TOPOLOGICAL RIGIDITY AND H_1-NEGATIVE INVOLUTIONS ON TORI

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Abstract. We show, for $n \equiv 0, 1 \pmod 4$ or $n = 2, 3$, there is precisely one equivariant homeomorphism class of C_2-manifolds (N^n, C_2) for which N^n is homotopy equivalent to the n-torus and $C_2 = \{1, \sigma\}$ acts so that $\sigma_*(x) = -x$ for all $x \in H_1(N)$. If $n \equiv 2, 3 \pmod 4$ and $n > 3$, we show there are infinitely many such C_2-manifolds. Each is smoothable with exactly 2^n fixed points.

The key technical point is that we compute, for all $n \geq 4$, the equivariant structure set $\mathscr{S}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n)$ for the corresponding crystallographic group Γ_n in terms of the Cappell UNil-groups arising from its infinite dihedral subgroups.

1. Introduction

1.1. **Statement of results.** Our goal here is to analyze topological rigidity for a sequence of crystallographic groups containing 2-torsion. For each n, we define the group: $\Gamma_n = \mathbb{Z}^n \rtimes_{-1} C_2$, where C_2 acts on $\mathbb{Z}^n$ by negation: $v \mapsto -v$.

We classify the proper actions of Γ_n on contractible n-manifolds.

The most powerful inspiration for our work is the remarkable rigidity theorem of Farrell and Jones concerning a discrete cocompact group of isometries of a simply connected non-positively curved manifold (M, Γ). They classify the cocompact proper actions of such a Γ on a contractible manifold, if Γ is torsion free.

The second major inspiration for our paper is the work of Cappell on UNil. If Γ as above has elements of order 2, then the nontrivial elements of UNil groups coming from virtually cyclic subgroups of Γ can provide examples of cocompact Γ-manifolds (M', Γ) which are isovariantly homotopy equivalent to, but not homeomorphic to (M, Γ). So how do we classify such actions?

The *Topological Rigidity Conjecture* stated below does this. We view it as a version of an old conjecture of Quinn, sharpened through the precision afforded by the work of [DL98]. We then prove this conjecture for Γ_n using [BL, CD04].

We can cast our results in terms of an action of a group $C_2 := \{1, \sigma\}$. We say an involution $\sigma : N \to N$ is H_1*-negative* if $\sigma_*(x) = -x$, for all $x \in H_1(N)$. We prove:

Theorem 1.1. *Let $\sigma : N \to N$ be an H_1-negative involution on a closed manifold homotopy equivalent to the n-torus T^n. Consider the C_2-manifold (N, C_2).*

(1) *The fixed set N^{C_2} is discrete and consists of exactly 2^n points.*
(2) *If $n \equiv 0, 1 \pmod 4$ or $n = 2, 3$, then (N^n, C_2) is equivariantly homeomorphic to the standard example, (T^n, C_2).*
(3) *If $n \equiv 2, 3 \pmod 4$ and $n > 3$, there are infinitely many such C_2-manifolds, (N^n, C_2). All are isovariantly homotopy equivalent to (T^n, C_2), but no two are equivariantly homeomorphic. Each is smoothable hence locally linear.*

By the *standard example* (T^n, C_2) above, we mean the involution $\sigma : T^n \to T^n$ given by $\sigma([x]) = [-x]$, for all $[x] \in \mathbb{R}^n/\mathbb{Z}^n = T^n$. Recall that any n-manifold homotopy equivalent to the n-torus is homeomorphic to it (see [Wal99, FQ90, And04]).

The construction of the exotic involutions mentioned in the theorem uses surgery theory, specifically the Wall Realization Theorem [Wal99, Thms. 5.8, 6.5]. Write $X := (T^n - (T^n)^{C_2})/C_2$, an open n-manifold. Define $\overline{X}$ as the obvious manifold compactification of X obtained by adding a copy of $\mathbb{R}P^{n-1}$ at each end of X. Note for all $n > 2$ that $\pi_1(\overline{X}) = \Gamma_n$ and that $\overline{X}$ is orientable if and only if n is even. Let $w_n : \Gamma_n \to \{\pm 1\}$ be the orientation character of $\overline{X}$. Then, for $n \geq 5$, an element $\theta \in L_{n+1}(\Gamma_n, w_n)$ determines a compact smooth manifold $\theta \cdot \overline{X}$, homotopy equivalent to $\overline{X}$ relative to the boundary. Passing to the two-fold cover and gluing in 2^n copies of D^n with the antipodal action, we get a smooth involution on the torus. All the exotic involutions in the above theorem arise in this way.

Observe that Γ_n is isomorphic to a rank n crystallographic group. This isometric action of Γ_n on $\mathbb{R}^n$ is given by $\mathbb{Z}^n$ acting by translation and C_2 acting by reflection through the origin. We let $(\mathbb{R}^n, \Gamma_n)$ denote this Γ_n-manifold.

Let $\mathscr{S}(\Gamma_n)$, be the set of equivariant homeomorphism classes of n-dimensional contractible manifolds equipped with a proper Γ_n-action. We compute $\mathscr{S}(\Gamma_n)$.

To parametrize the set $\mathscr{S}(\Gamma_n)$, we will need to use the unitary nilpotent groups of Cappell. For D_∞, these have been computed recently by Banagl, Connolly, Davis, Koźniewski, and Ranicki [CK95, CD04, CR05, BR06], yielding:

$$\mathrm{UNil}_m(\mathbb{Z};\mathbb{Z},\mathbb{Z}) \cong \begin{cases} 0 & \text{if } m \equiv 0 \pmod 4 \\ 0 & \text{if } m \equiv 1 \pmod 4 \\ (\mathbb{Z}/2\mathbb{Z})^\infty & \text{if } m \equiv 2 \pmod 4 \\ (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z})^\infty & \text{if } m \equiv 3 \pmod 4. \end{cases} \tag{1.1}$$

Let $\mathrm{mid}(\Gamma_n)$ be the set of maximal infinite dihedral subgroups of Γ_n. Let $(\mathrm{mid})(\Gamma_n)$ be a subset of $\mathrm{mid}(\Gamma_n)$ chosen so that it contains exactly one maximal infinite dihedral subgroup from each conjugacy class. Let D be a maximal infinite dihedral subgroup of Γ_n. For any integer n, with $\varepsilon = (-1)^n$, there is a map

$$\mathrm{UNil}_{n+1}(\mathbb{Z};\mathbb{Z}^\varepsilon,\mathbb{Z}^\varepsilon) \longrightarrow L_{n+1}(\mathbb{Z}D, w_n) \longrightarrow L_{n+1}(\mathbb{Z}\Gamma_n, w_n).$$

If n is odd, then there is an isomorphism $\mathrm{UNil}_{n-1}(\mathbb{Z};\mathbb{Z},\mathbb{Z}) \xrightarrow{\cong} \mathrm{UNil}_{n+1}(\mathbb{Z};\mathbb{Z}^-,\mathbb{Z}^-)$.

Theorem 1.2. *Suppose $n \geq 4$. Write $\varepsilon := (-1)^n$. The Wall realization map induces a pointed bijection of sets, mapping the zero element to the basepoint $[\mathbb{R}^n, \Gamma_n]$:*

$$\partial_\oplus \ : \ \bigoplus_{D \in (\mathrm{mid})(\Gamma_n)} \mathrm{UNil}_{n+1}(\mathbb{Z};\mathbb{Z}^\varepsilon,\mathbb{Z}^\varepsilon) \ \xrightarrow{\approx} \ \mathscr{S}(\Gamma_n).$$

Consequently, $\mathscr{S}(\Gamma_n)$ consists of a single element if $n \equiv 0, 1 \pmod 4$, and $\mathscr{S}(\Gamma_n)$ is countably infinite if $n \equiv 2, 3 \pmod 4$.

We do not need to assume any conditions beyond continuity in order to obtain a full homeomorphism classification and to show all actions are smoothable. It turns out that Smith theory guarantees the fixed sets consist of isolated points (see Section 2). Also, local linearity is a consequence of our calculation (see Remark 4.3, which concludes that the forgetful map $\mathscr{S}_{\mathrm{TOP}}(\overline{X}, \partial\overline{X}) \to \mathscr{S}_{\mathrm{TOP}}(\partial\overline{X})$ is constant).

Recall that an action $\Gamma \times X \to X$ of a discrete group Γ on a Hausdorff space X is *proper* if, given a compact set K of X, the set $\{\gamma \in \Gamma \mid K \cap \gamma K \neq \varnothing\}$ is finite.

Note that, given $(M, \Gamma_n) \in \mathscr{S}(\Gamma_n)$, the quotient manifold $M^n/\mathbb{Z}^n$ is homotopy equivalent to, and hence homeomorphic to the n-torus. Therefore the universal cover, M, admits a homeomorphism to $\mathbb{R}^n$.

Outline of the argument. In Section 2, we show that any H_1-negative involution on an n-manifold homotopy equivalent to the n-torus has exactly 2^n fixed points. This allows one to deduce a correspondence between H_1-negative involutions on n-manifolds homotopy equivalent to the n-torus and contractible n-manifolds equipped with a proper Γ_n-action. In Section 2 we show that any compact C_2-manifold with finite fixed set has the C_2-homotopy type of a finite C_2-CW-complex. This allows one to conclude that any H_1-negative involution on a manifold homotopy equivalent to the n-torus is equivariantly homotopy equivalent to (T^n, C_2) and that any contractible n-manifold equipped with a proper Γ_n-action is equivariantly homotopy equivalent to $(\mathbb{R}^n, \Gamma_n)$.

We prove Theorem 1.2 in Section 4 and then deduce Theorem 1.1 in Section 5.

For $n \geq 4$, the six structure sets we use are introduced in Section 3. These are:

$$\mathscr{S}(\Gamma_n) \qquad \mathscr{S}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n) \qquad \mathscr{S}^{iso}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n)$$

$$\mathscr{S}_{\mathrm{TOP}}(\overline{X}, \partial\overline{X}) \qquad \mathscr{S}_{\mathrm{TOP}}(T^n, C_2) \qquad \mathscr{S}^{iso}_{\mathrm{TOP}}(T^n, C_2).$$

For example, the isovariant structure set $\mathscr{S}^{iso}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n)$ is the set of equivalence classes of proper Γ_n-manifolds (M^n, Γ_n), together with an isovariant homotopy equivalence $M \to \mathbb{R}^n$. We show all six structure sets are isomorphic, and compute the fourth one to prove Theorem 1.2. The isomorphisms between the first and second, between the second and fifth, and the third and sixth structure sets are formal and are shown in Section 3. The isomorphism between the fifth and sixth structure set requires a detailed discussion of equivariance versus isovariance and is discussed in Appendix A. The isomorphism between the fourth and fifth structure set requires the use of end theory, see Lemma 3.6. Finally, the computation of the classical surgery theoretic structure set $\mathscr{S}_{\mathrm{TOP}}(\overline{X}, \partial\overline{X})$ use the Farrell–Jones Conjecture and is presented at the end of Section 4. This computation also uses the main result of Appendix B, which identifies the assembly map in surgery theory with a corresponding map in equivariant homology.

The final bit of the paper, Section 6, is independent of [BL] and gives examples of non-standard structures on $(\mathbb{R}^n, \Gamma_n)$, hence of exotic H_1-negative involutions on tori. The intent is to show that Cappell's work, for straightforward reasons, gives obstructions to isovariant rigidity of a Γ-space when Γ has elements of order two. Shmuel Weinberger pointed out these counterexamples to simple isovariant rigidity some time ago. Since the argument was never published, we include it here.

1.2. **Equivariant Rigidity.** This paper represents the start of a systematic attack on Quinn's ICM conjecture and the closely related questions of equivariant, isovariant, and topological rigidity for a discrete group Γ. We take some time to formulate these questions precisely. Recall that two closed aspherical manifolds with the same fundamental group Γ are homotopy equivalent and that the Borel Conjecture for Γ states that any such homotopy equivalence is homotopic to a homeomorphism.

For any discrete group Γ, *a model, $E_{\texttt{fin}}\Gamma$, for the classifying space for Γ-CW-complexes with finite isotropy* is a space which is Γ-homotopy equivalent to a Γ-CW-complex so that for all subgroups H of Γ

$$(E_{\texttt{fin}}\Gamma)^H = \begin{cases} \varnothing & \text{if } |H| = \infty \\ \simeq \text{pt} & \text{if } |H| < \infty \end{cases}.$$

Given any Γ-CW-complex X with finite isotopy groups, there is an equivariant map $X \to E_{\texttt{fin}}\Gamma$, unique up to equivariant homotopy. It follows that any two models are Γ-homotopy equivalent. Furthermore, a model $E_{\texttt{fin}}\Gamma$ exists for any group Γ.

A *cocompact manifold model for $E_{\texttt{fin}}\Gamma$* is a a model M for $E_{\texttt{fin}}\Gamma$ so that M/Γ is compact and so that M^F is a manifold for all finite subgroups F of Γ. A geometric example is given by a discrete cocompact group Γ of isometries of a simply connected complete nonpositively curved manifold M. *Equivariant (respectively, isovariant) rigidity holds for* Γ if any Γ-homotopy equivalence (respectively, isovariant homotopy equivalence) $M \to M'$ between cocompact manifold models for $E_{\texttt{fin}}\Gamma$ is Γ-homotopic (respectively, Γ-isovariantly homotopic) to a homeomorphism. With this terminology, our results can be restated as showing that every proper Γ_n-action on a contractible manifold is a cocompact manifold model for $E_{\texttt{fin}}\Gamma$, that equivariant and isovariant rigidity for Γ_n holds when $n \equiv 0, 1 \pmod 4$ or $n = 2, 3$, and that equivariant and isovariant rigidity fail for all other n. Previous results on equivariant and isovariant rigidity are found in: [Ros88], [CK90], [CK91], [Wei94, Section 4.2], [PS00], and [MP04]. In particular, [CK91] gives the first example of groups where isovariant rigidity fails; this proceeded via a version of Whitehead torsion. The Corollary below shows that the relevant Whitehead group vanishes for Γ_n. Thus we give the first counterexamples to simple isovariant rigidity in print.

Remark 1.3. A key algebraic property of Γ_n is that it admits a split epimorphism

$$\varepsilon : \Gamma_n \longrightarrow \Gamma_1 = \mathbb{Z} \rtimes C_2 = C_2 * C_2$$

to the infinite dihedral group. The last equality results from noting that $\Gamma_1 = \langle (0,\sigma), (1,\sigma)\rangle = C_2 * C_2$. The existence of ε follows from the fact that every epimorphism $f : \mathbb{Z}^n \to \mathbb{Z}$ gives a split epimorphism $f \rtimes \mathrm{id} : \mathbb{Z}^n \rtimes C_2 \to \mathbb{Z} \rtimes C_2$. Thus Γ_n has an injective amalgated product decomposition:

$$\Gamma_n = \varepsilon^{-1}(C_2 * 1) *_{\varepsilon^{-1}(1)} \varepsilon^{-1}(1 * C_2) \cong \Gamma_{n-1} *_{\mathbb{Z}^{n-1}} \Gamma_{n-1}.$$

Proposition 1.4. *The group Γ_n is K-*flat*, that is,* $\mathrm{Wh}(\Gamma_n \times \mathbb{Z}^k) = 0$ *for all $k \geq 0$. Suppose $n \geq 5$. Then* $\mathrm{Wh}^{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n) = 0$. *Therefore, if M and M' are cocompact $E_{\texttt{fin}}\Gamma_n$-manifolds that are Γ_n-h-cobordant, then they are Γ_n-homeomorphic.*

Proof. We prove K-flatness by induction on n, as follows. First, note $\Gamma_0 = C_2$ and $\mathrm{Wh}(C_2 \times \mathbb{Z}^k) = 0$, by consideration of Rim's cartesian square of rings [Mil71, §3] and vanishing of lower NK-groups [BHS64] [Bas68, Ch. XII].

Next, by Remark 1.3 and Waldhausen's sequence [Wal73], we obtain:

$$\mathrm{Wh}(\Gamma_{m-1} \times \mathbb{Z}^k) \oplus \mathrm{Wh}(\Gamma_{m-1} \times \mathbb{Z}^k) \to \mathrm{Wh}(\Gamma_m \times \mathbb{Z}^k) \to \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}^{m-1+k}]) = 0.$$

This sequence is exact, since the ring $\mathbb{Z}[\mathbb{Z}^k]$ is regular coherent, which implies that the Nil term vanishes. Therefore, by induction, we are done proving Γ_n is K-flat.

Recall the exact sequence of Steinberger [Ste88, Thm. 3] (cf. [Wei94, p. 182]):

$$\mathrm{Wh}(\Gamma_n) \longrightarrow \mathrm{Wh}^{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\Gamma_n])_c.$$

We already have shown $\mathrm{Wh}(\Gamma_n) = 0$. Note that the link of each singular point in $\mathbb{R}^n/\Gamma_n$ has fundamental group C_2. Since $\widetilde{K}_0(\mathbb{Z}[C_2]) = 0$, by a spectral sequence argument we conclude $\widetilde{K}_0(\mathbb{Z}[\Gamma_n])_c = H_0^{\Gamma_n}(\mathbb{R}^n; \underline{\widetilde{\mathbf{K}}}) = 0$. The rest of the corollary follows from the equivariant h-cobordism theorem of Steinberger [Ste88]. □

1.3. **The Topological Rigidity Conjecture.** This section is motivated by the conjecture of F. Quinn at the 1988 ICM (see [Qui87]). It aims to say the same thing, but in a more explicit way, by employing the work of [DL98] and [BL].

Our *Topological Rigidity Conjecture* concerns a discrete cocompact group Γ of isometries of a simply connected complete nonpositively curved manifold X^n (that is, a Hadamard manifold). It says, roughly, that any topologically simple isovariant homotopy equivalence $f : M \to X$ should be isovariantly homotopic to a homeomorphism, except for the examples created by UNil-groups of virtually cyclic subgroups of Γ. But it does so by parametrizing the set of such f in terms of a homology group. The coefficient spectrum of this homology group, $\underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}}$, is an $\mathrm{Or}(\Gamma)$-spectrum in the sense of [DL98] (see Section 4). It only needs to be evaluated at virtually cyclic subgroups of Γ when calculating this homology group. For these virtually cyclic groups, the nonzero homotopy groups of the spectrum are just UNil-groups of amalgamated products of finite groups.

To formulate the conjecture, one must restrict to isovariant homotopy equivalences since there is no reason to expect equivariant homotopy equivalences to be well-behaved (see [Wei94, Section 14.2]). Furthermore, the proof of Theorem 1.1(2) shows that this conjecture cannot be extended to low dimensions ($n = 2, 3$).

Conjecture 1.5. *Let X^n be a Hadamard manifold of dimension $n > 3$. Let Γ be a discrete cocompact group of isometries of X. Assume the fixed set of each finite subgroup has codimension > 2 in X. Then there is a bijection*

$$\partial \ : \ H^{\Gamma}_{n+1}(E_{\mathtt{vc}}\Gamma; \underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}}) \xrightarrow{\approx} \mathscr{S}^{iso}_{rel}(X^n, \Gamma).$$

The present paper proves this conjecture for Γ_n. The definition of the map ∂, appearing in Conjecture 1.5, for the group Γ_n can be found in Section 4.

The elements of $\mathscr{S}^{iso}_{rel}(X, \Gamma)$ are equivalence classes of pairs (M, f), where M is a cocompact topological Γ-manifold whose fixed sets are locally flat submanifolds in M, and $f : M \to X$ is a simple Γ-isovariant homotopy equivalence that restricts to a homeomorphism on the singular set. Shmuel Weinberger has been a long-time proponent of this “rel sing” structure set in a very similar conjecture ([CWY, §3]).

If Γ has no element of order 2, the conjecture implies each such f is isovariantly homotopic to a homeomorphism. The left hand side is defined in [DL98, DQR].

The formulation of the above conjecture takes account of the recent proof of the Farrell–Jones Conjecture, in the sense that while geometry and controlled topology would lead to $E_{\mathtt{all}}\Gamma$ in the domain of the above map, the formulated domain is much more computable (for example, in terms of UNil-groups).

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2. Applications of Smith theory

Given a covering map $p : E \to B$, where B is connected and locally path-connected, and given an effective action of a group G on B, consider the group

$$\mathcal{D}(p, G) \ := \ \{h \in \text{Homeo}(E) \mid \exists g \in G \text{ such that } p \circ h = g \circ p\}.$$

There is an obvious exact sequence

$$1 \longrightarrow \mathcal{D}(p) \longrightarrow \mathcal{D}(p, G) \longrightarrow G \longrightarrow 1 \tag{2.1}$$

where $\mathcal{D}(p)$ is the deck transformation group of p. If $\mathcal{D}(p)$ is abelian, and E is 1-connected, the natural isomorphism

$$k : \mathcal{D}(p) \to \pi_1(B, b) \to H_1(B) \tag{2.2}$$

is G-equivariant relative to the G-action on $\mathcal{D}(p)$ induced by conjugation.

For the quotient map $q : \mathbb{R}^n \to T^n$, and the standard action (T^n, C_2), note $\mathcal{D}(q, C_2) = \Gamma_n$.

Theorem 2.1. *Let C_2 act on a manifold N^n homotopy equivalent to T^n so that $\sigma_*(\alpha) = -\alpha$ for all $\alpha \in H_1(N^n)$.*

1. *The fixed set, N^{C_2}, of the action consists of 2^n points and $\mathcal{D}(p, C_2) \cong \Gamma_n$, where p is the universal covering map of N^n. Moreover, if G is any non-trivial finite subgroup of $\mathcal{D}(p, C_2)$, then $\widetilde{N}^G$ consists of one point.*
2. *Fix an isomorphism $\mathcal{D}(p, C_2) \cong \Gamma_n$. There is a Γ_n-homotopy equivalence of the universal covers, $\widetilde{J} : (\widetilde{N}^n, \Gamma_n) \to (\mathbb{R}^n, \Gamma_n)$. Any two such Γ_n-homotopy equivalences are Γ_n-homotopic. $\widetilde{J}$ is the universal covering of a C_2-homotopy equivalence, $J : (N^n, C_2) \to (T^n, C_2)$.*

Lemma 2.2. *Let C_p be a cyclic group of prime order p.*

1. *The fixed set of a C_p-action on a manifold is locally path connected.*
2. *The fixed set of a C_p-action on a contractible manifold is mod p acyclic, locally path connected and path connected.*
3. *If the fixed set of a C_p-action on a contractible manifold is compact, then the fixed set is a point.*

The proof of this lemma involves Smith theory. Our primary reference is Borel's Seminar on Transformation Groups [Bor60]. Borel et. al. use Alexander–Spanier cohomology $\bar{H}^*(X; R)$ with coefficients in a commutative ring R. This is, in turn, isomorphic to Čech cohomology $\check{H}^*(X; R)$ for X paracompact Hausdorff [Spa81, p. 334]. Čech cohomology is defined (see Spanier [Spa81, p. 327]) as the colimit under refinement of the simplical cohomology of the nerves of open covers

$$\check{H}^*(X; R) = \operatorname{colim} H^*(\|\mathcal{U}\|; R).$$

We need that X is connected if $\check{H}^0(X; \mathbb{F}_p) \cong \mathbb{F}_p$. See Spanier [Spa81, p. 309].

Of course, if X is a CW complex then Alexander–Spanier and Čech and Singular cohomology coincide; but fixed sets of actions are far from CW complexes.

Proof of Lemma 2.2(1). Let M be a topological n-manifold. We first show that the fixed set of a C_p action on M is locally connected.

A space X is *cohomologically locally connected over a field* L (written clc_L) if each neighborhood U of each $x \in X$ contains a neighborhood V of x so that the restriction map from U to V is zero on reduced Čech cohomology with coefficients in L. (See [Bor60, I.1.3] for the definition.) Looking at degree zero, we see that if X is clc_L, then X is locally connected (see also [Bor60, I.1.3]). Borel [Bor60, Prop. V.1.4] asserts that if C_p acts on a finite-dimensional Hausdorff $\mathrm{clc}_{\mathbb{F}_p}$ space X, then X^{C_p} is also $\mathrm{clc}_{\mathbb{F}_p}$. In particular the fixed set M^{C_p} is locally connected.

Since M^{C_p} is a locally connected complete metric space, by a theorem of Moore–Menger–Mazurkiewicz [Kur68, p. 254], M^{C_p} must be locally path connected. □

Proof of Lemma 2.2(2). By a *mod p acyclic space*, we mean a Hausdorff space X with $\check{H}^*(X;\mathbb{F}_p) \cong \check{H}^*(\mathrm{pt};\mathbb{F}_p)$. A standard result from global Smith theory is that, if C_p acts on a finite dimensional mod p acyclic space M, then its fixed set is also mod p acyclic; see for example, [Bre72, Thm. III.7.11] or [Bor60, Cor. III.4.6]. In particular $\check{H}^0(M)^{C_p};\mathbb{F}_p) \cong \mathbb{F}_p$, so M^{C_p} is connected. By Part (1), it is locally path connected. Therefore M^{C_p} is path connected. □

Proof of Lemma 2.2(3). If C_p acts on a contractible manifold M^n, then by Smith theory the fixed set F is an orientable mod p cohomology manifold ([Bor60, Theorem V.2.2]) of dimension $d \leq n$. Recall that F is connected and mod p acyclic. If F is compact, then there is a fundamental cohomology class in dimension d, so $\check{H}^d(F;\mathbb{F}_p) \cong \mathbb{F}_p$ ([Bor60, Theorem I.4.3(1)]). But F is acyclic so $d = 0$. Also, for a connected compact mod p homology manifold X of dimension d, $\check{H}^d(A;\mathbb{F}_p) = 0$ for any closed proper subset of X by [Bor60, Theorem I.4.3(1)]. Therefore F must consist of a single point. □

In the literature, we have not seen the application of Smith theory to path connectedness. The above lemma seems to be the first occurrence.

Proof of Theorem 2.1(1). Let σ be an involution on a manifold N^n homotopy equivalent to the n-torus such that $\sigma_*\alpha = -\alpha$ for all $\alpha \in H_1(N)$.

We first note that σ has a fixed point, because its Lefschetz number is $2^n \neq 0$. Indeed, $\mathrm{Trace}(\sigma_k^*) = (-1)^k\binom{n}{k}$, for $\sigma_k^* \in \mathrm{Aut}\, H^k(N;\mathbb{Q})$; but $\sum_{k=0}^n \binom{n}{k} = 2^n$.

Let $p : \widetilde{N} \to N$ be the universal covering map. Choose a point $x \in N^{C_2}$ and a point $\widetilde{x} \in \widetilde{N}$ so that $p(\widetilde{x}) = x$. The involution σ lifts to an involution $\tilde{\sigma} : \widetilde{N} \to \widetilde{N}$ so that $\tilde{\sigma}(\widetilde{x}) = \widetilde{x}$ and $p \circ \tilde{\sigma} = \sigma \circ p$. This shows that the sequence (2.1) has a right splitting. But also the conjugation action of $\tilde{\sigma}$ on $\mathcal{D}(p) \cong H_1(N)$ satisfies $\tilde{\sigma}t = t^{-1}\tilde{\sigma}$ for all $t \in \mathcal{D}(p)$. Therefore $\mathcal{D}(p;C_2)$ is isomorphic to Γ_n.

Let $N_x^{C_2}$ be the path component of N^{C_2} containing x. We next show that $p^{C_2} : \widetilde{N}^{C_2} \longrightarrow N_x^{C_2}$ is a homeomorphism. By Lemma 2.2(2), the fixed set $\widetilde{N}^{C_2}$ is path connected. The map $p^{C_2} : \widetilde{N}^{C_2} \longrightarrow N^{C_2}$ is a covering map, because p is a covering map and N is locally connected. So $p^{C_2} : \widetilde{N}^{C_2} \longrightarrow N_x^{C_2}$ is surjective, continuous and open. To prove p^{C_2} is injective, let $y_0, y_1 \in \widetilde{N}^{C_2}$ be points such that $p(y_0) = p(y_1)$. Let $f : [0,1] \to \widetilde{N}^{C_2}$ be a path from y_0 to y_1. f is the lifting of a loop $p \circ f : [0,1] \to N$, fixed by C_2. Therefore there is a deck transformation $t \in \mathcal{D}(p)$, fixed under conjugation by C_2, such that $t(y_0) = y_1$. But $\mathcal{D}(p) \cong H_1(N)$ and $H_1(N)^{C_2} = 0$. So $t = 1$ and $y_0 = y_1$. Therefore p^{C_2} is a homeomorphism.

So $\widetilde{N}^{C_2}$ is homeomorphic to the component of x in N^{C_2} by Lemma 2.2(1). This component is a compact set, since N^{C_2} is. By Lemma 2.2(3), we conclude $\widetilde{N}^{C_2}$ is a single point. So the components of N^{C_2} are isolated points by Lemma 2.2(1).

Next we must show $|N^{C_2}| = 2^n$. Each involution $s \in \mathcal{D}(p, C_2) = \Gamma_n$ determines a point, $p(\widetilde{N}^s)$ in N^{C_2}. Moreover, for involutions s and s', we have: $p(\widetilde{N}^s) = p(\widetilde{N}^{s'})$, iff for some deck transformation $t \in \mathcal{D}(p)$, $t(\widetilde{N}^s) = \widetilde{N}^{s'}$, iff $ts't^{-1} = s$ for some $t \in \mathcal{D}(p)$, iff s and s' are conjugate in Γ_n by an element $t \in \mathbb{Z}^n$, iff s and s' are conjugate in Γ_n (because $\Gamma_n = \mathbb{Z}^n \cup s\mathbb{Z}^n$). Therefore this rule $s \mapsto p(\widetilde{N}^s)$ induces a bijection between the set of conjugacy classes of involutions in $\Gamma_n = \mathbb{Z} \rtimes C_2$ and N^{C_2}. Since each such conjugacy class is represented uniquely by an element of $\{(\varepsilon, \sigma) \mid \varepsilon \in \{0,1\}^n\}$, we obtain $|N^{C_2}| = 2^n$.

Finally we show $\widetilde{N}^G$ is a single point if G is a finite subgroup of Γ_n, and $G \neq 1$. Necessarily the map $\mathcal{D}(p, C_2) \to C_2$ maps G isomorphically onto C_2. Therefore G fixes some point $\widetilde{x} \in \widetilde{N}$, by Lemma 2.2(2). As seen just above $\widetilde{N}^G$ is homeomorphic to the component of $p(\widetilde{x})$ in N^{C_2}, which is a single point by Theorem 2.1(1). □

Jiang [Jia83] used fixed point theory to show that if $\widetilde{f} : \widetilde{X} \to \widetilde{X}$ is a lift of a self-map $f : X \to X$ of a finite complex, then $p(\mathrm{Fix}\ \widetilde{f})$ is compact. This can be used to give an alternate proof that $p(\widetilde{N}^{C_2})$, and hence $\widetilde{N}^{C_2}$, is compact.

Before proving Theorem 2.1(2), we need the following useful fact.

Lemma 2.3. *Let (N^n, C_2) be compact C_2-manifold with finite fixed set. If $n \leq 5$, assume $\pi_1((N - N^{C_2})/C_2) \cong \Gamma_n$. Then (N, C_2) has the equivariant homotopy type of a finite C_2-CW complex.*

Proof. This is implicit, certainly in Quinn [Qui88] but we will make it explicit here. First, the pair $(N/C_2, N^{C_2})$ is forward tame and reverse tame by Quinn [Qui88]. From this it follows from Siebenmann [Sie65], when $n \geq 6$, that $N - N^{C_2}$ is the interior of a compact, free C_2 manifold, $\overline{N}$, with 2^n boundary components and N is homeomorphic to $\overline{N}/\sim$, where $\sim$ is the equivalence relation identifying each boundary component to a separate point. But $\overline{N}$ and each of its boundary components are homotopy equivalent to finite free C_2 complexes. It follows that $\overline{N}/\sim$ is homotopy equivalent to a finite C_2 complex with 2^n fixed points.

If $n \leq 5$, we can instead argue that $(N - N^{C_2})/C_2$ and $\mathrm{Holink}(N/C_2, N^{C_2})$ are finitely dominated. Since the projective class groups vanish: $\widetilde{K}_0(\mathbb{Z}[C_2]) = 0 = \widetilde{K}_0(\mathbb{Z}[\Gamma_n])$, by Wall's finiteness theorem [Wal65, Thm. F], each is homotopy equivalent to a finite CW complex. So there is a finite C_2-CW pair (K, L), homotopy equivalent to the pair $(\mathrm{Cyl}(e_1), \mathrm{Holink}(N, N^{C_2}))$, where $e_1 : \mathrm{Holink}(N, N^{C_2}) \to N - N^{C_2}$ is the evaluation map at time 1. This map passes to a C_2-homotopy equivalence of pairs:

$$(K \cup_L \mathrm{Cyl}(p), N^{C_2}) \to (\mathrm{Cyl}(e_1) \cup_H \mathrm{Cyl}(e_0), N^{C_2})$$

where $H = \mathrm{Holink}(N, N^{C_2})$, and $p : L \to N^{C_2}$ is the composite C_2-map, $L \to H \xrightarrow{e_0} N^{C_2}$. But, as noted in Quinn [Qui88], $\mathrm{Cyl}(e_1) \cup_H \mathrm{Cyl}(e_0)$ is C_2-homotopy equivalent to N and $K \cup_L \mathrm{Cyl}(p)$ is a finite C_2-CW complex. □

Proof of Theorem 2.1(2). For a discrete group Γ, recall (see [CK86]) that a Γ-space (E, Γ) is a *classifying space for proper Γ-actions* if E^G is contractible for each finite subgroup $G \subset \Gamma$, and (E, Γ) has the Γ-homotopy type of a Γ-CW complex, and the Γ-space (E, Γ) is proper. Consider $\mathbb{R}^n$ with its Γ_n-action. The action is obviously proper. The only non-trivial finite subgroups have the form $\{1, \sigma\}$ for some involution σ; for this subgroup the fixed set is a single point. Finally, $(\mathbb{R}^n, \Gamma_n)$ admits the structure of a Γ_n-CW complex. For it is the universal covering of (T^n, C_2), and (T^n, C_2) is the n-fold cartesian product of (S^1, C_2), (with the diagonal action), which is a C_2-CW complex with exactly two (fixed) vertices.

Now let $[N, C_2]$ be as in the hypothesis of Theorem 2.1. After the choice of an isomorphism $\mathcal{D}(p, C_2) \cong \Gamma_n$, $(\widetilde{N}, \Gamma_n)$ is a proper Γ_n-manifold, and has the Γ_n-homotopy type of a Γ_n-CW complex by Lemma 2.3. For each finite subgroup G of Γ_n, $\widetilde{N}^G$ is contractible by Theorem 2.1(1). Therefore $(\widetilde{N}, \Gamma_n)$ is also universal, and there is a unique Γ_n-homotopy class of Γ_n-maps $g : \mathbb{R}^n \to \widetilde{N}$. By uniqueness, g and $\widetilde{J}$ are mutually Γ_n-homotopy inverse. Therefore $\widetilde{J} : (\widetilde{N}, \Gamma_n) \to (\mathbb{R}^n, \Gamma_n)$ and its quotient, $J : (N, C_2) \to (T^n, C_2)$ are equivariant homotopy equivalences. □

This paper focuses on actions of $C_2 = \{1, \sigma\}$ on a torus N for which $\sigma_*(x) = -x$ for all $x \in H_1(N)$. But the following lemma (with Theorem 2.1) shows that this is *equivalent* to saying that the torus has at least one isolated fixed point:

Lemma 2.4. *Suppose $C_2 = \{1, \sigma\}$ acts on a torus N and there is at least one fixed point which is isolated in N^{C_2}. Then $\sigma_*(x) = -x$ for all $x \in H_1(N)$.*

Proof. Lift the involution to an involution $\tilde{\sigma} : \widetilde{N} \to \widetilde{N}$ with an isolated fixed point $\widetilde{x} \in \widetilde{N}$. The group $G = \{1, \tilde{\sigma}\}$ fixes only $\widetilde{x}$ since the fixed set is mod 2 acyclic. But the centralizer, $Z_{\mathcal{D}(p)}(G)$ acts freely on this fixed set $\{\widetilde{x}\}$, since the action is proper. So if $t \in \mathcal{D}(p)$, and $t\tilde{\sigma} = \tilde{\sigma}t$, then $t = 1$. Therefore, if $x \in H_1(N) \cong \mathcal{D}(p)$, and $\sigma_*(x) = x$, then $x = 0$. This implies that $\sigma_*(x) = -x$ for all $x \in H_1(N)$. □

3. Equivariant and isovariant structures

Definition 3.1. The *equivariant structure set* $\mathscr{S}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n)$, is the set of equivalence classes of pairs $((M, \Gamma_n), f)$, where (M, Γ_n) is a manifold with a cocompact proper Γ_n-action, and $f : M \to \mathbb{R}^n$ is a Γ_n-equivariant homotopy equivalence. Often we write such pairs as (M, f). Two such pairs (M, f) and (M', f') are equivalent if there is an equivariant homeomorphism $h : M \to M'$ and an equivariant homotopy H from f to $f' \circ h$. One defines the *isovariant structure sets* $\mathscr{S}^{iso}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n)$ and $\mathscr{S}^{iso}_{\mathrm{TOP}}(T^n, C_2)$ similarly, except one requires that f is an isovariant homotopy equivalence and that H is an isovariant homotopy.

Remark 3.2. We make no requirements on the fixed sets of subgroups in M and N above, because we have seen (in Section 2) that these fixed sets are discrete. A result of Quinn [Qui88, Lem. 2.6, Prop. 3.6] then guarantees that there are no local pathologies in such manifolds.

The universal covering of a C_2-homotopy equivalence, $f : N^n \to T^n$ is a Γ_n-homotopy equivalence $\widetilde{f} : \widetilde{N}^n \to \mathbb{R}^n$. Moreover $\widetilde{f}$ is isovariant if f is isovariant. This gives obvious bijections:

$$u \ : \ \mathscr{S}_{\mathrm{TOP}}(T^n, C_2) \xrightarrow{\approx} \mathscr{S}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n) \tag{3.1}$$

$$u^{iso} \ : \ \mathscr{S}^{iso}_{\mathrm{TOP}}(T^n, C_2) \xrightarrow{\approx} \mathscr{S}^{iso}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n). \tag{3.2}$$

Consider the closed C_2-manifold $T := T^n$. Write $X := (T - T^{C_2})/C_2$, an open n-manifold. Define $\overline{X}$ as the obvious manifold compactification of X obtained by adding a copy of $\mathbb{RP}^{n-1}$ at each end of X. Label the boundary components:

$$\partial\overline{X} = \bigsqcup_{i=1}^{2^n} \partial_i \overline{X}.$$

Definition 3.3. For $n \geq 5$, the *structure set* $\mathscr{S}_{\mathrm{TOP}}(\overline{X}, \partial\overline{X})$ is the set of equivalence classes of triples $(\overline{Y}, \overline{h}, \partial\overline{h})$, where $\overline{Y}$ is a compact n-manifold and $(\overline{h}, \partial\overline{h}) : (\overline{Y}, \partial\overline{Y}) \to (\overline{X}, \partial\overline{X})$ is a homotopy equivalence of pairs. Such triples $(\overline{Y}^0, \overline{h}^0, \partial\overline{h}^0)$ and $(\overline{Y}^1, \overline{h}^1, \partial\overline{h}^1)$ are equivalent if there is a homeomorphism $\varphi : \overline{Y}^0 \to \overline{Y}^1$ such that $(\overline{h}^1, \partial\overline{h}^1)\circ\varphi$ is homotopic to $(\overline{h}^0, \partial\overline{h}^0)$. Compare [Wal99, §10]; we used $\mathrm{Wh}(\Gamma_n) = 0$.

For the four-dimensional case, some modifications are required.

Definition 3.4. For $n = 4$, the *structure set* $\mathscr{S}_{\mathrm{TOP}}(\overline{X}, \partial\overline{X})$ is the set of equivalence classes of triples $(\overline{Y}, \overline{h}, \partial\overline{h})$, where $\overline{Y}$ is a compact topological 4-manifold and $(\overline{h}, \partial\overline{h}) : (\overline{Y}, \partial\overline{Y}) \to (\overline{X}, \partial\overline{Y})$ is a $\mathbb{Z}[\Gamma_4]$-homology equivalence of pairs. Such triples $(\overline{Y}^0, \overline{h}^0, \partial\overline{h}^0)$ and $(\overline{Y}^1, \overline{h}^1, \partial\overline{h}^1)$ are equivalent if there is a $\mathbb{Z}[\Gamma_4]$-homology h-bordism $(\overline{W}; \overline{Y}^0, \overline{Y}^1) \to \overline{X} \times (I; 0, 1)$ between them. Compare [FQ90, §11.3].

For our application, we give a more explicit form of this general notion.

Remark 3.5. Since $\mathrm{Wh}(\Gamma_4) = 0$ and Γ_4 is "good" in the sense of [FQ90], we can simplify Definition 3.4. First, for a representative $(\overline{Y}, \overline{h}, \partial\overline{h})$, by the manifold-theoretic plus-construction *rel* boundary [FQ90, §11.1], we can assume that $\overline{h} : \partial\overline{Y} \to \partial\overline{X}$ is a homotopy equivalence. Second, two triples $(\overline{Y}^0, \overline{h}^0, \partial\overline{h}^0)$ and $(\overline{Y}^1, \overline{h}^1, \partial\overline{h}^1)$ are equivalent, by plus-construction on $\overline{W}$, if and only if there are:

- a non-orientable closed 3-manifold $P = \bigsqcup_{i=1}^{16} P_i$
- $\mathbb{Z}[\Gamma_4]$-homology h-cobordisms $(\overline{Z}^j; \partial\overline{Y}^j, P)$ for both $j = 0, 1$
- $\mathbb{Z}[\Gamma_4]$-homology equivalences $g^j : \overline{Z}^j \to \partial\overline{X}$ extending $\partial\overline{h}^j : \partial\overline{Y}^j \to \partial\overline{X}$
- a homeomorphism $\varphi : \overline{Y}^0 \cup_{\partial\overline{Y}^0} \overline{Z}^0 \to \overline{Y}^1 \cup_{\partial\overline{Y}^1} \overline{Z}^1$ relative to P

such that $(\overline{h}^1 \cup g^1) \circ \varphi$ is homotopic to $(\overline{h}^0 \cup g^0)$ relative to P. See [FQ90, §11.1].

Lemma 3.6. *Suppose $n \geq 4$. There is a bijection:*

$$\Phi \ : \ \mathscr{S}_{\mathrm{TOP}}(\overline{X}, \partial\overline{X}) \ \xrightarrow{\approx} \ \mathscr{S}^{iso}_{\mathrm{TOP}}(T^n, C_2).$$

Proof. First, suppose $n \geq 5$. Let $(\overline{h}, \partial\overline{h}) : (\overline{Y}, \partial\overline{Y}) \to (\overline{X}, \partial\overline{X})$ be a homotopy equivalence of pairs, where $\overline{Y}$ is a compact n-dimensional topological manifold. So $[\overline{h}, \partial\overline{h}]$ is an element of $\mathscr{S}_{\mathrm{TOP}}(\overline{X}, \partial\overline{X})$. Denote $\hat{X}$ and $\hat{Y}$ as the corresponding double cover of $\overline{X}$ and $\overline{Y}$. Passage to double covers induces a C_2-equivariant homotopy equivalence $(\hat{h}, \partial\hat{h}) : (\hat{Y}, \partial\hat{Y}) \to (\hat{X}, \partial\hat{X})$. Each component $\partial_i\hat{Y}$ of $\partial\hat{Y}$ is homotopy equivalent to, and therefore homeomorphic to the sphere S^{n-1} (see [Sma61, FQ90]). Hence the cone $c(\partial_i\hat{Y})$ is homeomorphic to the disc D^n. So $N := \hat{Y} \cup \bigsqcup_i c(\partial_i\hat{Y})$ is a topological C_2-manifold. Thus, by coning off each map $\partial_i\hat{h}$, we obtain a function

$$\Phi : \mathscr{S}_{\mathrm{TOP}}(\overline{X}, \partial\overline{X}) \ \longrightarrow \ \mathscr{S}^{iso}_{\mathrm{TOP}}(T, C_2) \ ; \ (\overline{h}, \partial\overline{h}) \ \longmapsto \ \hat{h} \cup \bigsqcup_i c(\partial_i\hat{h}).$$

Now we show that Φ is a bijection by exhibiting its inverse. Let $f : N \to T^n$ be a C_2-isovariant homotopy equivalence. This gives a proper homotopy equivalence:

$$f/C_2 : (N - N^{C_2})/C_2 \ \longrightarrow \ (T - T^{C_2})/C_2.$$

Then, since all the ends of $(T - T^{C_2})/C_2$ are tame, the ends of $(N - N^{C_2})/C_2$ are also tame. Note $\mathrm{Wh}(C_2) = \widetilde{K}_0(\mathbb{Z}[C_2]) = 0$. Then, by a theorem of Siebenmann [Sie65] (or of Freedman if $n = 5$, see [FQ90]), we can fit a unique boundary, $\partial\overline{N}$ onto $(N - N^{C_2})/C_2$, thereby creating a compact manifold $\overline{N}$, unique up to homeomorphism. So we can extend f/C_2 to $\partial\overline{N}$. (Here a small proper equivariant homotopy of f/C_2 may be needed before the extension.) This construction, $[N, f] \mapsto [\overline{N}, \overline{f}]$ is clearly inverse to Φ. We conclude that Φ is both surjective and injective.

Finally, it remains to consider Φ for $n = 4$. Let $(\overline{h}, \partial\overline{h}) : (\overline{Y}, \partial\overline{Y}) \to (\overline{X}, \partial\overline{X})$ be a map of pairs such that $\overline{h} : \overline{Y} \to \overline{X}$ is a homotopy equivalence of 4-manifolds and each $\partial_i\overline{h} : \partial_i\overline{Y} \to \partial_i\overline{X}$ is a $\mathbb{Z}[C_2]$-homology equivalence of 3-manifolds. Recall the notation $\hat{(\ )}$ for the double cover of $\overline{(\ )}$, used above for $n \geq 5$. For each i, by [FQ90, Proposition 11.1C], there is a compact contractible 4-manifold $c^*(\partial_i\hat{Y})$ with a C_2-action such that its C_2-equivariant boundary is the homology 3-sphere $\partial_i\hat{Y}$ and it has a single fixed point. It is unique up to C_2-homeomorphism. Using that isolated fixed point, one can construct a C_2-isovariant homotopy equivalence

$$c^*(\partial_i\hat{h}) \ : \ c^*(\partial_i\hat{Y}) \ \longrightarrow \ c(\partial_i\hat{X}).$$

Suppose $(\overline{Y}^0, \overline{h}^0, \partial\overline{h}^0)$ is equivalent to $(\overline{Y}^1, \overline{h}^1, \partial\overline{h}^1)$, in the sense of Definition 3.4. In the setting of Remark 3.5, there is a C_2-homeomorphism $\hat{Y}^0 \cup \hat{Z}^0 \to \hat{Y}^1 \cup \hat{Z}^1$. For each $j = 0, 1$, by uniqueness in [FQ90, Prop. 11.1C], there are C_2-homeomorphisms

$$c^*(\partial_i\hat{Y}^j) \ \longrightarrow \ \hat{Z}^j_i \cup c^*(\hat{P}_i).$$

Using the identity map on each $c^*(\hat{P}_i)$, these produce a C_2-homeomorphism

$$N^{0*} := \hat{Y}^0 \cup \bigsqcup_i c^*(\partial_i\hat{Y}^0) \ \longrightarrow \ N^{1*} := \hat{Y}^1 \cup \bigsqcup_i c^*(\partial_i\hat{Y}^1).$$

Thus we may define Φ on equivalence classes similarly to the high-dimensional case, except we use the "homotopy cones" c^* instead of the "honest cones" c. The argument for showing that Φ is a bijection as in the high-dimensional case, except for surjectivity we invoke the Weak End Theorem [FQ90, Thm. 11.9B] and for injectivity we invoke the Classification of Weak Collars [FQ90, Thm. 11.9C(3)]. □

Lemma 3.7. *Suppose $n \geq 4$. Consider the above-defined structure sets.*

(1) *The following forgetful maps are bijections:*

$$\begin{aligned}\Psi \quad &: \quad \mathscr{S}^{iso}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n) \xrightarrow{\approx} \mathscr{S}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n)\\ \psi \quad &: \quad \mathscr{S}^{iso}_{\mathrm{TOP}}(T^n, C_2) \xrightarrow{\approx} \mathscr{S}_{\mathrm{TOP}}(T^n, C_2).\end{aligned}$$

(2) *The following forgetful map is a bijection:*

$$\chi \quad : \quad \mathscr{S}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n) \xrightarrow{\approx} \mathscr{S}(\Gamma_n).$$

Proof. For Part (1), in view of the bijections u and u^{iso}, it suffices to prove that ψ is a bijection. We do this in Appendix A.

For Part (2), it is immediate from Theorem 2.1(2) that χ is injective. We must prove it is surjective. Let $[M, \Gamma_n]$ be in $\mathscr{S}(\Gamma_n)$. Let $\widetilde{J} : M \to \mathbb{R}^n$ be its classifying map. Since $M/\mathbb{Z}^n$ is an Eilenberg–MacLane space of form $K(\mathbb{Z}^n, 1) \simeq T^n$ and also an n-manifold, it follows that $M/\mathbb{Z}^n$ is compact. So M/Γ_n is compact. The fixed set in M of each finite subgroup of Γ_n is contractible, so $\widetilde{J}$ is a homotopy equivalence. Therefore $[(M, \Gamma_n), \widetilde{J}] \in \mathscr{S}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n)$ and so χ is surjective. □

Proposition 3.8. *Suppose $n \geq 4$. The following map α is a bijection:*

$$\alpha \ := \ \chi \circ u \circ \psi \circ \Phi \ : \ \mathscr{S}_{\mathrm{TOP}}(\overline{X}, \partial\overline{X}) \xrightarrow{\approx} \mathscr{S}(\Gamma_n) \ ; \qquad [\overline{X}, \mathrm{id}] \longmapsto [\mathbb{R}^n, \Gamma_n].$$

Proof. This follows immediately from (3.1), Lemma 3.6, and Lemma 3.7. □

4. Calculation of the isovariant structure set

Our ultimate goal here is to prove Theorem 1.2. We also establish the Topological Rigidity Conjecture (of Section 1.3) for the crystallographic groups Γ_n.

Throughout this section, we assume $n \geq 4$ and shall use the shorthand $\Gamma := \Gamma_n$. For each family $\mathcal{F}$ of subgroups of Γ, we write $E_{\mathcal{F}}\Gamma$ for the classifying space for Γ-CW complexes whose isotropy groups are in $\mathcal{F}$. We use the families $\mathfrak{fin}$, $\mathfrak{vc}$, and $\mathfrak{all}$, consisting of finite, virtually cyclic and all subgroups respectively. For the remainder of this section, since the subgroups of $\Gamma = \Gamma_n$ have trivial reduced lower K-theory, for ease of reading, we shall simply write $\underline{\mathbf{L}}$ for the $\mathrm{Or}(\Gamma)$-spectrum $\underline{\mathbf{L}}^h$.

Recall the Wall realization map [Wal99, Thms. 5.8, 6.5], relative to the boundary:

$$\partial^{Wall} \ : \ L^h_{n+1}(\Gamma, w_n) \longrightarrow \mathscr{S}_{\mathrm{TOP}}(\overline{X}) \longrightarrow \mathscr{S}_{\mathrm{TOP}}(\overline{X}, \partial\overline{X}).$$

Using Cappell's map [Cap74b], define a composite homomorphism

$$\beta \ : \bigoplus_{D \in (\mathrm{mid})(\Gamma)} \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\varepsilon, \mathbb{Z}^\varepsilon) \longrightarrow \bigoplus_{D \in (\mathrm{mid})(\Gamma)} L^h_{n+1}(D, w_n) \longrightarrow L^h_{n+1}(\Gamma, w_n).$$

Now we can define the desired basepoint-preserving function

$$\partial_\oplus \ := \ \alpha \circ \partial^{Wall} \circ \beta \ : \bigoplus_{D \in (\mathrm{mid})(\Gamma)} \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\varepsilon, \mathbb{Z}^\varepsilon) \longrightarrow \mathscr{S}(\Gamma).$$

It remains to show that $\partial_\oplus$ is a bijection of sets. This will span several lemmas.

4.1. **Algebraic structure groups and equivariant homology.** For cofibrant pairs (A, B) of topological spaces, A. Ranicki [Ran79] defined the algebraic structure groups $\mathcal{S}\langle 1\rangle^h_*$ as the homotopy groups of the homotopy cofiber of an assembly map $\alpha\langle 1\rangle$, so that there is a long exact sequence

$$\cdots \to H_*(A, B; \mathbf{L}\langle 1\rangle) \xrightarrow{\alpha\langle 1\rangle} L^h_*(A, B) \to \mathcal{S}\langle 1\rangle^h_*(A, B) \to H_{*-1}(A, B; \mathbf{L}\langle 1\rangle) \xrightarrow{\alpha\langle 1\rangle} \cdots .$$

Here $\mathbf{L}\langle 1\rangle$ is the 1-connective cover of the 4-periodic surgery spectrum $\mathbf{L}$ algebraically defined in [Ran79]. (The homotopy invariant functor $\mathcal{S}\langle 1\rangle^h_*$ is a desuspended chain-complex analogue of the geometric structure groups $\mathscr{S}^h_*$ originally defined by F. Quinn.) When a map $i : B \to A$ is understood, we shall write (A, B) for the cofibrant pair $(\mathrm{Cyl}(i), B)$. The relative L-groups $L^h_*(A, B) = L^h_*(i : B \to A)$ were defined algebraically by Ranicki ([Ran81]), following C.T.C. Wall [Wal99].

For computational purposes, we employ the non-connective version $\mathcal{S}^h_*$ of $\mathcal{S}\langle 1\rangle^h_*$. It is the homotopy groups of a homotopy cofiber of an assembly map α:

$$\cdots \to H_*(A, B; \mathbf{L}) \xrightarrow{\alpha} L^h_*(A, B) \to \mathcal{S}^h_*(A, B) \to H_{*-1}(A, B; \mathbf{L}) \xrightarrow{\alpha} \cdots .$$

Remark 4.1. Suppose $B = \varnothing$ and that A is the quotient of a free Γ-action on a space $\widetilde{A}$ each of whose components is simply connected. Write $\Pi_0(\widetilde{A})$ for the Γ-set of components of $\widetilde{A}$; there is a canonical Γ-map $\widetilde{A} \to \Pi_0(\widetilde{A})$. By Theorem B.1, the Quinn–Ranicki assembly map can be naturally identified with the Davis–Lück assembly map, at the spectrum level. Then the cofibers of these assembly maps agree in a functorial manner. Specifically, Appendix B constructs an isomorphism in $\operatorname{Ho}\operatorname{sc}(\Gamma,1)\mathcal{CW}\text{-}\mathsf{Spectra}$, whose value on $\widetilde{A}$ after the application of homotopy groups gives an isomorphism:

$$H_*^\Gamma(\Pi_0(\widetilde{A}),\widetilde{A};\underline{\mathbf{L}}) \xrightarrow{\cong} \mathcal{S}_*^h(A). \tag{4.1}$$

Write $\mathbb{R}^n_{\text{free}} := \{x \in \mathbb{R}^n \mid \Gamma_x = 1\}$ for those points with trivial isotropy group. Observe that $\mathbb{R}^n_{\text{free}}$ equivariantly deformation retracts to the universal cover of $\overline{X}$. There is a canonical Γ-map from $\mathbb{R}^n$ to its singleton $\{\mathbb{R}^n\}$ with trivial Γ-action.

Lemma 4.2. *There is a commutative diagram with long exact rows:*

$$\begin{array}{ccccccc}
H_*^\Gamma(\mathbb{R}^n,\mathbb{R}^n_{\text{free}};\underline{\mathbf{L}}) & \longrightarrow & H_*^\Gamma(\{\mathbb{R}^n\},\mathbb{R}^n_{\text{free}};\underline{\mathbf{L}}) & \longrightarrow & H_*^\Gamma(\{\mathbb{R}^n\},\mathbb{R}^n;\underline{\mathbf{L}}) & \longrightarrow & H_{*-1}^\Gamma(\mathbb{R}^n,\mathbb{R}^n_{\text{free}};\underline{\mathbf{L}}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{S}_*^h(\partial\overline{X}) & \longrightarrow & \mathcal{S}_*^h(\overline{X}) & \longrightarrow & \mathcal{S}_*^h(\overline{X},\partial\overline{X}) & \longrightarrow & \mathcal{S}_{*-1}^h(\partial\overline{X}).
\end{array}$$

Furthermore, the vertical maps are isomorphisms of abelian groups.

Proof. The top row is the long exact sequence of the triple $(\{\mathbb{R}^n\},\mathbb{R}^n,\mathbb{R}^n_{\text{free}})$ in Γ-equivariant $\mathbf{L}$-homology [DL98]. The bottom row is the long exact sequence of the pair $(\overline{X},\partial\overline{X})$ in algebraic structure groups [Ran92]. The inner left vertical map is induced on homotopy groups from (4.1) for $\widetilde{A} = \mathbb{R}^n_{\text{free}}$. Write $D(\mathbb{R}^n_{\text{sing}})$ for the Γ-subset of points in $\mathbb{R}^n$ of distance ≤ 0.2 from $\mathbb{R}^n_{\text{sing}}$. Define $S(\mathbb{R}^n_{\text{sing}}) := \partial D(\mathbb{R}^n_{\text{sing}})$. The outer vertical maps are induced on homotopy groups from (4.1) for $\widetilde{A} = S(\mathbb{R}^n_{\text{sing}})$, precomposed with the inverse of the induced excision map

$$H_*^\Gamma(D(\mathbb{R}^n_{\text{sing}}),S(\mathbb{R}^n_{\text{sing}});\underline{\mathbf{L}}) \longrightarrow H_*^\Gamma(\mathbb{R}^n,\mathbb{R}^n_{\text{free}};\underline{\mathbf{L}}).$$

By functoriality of (4.1), the left square's diagram of spectra homotopy-commutes. In particular, the left square itself commutes. Then the inner right map is induced from a well-defined homotopy class of map of spectra. So the middle square and right square are defined and commute. Therefore, since (4.1) implies the outer maps and inner left map are isomorphisms, by the Five Lemma, we conclude that the inner right map is an isomorphism also. □

Remark 4.3. By the Isomorphism Conjecture [BL] and Bartels' splitting theorem [Bar03] on the top row, we conclude that the connecting homomorphism $\mathcal{S}_*^h(\overline{X},\partial\overline{X}) \to \mathcal{S}_{*-1}^h(\partial\overline{X})$ is zero. Therefore, using Ranicki's natural bijection (see Remark 4.7 below), the forgetful map $\mathscr{S}_{\text{TOP}}(\overline{X},\partial\overline{X}) \to \mathscr{S}_{\text{TOP}}(\partial\overline{X})$ is constant.

We now calculate $H_*^\Gamma(E_{\mathtt{vc}}\Gamma, E_{\mathtt{fin}}\Gamma;\underline{\mathbf{L}})$ by using a specific model for the spaces involved. Models of $E_{\mathtt{vc}}G$ for crystallographic groups G are due to Connolly–Fehrman–Hartglass [CFH]. For any group G, Lück–Weiermann [LW12] built models of $E_{\mathtt{vc}}G$ from $E_{\mathtt{fin}}G$. However, the following lemma is shown directly for our Γ.

Let C be an infinite cyclic subgroup of Γ. Let $\mathcal{P}_C$ denote the collection of all affine lines $\ell \subset \mathbb{R}^n$ which are stablized by C. Endow $\mathcal{P}_C$ with the affine structure and topology of a copy of $\mathbb{R}^{n-1}$. Since $\mathcal{P}_C$ is a partition of $\mathbb{R}^n$, there is a continuous quotient map $\pi_C : \mathbb{R}^n \to \mathcal{P}_C$. Since C is normal in Γ, the Γ-action on $\mathbb{R}^n$ extends to a Γ-action on the mapping cylinder, $\operatorname{Cyl}(\pi_C)$.

Let $\operatorname{mic}(\Gamma)$ denote the collection of maximal infinite cyclic subgroups of Γ.

Lemma 4.4. *A model E for the classifying space $E_{\mathrm{vc}}\Gamma$ (classifying Γ-CW complexes with virtually cyclic isotropy) is the union along $\mathbb{R}^n$ of mapping cylinders:*

$$E := \bigcup_{C \in \mathrm{mic}(\Gamma)} \mathrm{Cyl}(\pi_C : \mathbb{R}^n \longrightarrow \mathcal{P}_C).$$

Proof. If H is a finite nontrivial subgroup of Γ, then E^H is a tree with one edge in $\mathrm{Cyl}(\pi)^C$ for each $C \in \mathrm{mic}(\Gamma)$. So E^H is contractible. If H is infinite cyclic or infinite dihedral, there is just one $C \in \mathrm{mic}(\Gamma)$ for which $\mathrm{Cyl}(\pi_C)^H$ is non-empty. For this C, observe that $\mathrm{Cyl}(\pi_C)^H$ is a single point when H is dihedral and is all of $\mathrm{Cyl}(\pi_C)$ when H is cyclic. Also $E = E^{\{1\}}$ is contractible, and E^H is empty if H is not virtually cyclic.

Finally, we must prove that E has the structure of a Γ-CW complex. We begin by assuming K is a Γ-CW structure on $\mathbb{R}^n$ which is *convex*. By this we mean each closed cell is convex, and its boundary is a subcomplex. It suffices to show how to extend K to a Γ-CW structure over each mapping cylinder, $\mathrm{Cyl}(\pi_C)$ in E.

So fix C and parametrize $\mathrm{Cyl}(\pi_C)$ as

$$\mathrm{Cyl}(\pi_C) = \mathbb{R}^n \times [-1,1] \cup_{\pi_C} \mathcal{P}_C, \qquad \text{where } (x,1) = \pi_C(x) \text{ for all } x \in \mathbb{R}^n.$$

There are convex Γ-CW structures, L on $\mathcal{P}_C$, and $\hat{L}$ on $\mathbb{R}^n$, so that each j-cell f of L has the form $\pi_C(\hat{f})$ for some $(j+1)$-cell $\hat{f}$ of $\hat{L}$. This endows $\mathbb{R}^n \times [0,1] \cup_{\pi_C} \mathcal{P}_C$ with the structure of a Γ-CW complex, K_+ so that $\mathbb{R}^n \times 0$ is the complex $\hat{L}$. Now $\hat{L}$ and K have a common subdivision K', since each is convex. There is then a CW structure K_- on $\mathbb{R}^n \times [-1,0]$ in which K, K' and $\hat{L}$ are identified with $\mathbb{R}^n \times \{-1\}, \mathbb{R}^n \times \{-\frac{1}{2}\}$ and $\mathbb{R}^n \times \{0\}$ respectively as subcomplexes. (Also, $e \times [-1, -\frac{1}{2}]$ and $f \times [-\frac{1}{2}, 0]$ are cells of K_- if e and f are cells of K and $\hat{L}$ respectively.) Then $K_+ \cup K_-$ is the required Γ-CW structure on $\mathrm{Cyl}(\pi_C)$. □

Each infinite dihedral subgroup D of Γ contains a unique maximal infinite cyclic subgroup C. Moreover, D has a unique invariant line, $\ell_D \subset \mathbb{R}^n$. The image of ℓ_D in $\mathcal{P}_C$ is a single point, which we denote by the singleton $\{\ell_D\} = \pi_C(\ell_D)$.

Lemma 4.5. *The inclusion-induced map is an isomorphism of abelian groups:*

$$\bigoplus_{D \in (\mathrm{mid})(\Gamma)} H_*^D(\{\ell_D\}, \ell_D; \underline{\mathbf{L}}) \;\longrightarrow\; H_*^\Gamma(E, \mathbb{R}^n; \underline{\mathbf{L}}).$$

Proof. Lemma 4.1 of [DQR] allows one to translate between maps induced by Γ-maps of classifying spaces for actions with isotropy in a family to maps induced by maps of $\mathrm{Or}(\Gamma)$-spectra. There is a homotopy cofiber sequence of $\mathrm{Or}(\Gamma)$-spectra:

$$\mathbf{L}_{\mathtt{fin}} \longrightarrow \mathbf{L} \longrightarrow \mathbf{L}/\mathbf{L}_{\mathtt{fin}}.$$

By [DQR, Lemma 4.1(ii)], the following absolute homology group vanishes:

$$H_*^\Gamma(\mathbb{R}^n; \underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}}) \;=\; 0.$$

Also, by [DQR, Lemma 4.1(iii)], the following relative homology group vanishes:

$$H_*^\Gamma(E, \mathbb{R}^n; \underline{\mathbf{L}}_{\mathtt{fin}}) \;=\; 0.$$

So we obtain a composite isomorphism, informally first observed by Quinn:

$$H_*^\Gamma(E, \mathbb{R}^n; \underline{\mathbf{L}}) \xrightarrow{\cong} H_*^\Gamma(E, \mathbb{R}^n; \underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}}) \xleftarrow{\cong} H_*^\Gamma(E; \underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}}). \tag{4.2}$$

Now, since $N_\Gamma(C) = \Gamma$, by Lemma 4.4 and excision, we obtain:

$$\bigoplus_{C \in \mathrm{mic}(\Gamma)} H_*^\Gamma(\mathcal{P}_C; \underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}}) \xrightarrow{\cong} H_*^\Gamma(E; \underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}}). \tag{4.3}$$

Fix $C \in \mathrm{mic}(\Gamma)$. Observe that the action of the group Γ/C on the parallel pencil $\mathcal{P}_C$ has a discrete singular set:

$$\mathrm{sing}\,\mathcal{P}_C \;:=\; \{\ell_D \in \mathcal{P}_C \mid \exists D \in \mathrm{mid}(\Gamma) : D \supset C\}.$$

Let U be a Γ-tubular neighborhood of $\mathrm{sing}\,\mathcal{P}_C$ in $\mathcal{P}_C$. Write $V := \mathcal{P}_C - \mathrm{sing}\,\mathcal{P}_C$. Recall, by a theorem of J. Shaneson [Sha69], that the following assembly map is a homotopy equivalence:

$$A_C \;:\; S^1_+ \wedge \mathbf{L}(1) \xrightarrow{\simeq} \mathbf{L}(C).$$

That is, the spectrum $(\underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}})(\Gamma/C)$ is contractible. So, since V has isotropy C:

$$H_*^\Gamma(U \cap V; \underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}}) \;=\; 0 \;=\; H_*^\Gamma(V; \underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}}).$$

Since $\mathrm{sing}\,\mathcal{P}_C$ is discrete, there is a Γ-homotopy equivalence:

$$\bigsqcup_{\substack{D \supset C \\ D \in (\mathrm{mid})(\Gamma)}} \Gamma \times_D \{\ell_D\} \xrightarrow{\cong} \mathrm{sing}\,\mathcal{P}_C \xrightarrow{\simeq} U.$$

So the homotopy and excision axioms of equivariant homology imply:

$$\bigoplus_{\substack{D \supset C \\ D \in (\mathrm{mid})(\Gamma)}} H_*^\Gamma(\Gamma \times_D \{\ell_D\}; \underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}}) \xrightarrow{\approx} H_*^\Gamma(U; \underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}}) \xrightarrow{\cong} H_*^\Gamma(\mathcal{P}_C; \underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}}).$$

Thus (4.3) and the induction axiom of equivariant homology imply:

$$\bigoplus_{D \in (\mathrm{mid})(\Gamma)} H_*^D(\{\ell_D\}; \underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}}) \xrightarrow{\cong} H_*^\Gamma(E; \underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}}).$$

Finally, since ℓ_D is a model for $E_{\mathtt{fin}}D$, by [DQR, Lemma 4.1(ii)] again, we obtain:

$$H_*^D(\{\ell_D\}, \ell_D; \underline{\mathbf{L}}) \xrightarrow{\cong} H_*^D(\{\ell_D\}, \ell_D; \underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}}) \xleftarrow{\cong} H_*^D(\{\ell_D\}; \underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}}).$$

This completes the proof. □

Lemma 4.6. *Recall $\varepsilon = (-1)^n$. Let D be an infinite dihedral subgroup of Γ. Then the following composite map is an isomorphism of abelian groups:*

$$\mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\varepsilon, \mathbb{Z}^\varepsilon) \longrightarrow L_{n+1}(D, w_n) = H_{n+1}^D(\{\ell_D\}; \underline{\mathbf{L}}) \longrightarrow H_{n+1}^D(\{\ell_D\}, \ell_D; \underline{\mathbf{L}}).$$

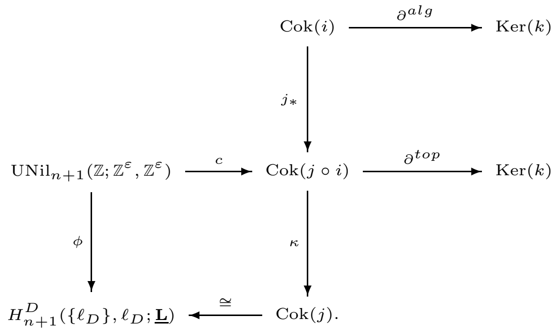
Proof. Denote the map under consideration by ϕ. Consider the maps

$$L_{n+1}(C_2, \varepsilon) \oplus L_{n+1}(C_2, \varepsilon) \xrightarrow{i} H_{n+1}^D(\ell_D; \underline{\mathbf{L}}) \xrightarrow{j} L_{n+1}(D, w_n)$$

and

$$L_n(1) \xrightarrow{k} L_n(C_2, \varepsilon) \oplus L_n(C_2, \varepsilon).$$

Observe that ϕ has a factorization given by the commutative diagram



Here, the map ∂^{alg} is a monomorphism induced from the connecting map in the Mayer–Vietoris sequence for D-equivariant $\mathbf{L}$-homology. The map ∂^{top} is induced from the connecting map in Cappell's exact sequence in L-theory. By [Cap74b, Theorem 2], the map c is injective. By [Cap74b, Theorem 5(ii)], the middle row is exact. By Bartels' theorem [Bar03], the bottom horizontal map is an isomorphism. For general group-theoretic reasons, the middle column is exact and κ is surjective.

Note, by the calculation in [Wal99, Theorem 13A.1], that $\mathrm{Ker}(k) = 0$ for all n. Then c is surjective and $\mathrm{Cok}(i) = 0$. Therefore κ, hence ϕ, is an isomorphism. □

Remark 4.7. In the context of the surgery fibration sequence [Ran92, Thm. 18.5], Ranicki's total surgery obstruction for homotopy equivalences is a bijection:

$$s \ : \ \mathscr{S}_{\mathrm{TOP}}(\overline{X}, \partial\overline{X}) \xrightarrow{\approx} \mathcal{S}\langle 1\rangle^h_{n+1}(\overline{X}, \partial\overline{X}).$$

Indeed this holds for $n = 4$, since $\Gamma_4 = \mathbb{Z}^4 \rtimes_{-1} C_2$ satisfies the Null Disc Lemma [FQ90], and since we use homology equivalences on the 3-dimensional boundary. Since $\overline{X}$ is n-dimensional, by the Atiyah–Hirzebruch spectral sequence, we obtain:

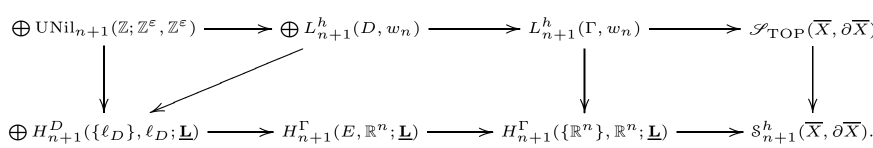
$$H_m(\overline{X}, \partial\overline{X}; \mathbf{L}/\mathbf{L}\langle 1\rangle) = 0 \quad \text{for all } m > n.$$

Hence the following induced map is an isomorphism:

$$\mathcal{S}\langle 1\rangle^h_{n+1}(\overline{X}, \partial\overline{X}) \xrightarrow{\cong} \mathcal{S}^h_{n+1}(\overline{X}, \partial\overline{X}).$$

Proposition 4.8. *The composite $\partial^{Wall} \circ \beta$ is a bijection of pointed sets.*

Proof. Consider the commutative diagram



The composition of the three maps in the top row is $\partial^{Wall} \circ \beta$. By Lemma 4.6, the leftmost vertical map is an isomorphism. By Lemma 4.5, the leftmost map of the bottom row is an isomorphism. By Lemma 4.4 and the Farrell–Jones Conjecture [BL], the middle map of the bottom row is an isomorphism. By Lemma 4.2, the rightmost map of the bottom row is an isomophism. By Remark 4.7, the rightmost vertical map is an bijection. This completes the proof. □

Proof of Theorem 1.2. This follows from Propositions 3.8 and 4.8, and (1.1). □

4.2. **Verification of the Topological Rigidity Conjecture.** Lastly, we show that our precise conjecture is satisfied for our family of crystallographic examples.

Proof of Conjecture 1.5 for $(X^n, \Gamma) = (\mathbb{R}^n, \Gamma_n)$. Consider the commutative diagram

$$\begin{array}{ccccccc} H^\Gamma_{n+1}(\mathbb{R}^n; \underline{\mathbf{L}}) & \xrightarrow{A^{\mathtt{vc}}_{\mathtt{fin}}} & H^\Gamma_{n+1}(E; \underline{\mathbf{L}}) & \xrightarrow{A^{\mathtt{all}}_{\mathtt{vc}}} & L^h_{n+1}(\Gamma, w_n) & \xrightarrow{\partial^{Wall}} & \mathscr{S}_{\mathrm{TOP}}(\overline{X}, \partial\overline{X}) \\ & & \downarrow & & \downarrow & & \downarrow \approx \\ & & H^\Gamma_{n+1}(E, \mathbb{R}^n; \underline{\mathbf{L}}) & \xrightarrow{\cong} & H^\Gamma_{n+1}(\{\mathbb{R}^n\}, \mathbb{R}^n; \underline{\mathbf{L}}) & \xrightarrow{\cong} & \mathcal{S}^h_{n+1}(\overline{X}, \partial\overline{X}). \end{array}$$

The three bijections hold as in the earlier diagram. Define a precursor map $\widehat{\partial}$ by

$$\widehat{\partial} \ := \ u^{iso} \circ \Phi \circ \partial^{Wall} \circ A^{\mathtt{all}}_{\mathtt{vc}} \ : \ H^\Gamma_{n+1}(E; \underline{\mathbf{L}}) \longrightarrow \mathscr{S}^{iso}_{rel}(\mathbb{R}^n, \Gamma).$$

By a theorem of A. Bartels [Bar03], there is a short exact sequence

$$0 \longrightarrow H^\Gamma_{n+1}(\mathbb{R}^n; \underline{\mathbf{L}}) \xrightarrow{A^{\mathtt{vc}}_{\mathtt{fin}}} H^\Gamma_{n+1}(E; \underline{\mathbf{L}}) \longrightarrow H^\Gamma_{n+1}(E, \mathbb{R}^n; \underline{\mathbf{L}}) \longrightarrow 0.$$

Then $\widehat{\partial}$ induces a map ∂ from $\mathrm{Cok}(A^{\mathtt{vc}}_{\mathtt{fin}})$. Using the identification (4.2), we obtain:

$$\partial \ : \ H^\Gamma_{n+1}(E_{\mathtt{vc}}\Gamma; \underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}}) \ \longrightarrow \ \mathscr{S}^{iso}_{rel}(\mathbb{R}^n, \Gamma).$$

Therefore, since u^{iso} and Φ are bijections, this desired map ∂ is also a bijection. ☐

5. Classification of involutions on tori

The goal of this section is to prove Theorem 1.1.

Proof of Theorem 1.1(1). This is immediate from Theorem 2.1(1). ☐

Proof of Theorem 1.1(2). The case $n = 0$, is trivial: $T^0 = \mathbb{R}^0/\mathbb{Z}^0 = pt$.

Assume $n = 1$. Set $D^1_\pm \ := \ \{z = x + iy \in S^1 \subset \mathbb{C} \mid \pm y \geq 0\}$.

Write $a, b \in N$ for the fixed points of σ. Let $f : D^1_+ \to N$ be a homeomorphism of D^1_+ onto either arc in N with endpoints a and b. Extend f to a continuous map $f : S^1 \to N$ by setting

$$f(z) = \sigma f(\overline{z}) \qquad \forall z \in D^1_-.$$

Then $f : (S^1, C_2) \to (N, C_2)$ is an equivariant homeomorphism.

Assume $n = 2, 3$. There is a homeomorphism $f : N \to T^n$ (by work of Perelman [And04] for $n = 3$). We want to show that each fixed point $x \in N^{C_2}$ has an invariant neighborhood D such that (D, C_2) is homeomorphic to (D^n, C_2), the orthogonal action fixing only 0.

To see this, note the involution σ of (N, x) lifts to an involution of the universal cover $(\widetilde{N}, \widetilde{x})$ (for any point $\widetilde{x}$ over x) whose one point compactification provides an involution $\widetilde{\sigma}$ with two fixed points on S^n. If this involution is standard, this yields arbitrarily small standard disk neighborhoods of $\widetilde{x}$ and the required invariant standard disk neighborhood (D, C_2) of x in N.

But this involution on S^n *is* standard. For, when $n = 2$ this was proved by Kérékjartò, Brouwer, and Eilenberg (see [CK94]); when $n = 3$ it was proved by Hirsch–Smale and Livesay (see [Rub76]).

Around each fixed point remove the interior of such an invariant standard disk, to obtain a compact manifold with a free involution, (N^n_0, σ_0) whose boundary consists of 2^n copies of S^{n-1} with the antipodal involution. This manifold with free involution is smooth. This is by Moise [Moi54, Thm. 9.1] and Whitehead

[Whi61] if $n = 3$, and by the classification of surfaces in $n = 2$. Gluing back the 2^n standard disks, we conclude N is smooth, and σ is smooth.

If $n = 3$, a theorem of Meeks–Scott [MS86] then proves there is a flat, invariant Riemannian metric on (N, C_2). So we may assume $N = T^3$ and C_2 acts by isometries, and the origin is an isolated fixed point. The group of all isometries fixing the origin is $O(3) \cap GL_3(\mathbb{Z})$. Only $-I$ acts with the origin as an isolated fixed point. This is the standard involution on T^3. This proves the theorem when $n = 3$.

If $n = 2$, we see by the Euler characteristic that N/C_2 must be S^2, and (N, C_2) must be the two-fold cover, branched at four points of S^2. This, again, is the standard involution on T^2. This proves the theorem when $n = 2$.

Assume $n \geq 4$ and $n \equiv 0, 1 \pmod 4$. By Theorem 2.1(2), there is a C_2-homotopy equivalence $J : N \to T^n$. Recall, from Section 3, that there is a bijection $u : \mathscr{S}_{\mathrm{TOP}}(T^n, C_2) \to \mathscr{S}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n)$. By Lemma 3.7(2), there is a bijection $\chi : \mathscr{S}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n) \to \mathscr{S}(\Gamma_n)$. By Theorem 1.2, $\mathscr{S}(\Gamma_n)$ is a singleton. Thus $\mathscr{S}_{\mathrm{TOP}}(T^n, C_2)$ is also. Therefore J is C_2-homotopic to a homeomorphism. □

The proof of Theorem 1.1(3) will take a little preliminary work. Let $\mathcal{T}_n$ denote the set of equivariant homeomorphism classes of H_1-negative C_2-manifolds (N^n, C_2) for which N^n has the homotopy type of T^n. We must prove $\mathcal{T}_n$ is infinite if $n \equiv 2, 3 \pmod 4$ and $n \geq 6$. Write $\mathrm{hAut}(T^n, C_2)$ for the group of C_2-homotopy classes of C_2-homotopy equivalences, $f : (T^n, C_2) \to (T^n, C_2)$. Note, by Theorem 2.1(2), that

$$\mathcal{T}_n \approx \mathscr{S}_{\mathrm{TOP}}(T^n, C_2)/\mathrm{hAut}(T^n, C_2). \tag{5.1}$$

We begin by constructing a homomorphism $\mathrm{Aut}(\Gamma_n) \to \mathrm{hAut}(T^n, C_2)$.

Recall $\Gamma_n = \mathbb{Z}^n \rtimes_{-1} C_2$. To avoid notational confusion we will write A_n for the subgroup of translations and σ_0 for the reflection through 0:

$$A_n = \{(x, 1) \in \mathbb{Z}^n \rtimes_{-1} C_2 \mid x \in \mathbb{Z}^n\}. \qquad \sigma_0 = (0, \sigma) \in \mathbb{Z}^n \rtimes_{-1} C_2$$

For each automorphism $a : \Gamma_n \to \Gamma_n$ choose an a-equivariant continuous map

$$\widetilde{J}_a : \mathbb{R}^n \longrightarrow \mathbb{R}^n. \text{ Therefore: } \widetilde{J}_a(\gamma \cdot v) = a(\gamma) \cdot \widetilde{J}_a(v), \ \ \forall (\gamma, v) \in \Gamma_n \times \mathbb{R}^n. \tag{5.2}$$

Note $\widetilde{J}_a$ is unique up to a-equivariant homotopy. Since $a(A_n) = A_n$, we see $\widetilde{J}_a$ descends to a map, $J_a : (T^n, C_2) \to (T^n, C_2)$. So $[J_a] \in \mathrm{hAut}(T^n, C_2)$. From (5.2) we see that for all $a, b \in \mathrm{Aut}(\Gamma_n)$, $\widetilde{J}_{ab}$ and $\widetilde{J}_a \widetilde{J}_b$ are ab-equivariantly homotopic.

For each $x \in \Gamma_n$, we write $c(x)$ for the automorphism:

$$c(x) : \Gamma_n \longrightarrow \Gamma_n \ ; \qquad \gamma \longmapsto x\gamma x^{-1}.$$

If $t \in A_n$ is any translation, a valid choice for $\widetilde{J}_{c(t)}$ is:

$$\widetilde{J}_{c(t)} : \mathbb{R}^n \longmapsto \mathbb{R}^n \ ; \qquad v \longmapsto t \cdot v$$

since (5.2) holds for this choice. So this construction specifies a homomorphism

$$J : \mathrm{Aut}(\Gamma_n)/c(A_n) \longrightarrow \mathrm{hAut}(T^n, C_2) \ ; \qquad [a] \longmapsto [J_a].$$

Write $\mathrm{Aut}(\Gamma_n)_{\sigma_0} := \{a \in \mathrm{Aut}(\Gamma_n) \mid a(\sigma_0) = \sigma_0\}$. For $a \in \mathrm{Aut}(\Gamma_n)_{\sigma_0}$, let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the unique linear isomorphism satisfying: $T(t \cdot 0) = a(t) \cdot 0$, $\forall t \in A_n$. Then $T(\gamma \cdot x) = a(\gamma) \cdot x$ for all $x \in \mathbb{R}^n$, and therefore a valid choice for $\widetilde{J}_a$ is: $\widetilde{J}_a = T$.

Proposition 5.1. *J is an isomorphism.*

Proof. Let $[a] \in \mathrm{Ker}(J)$. We show $[a] = 1$. The isomorphism $k : A_n \cong H_1(T^n)$ of (2.2), and the fact that $(J_a)_* = \mathrm{id} : H_1(T^n) \to H_1(T^n)$ imply that $a(t) = t$ for all $t \in A_n$. Also J_a fixes the discrete set $(T^n)^{C_2}$, since $[J_a] = 1$. Therefore $\widetilde{J}_a(0) \in \mathbb{Z}^n$. Replacing $a \in [a]$ with $a \cdot c(t)$ for a suitable $t \in A_n$, we conclude for our new representative a that $\widetilde{J}_a(0) = 0$. So $a(\sigma_0) = \sigma_0$. But $\Gamma_n = \langle A_n, \sigma_0 \rangle$. So $a = \mathrm{id}_{\Gamma_n}$ and J is injective.

Now we show J is surjective. Let $[f] \in \mathrm{hAut}(T^n, C_2)$. Here $f : T^n \to T^n$ is a C_2-map. Let $a'' \in \mathrm{Aut}(\Gamma_n)_{\sigma_0}$ satisfy: $a''(t) = k^{-1} f_*^{-1}(k(t)) \in A_n$ for all $t \in A_n$. Then $(J_{a''} f)_* = \mathrm{id}_{H_1(T^n)}$. Let $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ be a lift of f. Note $0 \in \mathbb{R}^n_{\mathrm{sing}}$, and so $p := \widetilde{J}_{a''} \widetilde{f}(0) \in \mathbb{R}^n_{\mathrm{sing}}$. There is an involution $\sigma_p \in \Gamma_n - A_n$ fixing p.

Let $\mathrm{res} : \mathrm{Aut}(\Gamma_n) \to \mathrm{Aut}(A_n)$ be the restriction homomorphism. Observe $\mathrm{Ker(res)}$ acts transitively on $\Gamma_n - A_n$, since $\Gamma_n = \langle A_n, \sigma \rangle$ for any $\sigma \in \Gamma_n - A_n$, and each such σ is an involution. Therefore there exists $a' \in \mathrm{Ker(res)}$ such that $a'(\sigma_p) = \sigma_0$. So $0 = \widetilde{J}_{a'}(p) = \widetilde{J}_{a'} \widetilde{J}_{a''} \widetilde{f}(0)$. Note $\widetilde{J}_{a'a''} \widetilde{f} : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is A_n equivariant. But $J_{a'a''} f$ is C_2-equivariant, so $\widetilde{J}_{a'a''} \widetilde{f}$ is σ_0 equivariant. Therefore $\widetilde{J}_{a'a''} \widetilde{f}$ is Γ_n-equivariant and so $\widetilde{J}_{a'a''} \widetilde{f}$ is Γ_n-homotopic to $\mathrm{id}_{\mathbb{R}^n}$. Therefore $[f] = [J_a]$, where $a = (a'a'')^{-1}$. So J is surjective. □

Proof of Theorem 1.1(3). Assume $n \equiv 2, 3 \pmod 4$ and $n \geq 6$. For any group G, we are going to abbreviate

$$H^G \ := \ H^G_{n+1}(E_{\mathrm{vc}} G; \underline{\mathbf{L}}/\underline{\mathbf{L}}_{\mathtt{fin}}).$$

From Proposition 5.1 and Section 4.2 and (5.1), we see that $H^{\Gamma_n}/\mathrm{Aut}(\Gamma_n) \approx \mathcal{T}_n$. So we must prove that this set $H^{\Gamma_n}/\mathrm{Aut}(\Gamma_n)$ is infinite. The proof is based on the fact that, for any maximal infinite dihedral subgroup D, we have $H^D \cong \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\varepsilon, \mathbb{Z}^\varepsilon)$ by Lemma 4.6, and so H^D is an infinite set by (1.1). We will produce an injective map from an infinite set:

$$H^D/\mathrm{Aut}(D) \longrightarrow H^{\Gamma_n}/\mathrm{Aut}(\Gamma_n).$$

and this will show that $\mathcal{T}_n$ is an infinite set.

Take $D \in (\mathrm{mid})(\Gamma_n)$, a maximal dihedral subgroup. The inclusion $i_D : D \to \Gamma_n$ induces a map, $i_{D*} : H^D \to H^{\Gamma_n}$. It is easy to see that each automorphism of D extends to an automorphism of Γ_n. For any $a \in \mathrm{Aut}(\Gamma_n)$, we have

$$a_* \circ i_{D*} \ = \ i_{a(D)*} \circ (a|_D)_* \ : \ H^D \longrightarrow H^{\Gamma_n},$$

where $a|_D : D \to a(D)$ denotes the restriction of a.

By Lemma 4.5, the induced map $i_{D*} : H^D \to H^{\Gamma_n}$ is a monomorphism, and $i_{D*}(H^D) \cap i_{a(D)*}(H^D) = 0$ if $a(D)$ is not conjugate to D. But if $a(D) = c_\gamma(D)$ for some $\gamma \in \Gamma_n$, then the following diagram commutes:

$$\begin{array}{ccc} H^D & \xrightarrow{\ i_{D*}\ } & H^{\Gamma_n} \\ \Big\downarrow {\scriptstyle (c_{\gamma^{-1}}\, a|_D)_*} & & \Big\downarrow {\scriptstyle (c_{\gamma^{-1}})_* a_*} \\ H^D & \xrightarrow{\ i_{D*}\ } & H^{\Gamma_n}. \end{array} \tag{5.3}$$

By a theorem of Taylor [Tay73], for any $\gamma \in D$, the map $c_{\gamma*} : H^D \to H^D$ is just multiplication by $(-1)^k$, if $w(\gamma) = (-1)^k$. Now if $n \equiv 3 \pmod 4$, then $w(\gamma) = 1$ for all γ, and if $n \equiv 2 \pmod 4$, then $2 \cdot H^D = 0$, so, in all cases $c_{\gamma*} = \mathrm{id}$. For the same reason, if $\gamma \in \Gamma_n$, then $c_{\gamma*} = \mathrm{id} : H^{\Gamma_n} \to H^{\Gamma_n}$.

This together with (5.3) proves, first, that the induced map

$$(i_D)_* : H^D/\mathrm{Aut}(D) \to H^{\Gamma_n}/\mathrm{Aut}(\Gamma_n)$$

is injective, and second that $\mathrm{Inn}(D)$ acts trivially on the infinite group H^D. But $\mathrm{Aut}(D)/\mathrm{Inn}(D) \cong C_2$, so $H^D/\mathrm{Aut}(D)$ is an infinite set. Therefore $H^{\Gamma_n}/\mathrm{Aut}(\Gamma_n)$ and $\mathcal{T}_n$ are also infinite sets, as required. □

6. A non-trivial element of $\mathscr{S}(\Gamma_n)$

In this section, which is independent of the rest of the paper, we give a classical argument for the existence of non-trivial elements of $\mathscr{S}(\Gamma_n)$ for some n. Indeed the argument for the case $n \equiv 2 \pmod 4$, could have been written in 1976. It was in fact pointed out by Weinberger to the first author many years ago.

Theorem 6.1. *Suppose $n \equiv 2$ or $3 \pmod 4$ and $n \geq 6$. There exists a cocompact action of the group Γ_n on a manifold M^n such that (M^n, Γ_n) is simply isovariantly homotopy equivalent to $(\mathbb{R}^n, \Gamma_n)$ but is not equivariantly homeomorphic to $(\mathbb{R}^n, \Gamma_n)$.*

The proof depends only on an idea of Farrell [Far79] and Cappell's Splitting Theorem ([Cap74a, Theorem 6], [Cap76]). It does not depend on [BL].

Let $w_n : \Gamma_n = \mathbb{Z}^n \rtimes C_2 \to \{\pm 1\}$ be the homomorphism such that $\ker(w_n) = \mathbb{Z}^n$ if n is odd, and $w_n(\Gamma_n) = \{1\}$ if n is even. By Remark 1.3, there is a group isomorphism $\Gamma_{n-1} *_{\mathbb{Z}^{n-1}} \Gamma_{n-1} \to \Gamma_n$. By Cappell [Cap74b], this decomposition defines a split monomorphism

$$\rho : \mathrm{UNil}^h_{n+1}(R; \mathcal{B}, \mathcal{B}') \to L^h_{n+1}(\mathbb{Z}[\Gamma_n], w_n)\,;\ R = \mathbb{Z}[\mathbb{Z}^{n-1}]\,,\ \mathcal{B} = \mathcal{B}' = \mathbb{Z}[\Gamma_{n-1} - \mathbb{Z}^{n-1}].$$

Here R is a ring with involution given by: $\overline{a} = a^{-1}$ for all $a \in \mathbb{Z}^{n-1} \subset \Gamma_{n-1}$. Also $\mathcal{B}$ and $\mathcal{B}'$ are R-bimodules with involution: $\overline{b} = (-1)^n b^{-1}$ for all $b \in \Gamma_{n-1} - \mathbb{Z}^{n-1}$.

Lemma 6.2 (Cappell)**.** *The action of the abelian group $L^h_{n+1}(\mathbb{Z}[\Gamma_n], w_n)$ on the set $\mathscr{S}_{\mathrm{TOP}}(\overline{X}, \partial\overline{X})$ restricts to a free action of $\mathrm{UNil}^h_{n+1}(R; \mathcal{B}, \mathcal{B}')$ on $\mathscr{S}_{\mathrm{TOP}}(\overline{X}, \partial\overline{X})$.*

Proof. Cappell's Splitting Theorem (see [Cap74a, Thm. 6], [Cap76]) applies only to a *closed* manifold X with $\Gamma_n = \pi_1(X)$, if X admits a splitting $X = X_1 \cup_Y X_2$ consistent with the decomposition $\Gamma_n \cong \Gamma_{n-1} *_{\mathbb{Z}^{n-1}} \Gamma_{n-1}$. As stated, it does not apply to $\overline{X}$, since $\partial\overline{X}$ is non-empty. But $\overline{X}$ does split along a *closed submanifold*:

$$\overline{X} = X_1 \cup_Y X_2 \quad \text{where} \quad Y := \left\{ [t_1, \ldots, t_n] \in T^n \,\middle|\, t_1 = \pm\tfrac{1}{4} \right\} / C_2\,.$$

Here X_1 (and X_2) is defined similarly in $\overline{X}$ but with $t_1 \in [-\frac{1}{4}, \frac{1}{4}]$ (respectively, $t_1 \in [\frac{1}{4}, \frac{3}{4}]$). The fundamental groups of X_1, X_2, Y are the groups appearing in Remark 1.3 with $f(a_1, \ldots, a_n) = a_1$, So, by Theorem 6.4, we are done. Again, the key point is $\partial Y = \varnothing$. □

Lemma 6.3. *$\mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}^\varepsilon, \mathbb{Z}^\varepsilon)$ is a summand of $\mathrm{UNil}_n(R; \mathcal{B}, \mathcal{B}')$, if $\varepsilon = (-1)^n$. Furthermore, the groups $\mathrm{UNil}_{4k+2}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ and $\mathrm{UNil}_{4k+3}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ are non-zero.*

Proof. The first claim is immediate from the split epimorphism

$$\varepsilon : \Gamma_n \to \Gamma_1 = C_2 * C_2$$

of Remark 1.3, which induces a split epimorphism

$$\mathrm{UNil}_n(R; \mathcal{B}, \mathcal{B}') \to \mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}^\varepsilon, \mathbb{Z}^\varepsilon).$$

The second claim has been known for many years. See [Cap73] or [CK95]. But for the reader's convenience, here is a very easy proof. Farrell (see [Far79]) extended Cappell's homomorphism ρ, mentioned above, to a homomorphism,

$$\rho' : \mathrm{UNil}_{2k}(R; R, R) \to L_{2k}(R)$$

for any ring with involution R. But the non-zero element of $L_2(\mathbb{Z})$ is the class of the rank-two (-1)-quadratic form with Arf invariant 1. This element is $\rho'([\zeta])$ where $[\zeta] \in \mathrm{UNil}_2(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ is the class of the unilform, $\zeta = (P, \lambda, \mu, P', \lambda', \mu')$, where,

$$P = \mathbb{Z}e,\ P' = \mathbb{Z}f,\ \lambda = 0,\ \lambda' = 0; \quad \mu(e) = \mu'(f) = 1 \pmod 2.$$

Finally a quick proof that $\mathrm{UNil}_{4k+3}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ has an element of order 4 can be found in [CD04, Corollary 1.9]. It uses almost no machinery. □

Proof of Theorem 6.1. By Lemmas 6.3, 6.2 and 3.6, there is an element $[M, f] \neq [\mathbb{R}^n, \mathrm{id}]$ in $\mathscr{S}^{iso}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n)$. By Lemma 3.7, we conclude that (M, Γ_n) is not equivariantly homeomorphic to $(\mathbb{R}^n, \Gamma_n)$. □

6.1. **Free Action Of** UNil **On The Structure Set Of A Pair.**

Our purpose here is to show that our Lemma 6.2 is a formal consequence of the L-theoretic exact sequence of Cappell, appearing in [Cap74b, Cap76].

Let X be a compact, connected topological manifold of dimension $n \geq 6$. Let Y a connected, separating, codimension-one submanifold of X without boundary (that is, $\partial Y = \varnothing$). Assume the induced map $\pi_1(Y) \to \pi_1(X)$ of groups is injective. Furthermore, assume the induced map $\pi_1(\partial X) \to \pi_1(X)$ of groupoids is injective.

Write $X = X_1 \cup_Y X_2$ for the induced decomposition of manifolds. Write $G = G_1 *_F G_2$ for the induced injective amalgam of fundamental groups, where $F := \pi_1(Y)$. Finally, write $H := \pi_1(\partial X)$ as the fundamental groupoid of the boundary.

For simplicity of notation, we shall suppress all the orientation characters. Furthermore, to avoid K-theoretic difficulties, *we assume throughout this subsection* that the projective class group for the codimension-one submanifold Y vanishes:

$$\widetilde{K}_0(\mathbb{Z}[F]) \ = \ 0.$$

Theorem 6.4. *On the structure set $\mathscr{S}^h_{\mathrm{TOP}}(X, \partial X)$ of the pair, Wall's action of the group $L^h_{n+1}(G)$ restricts to a free action of Cappell's subgroup*

$$\mathrm{UNil}^h_{n+1} \ := \ \mathrm{UNil}^h_{n+1}(\mathbb{Z}[F]; \mathbb{Z}[G_1 - F], \mathbb{Z}[G_2 - F]).$$

Thus we slightly generalize the case of $\partial X = \varnothing$ of Cappell [Cap74a, Thm. 2]. Our proof relies only on his algebraic results [Cap74b, Thm. 2, Thm. 5] [Cap76].

Theorem 6.5 (Cappell)**.** *There is a homomorphism*

$$\iota \ : \ \mathrm{UNil}^h_* \ \longrightarrow \ L^h_*(G)$$

whose composite with a map of Wall (see [Wal99, Thm. 9.6]*) is an isomorphism:*

$$\mathrm{UNil}^h_* \ \xrightarrow{\iota} \ L^h_*(G) \ \longrightarrow \ L^h_* \begin{pmatrix} F & \longrightarrow & G_1 \\ \downarrow & & \downarrow \\ G_2 & \longrightarrow & G \end{pmatrix}.$$

Furthermore, there is an exact sequence

$$\cdots \to \mathrm{UNil}^h_{*+1} \oplus L^h_*(F) \to L^h_*(G_1) \oplus L^h_*(G_2) \to L^h_*(G) \xrightarrow{\binom{s}{\partial}} \mathrm{UNil}^h_* \oplus L^h_{*-1}(F) \to \cdots$$

such that $s \circ \iota = \mathrm{id}$. *In particular,* ι *is split injective with a preferred left-inverse.* □

Using the first part of Theorem 6.5, the first two authors formally identify UNil with relative equivariant homology [CD]. Write $\mathfrak{all}$ as the family of all subgroups of G. Write $\mathfrak{fac}$ as the family of subgroups of G conjugate into either G_1 or G_2.

Theorem 6.6 (Connolly–Davis)**.** *The following composite is an isomorphism:*

$$\phi \;:\; \mathrm{UNil}^h_* \xrightarrow{\iota} L^h_*(G) \;=\; H^G_*(E_{\mathfrak{all}}G; \underline{\mathbf{L}}^h) \xrightarrow{h} H^G_*(E_{\mathfrak{all}}G, E_{\mathfrak{fac}}G; \underline{\mathbf{L}}^h).$$

□

This implies a relative version; recall $H = \pi_1(\partial X)$. Consider the homomorphism

$$j \;:\; L^h_*(G) \;\longrightarrow\; L^h_*(G, H).$$

Write the fundamental groupoid $H = H_1 \sqcup \cdots \sqcup H_m$ as the disjoint union of its vertex groups H_i. The associated G-set is defined by

$$G/H \;:=\; G/H_1 \sqcup \cdots \sqcup G/H_m.$$

Observe, since $Y \cap \partial X = \varnothing$, for each i that $H_i \subset G_1$ or $H_i \subset G_2$; hence $H_i \in \mathfrak{fac}$. Therefore there is a canonical G-map $G/H \to E_{\mathfrak{fac}}G$.

Corollary 6.7. *There is a split short exact sequence*

$$0 \longrightarrow H^G_*(E_{\mathfrak{fac}}G, G/H; \underline{\mathbf{L}}^h) \xrightarrow{A_{\mathfrak{fac}}} L^h_*(G, H) \xrightarrow{s'} \mathrm{UNil}^h_* \longrightarrow 0.$$

The preferred right-inverse for s' *is the composite* $j \circ \iota$.

Proof. There is a long exact sequence of the triple:

$$\cdots \to H^G_*(E_{\mathfrak{fac}}G, G/H; \underline{\mathbf{L}}^h) \to H^G_*(E_{\mathfrak{all}}G, G/H; \underline{\mathbf{L}}^h) \xrightarrow{k} H^G_*(E_{\mathfrak{all}}G, E_{\mathfrak{fac}}G; \underline{\mathbf{L}}^h) \to \cdots.$$

So, by Theorem 6.6, we may define a homomorphism

$$s' \;:\; L^h_*(G, H) \;=\; H^G_*(E_{\mathfrak{all}}G, G/H; \underline{\mathbf{L}}^h) \xrightarrow{k} H_*(E_{\mathfrak{all}}G, E_{\mathfrak{fac}}G; \underline{\mathbf{L}}^h) \xrightarrow{\phi^{-1}} \mathrm{UNil}^h_*.$$

That is, $s' := \phi^{-1} \circ k$. Note $h = k \circ j$. Recall $\phi = h \circ \iota$. Then

$$s' \circ (j \circ \iota) \;=\; (\phi^{-1} \circ k) \circ (j \circ \iota) \;=\; \phi^{-1} \circ (h \circ \iota) = \phi^{-1} \circ \phi = \mathrm{id}.$$

Therefore, k has right-inverse $j \circ \iota \circ \phi^{-1}$, and the above exact sequence splits. □

Now we are ready to prove the main theorem of this subsection.

Proof of Theorem 6.4. Ranicki defined algebraic structure groups $\mathcal{S}\langle 1\rangle^h_*(X, \partial X)$, a homomorphism $L^h_*(G, H) \to \mathcal{S}\langle 1\rangle^h_*(X, \partial X)$, and a pointed bijection

$$\mathscr{S}^h_{\mathrm{TOP}}(X, \partial X) \xrightarrow{\approx} \mathcal{S}\langle 1\rangle^h_{n+1}(X, \partial X)$$

such that it is equivariant with respect to the actions of $L^h_{n+1}(G, H)$ [Ran92]. Also observe, from Remark 4.7, that there is an induced isomorphism

$$\mathcal{S}\langle 1\rangle^h_{n+1}(X, \partial X) \xrightarrow{\cong} \mathcal{S}^h_{n+1}(X, \partial X).$$

Write W as the composite homomorphism $L^h_{n+1}(G,H) \to \mathbb{S}^h_{n+1}(X,\partial X)$, which is compatible with Wall's action of $L^h_{n+1}(G,H)$ on the structure set $\mathscr{S}^h_{\mathrm{TOP}}(X,\partial X)$. Thus it suffices to show that the following composite is a monomorphism:

$$\mathrm{UNil}^h_{n+1} \xrightarrow{\iota} L^h_{n+1}(G) \xrightarrow{j} L^h_{n+1}(G,H) \xrightarrow{W} \mathbb{S}^h_{n+1}(X,\partial X).$$

By definition of the algebraic structure groups, there is an exact sequence

$$H_{n+1}(X,\partial X;\mathbf{L}) \xrightarrow{A} L^h_{n+1}(G,H) \xrightarrow{W} \mathbb{S}^h_{n+1}(X,\partial X).$$

Also, using Theorem B.1, there is a commutative diagram of assembly maps:

$$\begin{array}{ccc} H^G_{n+1}(\widetilde{X}, G\times_H \widetilde{\partial X}; \underline{\mathbf{L}}^h) & \longrightarrow & H^G_{n+1}(E_{\mathfrak{fac}}G, G/H; \underline{\mathbf{L}}^h) \\ \Big\downarrow = & & \Big\downarrow A_{\mathfrak{fac}} \\ H_{n+1}(X,\partial X;\mathbf{L}) & \xrightarrow{\quad A \quad} & L^h_{n+1}(G,H). \end{array}$$

Then, by Corollary 6.7, note:

$$\mathrm{Ker}(W) = \mathrm{Im}(A) \subseteq \mathrm{Im}(A_{\mathfrak{fac}}) = \mathrm{Ker}(s') \quad \text{and} \quad \mathrm{Im}(j\circ\iota)\cap \mathrm{Ker}(s') = 0.$$

So $W\circ j\circ\iota$ is a monomorphism. Therefore UNil^h_{n+1} acts freely on $\mathscr{S}^h_{\mathrm{TOP}}(X,\partial X)$. □

Appendix A. From equivariance to isovariance

We want to prove that the forgetful map $\psi : \mathscr{S}^{iso}_{\mathrm{TOP}}(T^n, C_2) \to \mathscr{S}_{\mathrm{TOP}}(T^n, C_2)$ is bijective when $n \geq 4$. It seems best to approach this from a general study of isovariance. It shall be immediate from Theorem A.3 below that ψ is bijective.

Note we do not use any end theorems, and the Strong Gap Hypothesis is satisfied.

Let G be a finite group. For any G-spaces X and Y, write $[X,Y]_G$ and $[X,Y]^{iso}_G$ for the set of G-equivariant and G-isovariant homotopy classes of maps, respectively.

Let X be a G-space with a fixed point p. The *homotopy link* of p in X, denoted t^pX, and the *homotopy tangent space* of X at p, denoted t_pX, are defined by:

$$\begin{aligned} t^pX &:= \mathrm{Holink}(X,p) = \{\sigma : [0,1] \to X \mid \sigma^{-1}(p) = \{0\}\} \\ t_pX &:= t^pX \cup \{\sigma_p\}. \end{aligned}$$

Here σ_p denotes the constant path at p, and t_pX has the compact-open topology. This is the metric topology of uniform convergence if X is a metric space.

There is a G-subspace $X_{(p)} \subset X$ and isovariant evaluation map e_1 defined by

$$X_{(p)} := (X - X^G) \cup \{p\}, \qquad e_1 : t_pX_{(p)} \longrightarrow X \ ; \ \sigma \longmapsto \sigma(1).$$

Let U be a neighborhood of p in X. If U is homeomorphic to $\mathbb{R}^n$, then

(A.1) $\qquad e_1$ restricts to a homotopy equivalence, $t^pU \simeq \mathbb{R}^n - \{0\}$.

If U is G-invariant and p is an isolated fixed point, then the inclusion $\iota : t_pU \to t_pX$ is an isovariant homotopy equivalence.

Lemma A.1. *Let X and Y be metric spaces on which G acts semifreely and isometrically. Assume q is an isolated fixed point of Y. The rule $f \mapsto f|_{X-X^G}$ gives a bijection between isovariant and equivariant homotopy classes,*

$$[X, t_qY]^{iso}_G \;\cong\; [X - X^G, t^qY]_G.$$

Proof. Let $f : X - X^G \to t^q Y$ be a G-map. We first show f is G-homotopic to an*extendible* map, f'. (Here f is *extendible*, if $\lim_{x\to x_0} f(x) = \sigma_q$ for any $x_0 \in X^G$.) This will prove that the restriction map is a surjection. We assume X and Y have metrics, d_X and d_Y, bounded by 1. Write d_t for the induced metric on $t_q Y$.

For $x \in X - X^G$, set $\|x\| := d_X(x, X^G)$. For $\sigma \in t^q Y$, set $\|\sigma\| := d_t(\sigma, \sigma_q)$. If, for all $x \in X - X^G$, $\|f(x)\| \le \|x\|$, then f is obviously extendible. In $(X - X^G) \times I$, the subset $(X - X^G) \times 0$ is disjoint from the closed subset

$$B := \left\{(x,t) \in (X - X^G) \times I \mid d_Y(f(x)(t), q) \ge \|x\|\right\}.$$

Let $\phi : X - X^G \to (0,1]$ be the continuous map, $\phi(x) = dist((x,0), B)$, where $dist$ denotes the product metric on $(X - X^G) \times I$. Since $\phi(x) = dist((x,0)(x,\phi(x)))$, we see that $(x,t) \notin B$, if $0 \le t < \phi(x)$. Therefore $d_Y(f(x)(t), q) \le \|x\|, \forall\, t \in [0, \phi(x)]$.

Let $f' : X - X^G \to t^q Y$ be the map whose adjoint, $\mathcal{A}f' : (X - X^G) \times I \to Y$, is the composition

$$(X - X^G) \times I \xrightarrow{\Phi} (X - X^G) \times I \xrightarrow{\mathcal{A}f} Y; \quad \Phi(x,t) = (x,\ t \cdot \phi(x)).$$

By construction, $\|f'(x)\| \le \|x\|$ for all $x \in X - X^G$. So f' is extendible. But a G-homotopy, $f_s,\ 0 \le s \le 1$, from f to f' is defined by: $\mathcal{A}f_s = \mathcal{A}f \circ \Phi_s$, where Φ_s is defined by : $\Phi_s(x,t) = (x,\ t \cdot \phi_s(x))$ and $\phi_s(x) := s\phi(x) + (1 - s)$.

Note that if f is extendible, then each f_s is extendible too.

The same simple argument shows that if $f : (X - X^G) \times I \to t^q Y$, is a homotopy between two extendible G-homotopy equivalences, then f' supplies an extendible homotopy. One merely changes the definition of B to:

$$B := \left\{(x,t) \in (X - X^G) \times I \mid \text{for some } \ s \in [0,1], d_Y(f(x,s)(t), q) \ge \|x\|\right\}.$$

This proves that $[X - X^G, t^q Y]_G \ \cong\ [X, t_q Y]_G^{iso}$, as required. □

Because our next lemma employs a somewhat unusual form of Poincaré duality, (see (A.5)) we spend some space introducing it here.

A *tamed neighborhood* of a G-invariant closed subset A in a G-space X, is a G-invariant neighborhood U of A, together with a *strict* G-map:

$$(U \times I,\ U \times \{0\} \cup A \times I) \xrightarrow{h} (X, A)$$
$$\text{with: } h(x,t) = x \text{ if } (x,t) \in U \times \{1\} \cup A \times I.$$

(A map $f : (X, A) \to (Y, B)$ is *strict* if $f^{-1}(B) = A$. A homotopy equivalence between (X, A) and (Y, B) is strict if each of the homotopies, $(X \times I, A \times I) \to (Y, B)$, and $(Y \times I, B \times I) \to (X, A)$ is strict.)

We say A is *tame* in X, or (X, A) is *tame* if A has a tamed neighborhood in X.

If $A = \{p\}$ is an isolated fixed point, this implies $U^G = \{p\}$. Also if $A = \{p\}$, a point, the adjoint of $h : U \times I \to X$ is an isovariant map $\lambda_p : U \to t_p X$. The map

$$U \xrightarrow{\lambda_p} t_p X \xrightarrow{e_1} X$$

is the inclusion $U \to X$. Following Quinn [Qui88, Prop. 3.6 or Prop. 2.6], we note that each neighborhood of an isolated fixed point p in a G-manifold contains a tamed neighborhood.

Let (X, A) be a compact Hausdorff pair. Assume (X, A) is tame and assume $M := X - A$ is an n-manifold. Let $(\mathcal{U}, h)$ be a tamed neighborhood of A in X. Choose a sequence of invariant open neighborhoods in X of A: $\mathcal{U}_1 \supset \mathcal{U}_2 \supset \mathcal{U}_3 \ldots$, such that:

(1) $X - \mathcal{U}_q$ is a compact manifold with bicollared boundary in M.
(2) $h(\mathcal{U}_q \times I) \subset \mathcal{U}_{q-1}$. Therefore $\lambda : U_q \to \mathrm{Holink}(\mathcal{U}_{q-1}, A)$ where λ is the adjoint of h. Note the map, $\lambda \circ e_1 : \mathrm{Holink}(\mathcal{U}_q, A) \to U_q \to \mathrm{Holink}(\mathcal{U}_{q-1}, A)$ is homotopic to the inclusion map.
(3) Set $U_q = \mathcal{U}_q - A$. Then M deforms into $M - U_q$, rel $M - U_{q-1}$ for $q \geq 2$.

Now let

$$G_1 \xleftarrow{i_1} G_2 \xleftarrow{i_2} G_3 \xleftarrow{i_3} \cdots \tag{A.2}$$

be an inverse sequence of groups defined by: $G_q = H_p(\mathcal{U}_q)$, where i_q is induced by inclusion $\mathcal{U}_q \supset \mathcal{U}_{q+1}$. Set $G_\infty = H_p(A)$. If $G'_q := i_q(G_{q+1})$, and $i'_q := i_q|_{G'_{q+1}}$. Then, by (2) above, each i'_q is an isomorphism and the map $i^\infty_q : G_\infty \to G'_q$, induced by the inclusion $\mathcal{U}_q \supset A$, is an isomorphism. That is to say, the inverse sequence (A.2) is *stable*. The same argument, this time using (3), shows that

$$H^p(M - U_1) \xleftarrow{j_1} H^p(M - U_2) \xleftarrow{j_2} H^p(M - U_3) \xleftarrow{j_3} \tag{A.3}$$

is stable with inverse limit $H^p(M) = \varprojlim H^p(M - U_q)$. The binding maps are induced by inclusions. Finally, using (2) again we see that the sequence of inclusion induced maps,

$$H_p(U_1) \xleftarrow{k_1} H_p(U_2) \xleftarrow{k_2} H_p(U_3) \xleftarrow{k_3} \cdots \tag{A.4}$$

is stable, with inverse limit, $H_p(\mathrm{Holink}(X, A))$. The evaluation maps:

$$H_p(\mathrm{Holink}(X, A)) \equiv H_p(\mathrm{Holink}(\mathcal{U}_q, A)) \xrightarrow{e_1} U_q$$

induce the isomorphisms $k^\infty_q : H_p(\mathrm{Holink}(X, A)) \to k_q(H_p(U_{q+1}))$.

The Poincaré duality isomorphisms,

$$H^*(M - U_q) \cong H_{n-*}(M, \mathrm{Cl}_M(U_q)) = H_{n-*}(X, \mathrm{Cl}_X(\mathcal{U}_q))$$

then pass to an isomorphism of limiting groups

$$\cap[M] : H^*(M) \xrightarrow{\cong} \varprojlim H_{n-*}(X, \mathcal{U}_q) = H_{n-*}(X, A).$$

Now suppose X_0 is a closed set of X, and $A_0 = X_0 \cap A$, $\mathcal{U}_0 = X_0 \cap \mathcal{U}$ and $\lambda(\mathcal{U}_0 \times I) \subset X_0$. But this time suppose $M := X - A$ is a manifold with boundary, and $\partial M = X_0 - A_0$. Then by repeating the above argument, we get an isomorphism:

$$\cap[M] : H^*(M, \partial M) \longrightarrow H_{n-*}(X, A).$$

If, in addition, X is a semifree G-space and $A = X^G$, we work in the orbit space, $(X/G, X^G)$, with $M/G = X/G - A^G$ and express the resulting isomorphism as

$$\cap\,[M] : H^*_G(X - X^G, \partial(X - X^G)) \xrightarrow{\cong} H^G_{n-*}(X, X^G). \tag{A.5}$$

Lemma A.2. *Let U^m be a compact semifree G-manifold with G-collared boundary. Assume U^G is tame in U. Let V^n be a G-manifold, and q an isolated fixed point of V. Let $f : U \to t_qV$ be a G-map for which $f|_{\partial U}$ is isovariant. If (U, U^G) is 1-connected and $m \leq n + 1$, then f is G-homotopic, rel ∂U to an isovariant map.*

Proof. The map $f|\partial(U - U^G) : \partial(U - U^G) \to t^qV$ extends to a G-map $F : U - U^G \to t^qV$ because the obstructions to this extension are in $H^i_G(U - U^G, \partial(U - U^G); \pi_{i-1}(t^q(V))) \cong H^G_{m-i}(U, U^G; \pi_{i-1}(t^q(V)))$. These groups are zero if $i < n$ by (A.1), and they are zero for $i \geq n$ because (U, U^G) is a 1-connected pair and $m \leq n + 1$. Therefore, by Lemma A.1, there is an isovariant map $f' : U \to t_qV$ such that $f'|_{\partial U}$ is isovariantly homotopic to $f|_{\partial U}$. But since ∂U has a G-collar in U, the isovariant G-homotopy extension property applies, and we can choose f' so that $f'|_{\partial U} = f|_{\partial U}$. Finally since t_qV is equivariantly contractible, the maps $f, f' : U \to t_qV$ are G-homotopic. □

Let X be a G-manifold with boundary. A neighborhood U of X^G in X is a *k-neighborhood* if it is an invariant codimension zero submanifold with bicollared frontier in X, such that the pair (U, U^G) is k-connected.

Theorem A.3. *Let X^n and Y^n be compact semifree G-manifolds, with finite fixed sets. Assume $n \geq 4$. Let $f : X \to Y$ be a G-map such that the restriction $f^G : X^G \to Y^G$ is bijective.*

1. *If f is 1-connected, then f is G-homotopic to an isovariant map.*
2. *If $f = f_0$ is already isovariant, and 2-connected, and f_0 is G-homotopic to an isovariant map $f_1 : X \to Y$, then f_0 is isovariantly homotopic to f_1.*

Proof of Theorem A.3(1). From Siebenmann's thesis [Sie65], we note that each neighborhood of Y^G contains a 0-neighborhood if $n = 4$, and a 1-neighborhood if $n \geq 5$. Therefore we may as well assume V is a 0-neighborhood . We choose V so small that the component V_q containing a fixed point $q \in Y^G$, admits the structure of a tamed neighborhood of q, say (V_q, h_q). We may assume f is transverse to ∂V. We set $U := f^{-1}(V)$, a manifold neighborhood of $f^{-1}(Y^G)$. Let $N = f^{-1}(\partial V)$, the frontier of U in X. N is a manifold and N is bicollared in X.

We wish that U be a 0-neighborhood. We plan to accomplish this by handle exchanges along N realized through a homotopy of f.

Set $X_0 = \mathrm{Cl}_X(X - U)$; $Y_0 = \mathrm{Cl}_Y(Y - V)$. X_0 is a manifold with boundary, and $N = \partial X_0$. Also $X = U \cup_N X_0$, $Y = V \cup_{\partial V} Y_0$. We now recall this notion of handle exchange along N.

Suppose we can find a map, $i : (D^k, \partial D^k) \times \{0\} \to (X_0, N)$ (or alternatively, a map, $i : (D^k, \partial D^k) \times \{0\} \to (U, N)$), together with an extension of $f \circ i$ to a map:

$$(D^k, \partial D^k) \times (I, 1) \xrightarrow{j} (Y_0, \partial Y_0) \qquad \text{(or to } (V, \partial V)).$$

If $k < n/2$, we can, after a homotopy, thicken these to an equivariant embedding and an equivariant extension still called i and j:

$$G \times (D^k, \partial D^k) \times D^{m-k} \xrightarrow{i} (X_0, N) \qquad \text{(or to } (U - U^G, N)).$$

$$G \times (D^k, \partial D^k) \times D^{m-k} \times (I, 1) \xrightarrow{j} (Y_0, N) \qquad \text{(or to } (V - V^G, \partial V)).$$

Now deform f by a G-homotopy, stationary off $i(G \times \mathrm{Int}(D^k \times D^{m-k}))$, to a map f' so that f' is still transverse to N, but

$$f'^{-1}(V) = U \cup i(G \times D^k \times (1/2) D^{m-k})$$
$$(\text{or } f'^{-1}(Y_0) = X_0 \cup i(G \times D^k \times (1/2) D^{m-k})).$$

Note this homotopy is $rel\, \partial X$. If $q \in Y^G$ set $U_q = f^{-1}(V_q)$ and $N_q = N \cap U_q$.

If $k = 1$, this exchange process decreases the number of components of N provided that $i(\partial D^1 \times 0)$ is chosen to meet two components of some N_q. After finitely many such handle exchanges then, we arrive at a map f' for which N_q is connected for each $q \in Y^G$. Therefore U_q is connected too. So U is a 0-neighborhood of X^G.

We have: $f^{-1}(Y^G) \subset U \subset f^{-1}(V)$.

For each $q \in Y^G$, we need only show how to deform $f|_{U_q}$ $rel\ \partial U_q$ to an isovariant map $f'_q : U_q \to Y$. Let $Y_{(q)} = (Y - Y^G) \cup \{q\}$. Recall, $f|_{U_q} = e_1 \circ \lambda_q \circ f|_{U_q}$, where

$\lambda_q : V_q \to t_q(Y_{(q)})$, and $e_1 : t_q(Y_{(q)}) \to Y$ is equivariant. By Lemma A.2, there is an isovariant map $F_q : U_q \to t_q Y_{(q)}$ for which $(\lambda_q \circ f|_{U_q}) \simeq_G F_q \, rel \, \partial U_q$.

Therefore $f'_q := e_1 \circ F_q$ is isovariant and $f|_{U_q} \simeq_G f'_q$ as required. □

Proof of Theorem A.3(2). The argument is similar to that for the proof of Theorem A.3(1). Realize the homotopy from f_0 to f_1 by a G-map

$$X \times (I, \partial I) \xrightarrow{(F,\partial F)} Y \times (I, \partial I)$$

with $\partial F = f_0 \sqcup f_1$. Choose a tamed neighborhood (W, h) of Y^G in Y. Let W_q be the component of W containing q, for each $q \in Y^G$. Since $\dim(Y \times I) \geq 5$, by Siebenmann's thesis [Sie65], there exists a 1-neighborhood V of $Y^G \times I$ in $Y \times I$ such that $V \subset W \times I$.

After making F transverse to ∂V, let $U = F^{-1}(V)$. Let $N = f^{-1}(\partial V)$, the frontier of U in $X \times I$. N is a manifold with boundary and $\partial N \subset X \times \partial I$. Also $(N, \partial N)$ is bicollared in $(X \times I, \partial(X \times I))$.

Proceed as in the proof of Theorem A.3(1) to make U a 0-neighborhood of $(X \times I)^G$. As before let U_q be the component of U containing $(F^G)^{-1}(q \times I)$, and $N_q = N \cap U_q$.

We plan to make N_q simply connected for each q. We repeat the "innermost circles" argument of Browder [Bro65] doing handle exchanges along N using 2-handles to kill off the finitely many generators of each $\pi_1(N_q), q \in Y^G$. In the end we get a new $F : X \times I \to Y \times I$ with its new N for which N_q is 1-connected for each $q \in Y^G$. This implies that $\pi_1(X \times I) = \pi_1(U_q) * \pi_1(X \times I - \mathrm{Int}(U_q))$, and $\mathrm{incl}_* : \pi_1(U_q) \to \pi_1(X \times I)$ is injective. But V_q is simply connected and therefore U_q is simply connected by the following commutative diagram:

$$\begin{array}{ccc} \pi_1(U_q) & \xrightarrow{(f|_{U_q})_*} & \pi_1(V_q) = \{1\} \\ {\scriptstyle \mathrm{incl}_*}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{incl}_*} \\ \pi_1(X \times I) & \xrightarrow[\cong]{F_*} & \pi_1(Y \times I). \end{array}$$

This implies that U is a 1-neighborhood of $X^G \times I$ in $X \times I$, and $U = F^{-1}(V)$.

Let $H = p_1 \circ F : X \times I \to Y$. Note H is a G-homotopy from f_0 to f_1, with

$$H^{-1}(Y^G) \subset U \subset H^{-1}(W); \quad H(U_q) \subset W_q \quad \forall\, q \in Y^G.$$

As in the proof of Theorem A.3(1), for each $q \in Y^G$, $\lambda_q \circ H|_{U_q} : U_q \to t_q Y_{(q)}$ is homotopic $rel\ \partial U_q$ to an isovariant map, by Lemma A.2. Therefore F is homotopic $rel\ X \times I - \mathrm{Int}(U)$, to an isovariant map $H' : X \times I \to Y$. H' provides an isovariant homotopy from f_0 to f_1. □

Appendix B. Quinn–Ranicki = Davis–Lück

Let Ho $\mathsf{Spectra}$ be the *homotopy category*, given by formally inverting weak homotopy equivalences. There is a *localization functor* Ho : $\mathsf{Spectra} \to$ Ho $\mathsf{Spectra}$ sending weak equivalences to isomorphisms, and this functor is initial with respect to all such functors from $\mathsf{Spectra}$. The functor Ho is a bijection on objects. Homotopy groups $\pi_i : \mathsf{Spectra} \to \mathsf{Ab}$ factor through the functor Ho. Let $\mathcal{C}$ be a category. A $\mathcal{C}$-*spectrum* is a functor from $\mathcal{C}$ to $\mathsf{Spectra}$, a *map of* $\mathcal{C}$-*spectra* is a natural transformation, and a *weak equivalence of* $\mathcal{C}$-*spectra* is a map of $\mathcal{C}$-spectra $\mathbf{E} \to \mathbf{F}$ which induces a weak homotopy equivalence of spectra $\mathbf{E}(c) \to \mathbf{F}(c)$ for all objects c in $\mathcal{C}$. There is a localization functor $\mathrm{Ho} : \mathcal{C}\text{-}\mathsf{Spectra} \to \mathrm{Ho}\,\mathcal{C}\text{-}\mathsf{Spectra}$. A key property is that if $\mathbf{E}$ and $\mathbf{F}$ are $\mathcal{C}$-spectra which become isomorphic in $\mathrm{Ho}\,\mathcal{C}\text{-}\mathsf{Spectra}$, then there is a $\mathcal{C}$-spectrum $\mathbf{G}$ and weak equivalences $\mathbf{E} \leftarrow \mathbf{G} \to \mathbf{F}$.

For a groupoid $\mathcal{G}$, let $\mathbf{L}(\mathcal{G})$ be the corresponding L-spectrum, as in [DL98, Section 2]. This is a functor from the category of groupoids to the category of spectra which satisfies the additional property that an equivalence $F : \mathcal{G} \to \mathcal{G}'$ of groupoids induces a weak equivalence $\mathbf{L}(F) : \mathbf{L}(\mathcal{G}) \to \mathbf{L}(\mathcal{G}')$ of spectra.

Ranicki, motivated by earlier geometric work of Quinn, defined the *assembly map* [Ran92, Chapter 14], a natural transformation of functors from Top to $\mathsf{Spectra}$:

$$\mathbf{A}(Z) : \mathbf{H}(Z; \mathbf{L}(1)) \longrightarrow \mathbf{L}(\Pi_1 Z)$$

where $\Pi_1 Z$ is the fundamental groupoid of Z. When Z is a point, the assembly map is a weak equivalence. The *algebraic structure spectrum* $\mathcal{S}(Z)$ is defined to be the homotopy cofiber of $\mathbf{A}(Z)$. Its homotopy groups $\mathcal{S}_*(Z) := \pi_* \mathcal{S}(Z)$ are the algebraic structure groups used in Section 4; one can do this for pairs also.

Fix a group G. Consider the orbit category $\mathrm{Or}(G)$ and the $\mathrm{Or}(G)$-spectrum

$$\underline{\mathbf{L}} : \mathrm{Or}(G) \longrightarrow \mathsf{Spectra}\ ;\ G/H \longmapsto \mathbf{L}(\overline{G/H})$$

where $\overline{G/H}$ is the groupoid associated to the G-set G/H. For a G-CW-complex X, consider the spectrum

$$\mathbf{H}^G(X; \underline{\mathbf{L}}) \ := \ \mathrm{map}_G(-, X)_+ \wedge_{\mathrm{Or}(G)} \underline{\mathbf{L}}(-).$$

Then, by definition, $H^G_*(X; \underline{\mathbf{L}}) = \pi_* \mathbf{H}^G(X; \underline{\mathbf{L}})$.

Write $G\mathcal{CW}$ for the category whose objects are G-CW-complexes and whose morphisms are cellular G-maps. (Actually, for set-theoretic reasons we need to restrict our G-CW-complexes to a fixed universe; for our purposes it will suffice to assume that the underlying space of each CW-complex is embedded in $\mathbb{R}^\infty$.) A $(G, \mathcal{F})$-*CW-complex* is a G-CW-complex with isotropy in a family $\mathcal{F}$. Let $(G, \mathcal{F})\mathcal{CW}$ be the full subcategory of $G\mathcal{CW}$ whose objects are $(G, \mathcal{F})$-CW-complexes. Let $\mathrm{Or}(G, \mathcal{F})$ be the full subcategory of $(G, \mathcal{F})\mathcal{CW}$ whose objects are the discrete G-spaces G/H with $H \in \mathcal{F}$. The symbol 1 will denote both the trivial group and the family of subgroups of G consisting of the trivial group. Let $\mathrm{sc}(G, 1)\mathcal{CW}$ be the full subcategory of $(G, 1)\mathcal{CW}$ whose objects are free G-CW-complexes all of whose components are simply connected.

Let $\Pi_0 X$ be the G-set of path components of X. Here is the main theorem of this appendix.

Theorem B.1. *There is a commutative diagram in* $\mathrm{Ho}\,(G, 1)\mathcal{CW}\text{-}\mathsf{Spectra}$*:*

$$\begin{array}{ccc} \mathbf{H}(X/G; \mathbf{L}(1)) & \longrightarrow & \mathbf{L}(\Pi_1(X/G)) \\ \big\downarrow & & \big\downarrow \\ \mathbf{H}^G(X; \underline{\mathbf{L}}) & \longrightarrow & \mathbf{H}^G(\Pi_0 X; \underline{\mathbf{L}}). \end{array}$$

(1) *The top map is the assembly map* $\mathbf{A}(X/G)$ *and is a map of* $(G, 1)\mathcal{CW}\text{-}\mathsf{Spectra}$.
(2) *The bottom map is induced by the* G*-map* $X \to \Pi_0 X$ *and is a map of* $(G, 1)\mathcal{CW}\text{-}\mathsf{Spectra}$.

(3) *The right map is the composite of the formal inverse of the weak equivalence of* $(G,1)\mathcal{CW}$*-spectra* $\mathbf{L}(\Pi_1(EG\times_G X))\to\mathbf{L}(\Pi_1(X/G))$ *and the map of* $(G,1)\mathcal{CW}$*-spectra* $\mathbf{L}(\Pi_1(EG\times_G X))\to\mathbf{H}^G(\Pi_0 X;\underline{\mathbf{L}})$ *defined in Lemma B.2. This map is a weak equivalence when restricted to* $\mathrm{sc}(G,1)\mathcal{CW}$*-*$\mathsf{Spectra}$*.*

(4) *The left map is an isomorphism in* $\mathrm{Ho}\,(G,1)\mathcal{CW}$*-*$\mathsf{Spectra}$*.*

The proof of the theorem is quite formal and applies more generally. What is needed is a functor from groupoids to spectra which takes equivalences of groupoids to weak equivalences of spectra and an assembly map which is a weak equivalence when X is a point. So, for example, our theorem applies equally well to K-theory. See Remark B.4 below for the modifications necessary for the L-theory non-orientable case.

Let G be a discrete group. Let S be a G-set. Define the *action groupoid* $\overline{S}$ as the category whose object set is S, and whose morphisms from s to t are triples (t,g,s) such that $t=gs$, and whose composition law is $(t,g,s)\circ(s,f,r)=(t,gf,r)$. Define a functor

$$\mathbf{L}^G:(G,1)\mathcal{CW}\longrightarrow\mathsf{Spectra}\ ;\ X\longmapsto\mathbf{L}(\Pi_1(EG\times_G X)).$$

The next lemma relates $\mathbf{L}^G$ to the above functor $\underline{\mathbf{L}}:\mathrm{Or}(G)\to\mathsf{Spectra}$.

Lemma B.2. *Let G be a discrete group.*

(1) *For a discrete G-set S, there is a homeomorphism of spectra*

$$\mathbf{H}^G(S;\underline{\mathbf{L}})\cong\mathbf{L}(\overline{S}),$$

natural in S.

(2) *For a free G-CW-complex X, there is a map of groupoids*

$$\Phi(X):\Pi_1(EG\times_G X)\longrightarrow\overline{\Pi_0 X}$$

which is an equivalence of groupoids when all the components of X are simply connected. Furthermore, Φ is natural in X; that is, $\Phi(-)$ is a map of $(G,1)\mathcal{CW}$*-groupoids.*

(3) *There is a map of* $(G,1)\mathcal{CW}$*-spectra*

$$\Lambda(X):\mathbf{L}^G(X)\longrightarrow\mathbf{H}^G(\Pi_0 X;\underline{\mathbf{L}})$$

whose restriction to $\mathrm{sc}(G,1)\mathcal{CW}$ *is a weak equivalence.*

Proof. (1) The homeomorphism is given by

$$\mathbf{H}^G(S;\underline{\mathbf{L}})\longrightarrow\mathbf{L}(\overline{S})\ ;\ [(f,x)\in\mathrm{map}_G(G/K,S)_+\wedge L(\overline{G/K})_n]\longmapsto L(\overline{f})_n(x)\in L(\overline{S})_n.$$

If S is an orbit G/K, then the inverse is given by $x\in L(\overline{G/K})_n\mapsto[(\mathrm{id},x)\in\mathrm{map}_G(G/K,G/K)_+\wedge L(\overline{G/K})_n]$. The case of a general G-set follows since both $\mathbf{H}^G(-;\underline{\mathbf{L}})$ and $\mathbf{L}(-)$ convert disjoint unions to one-point unions of spectra.

(2) We first need some notation. For a subset A of a topological space Y, let $\Pi_1(Y,A)$ be the full subcategory of the fundamental groupoid $\Pi_1 Y$ whose objects are points in A. If $\Pi_0 A\to\Pi_0 Y$ is onto, then there is an equivalence of groupoids $\Pi_1 Y\to\Pi_1(Y,A)$ whose definition depends on a choice of a path from y to a point in A for every $y\in Y$.

Let $p:EG\times X\to EG\times_G X$ be the quotient map. We will define $\Phi(X)$ as a composite

$$\Pi_1(EG\times_G X)\xrightarrow{\Theta(X)}\Pi_1(EG\times_G X,p(\{e_0\}\times X))\xrightarrow{\Psi(X)}\overline{\Pi_0 X}.$$

We first define $\Theta(X)$ by making choices in the universal space EG. Choose a point $e_0 \in EG$. For each $e \in EG$, choose a path $\sigma_e : I \to EG$ from e to e_0, choosing the paths so that, for all $g \in G$ and $t \in I$, $g(\sigma_e(t)) = \sigma_{ge}(t)$. This can be accomplished by choosing a set-theoretic section $s : BG \to EG$ of the covering map, and defining the remaining σ_e by equivariance. Then for $p(e,x) \in EG \times_G X$, define the path $\theta_{p(e,x)}(t) := p(\sigma_e(t), x)$. This path is independent of the choice of representative of $p(e,x)$. These paths give the equivalence of groupoids $\Theta(X)$, natural in X.

We now define $\Psi(X)$ using the fact that p is a covering map. On objects, define $\Psi(X)(p(e_0,x)) := C(x) \in \Pi_0 X$, where $C(x)$ is the path component of x in X. For a morphism represented by a path $\alpha : I \to EG \times_G X$ with $\alpha(0) = p(e_0,x)$ and $\alpha(1) = p(e_0,y)$, let $\widetilde{\alpha} : I \to EG \times X$ be the lift of α starting at (e_0,x). Then $\widetilde{\alpha}(1) = (ge_0, gy)$ some $g \in G$. Then define $\Psi(X)[\alpha] := (C(x), g, C(y))$. We leave the geometric details of verifying that this is a functor to the reader, but note that we follow that convention that a path α determines a morphism from $\alpha(1)$ to $\alpha(0)$ in the fundamental groupoid.

Suppose all the components of X are simply connected. We now show that $\Psi(X)$ is an equivalence of groupoids. Choose a base point for each component of X. Define a functor $\overline{\Pi_0 X} \to \Pi_1(EG \times_G X, p(\{e_0\} \times X))$ on objects by sending $C(x)$ to $p(e_0,x)$, and on morphisms by sending $(C(x), g, C(y))$ to $[p \circ \widetilde{\alpha}]$ where $\widetilde{\alpha} : I \to EG \times X$ is a path from (e_0,x) to (ge_0, gy). This $\widetilde{\alpha}$ is unique up to homotopy rel endpoints since X is simply connected. This ends the proof of (2).

(3) Define $\Lambda(X)$ as the composite of $\mathbf{L}(\Phi(X))$ and the isomorphism from (1). $\square$

Remark B.3. We next recast the axiomatic approach of [DL98, Section 6]. Our terminology is self-consistent but does not precisely match that of [DL98]; in particular we drop the adverb "weakly." A functor $\mathbf{E} : (G,\mathcal{F})\mathcal{CW} \to \mathsf{Spectra}$ is *homotopy invariant* if any homotopy equivalence induces a weak equivalence of spectra. A functor $\mathbf{E} : (G,\mathcal{F})\mathcal{CW} \to \mathsf{Spectra}$ is *excisive* if $\mathbf{E}(-)$ and $\mathbf{H}^G(-;\mathbf{E}|_{\mathrm{Or}(G,\mathcal{F})})$ are isomorphic objects in $\mathrm{Ho}\,(G,\mathcal{F})\mathcal{CW}$-$\mathsf{Spectra}$. This is equivalent to the notion of weakly $\mathcal{F}$-excisive given in [DL98].[1] By [DL98, Theorem 6.3(2)], a map $\mathbf{T} : \mathbf{E} \to \mathbf{F}$ of excisive $(G,\mathcal{F})$-CW-spectra is a weak equivalence if and only if $\mathbf{T}(G/H) : \mathbf{E}(G/H) \to \mathbf{F}(G/H)$ is a weak equivalence of spectra for all $H \in \mathcal{F}$. An *excisive approximation* of a homotopy invariant functor $\mathbf{E} : (G,\mathcal{F})\mathcal{CW} \to \mathsf{Spectra}$ is a map $\mathbf{T} : \mathbf{E}' \to \mathbf{E}$ of $(G,\mathcal{F})$-CW-spectra such that $\mathbf{E}'$ is excisive and $\mathbf{T}(G/H)$ is a weak equivalence for all orbits G/H with $H \in \mathcal{F}$.

We next assert existence and uniqueness of excisive approximations. Theorem 6.3(2) of [DL98] constructs a specific excisive approximation $\mathbf{E}^{\%} \to \mathbf{E}$ which is functorial in $\mathbf{E}$. Excisive approximations are unique in the sense that, given any two excisive approximations $\mathbf{T}' : \mathbf{E}' \to \mathbf{E}$ and $\mathbf{T}'' : \mathbf{E}'' \to \mathbf{E}$, there is an isomorphism $\mathbf{S} : \mathbf{E}' \to \mathbf{E}''$ in $\mathrm{Ho}\,(G,\mathcal{F})\mathcal{CW}$-$\mathsf{Spectra}$ such that $\mathbf{T}' = \mathbf{T}'' \circ \mathbf{S}$. Indeed, to verify that $\mathbf{S}$ exists, it suffices to compare any excisive approximation $\mathbf{T}' : \mathbf{E}' \to \mathbf{E}$ is equivalent to the functorial excisive approximation $\mathbf{T}^{\%} : \mathbf{E}^{\%} \to \mathbf{E}$. Consider the

[1] Indeed [DL98, Theorem 6.3(1,3)] implies that if $\mathbf{E}$ is weakly $\mathcal{F}$-excisive in the sense of [DL98], then there is a $(G,\mathcal{F})$-spectrum $\mathbf{E}^{\%}$ and weak equivalences $\mathbf{E} \leftarrow \mathbf{E}^{\%} \to \mathbf{H}^G(-;\mathbf{E}|_{\mathrm{Or}(G,\mathcal{F})})$. Conversely, if $\mathbf{E}(-)$ and $\mathbf{H}^G(-;\mathbf{E}|_{\mathrm{Or}(G,\mathcal{F})})$ are isomorphic objects in $\mathrm{Ho}\,(G,\mathcal{F})\mathcal{CW}$-$\mathsf{Spectra}$, then there are weak equivalences $\mathbf{E} \leftarrow \mathbf{F} \to \mathbf{H}^G(-;\mathbf{E}|_{\mathrm{Or}(G,\mathcal{F})})$ for some $(G,\mathcal{F})\mathcal{CW}$-spectrum $\mathbf{F}$. But [DL98, Theorem 6.3(1)] shows that $\mathbf{H}^G(-;\mathbf{E}|_{\mathrm{Or}(G,\mathcal{F})})$ is weakly $\mathcal{F}$-excisive, and hence so is any weakly equivalent $(G,\mathcal{F})\mathcal{CW}$-spectrum.

commutative diagram in $(G, \mathcal{F})\mathcal{CW}$-$\mathsf{Spectra}$:

$$\begin{array}{ccc} \mathbf{E}'^{\%} & \longrightarrow & \mathbf{E}' \\ \downarrow & & \downarrow \\ \mathbf{E}^{\%} & \longrightarrow & \mathbf{E}. \end{array}$$

Since the left and top maps are both weak equivalences, we obtain an isomorphism in the homotopy category $\mathbf{S} := (\mathbf{E}' \leftarrow \mathbf{E}'^{\%} \to \mathbf{E}^{\%})$ with $\mathbf{T}' = \mathbf{T}^{\%} \circ \mathbf{S}$, as desired.

Proof of Theorem B.1. The theorem will be proven by concatenating three commutative squares. The first is a commutative diagram in $\mathrm{Ho}\,(1,1)\mathcal{CW}$-$\mathsf{Spectra}$, which we will apply below in the case $Z = X/G$.

$$\begin{array}{ccc} \mathbf{H}(Z; \mathbf{L}(1)) & \longrightarrow & \mathbf{L}(\Pi_1 Z) \\ \downarrow & & \downarrow \\ (\mathbf{L}^1)^{\%}(Z) & \longrightarrow & \mathbf{L}^1(Z) \end{array}$$

The right map is the identity. The top map is the assembly map $\mathbf{A}(Z)$. The left map exists in the homotopy category (see end of Remark B.3) and is an isomorphism since both horizontal maps are $(1,1)$-excisive approximations of $\mathbf{L}(\Pi_1 Z) = \mathbf{L}^1(Z)$.

Next comes a commutative diagram in $\mathrm{Ho}\,(G,1)\mathcal{CW}$-$\mathsf{Spectra}$.

$$\begin{array}{ccc} (\mathbf{L}^1)^{\%}(X/G) & \longrightarrow & \mathbf{L}^1(X/G) \\ \uparrow & & \uparrow \\ (\mathbf{L}^G)^{\%}(X) & \longrightarrow & \mathbf{L}^G(X) \end{array}$$

The right map is induced by the homotopy equivalence $EG \times_G X \to X/G$, inducing an equivalence of fundamental groupoids, hence a weak equivalence of spectra. The left map exists and is an isomorphism, since the top map and the composite of the bottom and right maps are $(G,1)$-excisive approximations of $\mathbf{L}^G(X)$.

Our final commutative diagram is in $\mathrm{Ho}\,(G,1)\mathcal{CW}$-$\mathsf{Spectra}$.

$$\begin{array}{ccc} (\mathbf{L}^G)^{\%}(X) & \longrightarrow & \mathbf{L}^G(X) \\ \downarrow & & \downarrow \\ \mathbf{H}^G(X; \underline{\mathbf{L}}) & \longrightarrow & \mathbf{H}^G(\Pi_0 X; \underline{\mathbf{L}}) \end{array}$$

The top map is the functorial excisive approximation of $\mathbf{L}^G(X)$. The bottom map is induced by the G-map $X \to \Pi_0 X$ and is an excisive approximation of $\mathbf{H}^G(\Pi_0(-); \underline{\mathbf{L}})$. The right map is defined in Lemma B.2(3) and is an isomorphism when restricted to $\mathrm{Ho}\,\mathrm{sc}(G,1)\mathcal{CW}$-$\mathsf{Spectra}$. Functoriality gives a map $(\mathbf{L}^G)^{\%}(X) \to \mathbf{H}^G(\Pi_0(-); \underline{\mathbf{L}})^{\%}(X)$ and the fact that the bottom map is an excisive approximation gives a map $\mathbf{H}^G(\Pi_0(-); \underline{\mathbf{L}})^{\%}(X) \to \mathbf{H}^G(X; \underline{\mathbf{L}})$; define the left map as the composite.

Since the left map is a map of excisive functors and is a homotopy equivalence when $X = G/1$, the left map is an isomorphism in $\mathrm{Ho}\,(G,1)\mathcal{CW}$-$\mathsf{Spectra}$. □

Remark B.4. We next indicate the modifications needed for the statement and proof of Theorem B.1 in the non-orientable case. A *groupoid with orientation character* $\mathcal{G}^w$ is a groupoid $\mathcal{G}$ together with a functor $w : \mathcal{G} \to \{\pm 1\}$, where $\{\pm 1\}$ is the category with a single object and two morphisms $\{+1, -1\}$ where $-1 \circ -1 = +1$. A map of groupoids with orientation character is a map of groupoids which preserves the orientation character. Let GWOC denote the category of groupoids with orientation character. There is an L-theory functor $\mathbf{L} : \mathsf{GWOC} \to \mathsf{Spectra}$. (The definition in [DL98] can be easily modified to cover this case, see also [BL10].) Two maps $F_0, F_1 : \mathcal{G}^w \to \mathcal{G}'^{w'}$ of groupoids with orientation character are *equivalent* if there is a natural transformation which is orientation preserving in the sense that $w'(F_0(x) \to F_1(x)) = +1$ for all objects x of $\mathcal{G}$. A map $F : \mathcal{G}^w \to \mathcal{G}'^{w'}$ is an *equivalence of groupoids with orientation characters* if there is a map $F' : \mathcal{G}'^{w'} \to \mathcal{G}^w$ so that both composites $F \circ F'$ and $F' \circ F$ are equivalent to the respective identity. An equivalence of groupoids with orientation characters gives an weak equivalence of L-spectra.

Now suppose G is a group with orientation character $w : G \to \{\pm 1\}$. Following forthcoming work of Davis and Lindenstrauss, we discuss two related groupoids with orientation character. First, if S is a G-set, give the action groupoid $\overline{S}$ the orientation character $(t, g, s) \mapsto w(g)$. This gives a functor $\mathrm{Or}(G) \to \mathsf{GWOC}$ defined on objects by $G/H \mapsto (\overline{G/H})^w$ and hence a functor:

$$\underline{\mathbf{L}} : \mathrm{Or}(G) \longrightarrow \mathsf{Spectra}\ ;\ G/H \longmapsto \mathbf{L}((\overline{G/H})^w).$$

Suppose $\phi : \widehat{Y} \to Y$ is a double cover. Define the *fundamental groupoid with orientation character* $\Pi_1^w(Y)$, as follows. The objects are the points of $\widehat{Y}$. A morphism from $\hat{y}$ to $\hat{y}'$ is a path α from $w(\hat{y}')$ to $w(\hat{y})$. A morphism is assigned $+1$ if the unique lift of α starting at $\hat{y}'$ ends at $\hat{y}$; otherwise assign the morphism -1.

Recall G is a group with orientation character w. Given a free G-CW-complex X, let $w : EG \times_{\mathrm{Ker}(w)} X \to EG \times_G X$ be the corresponding double cover. Thus, for a fixed (G, w), there is a functor $\mathbf{L}^G$ defined by

$$\mathbf{L}^G : (G,1)\mathcal{CW} \longrightarrow \mathsf{Spectra}\ ;\ X \longmapsto \mathbf{L}(\Pi_1^w(EG \times_G X)).$$

Then, after modifying $\mathbf{L}$, $\mathbf{L}^G$ and $\overline{S}$ as indicated above, the statement and proof of Lemma B.2 remain valid. The same is true for Theorem B.1 after accounting for Ranicki's version of the assembly map in the non-orientable case [Ran92, App. A].

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