the inverse of its value on $s/1$. Thus the mapping

$$\lambda: R \to R_S \tag{7}$$

is not just an S-inverting homomorphism, but the most general such homomorphism, in the sense that each S-inverting homomorphism can be obtained by taking a uniquely determined homomorphism from R_S. This property is expressed by saying that (7) is the *universal S-inverting homomorphism*, and R_S itself is called the *universal S-inverting ring*. This universal property in effect determines R_S up to isomorphism.

We shall wish to know when λ is injective; more generally, let us determine the kernel of λ. Clearly $a^\lambda = 0$ if and only if $a/1 = 0/1$, and by definition this means that $at = 0$ for some $t \in S$. We can now sum up our results:

THEOREM 1.1. *Let R be a commutative ring and S a subsemigroup of R. Then* *there is a universal S-inverting homomorphism* $\lambda: R \to R_S$; *the elements of* R_S *can be written as fractions* a/s ($a \in R$, $s \in S$), *where* $a/s = a'/s'$ *if and only if* $as't = a'st$ *for some* $t \in S$. *Further,*

$$\ker \lambda = \{a \in R \mid at = 0 \text{ for some } t \in S\}.$$

REMARKS. 1. Note that R_S is again commutative.

2. The ring R_S reduces to 0 precisely when $0 \in S$; it is to avoid this trivial case that one usually assumes $0 \notin S$.

3. The mapping λ is injective if and only if S contains only nonzero-divisors. In that case R_S is called a *ring of fractions* of R; the largest such ring is obtained by taking S to be the set of all nonzero-divisors, a set which is always a subsemigroup. The ring obtained by inverting all nonzero-divisors is called the *total ring of fractions* of R. In case R is an integral domain, this is the universal R^*-inverting ring of R, which is of course the field of fractions of R.

An important special case of the theorem is obtained by taking S to be the complement of a prime ideal $\mathfrak{p}$ in R (a *prime* ideal is an ideal of R whose complement is a semigroup under multiplication; note that this does not allow R as prime ideal). In that case one often writes $R_{\mathfrak{p}}$ instead of R_S, somewhat inconsistently, but without risk of confusion, because S never contains 0, whereas $\mathfrak{p}$ always does.

The problem of constructing fractions already arises in a semigroup, and should really be considered in that setting. We have nevertheless treated the case of rings first, on account of its importance; it also happens to coincide with the historical order of its development [16]. In any case we can easily extract the answer for semigroups from our conclusion:

THEOREM 1.2. *Let* M *be a commutative semigroup and* S *a subsemigroup of* M. *Then there is a universal S-inverting homomorphism*

$$\lambda: M \to M_S,$$

the elements of M_S *can be written as fractions* a/s ($a \in M$, $s \in S$) *as in the ring case, and* a, a' *have the same image under* λ *if and only if* $at = a't$ *for some* $t \in S$.

2. Some observations on the general case. Let us return to our basic problem, which is to construct a field of fractions for a given ring, when possible. If a ring R is to be embedded in a field, then whether commutative or not, R must be entire. But this necessary condition, which in the commutative case was sufficient, in general is no longer so. The first example of an entire ring not embeddable in a field was given by Malcev [28]. He takes the ring R generated by eight elements a, b, c, d, x, y, u, v, with defining relations

$$ax = by, \qquad cx = dy, \qquad cu = dv. \tag{1}$$

To show that this ring is entire, one uses a normal form for its elements. In outline the argument goes as follows. Each element of R can, by use of (1), be expressed as a noncommutative polynomial in the given generators, in which there are no occurrences of by, dy, dv (the right-hand sides of the equations (1)), and such an expression is unique. The verification that the product of nonzero elements is nonzero is fairly straightforward, though care is needed to ensure that all possibilities are considered at each stage. The normal form also shows that $au \neq bv$, but if R were embeddable in a field, or even in a ring in which a, c, y, v have inverses, we could deduce from (1) that $a^{-1}b = xy^{-1}$, $xy^{-1} = c^{-1}d$, $c^{-1}d = uv^{-1}$, hence $a^{-1}b = uv^{-1}$, and so

$$au = bv, \tag{2}$$

which is a contradiction.

Malcev obtained his example in the course of studying conditions under which a semigroup is embeddable in a group [29]. In fact he was able to write down an infinite series of 'quasi-identities', i.e., conditions of the form

$$A_1, \cdots, A_n \text{ imply } B,$$

(where $A_1, \cdots, A_n, B$ are equations in a semigroup) which he proved necessary and sufficient for a semigroup to be embeddable in a group. The simplest of these quasi-identities are left and right cancellation: '$xy = xz$ implies $y = z$' and '$xz = yz$ implies $x = y$.' The next condition is of the form 'the equations (1) imply (2)', and the example just given shows it to be independent of cancellation (more generally, the infinite set of quasi-identities cannot be replaced by any finite subset). A detailed account of Malcev's Theorem can be found in [11]; it seems likely that any corresponding criterion for the embeddability of rings in skew fields is rather more complicated. But it follows from results in general algebra that the embeddability of a nonzero ring in a field can be expressed by a (possibly infinite) set of quasi-identities ([11], p. 235).

Recently, in [41], A. A. Klein has found an infinite set of quasi-identities which are necessary for embeddability in a field and which he conjectures to be sufficient.* They express that R is entire and for all n, each nilpotent $n \times n$ matrix C satisfies $C^n = 0$.

Malcev has asked whether rings exist whose nonzero elements are embeddable in groups, but which are not embeddable in fields. Such rings were found simultaneously and independently by three people in 1966 [4, 5, 24].

Let us now take the general situation and see whether anything found in the commutative case can be used here. If R is a ring and S any subset (not necessarily a subsemigroup), we can define an S-inverting homomorphism as before. Given an S-inverting homomorphism $f: R \to R'$, let $\bar{S}$ be the subset of R whose elements are mapped into invertible elements of R'. Clearly $\bar{S} \supseteq S$, but equality need not hold; in particular $\bar{S}$ contains all invertible elements of R, and if $u, v \in \bar{S}$, then $uv \in \bar{S}$, because $(uv)^f = u^f v^f$, and the latter is invertible when u^f, v^f are. This shows that $\bar{S}$ is always a subsemigroup, so nothing is lost by taking the set to be inverted as a semigroup.

* Added in proof (March 29, 1971): A counterexample to this Conjecture has just been found by G. M. Bergman.

We can again construct a universal S-inverting ring R_S: simply take a presentation of R (by generators and defining relations) and for each $s \in S$ adjoin a new generator s' and extra relations

$$ss' = s's = 1. \tag{3}$$

The ring R_S so obtained may no longer contain R as subring, e.g., if we apply this construction to Malcev's example, with $S = \{a, c, y, v\}$, we get a collapse because then $au = bv$. But we always have a natural homomorphism $R \to R_S$ which is S-inverting, and in fact this is the universal S-inverting homomorphism, because the relations holding in R_S, namely (3) together with the relations of R itself, must hold in any image of R under an S-inverting homomorphism. (This is essentially an application of Dyck's Theorem, see, e.g., [11], p. 183.) In this way we obtain the following result:

THEOREM 2.1. *Let R be any ring and S any subset of R. Then there is a universal S-inverting homomorphism $\lambda: R \to R_S$, where R_S is unique up to isomorphism. Moreover, λ is injective if and only if R can be embedded in a ring in which all elements of S have inverses.*

The assertion is quite general and, because of its very generality, rather easy to prove. It is also not hard to see that the correspondence $(R, S) \mapsto R_S$ is a *functor* (from the category of pairs R, S to the category of rings). This means that to each homomorphism of rings $f: R \to R'$ such that $S^f \subseteq S'$ for subsets S, S' of R, R' respectively, there corresponds a homomorphism $\bar{f}: R_S \to R'_{S'}$ such that $\overline{fg} = \bar{f}\bar{g}$ and the identity mapping on R corresponds to the identity on R_S: $\bar{1} = 1$. Moreover, the natural mappings $\lambda: R \to R_S$ and $\lambda': R' \to R'_{S'}$ have the property of making the following diagram commute:

$$\begin{array}{ccc} R & \xrightarrow{\quad f \quad} & R' \\ \lambda \downarrow & & \downarrow \lambda' \\ R_S & \xrightarrow{\quad \bar{f} \quad} & R'_{S'} \end{array}$$

This is expressed by saying that λ is a *natural transformation* (for details cf. e.g., [27]).

At first sight Th.2.1 looks deceptively like Th.1.1, but it has the serious drawback that no normal form for the elements of R_S is given. This makes it hard to decide when λ is injective; also we cannot be sure, even when R is embeddable in a field, that R_{R^*} will be the whole field of fractions. The trouble is that after adjoining inverses of all the nonzero elements to R, there may still be elements without inverses, e.g., elements of the form $ab^{-1}c + de^{-1}f$. So the process of adjoining inverses may have to be repeated, perhaps infinitely often. The following observation is sometimes useful:

THEOREM 2.2. *Let R be any ring. If there is an R^*-inverting homomorphism* *of R into a field K, then R is embeddable in a field.*

For by hypothesis, we have a homomorphism

$$f: R \to K, \tag{4}$$

and since any nonzero element of R maps to an invertible element of K, it cannot map to 0, i.e., (4) is injective.

The necessity of having to repeat the process of adjoining inverses does not arise for semigroups: If a semigroup M is embeddable in a group G, then the subsemigroup of G generated by the elements of M and their inverses already forms a group. Neither does the problem arise in the special case treated by Ore, to which we now turn.

3. Ore's Construction. In an attempt to carry over the results of Section 1 to the noncommutative case, let us examine the situation where every element of the universal S-inverting ring R_S can be written in the form of a fraction a/s. If this is to be possible, we must be able to express $(1/s)(a/1)$ in this form, say

$$(1/s)(a/1) = a'/s'. \tag{1}$$

Multiplying both sides by $s/1$ on the left and by $s'/1$ on the right, we get

$$as'/1 = sa'/1. \tag{2}$$

This gives us a clue to the extra condition required now.

THEOREM 3.1. *Let R be any ring, S a subsemigroup, and assume further that*

(i) *for any $a \in R$ and $s \in S$, $aS \cap sR \neq \emptyset$,*

(ii) *for any $a \in R$ and $s \in S$, if $sa = 0$, then $at = 0$ for some $t \in S$.*

Then the elements of the universal S-inverting ring R_S can be constructed as fractions a/s ($a \in R$, $s \in S$), where

$$a/s = a'/s' \Rightarrow au = a'u', \quad su = s'u' \in S \text{ for some } u, u' \in R. \tag{3}$$

The kernel of the natural mapping is then

$$\ker \lambda = \{a \in R \mid at = 0 \text{ for some } t \in S\}.$$

Of course here we must distinguish carefully between as^{-1} and $s^{-1}a$; the expression a/s corresponds to the former.

A subsemigroup S of R satisfying the conditions (i), (ii) of this theorem will be called a *right denominator set* in R. The proof of this result is largely an exercise in patience, and the reader is recommended to verify at least some of the steps. The basic observation is that any two fractions can be brought to a common denominator in S, using (i), and they represent the same element of R_S if and only if, over a suitable common denominator, their numerators are equal (cf. (2)). Addition of fractions over the same denominator is straightforward, and the multiplication of fractions is based on the rule (1).

REMARKS: 1. Again $R_S = 0$ if and only if $0 \in S$; one usually excludes this case.

2. There is a left-right analogue of the theorem, obtained by switching sides; it shows how to form *left* fractions, starting from a *left* denominator set.

3. There is a corresponding theorem for constructing fractions in a semigroup. More generally, the construction can be performed in any category (cf. [**20**], p. 28).

4. Any subsemigroup S consisting of invertible elements of R is a right (and left) denominator set, and the universal S-inverting homomorphism is then an isomorphism.

5. Any *central* subsemigroup S (i.e., satisfying $as=sa$ for all $a\in R$, $s\in S$) is a right (and left) denominator set.

6. The universal S-inverting mapping is injective if and only if S contains only nonzero-divisors. In that case condition (ii) becomes superfluous. This case is sufficiently important to be stated separately.

COROLLARY 1. *Let R be a ring and S a subsemigroup of R consisting of nonzerodivisors, such that $aS\cap sR\neq\emptyset$ for any $a\in R$, $s\in S$. Then the universal S-inverting homomorphism is injective.*

When S is as in Corollary 1, R_S is again called a *ring of fractions*, *total* in case S consists of all nonzero-divisors. But this time we cannot be sure that there is a total ring of fractions, because the set of all nonzero-divisors need not satisfy the hypothesis of Corollary 1.

If R is entire and R^* is a right denominator set, we get the case originally treated by Ore (cf. [**33**]; the generalizations were given by Asano [**3**] and others).

COROLLARY 2. *Let R be an entire ring such that*

$$aR\cap bR\neq 0 \qquad \textit{for any } a,\, b\in R^*. \tag{4}$$

Then R can be embedded in a skew field K, whose elements have the form of fractions a/b ($a\in R$, $b\in R^*$).

Condition (4) is called the *right Ore condition*, and an entire ring satisfying (4) is called a *right Ore ring*. We observe that (4) is necessary as well as sufficient for the conclusion to hold. For if an entire ring R can be embedded in a field K in such a way that each element of K has the form ab^{-1} (a, $b\in R$), then in particular, for any a, $b\in R^*$ we can find a', $b'\in R^*$ such that $b^{-1}a=a'b'^{-1}$; hence $ab'=ba'\neq 0$.

4. Applications of Ore's Construction. An important class of Ore rings (which Ore himself had studied and clearly had in mind when making his construction, cf. [**34**]) is formed by the *skew polynomial rings*. Given any field K, passing to the polynomial ring $K[x]$ in an indeterminate x is a familiar construction, which still works even when K is skew. In that case it is usual to require x to be *central*, i.e., to commute with the field elements. But we often need a more general case: we still assume that each element of our ring can be written as a polynomial in just one way:

$$f = a_0 + xa_1 + \cdots + x^n a_n \qquad (a_i \in K); \tag{1}$$

we no longer assume that x is central, but instead that for each $a \in K$ there exist $\bar{a}$, $a' \in K$ satisfying

$$ax = x\bar{a} + a'. \tag{2}$$

It is not too hard to show that the mapping $\alpha: a \mapsto \bar{a}$ is an endomorphism of K, and that $\delta: a \mapsto a'$ is an additive mapping satisfying

$$(ab)' = a'\bar{b} + ab'. \tag{3}$$

Any additive mapping δ of a field into itself, satisfying (3) (for some endomorphism $\alpha: a \mapsto \bar{a}$) is called an *$\alpha$-derivation*. E.g., on the field of rational functions $F(t)$ over some commutative field F, the usual derivative $f' = df/dt$ defines a 1-derivation (associated with the identity automorphism of $F(t)$).

Conversely, given any endomorphism α of a field K and any α-derivation δ, we can define a multiplication on the set of polynomials (1) by using the commutation rule (2). This leads to a ring denoted by $K[x; \alpha, \delta]$ and called a *skew polynomial ring*. This ring is entire and is in fact a right Ore ring; for the proof we can use the Euclidean algorithm, as in ordinary polynomial rings, but we must take care here to perform all divisions on the right. It follows that we can form the field of fractions, denoted by $K(x; \alpha, \delta)$. Of course for $\alpha = 1$, $\delta = 0$ the skew polynomial ring reduces to the ordinary polynomial ring $K[x]$ with a central indeterminate and its field of fractions $K(x)$.

It is important to note that the construction just given is unsymmetric, and $K[x; \alpha, \delta]$ will not in general be a *left* Ore ring. The condition for it to be one is that α should be an automorphism, for this is the condition which enables us to rewrite (2) as a commutator formula in the other direction:

$$xa = a_1 x + a_2. \tag{4}$$

Explicitly, if $ax = xa^\alpha + a^\delta$ and $\alpha^{-1} = \beta$, then $a^\beta x = xa + a^{\beta\delta}$, hence (4) holds with $a_1 = a^\beta$, $a_2 = -a^{\beta\delta}$. Conversely, if α is not an automorphism, take a in K but not in the image under α. Then it is easily checked that x and xa have no common left multiple other than 0; so $K[x; \alpha, \delta]$ is a *left* Ore ring precisely when α is an automorphism.

We observe that the class of right Ore rings is closed under forming polynomial rings [**15**].

THEOREM 4.1. *If R is a right Ore ring, then so is the ring $R[x]$ of polynomials in a central indeterminate.*

Proof. Since R is right Ore, it has a field of fractions, K say, and $R[x]$ can be embedded in the field $K(x)$ of rational functions in x. Each element of $K[x]$ has the form fa^{-1}, where $f \in R[x]$ and $a \in R^*$ is a common denominator for the coefficients. Hence each element w of $K(x)$ can be written $w = fa^{-1}(gb^{-1})^{-1} = fa^{-1}bg^{-1}$, where f, $g \in R[x]$ and a, $b \in R^*$. Since R is right Ore, it contains a', b' such that $ab' = ba' \neq 0$; hence $a^{-1}b = b'a'^{-1}$, so $w = fb'a'^{-1}g^{-1} = fb'(ga')^{-1}$. This shows that each element of $K(x)$ is a fraction of elements of $R[x]$; therefore the latter is a right Ore ring, as asserted.

It is a remarkable fact, first observed by Goldie [18], that every right Noetherian entire ring is a right Ore ring. (A ring is *right Noetherian* if all its right ideals are finitely generated.) For if R is entire and right Noetherian, take $x, y \neq 0$, and consider the right ideal generated by the elements $x^n y$ $(n = 0, 1, \cdots)$. This must be finitely generated, say $x^n y = ya_0 + xya_1 + \cdots + x^{n-1}ya_{n-1}$. If $a_0 = 0$, we can cancel a power of x from the left, so we may assume that $a_0 \neq 0$, and then

$$x(x^{n-1}y - ya_1 - \cdots - x^{n-2}ya_{n-1}) = ya_0 \neq 0.$$

The result is sometimes called the "little Goldie theorem":

THEOREM 4.2. *Every right Noetherian entire ring is a right Ore ring.*

Similarly the total ring of fractions introduced earlier plays a role in the "big Goldie theorem." To state it we recall that a ring R is said to be *prime* if the product of nonzero ideals in R is nonzero and *semiprime* if the square of each nonzero ideal is nonzero. A ring is *right Artinian* if its right ideals satisfy the descending chain condition; this is a very much stronger condition than being right Noetherian, and much more is known about the Artinian rings (cf. e.g., [26]). Goldie's theorem provides a connection between the two; in one form it states:

If a right Noetherian ring R is prime (respectively semiprime), then R has a total ring of fractions which is right Artinian and simple (respectively semisimple).

For a brief proof see [19].

Let us return to Th.4.2 and consider an entire ring R which is not right Ore. For simplicity we take R to be an algebra over a commutative field F. By hypothesis we can find $x, y \in R^*$ such that $xR \cap yR = 0$. It follows that there is no polynomial in x and y (treated as noncommuting variables) which is zero, except the one whose coefficients are all 0. For each such polynomial is of the form $f = \alpha + xf_1 + yf_2$, where $\alpha \in F$ and f_1, f_2 are polynomials of lower degree than f. Suppose that f is the polynomial of least degree in two noncommuting indeterminates that vanishes for x and y. If $\alpha \neq 0$, then f_1, f_2 cannot both vanish; say $f_2 \neq 0$, hence $\alpha x + xf_1x + yf_2x = 0$, i.e., $x(f_1x + \alpha) = -yf_2x \neq 0$, a contradiction. Hence $\alpha = 0$, so $xf_1 = -yf_2$; by the choice of x and y this implies $f_1 = f_2 = 0$, which contradicts the choice of f. So we have proved that there is no polynomial f other than the zero polynomial such that $f(x, y) = 0$. In other words, the subalgebra generated by x and y is the *free associative algebra* on these generators. The restriction on the coefficients is easily lifted, so one has the following result ([23], [12], [25]):

THEOREM 4.3. *An entire ring is either a left and right Ore ring, or it contains a free algebra on two generators.*

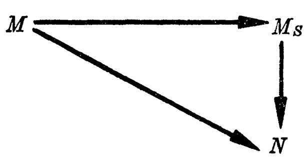
By definition a *polynomial identity* is an identical relation not holding in all rings, in particular not in free rings. Thus Th.4.3 has the following immediate consequence [1]:

COROLLARY 1. *An entire ring with a polynomial identity is a (left and right) Ore ring.*

In analogy with Goldie's theorem, Posner [35] has generalized this result to show that any prime ring with a polynomial identity has a total ring of fractions which is a central simple algebra of finite dimension over its centre.

Jategaonkar who first proved Th.4.3 has also shown how to use it to embed the free algebra in a field [23]. Take a ring R which is a right but not left Ore ring. (E.g., $K[x;\ \alpha,\ 0]$ with a non-surjective endomorphism α, say $K=F(t)$ with $\alpha: f(t)\mapsto f(t^2)$.) By Th.4.3 R contains a free algebra and by Th.3.1, Cor.2 it has a field of (right) fractions. So the free algebra is embedded in a field. This is of interest because the free algebra is very far from being an Ore ring. However, this embedding is rather artificial; indeed most automorphisms of the free algebra cannot be extended to the field of fractions just constructed. Later, in Section 7, we shall meet fields of fractions which do not suffer from this defect.

The process of forming fractions can be applied to modules as well as to rings. If R is a ring with a subsemigroup S, and $\lambda: R\rightarrow R_S$ is the universal S-inverting homomorphism, then to each right R-module M there corresponds a right R_S-module M_S with an R-module homomorphism $\mu: M\rightarrow M_S$ (where M_S is regarded as R-module by means of $\lambda: x\cdot a = xa^\lambda$ for $x\in M_S$, $a\in R$), and μ is the universal mapping with this property, i.e., given any R-module homomorphism of M into an R_S-module N, there exists a unique R_S-module homomorphism of M_S into N such that the accompanying diagram commutes:



This much is general theory, proved in the same way as Th.2.1. There is even a formula for M_S if we are willing to use tensor products (cf. e.g., [27]):

$$M_S = M \otimes R_S,$$

but this formula makes it no easier to study M_S in detail. Now let us assume that S is a right denominator set in R; then the elements of M_S can be written as fractions m/s, where $m\in M$, $s\in S$, and two fractions represent the same element of M_S if and only if, over a suitable common denominator, they have the same numerator. The kernel of the natural mapping $\mu: M\rightarrow M_S$ is a submodule tM of M, called the *S-torsion submodule* of M. It consists of all $m\in M$ such that $ms=0$ for some $s\in S$. If $tM=0$, the module M is said to be *S-torsionfree*; e.g., the quotient M/tM is always S-torsionfree.

When R is the ring $\mathbf{Z}$ of integers and $S=\mathbf{Z}^*$ the set of all nonzero integers, tM reduces to the usual torsion subgroup of an abelian group.

5. Strongly regular rings. There have been many attempts to generalize the notion of 'inverse' of an element, to take account of zero-divisors, usually in the form of a 'relative inverse' a', satisfying $aa'a=a$. We shall present the part of this theory that is relevant to the embedding problem.

A ring R is said to be *regular* if to each $a\in R$ there corresponds $x\in R$ such that $axa=a$; if R is such that for each $a\in R$ there exists $x\in R$ satisfying $a^2x=a$, it is called *strongly regular*. In the commutative case this is the same as requiring R to be regular, but in general it is stronger. This is not apparent at first sight, but it will follow from the structure theorems given below. We shall need some of the standard theory of the Jacobson radical; this can be found in [**22**], to which we refer when necessary. Let us recall that from any family R_λ of rings we can form a *direct product* $P=\prod R_\lambda$ by taking the Cartesian (set-theoretical) product and performing all the operations componentwise. (In the older books this is also called the direct sum.) A subring R of the direct product P is said to be a *subdirect* product if the canonical projections ϵ_λ on the factors R_λ, when restricted to R, are still surjective. E.g., $\mathbf{Z}$ can be expressed as subdirect product of fields $\mathbf{Z}_p$, where p ranges over all primes. The projection of $\mathbf{Z}$ on $\mathbf{Z}_p$ maps each integer n to its residue class (mod p).

As an example of a strongly regular ring we have any field, or more generally, any direct product of fields. However, a subdirect product of fields need not be strongly regular, as the example of the integers shows. Nevertheless, these two notions are closely related:

THEOREM 5.1. *Every strongly regular ring is a subdirect product of fields.*

Proof. Let R be strongly regular. Then its Jacobson radical J is 0. For an element $a\in R$ lies in J precisely when $1-ax$ is invertible for all $x\in R$. By hypothesis we can find $x\in R$ such that $a(1-ax)=a-a^2x=0$ and $1-ax$ is invertible; hence $a=0$. It follows ([**22**], p. 14, [**26**], p. 58) that R is a subdirect product of primitive rings, each a homomorphic image of R and therefore again strongly regular, so it only remains to show that a strongly regular ring which is also primitive is a field. Now any primitive ring is a dense ring of linear transformations in a vector space V over a field ([**22**], p. 28, [**26**], p. 54), and we shall be done if we show that V is 1-dimensional. Assume that V contains two linearly independent elements v_1, v_2. By density there exists $a\in R$ such that $v_1a=v_2$, $v_2a=0$, and by strong regularity we can find $x\in R$ such that $a^2x=a$; hence $v_2=v_1a=v_1a^2x=v_2ax=0$, a contradiction. This completes the proof.

COROLLARY. *A ring is strongly regular if and only if it is regular and has no nilpotent elements other than* 0.

Proof. Let R be strongly regular; then it is a subdirect product of fields and therefore cannot have any nonzero nilpotent elements. Further, if $a^2x=a$, then for each projection ϵ_λ of R on a factor K_λ of the product, either $a\epsilon_\lambda = 0$ or $a\epsilon_\lambda \cdot x\epsilon_\lambda = 1 = x\epsilon_\lambda \cdot a\epsilon_\lambda$. In all cases, $a\epsilon_\lambda \cdot x\epsilon_\lambda \cdot a\epsilon_\lambda = a\epsilon_\lambda$; hence $axa = a$ and R is regular.

Conversely, let R be regular without nonzero nilpotent elements, and take $x \in R$ to satisfy $axa = a$. Then

$$(a^2x - a)^2 = a^2xa^2x - a^2xa - a^3x + a^2 = a^3x - a^2 - a^3x + a^2 = 0,$$

hence $a^2x - a = 0$ and R is strongly regular.

The connection with fields of fractions is provided by the following result [**7**]:

THEOREM 5.2. *A subring of a strongly regular ring is embeddable in a field if and only if it is entire.*

Proof. The condition is clearly necessary; so assume that R is entire and is a subring of a strongly regular ring. The latter is a subdirect product of fields, so R is itself a subring of a direct product of fields, say

$$R \subseteq P = \prod K_\lambda.$$

Let $\epsilon_\lambda : P \to K_\lambda$ be the canonical projection and define for each $x \in P$,

$$\Gamma_x = \{\lambda \in \Lambda \mid x\epsilon_\lambda = 0\}, \qquad I_x = \prod_{\lambda \in \Gamma_x} K_\lambda.$$

Each I_x is an ideal in P. Let I be the ideal generated by all the I_x such that $x \in R^*$. Then $R \cap I = 0$; for if $x \in R \cap I$, then $x \in I_{y_1} + \cdots + I_{y_n}$, where $y_i \in R^*$, and hence $xy_1y_2 \cdots y_n = 0$. Therefore $x = 0$.

Let $f: P \to P/I$ be the natural homomorphism. Then by the construction of I, f is R^*-inverting. Since P/I, like P, is strongly regular, there is a homomorphism g of P/I into a field; now fg is an R^*-inverting homomorphism of P into a field, and (by Th.2.2) this provides an embedding of R in a field.

The following consequence was first proved using ultraproducts [**36**]:

COROLLARY. *An entire subring of a direct product of fields is embeddable in a field.*

Th.5.2 shows that an entire ring can be embedded in a strongly regular ring if and only if it can be embedded in a field. By contrast, any entire ring can be embedded in a regular ring; for there is a construction which associates with any ring R its 'total (right) quotient ring' $Q(R)$, and when R is entire (more generally, for any ring with 'zero singular ideal') $Q(R)$ is regular (cf. [**17**, **26**]). Moreover, this total quotient ring agrees with the total ring of fractions when the latter exists, i.e., by Th.3.1, when the nonzero-divisors of R form a right denominator set. But in general the total quotient ring of R gives no clue about the embeddability of R in a field. E.g., though for an entire ring $Q(R)$ is always regular, it is not strongly regular unless R is a right Ore ring.

For a very thorough survey of quotient rings, see [**40**].

6. Topological embedding methods. Besides the time-honoured way of forming fractions there is another method of embedding rings in fields, which is also taught at school and goes back to the 16th century [38]. This is the method of decimal fractions; it consists of taking all expressions of the form

$$\sum_{-n}^{\infty} a_\nu t^\nu, \tag{1}$$

where $a_\nu = 0, 1, \cdots, 9$ and $t = 1/10$, and adding and multiplying in the usual way. Division is possible because if in (1), $a_{-n} \neq 0$ say, the series (1) can be written as $t^{-n}a_{-n}(1 - \sum_1^\infty b_\nu t^\nu)$, where each factor is invertible. Of course all the series are convergent, as Laurent series, because $|t| < 1$. This method can be generalized; the general form is even simpler in some respects, because the usual absolute value is replaced by a non-Archimedean valuation.

Let R be a ring with a *valuation*, i.e., a function $v(x)$ taking the integers or $+\infty$ as values, such that

V.1. $v(x) = \infty$ if and only if $x = 0$,

V.2. $v(xy) = v(x) + v(y)$,

V.3. $v(x - y) \geqq \min\{v(x),\ v(y)\}$.

Such a valuation may be thought of as defining a topology on R; since our aim is to embed R in a field (if possible), we shall specify the topology by its neighbourhoods of 1. We shall limit ourselves to the case where the set

$$P = \{ab^{-1} \mid a \in R,\ b \in R^*\} \tag{2}$$

is dense in the field to be constructed, so we must say when ab^{-1} is close to 1. The natural condition for this is to require $v(ab^{-1} - 1)$ to be large. Of course v is only defined on R in the first instance, but if we assume that it can be extended to the field of fractions in such a way as to satisfy V.1–3, we have

$$v(a - b) = v([ab^{-1} - 1]b) = v(ab^{-1} - 1) + v(b);$$

hence ab^{-1} is close to 1 precisely when $v(a - b) - v(b)$ is large.

We now have a topology (in fact a uniformity, cf. [6]) on the set P of fractions ab^{-1}. What is still needed to make it into a ring? We shall not make P itself into a ring, but its completion in the given topology. To enable us to add and multiply we need an "asymptotic Ore condition":

A. For any a, b in R^* the function

$$f(x, y) = v(ax - by) - v(by)$$

is unbounded above.

This enables us to embed R in a field, by defining the ring operations in the completion of P, much in the same way as the usual Ore condition was used before. The details are somewhat technical, so we omit them (cf. [8]), but there is a simple way of restating the result in terms of graded rings.

Let us define

$$R_n = \{x \in R \mid v(x) \geqq n\}.$$

Then the R_n form a descending series of additive subgroups of R such that $\bigcap R_n=0$, $\bigcup R_n=R$ and $R_iR_j\subseteq R_{i+j}$. In other words, we have a *filtered ring*. With each such filtered ring R one associates another ring, its *graded ring* gr R, as follows: the additive group of gr R is the direct sum of the terms

$$\text{gr}_n R = R_n/R_{n+1}. \tag{3}$$

To define multiplication it is enough, by the distributive law, to specify the product of an element of $\text{gr}_i R$ and one of $\text{gr}_j R$. Let $\alpha\in\text{gr}_i R$, $\beta\in\text{gr}_j R$; according to (3), these are cosets, say $\alpha=a+R_{i+1}$, $\beta=b+R_{j+1}$. We define $\alpha\beta=ab+R_{i+j+1}$; it is easily verified that this definition does not depend on the choice of a, b within their cosets. Associativity is clear, so gr R becomes a ring in this way. Loosely speaking, it is the ring formed by taking 'leading terms' in R.

We shall get a graded ring even if the function v satisfies, instead of V.2, only $v(xy)\geqq v(x)+v(y)$. The stronger condition V.2 merely ensures that gr R is entire; further the asymptotic Ore condition can be shown to be equivalent to the Ore condition for gr R. The result may be summed up as follows [8]:

THEOREM 6.1. *Let R be a filtered ring whose associated graded ring is a right Ore ring. Then R can be embedded in a field K. In fact K can be taken as a complete topological field, with P given by* (2) *as a dense subset.*

The result can be used to embed the universal associative envelope of a Lie algebra (even infinite-dimensional) in a field. In particular it provides another embedding of the free algebra, because this can be regarded as the universal associative envelope of the free Lie algebra. (Cf. [8] and for other applications [9].) For a further generalization see [42].

Another 'topological' method of constructing fields of fractions consists of taking an ordered group and forming 'Laurent series': Given a totally ordered group G and a commutative field F, consider the direct power F^G of F indexed by G. With each $f\in F^G$ we associate a subset $D(f)$ of G, its *support*, defined as

$$D(f) = \{s\in G \mid f(s)\neq 0\}.$$

E.g., the group algebra FG of G may be identified with the set of elements of finite support. In general it will not be possible to define the multiplication of F^G in such a way as to extend the operation on FG, for this requires that

$$fg = (\Sigma f(s)s)(\Sigma g(t)t) = \sum_u \left[\sum_{st=u} f(s)g(t)\right]u, \tag{4}$$

and here the inner sum on the right will generally contain infinitely many nonzero terms to be added. However, if both f and g have a *well-ordered* support, then the equation $st=u$ has, for a given $u\in G$, only finitely many solutions (s, t) in $D(f)\times D(g)$. Moreover, the sum (4) itself will then have well-ordered support. We can therefore define a ring structure on the set A of all elements of F^G whose support is well-ordered, in such a way that the group algebra FG becomes a subalgebra of A.

Finally it can be shown that A is in fact a field, so that the group algebra of G has been embedded in a field. This result was obtained simultaneously and independently by Malcev [**30**] and Neumann [**32**]. Later Higman [**21**] proved a general result on ordered algebraic systems which includes the Malcev-Neumann construction as a special case. For a simplified presentation of Higman's result see [**11**], p. 123.

We now have another way of embedding free algebras in fields: Let G be the free group on a set X. Then G can be totally ordered (by writing elements as products of basic commutators and taking the lexicographic ordering of the exponents, cf. [**31**]). Hence its group algebra FG is embeddable in a field. Since FG clearly contains the free algebra $F\langle X\rangle$ as subalgebra, this provides an embedding of the latter in a field.

7. The matrix method. In Section 2 we observed that to embed a noncommutative ring R in a field it may not be enough to produce inverses of all nonzero elements of R. We can try to overcome this difficulty by adjoining inverses of suitable matrices. Given a set Σ of square matrices over R, we can formally adjoin inverses of these matrices as follows. For each $n\times n$ matrix $A=(a_{ij})$ in Σ, take a set of n^2 symbols $A'=(a'_{ij})$ and adjoin the a'_{ij} to R as extra generators with defining relations, in matrix form,

$$AA' = A'A = I.$$

The resulting ring is denoted by R_Σ, and we have a natural homomorphism $\lambda: R\to R_\Sigma$. This ring has properties entirely analogous to the ring R_S described in Th.2.1, to which it reduces when all the matrices in Σ are 1×1. So we again call R_Σ and λ the *universal Σ-inverting ring* and *homomorphism*, respectively.

It is of interest to note that under suitable conditions on Σ, each element of R_Σ is some a'_{ij}; thus the generating set of R_Σ described above is then the whole ring. To state the result, let us say that the set Σ of matrices is *admissible* if (i) $1\in\Sigma$, (ii) the result of applying elementary row (or column) transformations to any matrix of Σ again lies in Σ, and (iii) if $A, B\in\Sigma$, then $\begin{pmatrix} A & C \\ 0 & B\end{pmatrix}\in\Sigma$ for any matrix C and zero matrix 0 of the appropriate size.

THEOREM 7.1. *Let R be a ring, Σ an admissible set of matrices over R, and $f: R\to S$ any Σ-inverting homomorphism. Then the set $\overline{R}$ consisting of all components of inverses of matrices in $\Sigma^{\&}$ is a subring of S.*

When S is a field, this result applies in particular to the set Σ_1 of all matrices over R whose images are invertible in S, for then Σ_1 can easily be shown to be admissible. When $\Sigma=\Sigma_1$, the set $\overline{R}$ is also called the *rational closure* of R under the homomorphism f.

The question now is: Which matrices do we have to invert to get a field? In the commutative case the answer was easy: we had to invert all nonzero elements, and this ensured that all matrices that are nonzero-divisors also become invertible. But in the general case there may well be matrices that are nonzero-divisors and yet are not invertible in any larger ring. Thus let R be any entire ring that is neither a right nor a left Ore ring. Then there exist $a, b, c, d \in R^*$ such that $Ra \cap Rb = 0$, $cR \cap dR = 0$, and it follows easily that the matrix

$$\begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} (c\ d) \tag{1}$$

is a nonzero-divisor. But this matrix cannot be invertible in any field; more generally, no homomorphism of R into a field can map (1) to an invertible matrix. This example suggests the following definition:

A matrix A over a ring R is said to be *full* if it is square, say $n \times n$, and it cannot be written as a product $A = PQ$, where P is $n \times r$, Q is $r \times n$, and $r < n$. Clearly, the most we can hope for, in mapping a ring R into a field, is to invert the full matrices. The next result goes some way towards saying when this can be done [**13**].

THEOREM 7.2. *Let R be a ring such that the set Φ of all full matrices over R is admissible. Then the universal Φ-inverting ring R_Φ is either* 0 *or a field; moreover, when R_Φ is a field, the universal Φ-inverting homomorphism $\lambda: R \to R_\Phi$ is injective.*

This is proved by exhibiting each element of R_Φ as a component of the solution of a matrix equation with a full matrix of coefficients, and showing that the inverse element satisfies a similar equation. The last part of the theorem follows from Th.2.2, because any nonzero element of a ring is full.

The hypotheses of Th.7.2 are satisfied if R_Φ is a nonzero ring in which each one-sided matrix inverse is two-sided (i.e., $AB = I$ implies $BA = I$, cf. [**14**]), but it is more difficult to find conditions in terms of R itself. Here is one case where this has been done.

A *free ideal ring*, or *fir* for short, is a ring R in which each right ideal (and each left ideal) is free as an R-module, and all bases of a free module have the same number of elements [**10**]. Examples of firs are: (i) free algebras over a commutative field (on any free generating set), (ii) group algebras of free groups, and (iii) free products of fields, over a common subfield, [**10**]. For a fir one can show that the set Φ of full matrices is admissible and that $\lambda: R \to R_\Phi$ is an embedding. Hence using Th.7.2 we see that each fir can be embedded in a field. Since the class of full matrices is preserved under automorphisms, each automorphism of the fir can be extended (in just one way) to an automorphism of its field of fractions.

A final point concerns the uniqueness. The field of fractions of an integral domain or, more generally, of a right Ore ring is unique up to isomorphism. For any ring isomorphism $a \mapsto a'$ extends to an isomorphism of the field of fractions by the formula $(a/s)' = a'/s'$. In the general case this is no longer so: there may be several nonisomorphic fields of fractions of a noncommutative ring. (E.g., for the free algebra, the field of fractions obtained from Th.7.2, using the fact that the free algebra is a fir, can be shown to be different from the field obtained by Jategaonkar's construction in Section 4.) Given two fields of fractions K_1, K_2 of a ring R, we define a *specialization* from K_1 to K_2 as a homomorphism f from a subring R_1 of K_1 to K_2 that reduces to the identity map on R and such that any nonunit of R_1 is mapped to 0 by f. Of course no unit can be mapped to 0, so ker f is the precise set of nonunits. This means that the nonunits of R_1 form an ideal, $\mathfrak{m}$ say; a ring with this property is said to be *local*. Clearly $R_1/\mathfrak{m}$ is a field, isomorphic to the image of R_1 under f. This image is a subfield of K_2 containing R, but since K_2 is generated (as a field) by R, the image is all of K_2, i.e., f is surjective. This then shows each specialization to be surjective.

A field of fractions K of R is said to be *universal* if for each field of fractions K' of R there is a unique specialization from K to K'. This property determines K up to isomorphism, and in looking for fields of fractions, we are naturally interested in finding the universal one, if it exists. Any free algebra (over a commutative field) has a universal field of fractions [2] and also has other non-universal ones. More generally one can show [**14**]:

THEOREM 7.3. *Let R be a ring such that the set Φ of full matrices is admissible. If $R_\Phi \neq 0$ (so that R_Φ is a field, by Th.7.2), then R_Φ is the universal field of fractions of R.*

In particular this shows that each fir has a universal field of fractions.

These notions will be useful when one tries to do noncommutative algebraic geometry, which might be defined as the study of zero-sets of rational functions in skew fields, just as the usual kind is the study of zero-sets of polynomials in commutative fields. In the commutative case polynomial zero-sets and rational zero-sets are the same, which is why we could confine ourselves to the former. In general this may not be so (cf. the examples in [**39**]), and some thought is needed even to construct rational functions. In [2] Amitsur classified rational function fields according to the (rational) identities they satisfy; this is taken up by Bergman [**39**] from a more general view-point; he also shows that affine space over a field with infinite center is irreducible and describes a universal field of functions.

Clearly many problems remain; we end by listing a few:

1. Find criteria for the existence of a homomorphism of a ring into a field (remember that the zero-mapping is not a homomorphism).
2. Which rings have fields of fractions?
3. Which rings have more than one field of fractions?
4. Which rings have a universal field of fractions?

References

1. S. A. Amitsur, On rings with identities, J. London Math. Soc., 30 (1955) 464–470.
2. ———, Rational identities and applications to algebra and geometry, J. Algebra, 3 (1966) 304–359.
3. K. Asano, Über die Quotientenbildung von Schiefringen, J. Math. Soc. Japan, 1 (1949) 73–78.
4. L. A. Bokut', On Malcev's problem, Sibirsk. Mat. Zh., 10 (1969) 965–1005.
5. A. J. Bowtell, On a question of Malcev, J. Algebra, 6 (1967) 126–139.
6. N. Bourbaki, Topologie générale, Ch. I–II, Hermann, Paris 1949.
7. I. E. Burmistrovič, On the embedding of rings in fields, Sibirsk. Mat. Zh., 4 (1963) 1235–1240.

8. P. M. Cohn, On the embedding of rings in skew fields, Proc. London Math. Soc., (3) 11 (1961) 511–530.

9. ———, Quadratic extensions of skew fields, Proc. London Math. Soc., (3) 11 (1961) 531–556.

10. ———, Free ideal rings, J. Algebra, 1 (1964) 47–69.

11. ———, Universal Algebra, Harper & Row, (New York, London, Tokyo 1965).

12. ———, Free associative algebras, Bull. London Math. Soc., 1 (1969) 1–39.

13. ———, The embedding of firs in skew fields, to appear.

14. ———, Universal skew fields of fractions, to appear.

15. C. W. Curtis, A note on noncommutative polynomials, Proc. Amer. Math. Soc., 3 (1952) 965–969.

16. M.-L. Dubreil-Jacotin, Sur l'immersion d'un semigroupe dans un groupe, C. R. Acad. Sci. Paris, 225 (1947) 787–788.

17. C. C. Faith, Lectures on injective modules and quotient rings, Springer Lecture notes in Math. No. 49 (Berlin 1967).

18. A. W. Goldie, The structure of prime rings under ascending chain conditions, Proc. London Math. Soc., (3) 8 (1958) 589–608.

19. ———, Some aspects of ring theory, Bull. London Math. Soc., 1 (1969) 129–154.

20. R. Hartshorne, Residues and duality, Springer Lecture notes in Math., No. 20 (Berlin 1966).

21. G. Higman, Ordering by divisibility in abstract algebras, Proc. London Math. Soc., (3) 2 (1952) 326–336.

22. N. Jacobson, Structure of rings, Providence, R. I., 1956, 1964.

23. A. V. Jategaonkar, Ore domains and free algebras, Bull. London Math. Soc., 1 (1969) 45–46.

24. A. A. Klein, Rings nonembeddable in fields with multiplicative semigroups embeddable in groups, J. Algebra, 6 (1967) 101–125.

25. E. G. Koševoi, On certain associative algebras with transcendental relations, Algebra i Logika, Sem. 9, No. 5 (1970).

26. J. Lambek, Lectures on rings and modules, Blaisdell, Waltham, Mass., 1966.

27. S. MacLane, Homology, Springer, Berlin-New York, 1963.

28. A. I. Malcev, On the immersion of an algebraic ring into a field, Math. Ann., 113 (1937) 686–691.

29. ———, Über die Einbettung von assoziativen Systemen in Gruppen I, Mat. Sbornik, 6 (48) (1939) 331–336, II. dto 8 (50) (1950) 251–264.

30. ———, On the embedding of group algebras in division algebras, Dokl. Akad. Nauk SSSR, 60 (1948) 1499–1501.

31. B. H. Neumann, On ordered groups, Amer. J. Math., 71 (1949) 1–18.

32. ———, On ordered division rings, Trans. Amer. Math. Soc., 66 (1949) 202–252.

33. O. Ore, Linear equations in noncommutative fields, Ann. of Math., 32 (1931) 463–477.

34. ———, Theory of noncommutative polynomials, Ann. of Math., 34 (1933) 480–508.

35. E. C. Posner, Prime rings satisfying a polynomial identity, Proc. Amer. Math. Soc., 11 (1960) 180–184.

36. A. Robinson, A note on embedding problems, Fund. Math., 50 (1961/62) 167–309; 455–461.

37. E. Steinitz, Algebraische Theorie der Körper, J. Reine Angew. Math., 137 (1910) new edition by R. Baer and H. Hasse, Berlin, 1930.

38. S. Stevin, Arithmétique (Amsterdam 1585).

39. G. M. Bergman, Skew fields of noncommutative rational functions, after Amitsur, to appear.

40. V. P. Elizarov, Rings of fractions, Algebra i Logika Sem., 8, no. 4 (1969) 381–424.

41. A. A. Klein, A conjecture concerning embeddability of rings into fields, J. Algebra, (to appear).

42. J. Dauns, Embeddings in division rings, Trans. Amer. Math. Soc., 150 (1970) 287–299.