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Completing the Riesz-Dunford functional calculus

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ABSTRACT

Let f be any analytic function defined in a neighborhood of the non-empty set E and let $\mathcal{S}(E)$ denote the set of all operator having spectrum included in E . In this paper the closure and interior of the set $f(\mathcal{S}(E)) \equiv \{ f(A) : A \in \mathcal{S}(E) \}$ are characterized. In addition the sets cl[int{ $f(\mathcal{S}(E))$ }] , int{ cl[$f(\mathcal{S}(E))$] } , cl[int{ cl[$f(\mathcal{S}(E))$] }] , and int{ cl[int{ $f(\mathcal{S}(E))$ }] } are characterized. Several examples and applications are given.

Key words and phrases: Functional calculus, analytic function, spectrum, Fredholm index, non-abelian approximation.

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INTRODUCTION

Given an operator A on a Hilbert space $\mathcal{H}$ and an analytic function f defined in a neighborhood of $\sigma(A)$, the spectrum of A , the operator f(A) is defined via the Cauchy integral of the operator valued function $f(z)(z - A)^{-1}$ around a suitably chosen system of contours enclosing $\sigma(A)$. This mapping $f \to f(A)$ is an algebraic homomorphism from the algebra of all analytic functions defined in some neighborhood of $\sigma(A)$ into the double commutant of A such that 1 is mapped to the identity operator and z is mapped to A . Moreover this map has a certain continuity property which makes it unique. For details, the reader can consult pages 203-210 of [12], where this is worked out in the framework of Banach algebras.

This mapping $f \to f(A)$ is called the Riesz functional calculus or the Riesz-Dunford functional calculus. The first appearance of these ideas is [27], where only compact operators are considered and the only analytic function considered is the characteristic function of an isolated point of the spectrum. Though the topic, with the near simultaneous appearance of [16], [24], and [30], all of which extended Riesz's ideas, takes on all the aspects of one whose time had come, it is the work of Dunford [16] which is the most complete in its treatment. In particular, it was Dunford who first proved the Spectral Mapping Theorem. For this reason it is the custom of many, including the authors, to call this the Riesz-Dunford functional calculus.

Whereas in the discussion of the Riesz-Dunford functional calculus the idea is to fix the operator A and let f vary through a collection of analytic functions, the attitude taken here is to fix the analytic function f and allow the operator A to vary through the collection of all operators for which it makes sense to define f(A) . Specifically, for an arbitrary (not necessarily open) non-empty subset E of the complex plane $\mathbb{C}$, let $\mathcal{S}(E)$ denote the collection of all operators A defined on some separable Hilbert space with $\sigma(A) \subseteq E$. (Note that in the definition of $\mathcal{S}(E)$ neither the Hilbert space nor the dimension of the Hilbert space is specified or restricted, save the restriction that the space is separable. This is done to ensure that such statements as " $A \oplus B \in \mathcal{S}(E)$ if and only if A and $B \in \mathcal{S}(E)$ " are valid without modification or qualification.) If f is analytic in a neighborhood of E , what is the characterization of the operators that belong to $f(\mathcal{S}(E)) \equiv \{ f(A) : A \in \mathcal{S}(E) \}$? As discussed in [13] , which can be considered as the predecessor of this paper, such a question seems beyond the present capabilities of operator theory even for such nice functions as z^p and the exponential. In particular, such a description in terms of spectral properties alone is impossible as two operators can be found with the same spectral picture, only one of which has a square root.

Instead, the characterization of cl[$f(\mathcal{S}(E))$] , the closure of $f(\mathcal{S}(E))$, is obtained. The methods used are those of non-abelian approximation as presented in [20] and [3]. Some background material will be presented to ease the reader's burden.

These results provide an important illustration of the Closure Theorem [22] that states that a closed set of operators on Hilbert space that is similarity invariant and has "sufficient structure" (in a certain technical sense described in [22]) can be characterized in terms of spectral properties alone. Indeed, if $A \in \mathcal{S}(E)$ and R is an invertible operator, then $RAR^{-1} \in \mathcal{S}(E)$ and $f(RAR^{-1}) = Rf(A)R^{-1}$. Thus cl[$f(\mathcal{S}(E))$] is a closed similarity invariant set and membership in this set is characterized solely in terms of spectral properties (Theorem 2.1).

Some ruminations and reflections seem appropriate here as a caution and encouragement for the reader. The results of this paper may strike our audience as extremely complicated. This is, undoubtedly, a correct perception. In order to achieve total generality, something must be

sacrificed and in this case, as in many similar cases, it is simplicity. A comparison of Theorem 2.1 here with the clean characterization of cl { A^p : $A \in \mathcal{B}(\mathcal{H})$ } from [13] (see Corollary 2.4 of this paper for the statement) will certainly confirm this. It is not generally the case that these complications arise from allowing analytic functions to be constant on some of the components of their domain. To be sure, this extra generality does introduce some complexity, but the principal source of the difficulty is the complex nature of an analytic function, especially if it is defined on a non-connected open set whose components are not simply connected. But complications may arise even if the function is a polynomial (see Example 3.8).

In Theorem 2.2, the characterization of cl[$f(\mathcal{S}(E))$] for a completely non-constant analytic function (one that is not constant on any component of its domain) is stated, though the set E is not assumed to be open. The statement may be simplified by requiring that E be open, since f(E) can then have no isolated points. This eliminates condition (e); conditions (a) through (d) may not, however, be simplified.

In contrasting the results here with those from [13] for the functions z^p and e^z , two facts emerge. First, both z^p and e^z have uniform valence. Second, and more important, is that a beautiful geometric condition may be imposed on an open subset Ω of $\mathbb{C}$ to ensure that there exists an analytic inverse $g : \Omega \to \mathbb{C}$ for these functions. A necessary and sufficient condition for this is that Ω does not separate 0 from ∞ . Nothing like this is is possible in general, even for an analytic function defined on a disk.

Because of the nature of this material, a leisurely style has been adopted. We include several details that might be eliminated in another presentation. Many corollaries of special cases have been included. In particular, we have included statements of these results for the special functions z^p and e^z defined on all of $\mathbb{C}$ as well as the function $f(z) = z$ defined in a neighborhood of an arbitrary set E . This last special case allows us to

§1 SPECTRAL PRELIMINARIES

In this section several results that will be used in the remainder of the paper are presented. Some of these are new or at least do not seem to be stated in the literature; some are here only for the reader's convenience. Proofs are given where it is appropriate.

The convention will be adopted in this paper that elements and subsets of the domain of a function will be denoted by Roman letters, while elements and subsets of the range will be Greek. This convention will be abandoned if the circumstances warrant it.

The reader is assumed to be familiar with spectral theory, including the properties of the Fredholm index as contained in the last chapter of [12]. A few concepts are recalled here.

For an operator T, $\sigma(T)$, $\sigma_e(T)$, $\sigma_{le}(T)$, and $\sigma_{re}(T)$ denote the spectrum, essential spectrum, left essential spectrum, and right essential spectrum, respectively, of T. Let $\sigma_{lre}(T) = \sigma_{le}(T) \cap \sigma_{re}(T)$. For any operator T, $\text{nul } T \equiv \dim [\ker T]$ and for $\lambda \notin \sigma_{lre}(T)$, $\text{ind}(\lambda - T) = \text{nul}(\lambda - T) - \text{nul}(\lambda - T)^*$.

For an operator T, $\sigma_0(T)$ denotes the isolated eigenvalues of T such that the corresponding Riesz idempotent has finite rank. If n is an extended integer, $P_n(T) \equiv \{ \lambda \in \mathbb{C} : \lambda - T$ is semi-Fredholm and $\text{ind}(\lambda - T) = n \}$; let $P_{\pm}(T) = \bigcup \{ P_n(T) : n \neq 0 \}$ and $P_{\pm\infty}(T) = P_{-\infty}(T) \cup P_{+\infty}(T)$.

For any subset F of $\mathbb{C}$ let $F_\delta \equiv \{ z \in \mathbb{C} : \text{dist}(z, F) \le \delta \}$.

The proof of the next theorem, a refinement of the Spectral Mapping Theorem, is left to the reader.

1.1 Theorem Let $A \in \mathcal{B}(\mathcal{H})$ and let f be an analytic function defined in a

neighborhood of $\sigma(A)$. If $T = f(A)$, then the following statements are true.

(a) $\sigma(T) = f(\sigma(A))$.

(b) $\sigma_e(T) = f(\sigma_e(A))$.

(c) $\sigma_{lre}(T) = f[\sigma_{lre}(A)] \cup \{ f(a) : a \in P_{+\infty}(A)$ and there exists a point b in $P_{-\infty}(A)$ with $f(b) = f(a) \}$.

(d) $\sigma_{le}(T) = f[\sigma_{le}(A)] \cup \sigma_{lre}(T)$.

(e) $\sigma_{re}(T) = f[\sigma_{re}(A)] \cup \sigma_{lre}(T)$.

(f) If $\lambda \in P_{\pm}(T)$, then $f^{-1}(\lambda) \cap \sigma(A)$ is a finite subset $\{ a_1, \ldots, a_n \}$ of $\sigma(A) \setminus \sigma_{lre}(A)$. Moreover, if $\{ a_1, \ldots, a_n \} \cap Z(f') = \emptyset$, then

(i) $\text{nul}(\lambda - T) = \sum_i \text{nul}(a_i - A)$;

(ii) $\text{nul}(\lambda - T)^* = \sum_i \text{nul}(a_i - A)^*$;

(iii) $\text{ind}(\lambda - T) = \sum_i \text{ind}(a_i - A)$.

An *analytic Cauchy domain* is a bounded open set Ω contained in $\mathbb{C}$ whose boundary consists of a finite number of pairwise disjoint analytic Jordan curves. An *analytic Cauchy region* is a connected analytic Cauchy domain. Note that if K is a compact subset of an open set G contained in $\mathbb{C}$, then there is an analytic Cauchy domain Ω with $K \subseteq \Omega \subseteq \text{cl } \Omega \subseteq G$. Moreover, if F is any countable set in $\mathbb{C}$ (for example, if, as will often be the case in the paper, F is $f(Z(f')) \equiv$ the image under f of the zeros of the derivative of f), then Ω can be chosen such that $\partial\Omega \cap F = \emptyset$. To see this assume that Ω is connected and let D be a circle domain (a region bounded by a finite number of pairwise disjoint circles) and $\phi : D \to \Omega$ a conformal equivalence. For all small $\varepsilon > 0$ let D_ε be a circle domain with $\text{cl } D_\varepsilon \subseteq D$ and so that $D \subseteq$ an ε- neighborhood of D_ε. Then $K \subseteq \phi(D_\varepsilon) \subseteq \phi(D)$ for small ε. Since there are uncountably many ε's, one can be chosen with $\partial\phi(D_\varepsilon)$ disjoint from F.

If $f : G \to \mathbb{C}$ is an analytic function and p is a natural number, say that f is a *strictly p-valent* function if for every α in $f(G)$, the equation $f(z) = \alpha$ has p solutions in G counting multiplicities. Because the concept is frequently used in this paper, call an analytic function f *completely non-constant* if f is not constant on any component of its domain.

1.2 Proposition Assume that f is analytic in a neighborhood of $\sigma(A)$ and completely non-constant ; put $T = f(A)$. If Ω is an analytic Cauchy region with cl $\Omega \subseteq P_{\pm}(T)$ and $\partial\Omega \cap f(Z(f')) = \varnothing$, then $f^{-1}(\Omega) \cap P_{\pm}(A)$ consists of a finite non-zero number of components $H_1, \dots, H_d$ and for each j , $f(H_j) = \Omega$, $f(\partial H_j) = \partial\Omega$, H_j is an analytic Cauchy region, and there is a natural number p_j such that f is a strictly p_j-valent map of H_j onto Ω .

Proof. Since Ω is connected, there is an extended integer N such that ind $(\lambda - T) = N$ for λ in cl Ω . Let $H = f^{-1}(\Omega) \cap \sigma(A)$; by (1.1) , $H \subseteq \sigma(A) \setminus \sigma_{lre}(A)$ and $H \cap P_{\pm}(A) \neq \varnothing$. Note that H is bounded.

Claim 1. If D is a component of H , then $f(D) = \Omega$.

If $\omega \in \Omega \setminus f(D)$, then there is a path in Ω from ω to a point ζ in f(D) . Look at the first point ζ_0 on this curve that is not in f(D). So $\zeta_0 \in \Omega \cap \partial f(D)$. Let $\{ z_n \} \subseteq D$ such that $f(z_n) \to \zeta_0$. Since $\{ z_n \} \subseteq \sigma(A)$, we may assume that $z_n \to z_0$ in $\sigma(A)$. Hence $z_0 \in f^{-1}(\Omega) \cap \sigma(A) = H$. But $D \cup \{ z_0 \}$ is connected and included in H . Since D is a component of H , $z_0 \in D$. This implies $\zeta_0 = f(z_0) \in f(D) \cap \partial f(D)$, a contradiction.

Claim 2. H has only a finite number of components.

Indeed, if not, then there is an ω in Ω such that $f(z) = \omega$ has an infinite number of solutions in the compact set $\sigma(A)$.

Claim 3. If D is a component of H , then $f(\partial D) = \partial\Omega$.

By Claim 1, $f(D) = \Omega$. It is clear that $f(\partial D) \subseteq$ cl Ω . If $\omega \in \partial\Omega = \partial f(D) \subseteq$ cl f(D) $\subseteq$ f(cl D) , then $\omega = f(z)$ for some z in cl D . If $z \in D$, $\omega \in \Omega$, a contradiction. Hence $\partial\Omega \subseteq f(\partial D)$.

Now let $\omega_0 = f(z_0)$ for some z_0 in ∂D . If $\omega_0 \in \Omega$, then there is an $\varepsilon > 0$ and a $\delta > 0$ such that $f[B(z_0 ; \delta)] \subseteq B(\omega_0 ; \varepsilon) \subseteq \Omega$. Hence $D \cup B(z_0 ; \delta)$ is a connected subset of H that includes D . Since D is a component of H , $B(z_0 ; \delta) \subseteq D$, a contradiction to the fact that $z_0 \in \partial D$. Hence $f(\partial D) \subseteq \partial\Omega$.

Claim 4. If K is a compact subset of Ω , then $f^{-1}(K) \cap H$ is compact.

If $\{ z_k \} \subseteq f^{-1}(K) \cap H$, then, by passing to a subsequence if necessary, it may be assumed that $z_k \to z_0$ in cl H . Hence $f(z_k) \to f(z_0)$. Thus $f(z_0) \in K \subseteq \Omega$ and so $z_0 \in f^{-1}(K) \cap \sigma(A) \subseteq H$. So $z_0 \in f^{-1}(K) \cap H$ and this set is compact.

Claim 5. If D is a component of H and ω_0 and $\omega_1 \in \Omega$, then the equations $f(z) = \omega_0$ and $f(z) = \omega_1$ have the same number of solutions in D , counting multiplicities.

By Claim 4 these equations have only a finite number of solutions in D . Indeed, $f^{-1}(\omega_j) \cap H$ is a compact set and f is a non-constant analytic function. Let p_j = the number of solutions of $f(z) = \omega_j$ in D (counting multiplicities). Let γ be a path in Ω from ω_0 to ω_1 . Again Claim 4 implies that $f^{-1}(\gamma) \cap D$ is compact. Thus there is a smooth Jordan system Γ in D such that $f^{-1}(\gamma) \cap D \subseteq$ the inside of Γ . Hence

$$p_i = \frac{1}{2\pi i} \int_\Gamma \frac{f'}{f - \omega_i} = n(f \circ \Gamma ; \omega_i) .$$

But the winding number $n(f \circ \Gamma ; \zeta)$ is constant on components of $\mathbb{C} \setminus f \circ \Gamma$ and ω_0 and ω_1 are both contained in γ which is contained in the same component of $\mathbb{C} \setminus f \circ \Gamma$. Hence $p_0 = p_1$.

Claim 6 If D is a component of H , then there is a natural number p such that f is a strictly p-valent map of D onto Ω .

This is a direct consequence of combining claims 5 and 1.

Claim 7 If D is a component of H, then D is an analytic Cauchy region. In fact, this is clear since Ω is an analytic Cauchy region and f' does not vanish on ∂H.

This completes the proof of the proposition. ♦

Some remarks and examples relative to the preceding proposition might be helpful for the reader. Let f and Ω be as in Proposition 1.2 and let D be one of the components of $f^{-1}(\Omega) \cap P_{\pm}(A)$ such that f is a strictly p-valent mapping of D onto Ω. Suppose $\partial\Omega$ consists of m pairwise disjoint analytic Jordan curves $\gamma_1, \ldots, \gamma_m$ and ∂D consists of n pairwise disjoint analytic Jordan curves $g_1, \ldots, g_n$. Then f maps the boundary curve g_j onto some component curve of $\partial\Omega$ in an r_j -to-one fashion and, moreover, for $1 \le i \le m$

$$\sum \{ r_j : f(g_j) = \gamma_i \} = p ;$$

consequently,

$$\sum_{j=1}^{n} r_j = mp .$$

The following examples will convince the reader that, except for these equalities, everything else is possible.

If $f(z) = z^p$ and $\Omega = D = \mathbb{D}$, then f maps ∂D p-to-one onto $\partial\Omega$. At the other extreme, if D is an analytic Cauchy region whose boundary consists of p pairwise disjoint analytic curves $g_1, \ldots, g_p$, let f be the Ahlfors function mapping D onto $\Omega = \mathbb{D}$. (See [1].) Then f is a strictly p-valent function on Ω and f is a bijection of each g_j onto $\partial\mathbb{D}$.

For a further example, let f be the monic polynomial with distinct zeros $\lambda_1, \ldots, \lambda_s$ and let λ_i have multiplicity d_i ; put $p = d_1 + \cdots + d_s =$ degree of f. Let

$$\Omega = \{ \lambda : |\lambda| < r, |\lambda| > \varepsilon, \text{ and } |\lambda - 3\varepsilon| > \varepsilon \},$$

where $r >> 1 >> \varepsilon > 0$ are chosen so that $\partial\Omega$ consists of three circles: γ_r of radius r and center 0, γ_ε of radius ε and center 0, and γ_ε' of radius ε and center 3ε. If r is sufficiently large and ε is sufficiently small, then $D = f^{-1}(\Omega)$ is an analytic Cauchy region and $\partial D = g_r \cup g_{\varepsilon,1} \cup \cdots \cup g_{\varepsilon,s} \cup g'_{\varepsilon,1} \cup \cdots \cup g'_{\varepsilon,p}$, where these sets in the union are pairwise disjoint analytic Jordan curves, g_r is the boundary of the unbounded component of $\mathbb{C} \setminus D$ and f is a p-to-one map of g_r onto γ_r ; $g_{\varepsilon,k}$ is the boundary of some neighborhood of λ_k and f is a d_k -to-one map of $g_{\varepsilon,k}$ onto γ_ε $(1 \le k \le s)$; and f is a bijection of each $g_{\varepsilon,j}'$ onto γ'_ε $(1 \le j \le p)$.

By combining this polynomial with the Ahlfors function mapping Ω strictly 3 -to-one onto $\mathbb{D}$, even more pathology can be obtained.

1.3 Proposition If $f : G \to \mathbb{C}$ is an analytic function that is completely non-constant, $A \in \mathcal{S}(G)$, and $T = f(A)$, then the following statements are true.

(a) If Ω is an analytic Cauchy region with $\partial\Omega \cap f(Z(f')) = \varnothing$ and $\text{cl}\,\Omega \subseteq P_{\pm\infty}(T)$, then there is an analytic Cauchy region H such that $\text{cl}\,H \subseteq P_{\pm\infty}(A)$, $f(H) = \Omega$, $f(\partial H) = \partial\Omega$, and there is a natural number p such that f is a strictly p-valent map of H onto Ω.

(b) If Ω is an analytic Cauchy region such that $\text{cl}\,\Omega \subseteq P_{\pm}(T) \setminus P_{\pm\infty}(T)$ and $\partial\Omega \cap f(Z(f')) = \varnothing$, then there are analytic Cauchy regions $H_1, \ldots, H_d$ such that for $i \le j \le d$, $\text{cl}\,H_j \subseteq P_{\pm}(A) \setminus P_{\pm\infty}(A)$, $f(H_j) = \Omega$, $f(\partial H_j) = \partial\Omega$, and there are positive integers $p_1, \ldots, p_d$ such that if $m_j = \text{ind}\,(\alpha - A)$ for α in H_j, then :

(i) f is a strictly p_j-valent map of H_j onto Ω ;

(ii) $\text{ind}\,(\lambda - T) = \sum_j p_j m_j$ for all λ in Ω ;

(iii) $\text{nul}\,(\lambda - T) \ge \sum \{ p_j m_j : m_j > 0 \}$ for all λ in Ω. .

Proof. (a) Let $H_1, \ldots, H_d$ be the components of $f^{-1}(\Omega) \cap P_{\pm}(A)$. By Proposition 1.2, each H_j is an analytic Cauchy region with cl $H_j \subseteq P_{\pm}(A)$, $f(H_j) = \Omega$, $f(\partial H_j) = \partial\Omega$, and f is a strictly p_j-valent map of H_j onto Ω for some positive integer p_j. Also Theorem 1.1 implies that for λ in Ω, $\pm\infty$ = ind $(\lambda - T)$ = $\sum$ { ind $(a - A)$: $a \in f^{-1}(\Omega) \cap P_{\pm}(A)$ such that $f(a) = \lambda$ }. Thus for at least one j, $1 \leq j \leq d$, there is a point a in H_j such that ind $(a - A)$ = $\pm\infty$; for this value of j, let $H = H_j$.

(b) Now assume that cl $\Omega \subseteq P_{\pm}(T) \setminus P_{\pm\infty}(T)$. Adopt the notation of the preceding paragraph. By Proposition 1.2 all the properties from part (b) hold, with the possible exception of (ii) and (iii). Let n = ind $(\lambda - T)$ for λ in Ω and m_j = ind $(a - A)$ for a in H_j. Since $f(Z(f'))$ is a countable set, there is a λ in Ω such that $\lambda \notin f(Z(f'))$. Thus part (f) of Theorem 1.1 implies n = $\sum_j m_j p_j$.

It remains to establish (iii). For this, it may be assumed that $\lambda \notin f(Z(f'))$ since such points are dense in Ω and so the general result will follow from this case by results of spectral theory (see Theorem 1.13 (iii) in [20]). Let { $a_{ij} : 1 \leq i \leq p_j$ } be the distinct points in H_j such that $f(a_{ij}) = \lambda$. Note that m_j = nul $(a_{ij} - A)$ - nul $(a_{ij} - A)^*$ for each j. By Theorem 1.1(f).

$$\begin{aligned} \text{nul}\,(\lambda - T) &= \textstyle\sum_{ij} \text{nul}\,(a_{ij} - A) \\ &= \textstyle\sum_j m_j p_j + \sum_{ij} \text{nul}\,(a_{ij} - A)^* \\ &= \textstyle\sum \{ m_j p_j : m_j > 0 \} + \sum_{ij} \{ \text{nul}\,(a_{ij} - A)^* : m_j > 0 \} \\ &\quad + \textstyle\sum \{ [m_j p_j + \sum_i \text{nul}\,(a_{ij} - A)^*] : m_j < 0 \} \end{aligned}$$

But each of these last two summands is positive. Hence (iii) holds. ♦

1.4 Proposition If f is an analytic function (not assumed to be completely non-constant) defined in a neighborhood of E and N is a normal operator, then $N \in f(\mathcal{S}(E))$ if and only if there is a bounded set $E_1 \subseteq E$ such that $\sigma(N) \subseteq f(E_1)$. If f is analytic and completely non-constant on the open set G, then a normal operator N belongs to $f(\mathcal{S}(G))$ if and only if $\sigma(N) \subseteq f(G)$.

Proof. If $N = f(A)$ for some A in $f(\mathcal{S}(E))$, then the Spectral Mapping Theorem implies that the set E_1 may be taken to be $\sigma(A)$. Conversely, if such a bounded set E_1 exists, then there is a Borel function $g: \sigma(N) \to E_1$ such that $g(\lambda) \in E_1 \cap f^{-1}(\lambda)$ for all λ in $\sigma(N)$. (This last statement follows by any of many measurable selection theorems, or, using the analyticity of f, the reader can give a direct proof.) So g is bounded and if $A = g(N)$, then $N = f(A)$.

In the second statement of the proposition, one implication is, again, an immediate consequence of the Spectral Mapping Theorem. If $\sigma(N) \subseteq f(G)$, then the assumptions about f imply that there is a compact set K contained in G such that $f(K) = \sigma(N)$. To see this note that $G = \bigcup_n K_n$, where each K_n is compact and $K_n \subseteq$ int K_{n+1}. Since f is completely non-constant on G, f(int K_n) is open for each n. Thus { f(int K_n) } is an open cover of $\sigma(N)$ and there is an integer n such that $\sigma(N) \subseteq f(K_n)$. The result now follows by the first part of the proposition. ♦

The stated condition on f and E in the preceding proposition is not always satisfied. For example, let $E = \{0\} \cup \{ n^{-1} + 2\pi n i : n \in \mathbb{N} \}$ and let $f(z) = e^z$. So $f(E) = \{ 1, e^{1/2}, e^{1/3}, \ldots \}$. If N is the diagonal operator with entries $1, e^{1/2}, e^{1/3}, \ldots$, then $N \notin f(\mathcal{S}(E))$. It is not too difficult to see that $N \in$ cl[$f(\mathcal{S}(E))$]. See Corollary 2.7 for a characterization of the normal operators belonging to cl[$f(\mathcal{S}(E))$].

1.5 Proposition (a) If X and Y are operators such that there is no non-zero operator S with XS = SY, then for any operator Q the spectrum of $T = \begin{bmatrix} X & 0 \\ Q & Y \end{bmatrix}$ is $\sigma(X) \cup \sigma(Y)$. Moreover, if $T \in f(\mathcal{S}(E))$, then X and Y belong to $f(\mathcal{S}(E))$.

(b) If X and Y are operators with $\sigma_l(X) \cap \sigma_r(Y) = \emptyset$ and $X \oplus Y$ belongs to int [$f(\mathcal{S}(E))$] , then X and Y $\in$ int $f(\mathcal{S}(E))$.

Proof. (a) The stated condition on X and Y implies that the commutant of T consists of lower triangular operator- valued matrices. Thus if λ - T is invertible, $(\lambda - T)^{-1}$ is lower triangular and the diagonal entries must be $(\lambda - X)^{-1}$ and $(\lambda - Y)^{-1}$. Conversely, if λ - X and λ - Y are both invertible, the lower triangular matrix with $(\lambda - X)^{-1}$ and $(\lambda - Y)^{-1}$ on the diagonal and $-(\lambda - X)^{-1} Q (\lambda - Y)^{-1}$ in the lower left corner is the inverse of λ - T .

(b) Let $\varepsilon > 0$ such that $\| X \oplus Y - T \| < \varepsilon$ implies T $\in f(\mathcal{S}(E))$. By a result of [14], there is a c > 0 such that $\| XS - SY \| \geq c \|S\|$ for all S in $\mathcal{B}(\mathcal{H})$. Let $\delta < \varepsilon$ be sufficiently small that $\| X_1S - SY \| \geq (c/2) \|S\|$ whenever $\|X - X_1\| < \delta$. Then $\| X \oplus Y - X_1 \oplus Y \| < \varepsilon$ whenever $\|X - X_1\| < \delta$ and so $X_1 \oplus Y \in f(\mathcal{S}(E))$. But $X_1T = TY$ implies T = 0 . Hence $X_1 \in f(\mathcal{S}(E))$. That is, $\| X - X_1 \| < \delta$ implies $X_1 \in f(\mathcal{S}(E))$. Thus X $\in$ int[$f(\mathcal{S}(E))$] . Similarly, Y $\in$ int $f(\mathcal{S}(E))$. ♦

For an analytic Cauchy domain Λ define the following operators.

$$A(\Lambda) = M_z \text{ on } H^2(\partial\Lambda) \qquad C(\Lambda) = A(\Lambda^*)^*.$$

1.6 Proposition Let Λ_1 and Λ_2 be analytic Cauchy domains. If X is a bounded operator, n and m are extended positive integers, and $A(\Lambda_1)^{(n)}X = XC(\Lambda_2)^{(m)}$, then X = 0 .

Proof. In fact, $A(\Lambda_1)$ is a subnormal operator and $C(\Lambda_2)$ is a cosubnormal operator. The result now follows by a standard result of subnormal operator theory [25] (also see [11], p. 199). ♦

1.7 Proposition Let Λ be an open subset of $\mathbb{C}$.

(a) $A(\Lambda) \in f(\mathcal{S}(E))$ if and only if there is an open set G with cl G included in E such that f is one-to-one on G and $f(G) = \Lambda$. Moreover, if $A(\Lambda)$ = f(R) and g is the inverse of the restriction of f to G, then R = multiplication by g on $H^2(\partial\Lambda)$.

(b) $C(\Lambda) \in f(\mathcal{S}(E))$ if and only if there is an open set G with cl G included in E such that f is one-to-one on G and $f(G) = \Lambda$. Moreover, if $C(\Lambda)$ = f(R) and g is the inverse of the restriction of f to G, then R^* is multiplication by $\overline{g(\bar{z})}$ on $H^2(\partial\Lambda^*)$.

Proof. (a) If there is an open set G with cl G included in E such that f is one-to-one on G , g is the inverse of the restriction of f to G, and R is multiplication by g , then R $\in \mathcal{S}(E)$ and $f(R) = A(\Lambda)$. Conversely, if $A(\Lambda)$ = f(R) for some R in $\mathcal{S}(E)$, then R must commute with $A(\Lambda)$. Hence (see, for example, [31] or page 147 of [11]) there is a g in $H^\infty(\Lambda)$ such that R = multiplication by g on $H^2(\partial\Lambda)$. It is routine to check that $f(g(\zeta)) = \zeta$ for all ζ in Λ . If $G = g(\Lambda)$, then f is one-to-one on G . The Spectral Mapping Theorem implies cl G = $\sigma(R) \subseteq E$.

(b) This follows from part (a) and the definition of $C(\Lambda)$. ♦

Before stating the next result, two additional pieces of notation are needed. For any operator T , let min ind T $\equiv$ min{ nul T , nul T^*} . Also, if σ is a closed and relatively open subset of $\sigma(T)$, let $\mathcal{H}(T; \sigma)$ denote the range of the Riesz idempotent, $E(T; \sigma)$, associated with σ . If σ consists of a single isolated point λ , let $E(T; \lambda) = E(T; \{\lambda\})$ and $\mathcal{H}(T; \lambda) = \mathcal{H}(T; \{\lambda\})$.

The next result is a special case of the Similarity Orbit Theorem from [5]. Also see [3], page 5.

1.8 Theorem (Special case of the Similarity Orbit Theorem) Assume the operator X has the property that if λ is an isolated point of $\sigma_e(X)$, $k_{\lambda,X}(z)$ is defined to be λ - z on a neighborhood of λ and 0 on a neighborhood of $\sigma_e(X) \setminus \{\lambda\}$, and $\tilde{X}$ is the image of X in the Calkin algebra, then $[k_{\lambda,X}(\tilde{X})]^m \neq 0$ for all $m \geq 1$. If the operator Y satisfies the conditions:

(a) $\sigma_0(Y) \subseteq \sigma_0(X)$ and each component of $\sigma_{lre}(Y)$ meets $\sigma_e(X)$;

(b) $P(Y) \subseteq P(X)$ and $\text{ind}(\lambda - Y) = \text{ind}(\lambda - X)$ and $\min \text{ind}(\lambda - Y)^k \geq \min \text{ind}(\lambda - X)^k$ for all λ in $P(Y)$ and all $k \geq 1$;

(c) $\dim \mathcal{H}(Y; \lambda) = \dim \mathcal{H}(Y; \lambda)$ for all λ in $\sigma_0(Y)$; then Y is in the closure of the similarity orbit of X ,

$$\mathcal{S}(X) \equiv \{ WXW^{-1} : W \text{ is invertible in } \mathcal{B}(\mathcal{H}) \} .$$

The next proposition is a variation on a result of [6] (also see page 136 of [20]). Essentially it is a special case of that result that will be needed in this paper.

1.9 Proposition If $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, then there is an operator S such that $\|S - T\| < \varepsilon$ and S is similar to a direct sum $S_1 \oplus \cdots \oplus S_n \oplus F$, where these direct summands satisfy the following properties:

(a) F is a finite rank operator with $\sigma(F) \subseteq \sigma_0(T)$ and for λ in $\sigma(F)$, $F \mid \mathcal{H}(F; \lambda)$ similar to $T \mid \mathcal{H}(T; \lambda)$;

(b) for each j , $\sigma(S_j)$ is connected;

(c) $\sigma(F) \cap \sigma(S_j) = \emptyset = \sigma(S_j) \cap \sigma(S_i)$ for $i \neq j$ and $1 \leq i, j \leq n$;

(d) for each j , $\sigma_{lre}(S_j)$ is the closure of an analytic Cauchy domain and $\sigma(S_j) \setminus \sigma_{lre}(S_j)$ is an analytic Cauchy domain;

(e) $P(S) \subseteq P(T)$, $P_{\pm}(S)$ has only a finite number of components, and for each λ in $P(S)$, $\text{nul}(\lambda - S) = \text{nul}(\lambda - T)$ and $\text{nul}(\lambda - S)^* = \text{nul}(\lambda - T)^*$.

(f) $\sigma_e(T) \subseteq \sigma_e(S)$, $\sigma_{lre}(T) \subseteq \sigma_{lre}(S)$, and each component of $\sigma_e(S)$ contains at least one component of $\sigma_e(T)$.

Proof. Let $\delta > 0$ and let Λ be an analytic Cauchy domain with $\sigma_{lre}(T) \subseteq \Lambda \subseteq [\sigma_{lre}(T)]_\delta$ and $\partial\Lambda \cap \sigma_0(T) = \emptyset$. Let N be a normal operator with $\sigma(N) = \text{cl}\,\Lambda$. By Proposition 1.4 of [6] (also see Chapter 3 in [20]) there is an operator S with $\|T - S\| < 2\delta$ such that S is similar to $T \oplus N$. (This could be obtained as a consequence of the Similarity Orbit Theorem, but this would be putting the cart before the horse.) Consequently

$$\sigma(S) = \sigma(T) \cup \text{cl}\,\Lambda ,$$
$$\sigma_{lre}(S) = \text{cl}\,\Lambda ,$$
$$\sigma_0(S) = \sigma_0(T) \setminus \Lambda .$$

Note that this implies that $\sigma_0(S)$ is finite and $\sigma(S)$ has only a finite number of components. Moreover S can be chosen so that $\mathcal{H}(S; \lambda) = \mathcal{H}(T; \lambda)$ and $S \mid \mathcal{H}(T; \lambda) = T \mid \mathcal{H}(T; \lambda)$ for λ in $\sigma_0(S)$. Let $\sigma_1, \ldots, \sigma_n$ be the components of $\sigma(S) \setminus \sigma_0(S)$. Using the Riesz-Dunford functional calculus, S is similar to $S_1 \oplus \cdots S_n \oplus F$, where each $S_j = S \mid \mathcal{H}(S; \sigma_j)$ and $F = S \mid \mathcal{H}(T; \sigma_0(S))$. It is routine to check that the properties listed in the statement of the proposition are verified. ♦

§2 THE CLOSURE OF $f(\mathcal{S}(E))$

In this section we will prove the main result of this paper and its corollaries. Note that the function in this theorem is not assumed to be completely non-constant. After the statement we will give the explicit statement for the completely non-constant case (Theorem 2.2), the case where f is constant on each component of its domain of definition (Theorem 2.3), as well as show how to obtain the results of [13] for the functions z^p ($p \geq 2$) and e^z (Corollaries 2.4 and 2.5), and the function $f(z) = z$ (Corollary 2.8), thus recapturing a result of [6] characterizing cl $\mathcal{S}(E)$.

2.1 Theorem Let f be an analytic function defined on an open set D that includes the non-empty set E. Put $D_0 = \text{int } f(Z(f'))$, $E_0 = E \cap D_0$, and $\Delta =$ the derived set of $f(E_0)$. An operator T belongs to the closure of $f(\mathcal{S}(E))$ if and only if the following conditions are satisfied.

(a) Every component of $\sigma_0(T) \cup \sigma_e(T)$ meets cl[f(E)] . Furthermore, if σ is a component of $\sigma_e(T)$ that is not a singleton, then σ meets cl[$f(E \setminus E_0)$] $\cup\ \Delta$.

(b) $P_{\pm}(T) \subseteq f(E \setminus E_0)$.

(c) If Ω is a connected analytic Cauchy region such that cl $\Omega \subseteq P_{\pm\infty}(T)$ and $\partial\Omega \cap f(Z(f')) = \varnothing$, then there is a component G of $f^{-1}(\Omega)$ with $G \subseteq E$ and a natural number p such that f is a strictly p-valent map of G onto Ω .

(d) If Ω is a connected analytic Cauchy region such that cl $\Omega \subseteq P_{\pm}(T) \setminus P_{\pm\infty}(T)$ and $\partial\Omega \cap f(Z(f')) = \varnothing$, then there are a finite number of components $G_1, \ldots, G_r$ of $f^{-1}(\Omega)$ that are contained in E , natural numbers $p_1, \ldots, p_r$, and non-zero integers $m_1, \ldots, m_r$ such that for $1 \leq j \leq r$, f is a strictly p_j-valent map of G_j onto Ω_j and for all λ in Ω ,

(i) $\quad \text{ind}(\lambda - T) = \sum_j p_j m_j$,

(ii) $\quad \text{nul}(\lambda - T) \geq \sum \{ p_j m_j : m_j > 0 \}$.

(e) If $\lambda \in \sigma_0(T)$ and λ is an isolated point of $f(E \setminus E_0)$ such that $\lambda \notin$ cl[$f(E_0)$] and $f^{-1}(\lambda) \subseteq Z(f')$, then for every positive integer m ,

$$\text{nul}(\lambda - T)^m \geq \min[\, mk_0 \,,\ \dim \mathcal{H}(T;\lambda) \,] ,$$

where

$$k_0 = \min \{ k : \text{there is a z in E with } f(z) = \lambda ,\ f'(z) = \cdots = f^{(k-1)}(z) = 0 ,\ \text{and } f^{(k)}(z) \neq 0 \} .$$

(f) If λ is an isolated point of $\sigma(T)$ as well as $f(E_0)$, but $\lambda \notin$ cl $f(E \setminus E_0)$, then $\mathcal{H}(T;\lambda) = \ker(\lambda - T)$.

(g) If λ is an isolated point of $\sigma_e(T)$ as well as $f(E_0)$, but λ does not belong to cl[$f(E \setminus E_0)$] , and if $k_{\lambda,T}(z)$ is defined to be $\lambda - z$ on some neighborhood of λ and 0 on a neighborhood of $\sigma_e(T) \setminus \{\lambda\}$, then $k_{\lambda,T}(\tilde{T}) = 0$, where $\tilde{T}$ denotes the image of T in the Calkin algebra.

Some comments on the conditions of the previous theorem may aid the reader's digestion. Conditions (a) through (d) arise from the nature of the problem and are not connected with the fact that f is allowed to be defined on a non-connected open set in $\mathbb{C}$ and constant on some components of its domain. Moreover, condition (b) is a consequence of (c) and (d) and is stated for the purpose of emphasis rather than substance.

Also note that if E is an open set and f is completely non-constant, then f(E) is open and thus can have no isolated points. Thus conditions (e), (f), and (g) are vacuously satisfied. (In this case, condition (a) can also be simplified since $\Delta = \varnothing$.) So in this case, only conditions (a), (c), and (d) are required.

If f is a constant function, $f(z) \equiv \lambda$ on D, then $f(\mathcal{S}(E))$ = the single operator λI. So, for a general f, the operators in $f(\mathcal{S}(E_0))$ tend to be uncomplicated. Operators in cl[$f(\mathcal{S}(E \setminus E_0))$] are more complicated. If cl[$f(E \setminus E_0)$] meets $f(E_0)$, then the simplicity that accrues from cl[$f(\mathcal{S}(E_0))$] is eliminated by the complications of cl[$f(\mathcal{S}(E \setminus E_0))$]. The statements of conditions (f) and (g) reflect this phenomenon.

In addition there are equivalent formulations of (f) and (g) that are useful. In fact, it is these equivalent formulations that will be shown to be necessary for membership in cl[$f(\mathcal{S}(E))$]. Specifically, condition (f) is equivalent to the requirement that $TE(T; \lambda) = \lambda E(T; \lambda)$, where $E(T; \lambda)$ is the Riesz idempotent associated with the closed and relatively open set $\{\lambda\}$ of $\sigma(T)$. Condition (g) is equivalent to the condition that $\tilde{T}E(\tilde{T}; \lambda) = \lambda E(\tilde{T}; \lambda)$, where $E(\tilde{T}; \lambda)$ is the Riesz idempotent associated with the closed and relatively open set $\{\lambda\}$ of $\sigma_e(T)$.

Even though f is completely non-constant, f(E) may still have isolated points since E is not assumed to be open. Condition (e) is present because E may not be open and, as is clear from its statement, is divorced from the fact that f is allowed to be constant on some components of its domain. Also note that the integer k_0 in part (e) must be at least 2.

The proof of Theorem 2.1 will be accomplished by combining the analogous result for the special (extreme) cases where f is completely non-constant and where f is constant on each component of its domain. We now state these special cases.

2.2 Theorem If f is a completely non-constant analytic function on a neighborhood of the set E, then an operator T belongs to the closure of $f(\mathcal{S}(E))$ if and only if the following conditions are satisfied.

(a) Every component of $\sigma_0(T) \cup \sigma_e(T)$ meets cl[f(E)].

(b) $P_{\pm}(T) \subseteq f(E)$.

(c) If Ω is a connected analytic Cauchy region such that cl $\Omega \subseteq P_{\pm\infty}(T)$ and $\partial\Omega \cap f(Z(f')) = \varnothing$, then there is a component G of $f^{-1}(\Omega)$ with $G \subseteq E$ and a natural number p such that f is a strictly p-valent map of G onto Ω.

(d) If Ω is a connected analytic Cauchy region such that cl $\Omega \subseteq P_{\pm}(T) \setminus P_{\pm\infty}(T)$ and $\partial\Omega \cap f(Z(f')) = \varnothing$, then there are a finite number of components $G_1, \ldots, G_r$ of $f^{-1}(\Omega)$ that are contained in E, natural numbers $p_1, \ldots, p_r$, and non-zero integers $m_1, \ldots, m_r$ such that for $1 \le j \le r$, f is a strictly p_j-valent map of G_j onto Ω_j and for all λ in Ω,

(i) $\quad \operatorname{ind}(\lambda - T) = \sum_j p_j m_j$,

(ii) $\quad \operatorname{nul}(\lambda - T) \ge \sum \{ p_j m_j : m_j > 0 \}$.

(e) If $\lambda \in \sigma_0(T)$ and λ is an isolated point of f(E) such that $f^{-1}(\lambda) \subseteq Z(f')$, then for every positive integer m,

$$\operatorname{nul}(\lambda - T)^m \ge \min [\, mk_0 ,\ \dim \mathcal{H}(T; \lambda)\,],$$

where

$$k_0 = \min \{ k : \text{there is a z in E with } f(z) = \lambda,\ f'(z) = \cdots = f^{(k-1)}(z) = 0, \text{ and } f^{(k)}(z) \ne 0 \}.$$

Observe that conditions (a) through (e) are (essentially) identical in both theorems.

2.3 Theorem Let f be an analytic function on a domain D and assume that f is constant on each component of D. If E is a non-empty subset of D and $T \in \mathcal{B}(\mathcal{H})$, then the following statements are equivalent.

(a) $T \in$ cl[$f(\mathcal{S}(E))$].

(b) T is the limit of a sequence of operators each of which is similar to a normal operator with spectrum contained in $f(E)$.

(c) T satisfies the following conditions:

(i) Every component of $\sigma_0(T) \cup \sigma_e(T)$ meets cl $f(E)$;

(ii) $P_{\pm}(T) = \emptyset$;

(iii) Every component of $\sigma_e(T)$ that is not a singleton meets Δ ;

(iv) If λ is an isolated point of $\sigma(T)$ as well as $f(E)$, then $\mathcal{H}(T;\lambda) = \ker(\lambda - T)$.

(v) If λ is an isolated point of $\sigma_e(T)$ as well as $f(E)$, and if $k_{\lambda,T}(z)$ is defined to be $\lambda - z$ on some neighborhood of λ and 0 on a neighborhood of $\sigma_e(T) \setminus \{\lambda\}$, then $k_{\lambda,T}(\tilde{T}) = 0$, where $\tilde{T}$ denotes the image of T in the Calkin algebra.

The proof of these three theorems will be postponed until later in this section. In fact, they will be proved in reverse order, with Theorem 2.1 being derived as a consequence of Theorems 2.2 and 2.3. Let us first consider some of the corollaries. First, the special case when $f(z) = z^p$ or e^z and $E = \mathbb{C}$ can be recaptured.

2.4 Corollary [13] If $\mathcal{R}_p = \{ A^p : A \in \mathcal{B}(\mathcal{H}) \}$, then $T \in$ cl $\mathcal{R}_p$ if and only if

$$\{ \lambda \in \mathbb{C} : \lambda - T \text{ is Fredholm and ind } (\lambda - T) \notin p\ \mathbb{Z} \}$$

does not separate 0 from ∞ .

Proof. Here $E = \mathbb{C}$ and $f(z) = z^p$, so that conditions (a) and (b) of Theorem 2.2 are trivially satisfied by every operator. Let Ω be any analytic Cauchy region with $0 \notin \partial\Omega$. Either $0 \in \hat{\Omega}$, the polynomially convex hull of Ω, or $0 \notin \hat{\Omega}$. If $0 \notin \hat{\Omega}$, then $f^{-1}(\Omega)$ has p pairwise disjoint components and f is a one-to-one map of each of these components onto Ω . Thus if $T \in \mathcal{B}(\mathcal{H})$ and Ω is such an analytic Cauchy region with $0 \notin \hat{\Omega}$, then conditions (c) and (d) are seen to hold.

Now suppose $0 \in \hat{\Omega}$. Hence there is a Jordan curve γ in Ω that surrounds 0 . It follows that $G = f^{-1}(\Omega)$ is connected and f is a strictly p-valent mapping of G onto Ω . Thus condition (c) of Theorem 2.2 is always satisfied for every operator in $\mathcal{B}(\mathcal{H})$. If cl $\Omega \subseteq P_{\pm}(T) \setminus P_{\pm\infty}(T)$, $n = \text{ind}(\lambda - T)$, and $0 \in \Omega$, then it is seen that condition (d) (i) of Theorem 2.2 is satisfied if and only if n is a multiple of p . Since, in this case, condition (d) (ii) is an easy consequence of (d) (i), this proves the corollary. ♦

The proof of the next corollary is similar to that of the preceding one.

2.5 Corollary [13] If $\mathcal{E} = \{ \exp(A) : A \in \mathcal{B}(\mathcal{H}) \}$, then $T \in$ cl $\mathcal{E}$ if and only if $P_{\pm}(T)$ does not separate 0 from ∞ .

Recall from [4] (also see [15]) that an operator T is biquasitriangular if and only if $P_{\pm}(T) = \emptyset$. Thus, if T is biquasitriangular, conditions (b), (c), and (d) of Theorem 2.1 are vacuously satisfied. Necessary and sufficient conditions for a biquasitriangular operator to belong to cl[$f(\mathcal{S}(E))$] can thus be formulated. If f is completely non-constant, this can be phrased in a way that is worth making explicit.

2.6 Corollary If T is a biquasitriangular operator on $\mathcal{H}$ and f is a completely non- constant analytic function defined in a neighborhood of E , then $T \in$ cl[$f(\mathcal{S}(E))$] if and only if every component of $\sigma_0(T) \cup \sigma_e(T)$ meets cl[$f(E)$] and condition (e) of Theorem 2.2 holds.

If T is a normal operator and λ is any point of $\sigma_0(T)$, then $\mathcal{H}(T;\lambda) = \ker(\lambda - T)$ and for every positive integer m , $\text{nul}(\lambda - T)^m = \text{nul}(\lambda - T) = \dim \mathcal{H}(T;\lambda)$. Thus conditions (e), (f), and (g) of Theorem 2.2 are always satisfied by every normal operator, irrespective of the set E and the function

f . Since normal operators are biquasitriangular, we get the following pleasant corollary.

2.7 Corollary If T is a normal operator and f is an analytic function defined in a neighborhood of the non- empty set E , then $T \in$ cl[$f(\mathcal{S}(E))$] if and only if every component of $\sigma(T)$ meets cl[$f(E)$] and if σ is a component of $\sigma_e(T)$ that is not a singleton, then σ meets [$\Delta \cup$ cl $f(E \setminus E_0)$].

It is also possible to recapture one of the results of [6] from Theorem 2.2, since $f(z) = z$ is completely non-constant.

2.8 Corollary [6] If E is a non-empty subset of $\mathbb{C}$, then $T \in$ cl[$\mathcal{S}(E)$] if and only if :

(a) every component of $\sigma_0(T) \cup \sigma_e(T)$ meets cl E ;

(b) $P_{\pm}(T) \subseteq E$.

2.9 Corollary If $A \oplus B \in$ cl[$f(\mathcal{S}(E))$] and $\sigma(A) \cap \sigma(B) = \varnothing$, then A and B belong to cl[$f(\mathcal{S}(E))$] .

We begin the process of proof of the three theorems by proving Theorem 2.3. For any non-empty subset Σ of $\mathbb{C}$, let $\mathcal{N}(\Sigma)$ be defined by

$$\mathcal{N}(\Sigma) \equiv \{ T \in \mathcal{B}(\mathcal{H}) : \text{T is the limit of a sequence of operators each of which is similar to a normal operator with spectrum contained in } \Sigma \} .$$

So condition (b) in Theorem 2.3 is that $T \in \mathcal{N}(f(E))$.

2.10 Lemma If f , D , and E are as in Theorem 2.3 , then $\mathcal{N}(f(E))$ is included in cl[$f(\mathcal{S}(E))$] .

Proof. If N is a normal operator with finite spectrum contained $f(E)$, then $N \in f(\mathcal{S}(E))$ by Proposition 1.4. If N is any normal operator with $\sigma(N) \subseteq f(E)$, then the Spectral Theorem implies that N can be approximated by normal operators with finite spectrum contained in $\sigma(N)$; hence, $N \in$ cl[$f(\mathcal{S}(E))$] . From here the lemma is immediate. ♦

Recall that a set is called perfect if it is closed and each point in the set is a limit point of the set.

2.11 Lemma Let f , D , and E be as in Theorem 2.3 and let Δ be the derived set of $f(E)$. If S is an operator with $\sigma(S) = \sigma_{lre}(S)$ = a connected set that is not a singleton and $\sigma(S) \cap \Delta \neq \varnothing$, then $S \in \mathcal{N}(f(E))$.

Proof. Let M be a normal operator with $\sigma(M) = \sigma(S)$. By The Similarity Orbit Theorem (Theorem 1.8), S belongs to the similarity orbit of M . Thus it suffices to show that $M \in \mathcal{N}(f(E))$.

Let $\lambda_0 \in \sigma(M) \cap \Delta$ and fix $\varepsilon > 0$. By definition there are distinct points $\{ \lambda_n \}$ in $f(E)$ such that $\lambda_n \to \lambda_0$ and $|\lambda_n - \lambda_0| < \varepsilon$ for all n . Let N_ε be a normal operator with $\sigma(N_\varepsilon) = \sigma_e(N_\varepsilon) = \{ \lambda_0 , \lambda_1 , \lambda_2 , \ldots \}$; clearly $N_\varepsilon \in \mathcal{N}(f(E))$. If Q is any nilpotent operator, Lemma 5.3 of [20] implies that $N_\varepsilon \oplus (\lambda_0 + Q) \in \mathcal{N}(f(E))$. But Theorem 5.1 of [20] implies there is a sequence $\{Q_k\}$ of nilpotents such that $\| Q_k - M \| \to 0$. Hence $N_\varepsilon \oplus M \in \mathcal{N}(f(E))$. But $\| N_\varepsilon - \lambda_0 \| < \varepsilon$ and, since ε was arbitrary, $\lambda_0 \oplus M \in \mathcal{N}(f(E))$. But $\lambda_0 \in \sigma(M)$ and $\sigma(M)$ is a perfect set, so an elementary argument using the Spectral Theorem shows that $M \in \mathcal{N}(f(E))$ and completes the proof. ♦

2.12 Lemma Suppose f , D , and E are as in Theorem 2.3, $T \in$ cl[$f(\mathcal{S}(E))$] , and λ is an isolated point of $f(E)$.

(a) If λ is also an isolated point of $\sigma(T)$, then T is similar to $\lambda \oplus T_0$, where $T_0 \in$ cl[$f(\mathcal{S}(E))$] and $\lambda \notin \sigma(T_0)$.

(b) If λ is also an isolated point of $\sigma_e(T)$ and $k_{\lambda,T}(z)$ is defined as in

Theorem 2.3, then $k_{\lambda,T}(\widetilde{T}) = 0$.

Proof. (a) Let $T_0 \equiv T \mid \mathcal{H}(T; \sigma(T) \setminus \{\lambda\})$. Since λ is isolated in both $\sigma(T)$ and $f(E)$, there are open sets Λ and Ω such that $\mathrm{cl}\,\Omega \cap \mathrm{cl}\,\Lambda = \emptyset$, $\{\lambda\} = \Lambda \cap \sigma(T) = \Lambda \cap f(E)$, $K = \Omega \cap \sigma(T) = \Omega \cap f(E)$. Let $T_n = f(A_n)$, $\sigma(A_n) \subseteq E$, such that $T_n \to T$. We can assume that $\sigma(T_n) \subseteq \Lambda \cup \Omega$ for all n . By an application of the Riesz- Dunford functional calculus, for $\Phi = \Lambda$ or Ω ,

$$T_n E(T_n ; \Phi) \to TE(T ; \Phi) .$$

It is easy to see that $T_n E(T_n ; \Phi) \in f(\mathcal{S}(E))$. In fact, the conditions on Λ and Ω imply that for $\Phi = \Lambda$ and Ω , $f^{-1}(\Phi)$ consists of the union of some collection of components of E . Thus $f^{-1}(\Phi) \cap \sigma(A_n)$ is a closed and relatively open subset of $\sigma(A_n)$. It is straightforward to see that $E(T_n ; \Phi) = E(A_n ; f^{-1}(\Phi))$ and $T_n E(T_n ; \Phi) = f(B_n)$, where $B_n = A_n \mid \mathcal{H}(A_n ; f^{-1}(\Phi))$. In particular, $T_0 \in \mathrm{cl}[\, f(\mathcal{S}(E))\,]$.

Because $\Lambda \cap f(E) = \{\lambda\}$, $T_n \in f(\mathcal{S}(E))$, and $\sigma(T_n) \subseteq \Lambda \cup \Omega$, it follows that $\sigma(T_n) \cap \Lambda = \{\lambda\}$. Thus $T_n E(T_n ; \Lambda) = T_n E(T_n ; \lambda) = f(A_n \mid \mathcal{H}_n(A_n ; f^{-1}(\Lambda))$. But f is constantly equal to λ on $f^{-1}(\Lambda)$ and so $T_n E(T_n ; \Lambda) = \lambda E(T_n ; \Lambda)$ and this sequence converges to $\lambda E(T; \Lambda) = TE(T ; \Lambda)$. That is, $T \mid \mathcal{H}(T; \lambda) = \lambda$. Part (a) is now immediate from standard elementary results.

The proof of part (b) is similar and is left to the reader. ♦

Proof of Theorem 2.3. From Lemma 2.10 we already know that (b) implies (a) .

(a) *implies* (c). By standard arguments using the semicontinuity of the parts of the spectrum, (i) holds. Also, the continuity of the index and the fact that each operator in $f(\mathcal{S}(E))$ has countable spectrum imply that $P_{\pm}(T)$ is empty; that is, (ii) holds. Conditions (iv) and (v) follow from Lemma 2.12 above. It remains to establish (iii).

Suppose C is a component of $\sigma_e(T)$ that is disjoint from Δ , the derived set of f(E) . By a standard topological argument (for example, see 16.15 of [19]), there is an open neighborhood U of C such that $\partial U \cap \sigma_e(T) = \emptyset$ and $U \cap \Delta = \emptyset$. Thus $U \cap \mathrm{cl}\, f(E)$ is a finite set that is non-empty by (i) . Let $U \cap \mathrm{cl}\, f(E) = \{ \lambda_1 , \ldots , \lambda_m \}$. Define k(z) in a neighborhood of $\sigma_e(T)$ by letting $k(z) = \prod_j (z - \lambda_j)$ on U and $k(z) = 0$ on $\mathbb{C} \setminus \mathrm{cl}\, U$. If $\{T_n\} \subseteq f(\mathcal{S}(E))$ and $\| T_0 - T \| \to 0$, we may assume that $\sigma_e(T_n) \cap \partial U = \emptyset$ for all n . As in the proof of Lemma 2.12, $k(\widetilde{T}) E(\widetilde{T}; U)$ is the limit of $\{ k(\widetilde{T}_n) E(\widetilde{T}_n ; U) \}$ and $k(\widetilde{T}_n) E(\widetilde{T}_n ; U) = 0$ for all n . Hence $k(\widetilde{T}) E(\widetilde{T}; U) = 0$ and condition (iii) holds.

(c) *implies* (b). Suppose T satisfies the conditions of (c) and fix $\varepsilon > 0$. We first eliminate int $P_0(T)$. In fact, by [2] (also see [20], Proposition 8.42) there is a compact operator K_ε with $\| K_\varepsilon \| < \varepsilon$ and such that $T_1 \equiv T - K_\varepsilon$ satisfies the conditions of(c) as well as the following:

(2.13 i) $\sigma(T_1) = \sigma_e(T_1) \cup \sigma_0(T)$;

(2.13 ii) if λ is an isolated point of $\sigma(T_1)$, then $\mathcal{H}(T_1 ; \lambda) = \ker (\lambda - T_1)$.

Let $K = \{ \lambda \in \sigma(T_1) : \lambda \in C$, where C is a component of $\sigma(T_1)$ and $C \cap \Delta \neq \emptyset \}$. Note that if $\lambda \in \sigma(T_1) \setminus K$, then (iii) implies $\{\lambda\}$ is a component of $\sigma(T_1)$ and, by definition, $\lambda \notin \Delta$, though $\lambda \in f(E)$ by The Spectral Mapping Theorem. We claim that K is closed. In fact, suppose $\{ \lambda_n \} \subseteq K$, $\lambda_n \to \lambda$, and $\lambda \notin K$. Since $\lambda \notin K$, there is an open neighborhood U of λ such that $\partial U \cap \sigma(T) = \emptyset$ and $U \cap f(E) = \{\lambda\}$. So $\lambda_n \in U$ for large n . But if C_n is the component of $\sigma(T_1)$ that contains λ_n , the fact that $\partial U \cap \sigma(T_1) = \emptyset$ implies $C_n \subseteq U$. Since C_n meets Δ , this contradicts the condition that $U \cap f(E) = \{\lambda\}$. Therefore K is closed. Let N be a normal operator with $\sigma(N) = K_\varepsilon$. As in the proof of Proposition 1.9, there exists an operator S that is similar to $T_1 \oplus N$ and also satisfies $\| S - T_1 \| < 2\varepsilon$. However (as in Proposition 1.9) $S \approx S_1 \oplus \cdots \oplus S_m \oplus F$, where $F = T_1 \mid \mathcal{H}(T_1 ; \sigma(T_1) \setminus K_\varepsilon)$ and for each j , $\sigma(S_j)$ is a component of K_ε . From (2.13 ii), $F \in \mathcal{N}(f(E))$. By construction, for $1 \le j \le m$, $\sigma(S_j)$ meets Δ . Therefore by Lemma 2.11, each $S_j \in \mathcal{N}(f(E))$. Hence $S \in$

$\mathcal{N}(f(E))$. Since ε was arbitrary, we get that $T \in \mathcal{N}(f(E))$, completing the proof. ♦

We now begin the proof of Theorem 2.2. Only one additional lemma is needed.

2.14 Lemma Let f be a completely non-constant analytic function defined on the open set D containing E . If $\varepsilon > 0$ and T satisfies conditions (a) through (e) of Theorem 2.2, then there is an operator S such that $\|S - T\| < \varepsilon$ and S satisfies the following strengthened form of conditions (a) through (e).

(a) (i) $\sigma_0(S)$ is a finite subset of f(E) .

(ii) $\sigma_{lre}(S) = \text{cl}\ \Lambda$ where Λ is an analytic Cauchy domain, each component of Λ meets f(E) , and $\partial\Lambda \cap f(Z(f')) = \emptyset$.

(b) $\text{cl}[\ P_{\pm}(S)\ ] \subseteq f(E)$, $P_{\pm}(S)$ has only a finite number of components $\Omega_1, \ldots, \Omega_q$, and each Ω_k is an analytic Cauchy region with $\partial\Omega_k \cap f(Z(f')) = \emptyset$.

(c) If $\Omega_k \subseteq P_{\pm\infty}(S)$, then there is a component G_k of $f^{-1}(\Omega_k)$ and a natural number p such that f is a strictly p-valent map of G_k onto Ω_k .

(d) If $\Omega_k \cap P_{\pm\infty}(S) = \emptyset$, then there are components $G_1^{(k)}, \ldots, G_r^{(k)}$ of $f^{-1}(\Omega_k)$, there are natural numbers $p_1^{(k)}, \ldots, p_r^{(k)}$ and there are non-zero integers $m_1^{(k)}, \ldots, m_r^{(k)}$ such that f is a strictly $p_i^{(k)}$ -valent map of $G_i^{(k)}$ onto Ω_k and for all λ in Ω_k ,

$$\text{ind}\ (\lambda - S) = \sum_{i=1}^{r(k)} p_i^{(k)} m_i^{(k)} ,$$

$$\text{nul}\ (\lambda - S) \geq \sum \left\{ p_i^{(k)} m_i^{(k)} : 1 \leq i \leq r(k) \text{ and } m_i^{(k)} > 0 \right\} .$$

(e) If $\lambda \in \sigma_0(S)$ and λ is an isolated point of f(E) such that $f^{-1}(\lambda) \subseteq Z(f')$, then for every positive integer m ,

$$\text{nul}\ (\lambda - S)^m \geq \min [\ mk_0\ , \dim \mathcal{H}(S;\lambda)\ ] ,$$

where

$$k_0 = \min \{\ k : \text{there is a z in E with } f(z) = \lambda\ ,\ f'(z) = \cdots = f^{(k-1)}(z) = 0\ , \text{ and } f^{(k)}(z) \neq 0\ \} .$$

Proof. This is a routine application of Proposition 1.9 ♦

Proof of Theorem 2.2. Let $\mathcal{X}$ be the set of all operators satisfying conditions (a) through (e) in Theorem 2.2. If $\|T - f(A_k)\| \to 0$ for some sequence $\{A_k\}$ in $\mathcal{S}(E)$ and if $\lambda \in \sigma_0(T)$ that is an isolated point of cl[f(E)] with $f^{-1}(\lambda) \subseteq Z(f')$, then for sufficiently large k, A_k is the algebraic direct sum of operators F_k and B_k , where F_k acts on a space of the same finite dimension as $\mathcal{H}(T;\lambda)$, $f(\sigma(F_k)) = \{\lambda\}$, and $\lambda \notin f(\sigma(B_k))$. Thus $\sigma(F_k) = \{ a_{k,1}, \ldots, a_{k,r} \} \subseteq Z(f')$ and $f(a_{k,j}) = \lambda$ for $1 \leq j \leq r$. If k_0 is as in the statement of (e), it is straightforward to check that

$$\text{nul}\ [\lambda - f(F_k)\ ]^m \geq \min [\ mk_0\ , \dim \mathcal{H}(T;\lambda)\ ]$$

for all $m \geq 1$. That is, T satisfies (e).

This is the first step in showing that the set $\mathcal{X}$ is closed in the norm topology. The remaining steps in establishing this fact can be accomplished by using standard stability properties of the Fredholm index and the various parts of the spectrum. The details are left to the reader. By using Proposition 1.3 and Theorem 1.1, it can also be deduced that $f(\mathcal{S}(E)) \subseteq \mathcal{X}$. Hence $\text{cl}[\ f(\mathcal{S}(E))\ ] \subseteq \mathcal{X}$.

Now for the converse. By the preceding lemma it suffices to show that if S is an operator satisfying conditions (a) through (e) there, then S ∈

cl[$f(\mathcal{S}(E))$] . Adopt the notation of Lemma 2.14. By (a) (i) $\sigma_0(S)$ is a finite subset $\{\lambda_1, \ldots, \lambda_m\}$ of f(E). For each j , $1 \le j \le m$, let $a_j \in E$ such that $f(a_j) = \lambda_j$ and either $f'(a_j) \neq 0$ or $f'(a_j) = \cdots = f^{(k_j-1)}(a_j) = 0$ and $f^{(k_j)}(a_j) \neq 0$; moreover, choose a_j so that k_j is the minimum possible integer arising in this way in the set $f^{-1}(\lambda_j)$. Let $J_{n(j)}$ be the nilpotent Jordan cell on a space of finite dimension equal to $n(j) \equiv \dim \mathcal{H}(S; \lambda_j)$.

Let $\Lambda_1, \ldots, \Lambda_n$ be the components of Λ and for for $1 \le \ell \le n$ let b_ℓ be any point in E such that $f(b_\ell) \in \Lambda_\ell$. Let Q be a universal quasinilpotent operator. That is, Q is a quasinilpotent operator such that Q^m is not compact for any positive integer m . (See [20], page 193.)

If $\Omega_k \subseteq P_{\pm\infty}(S)$, let $G_1^{(k)}$ = the component G_k of $f^{-1}(\Omega_k)$ as described in part (c) of Lemma 2.14 ; also let $m_1^{(k)} = \text{ind}(\lambda - S)$ for λ in Ω_k , r(k) =1, and $p_1^{(k)} = p$. Define the operator A by

$$A = \bigoplus_{j=1}^{m} (a_j + J_{n(j)}) \oplus \bigoplus_{i=1}^{n} (b_i + Q)$$

$$\oplus \bigoplus_{k=1}^{q} \left[\bigoplus \left\{ C(G_h^{(k)})^{(m_h^{(k)})} : 1 \le i \le r(k) \, , \, m_h^{(k)} > 0 \right\} \right]$$

$$\oplus \bigoplus_{k=1}^{q} \left[\bigoplus \left\{ A(G_h^{(k)})^{(-m_h^{(k)})} : 1 \le i \le r(k) \, , \, m_h^{(k)} < 0 \right\} \right] .$$

Hence

$$\sigma(A) = \{a_1, \ldots, a_m\} \cup \{b_1, \ldots, b_n\} \cup \bigcup_{k=1}^{q} \bigcup_{i=1}^{r(k)} \text{cl}\,[\,G_i^{(k)}\,] \subseteq E \, .$$

Also $\sigma_0(A) = \{a_1, \ldots, a_m\}$ and

$$\sigma_{lre}(A) = \{b_1, \ldots, b_n\} \cup \bigcup_{k=1}^{q} \bigcup_{i=1}^{r(k)} \partial G_i^{(k)} \, .$$

Thus if $a \in \sigma(A) \setminus [\sigma_{lre}(A) \cup \sigma_0(A)]$, then $a \in G_i^{(k)}$ for some k and $1 \le i \le r(k)$. Moreover, $\text{ind}(a - A) = m_i^{(k)}$ and $\min \text{ind}(a - A) = 0$. Put R = f(A) .

It follows that

$$R = \bigoplus_{j=1}^{m} (\lambda_j + F_j) \oplus \bigoplus_{i=1}^{n} f(b_i + Q_i)$$

$$\oplus \bigoplus_{k=1}^{q} \left[\bigoplus \left\{ f(C(G_h^{(k)}))^{(m_h^{(k)})} : 1 \le i \le r(k) \, , \, m_h^{(k)} > 0 \right\} \right]$$

$$\oplus \bigoplus_{k=1}^{q} \left[\bigoplus \left\{ f(A(G_h^{(k)}))^{(-m_h^{(k)})} : 1 \le i \le r(k) \, , \, m_h^{(k)} < 0 \right\} \right] ,$$

where

$$F_j = \sum_{r=1}^{n(j)-1} \frac{f^{(r)}(a_j)}{r!} J_{n(j)}^r$$

is nilpotent for $1 \le j \le m$ and

$$Q_i = \sum_{r=1}^{\infty} \frac{f^{(r)}(b_i)}{r!} Q^r$$

is a universal quasinilpotent operator for $1 \le i \le n$. Indeed, since f' does not vanish identically on a neighborhood of any point of E , for each i , $1 \le i \le n$, there is a first derivative, say $f^{(r(i))}(b_i)$, different from 0 ; thus we have

$$Q_i = \sum_{r=r(i)}^{\infty} \frac{f^{(r)}(b_i)}{r!} Q^r = \left(\frac{f^{(r(i))}(b_i)}{r(i)!} Q^{r(i)} \right) L_i$$

for some invertible operator L_i commuting with Q. Hence Q_i is quasinilpotent and for each $m \geq 1$, Q_i^m is the product of an invertible operator and a non-compact operator and so is not compact.

By Theorem 1.1

$$\sigma(R) = \{ \lambda_1, \ldots, \lambda_m, f(b_1), \ldots, f(b_n) \} \cup \bigcup_{k=1}^{q} \mathrm{cl}[\Omega_k] \subseteq E,$$

$$\sigma_{lre}(R) = \{ f(b_1), \ldots, f(b_n) \} \cup \bigcup_{k=1}^{q} \partial\Omega_k,$$

and

$$\sigma_e(R) = \{ f(b_1), \ldots, f(b_n) \} \cup \bigcup_{k=1}^{q} \partial\Omega_k \cup \bigcup_{k=1}^{q} \{ \Omega_k : m_1^{(k)} = \pm\infty \}.$$

By the choices that have been made,

(a) $\sigma_0(S) = \sigma_0(R) = \{ \lambda_1, \ldots, \lambda_m \}$ and each component of $\sigma_{lre}(S) = \mathrm{cl}[\Lambda]$ meets $\sigma_e(R)$;

(b) $P(S) = \mathbb{C} \setminus \mathrm{cl}[\Lambda] \subseteq P(R)$ and for each λ in Ω_k and each integer $p \geq 1$,

$$\mathrm{ind}(\lambda - R) = \sum_{i=1}^{r(k)} m_i^{(k)} p_i^{(k)} = \mathrm{ind}(\lambda - S),$$

$$\mathrm{nul}(\lambda - R)^p = p \sum \{ m_i^{(k)} p_i^{(k)} : m_i^{(k)} > 0 \} \leq \mathrm{nul}(\lambda - S)^p,$$

and

$$\mathrm{nul}(\lambda - R)^{*p} = p \sum \{ -m_i^{(k)} p_i^{(k)} : m_i^{(k)} < 0 \} \leq \mathrm{nul}(\lambda - S)^{*p},$$

so that

$$\min \mathrm{ind}(\lambda - R)^p \leq \min \mathrm{ind}(\lambda - S)^p$$

for all λ in $\Omega_1 \cup \cdots \cup \Omega_q$ and for all positive integers p. Also (see [20], Corollary 2.2)

$$\mathrm{nul}(\lambda_j - R)^p = \mathrm{nul}\, F_j^p \leq \mathrm{nul}(\lambda_j - S)^p$$

for all p and $1 \leq j \leq m$. Since, by construction,

(c) $\dim \mathcal{H}(R; \lambda_j) = \dim \mathcal{H}(S; \lambda_j)$ for $1 \leq j \leq m$,

we infer that

$$\min \mathrm{ind}(\lambda - R)^p \leq \min \mathrm{ind}(\lambda - S)^p$$

for all λ in $P(S)$ and all $p \geq 1$.

By the Similarity Orbit Theorem, S belongs to the closure of the similarity orbit of R. Hence $S \in \mathrm{cl}[f(\mathcal{S}(E))]$. ♦

Proof of Theorem 2.1. We will only sketch a proof.

Let $E_1 = \{ z \in E : f(z) \in cl[f(E \setminus E_0)] \}$ and $E_2 = E \setminus E_1$. It is not difficult to see that even though E_1 may be larger than $E \setminus E_0$, $cl[f(\mathcal{S}(E_1))] = cl[f(\mathcal{S}(E \setminus E_0))]$. But $cl[f(\mathcal{S}(E \setminus E_0))]$ is characterized by Theorem 2.2 and $cl[f(\mathcal{S}(E_2))]$ is characterized by Theorem 2.3. Let D_i ($i = 1, 2$) be the union of the components of D that meet E_i. From the definition of E_i, $D_1 \cap D_2 = \emptyset$. So if $A \in \mathcal{S}(E)$, then the Riesz-Dunford functional calculus implies that A is similar to $A_1 \oplus A_2$, where $\sigma(A_i) \subseteq E_i$. Hence each T in $f(\mathcal{S}(E))$ is similar to $T_1 \oplus T_2$, where $T_i \in f(\mathcal{S}(E_i))$. From here the necessity of the conditions can be deduced using elementary properties of the parts of the spectrum and the index.

For the proof of sufficiency, let E_1 and E_2 be as in the preceding paragraph, assume that T satisfies the conditions of Theorem 2.1, and fix $\varepsilon > 0$. By Proposition 3.47 of [20], there is a compact operator K with $\| K \| < \varepsilon$ such that if $R = T - K$, then $\sigma_0(R)$ is a finite subset of $\sigma_0(T)$, $R \mid \mathcal{H}(R ; \lambda) \approx T \mid \mathcal{H}(T; \lambda)$ for all λ in $\sigma_0(R)$, the only singular points (as defined on page 10 of [20]) in the semi- Fredholm domain of R belong to $\sigma_0(R)$, and, moreover, min ind $(\lambda - R)$ = min ind $(\lambda - T)$ for every λ in P(T) that is not a singular point in the semi- Fredholm domain of T.

Let $Isol_e(R) \equiv \{ \lambda : \lambda$ is an isolated point of $\sigma_e(R)$ and $f(E_2) \}$. Let Ω be an analytic Cauchy domain with

$$\sigma_{lre}(R) \setminus Isol_e(R) \subseteq \Omega \subseteq [\sigma_{lre}(R) \setminus Isol_e(R)]_\varepsilon$$

and each component of Ω meets $\sigma_{lre}(R) \setminus Isol_e(R)$. As in the proof of Proposition 1.9, there is an operator S with $\| S - R \| < 2\varepsilon$, $\sigma_{lre}(S) = cl\, \Omega \cup \{ \lambda_1, \ldots, \lambda_m \}$, where $\{ \lambda_1, \ldots, \lambda_m \} \subseteq Isol_e(R)$, $\sigma_0(S) = \sigma_0(R) \setminus cl\, \Omega$, $S \mid \mathcal{H}(S; \lambda) \approx T \mid \mathcal{H}(T; \lambda)$ for all λ in $\sigma_0(S)$, $P_n(S) \subseteq P_n(R) = P_n(T)$ for every non-zero extended integer n, and min ind $(\lambda - S)$ = min ind $(\lambda - R)$ for all λ in P(S). Let $\{ \mu_1, \ldots, \mu_s \}$ be the points of $\sigma_0(S)$ that are isolated points of $f(E_2)$ and let $\sigma_0(S) = \{ v_1, \ldots, v_r, \mu_1, \ldots, \mu_s \}$. It follows that

$$S \approx X \oplus (\textstyle\bigoplus_j \lambda_j I_{\mathcal{H}_j}) \oplus (\bigoplus_i v_i + J_i) \oplus (\bigoplus_k \mu_k I_{\mathcal{K}_k})$$

where each $\mathcal{H}_j$ is an infinite dimensional space, each $\mathcal{K}_k$ is a finite dimensional space, and each J_i is a nilpotent operator on a finite dimensional space. Put

$$S_1 = X \oplus \textstyle\bigoplus_i (v_i + J_i),$$

$$S_2 = (\textstyle\bigoplus_j \lambda_j I_{\mathcal{H}_j}) \oplus (\bigoplus_k \mu_k I_{\mathcal{K}_k}).$$

It is left to the reader to check that Theorem 2.2 implies that S_1 belongs to $cl[f(\mathcal{S}(E \setminus E_0))] = cl[f(\mathcal{S}(E_1))]$ and Theorem 2.3 implies that $S_2 \in cl[f(\mathcal{S}(E_2))]$. Since ε was arbitrary, $T \in cl[f(\mathcal{S}(E))]$. ♦

The next result answers a question put to the authors by Raúl E Curto. The question is "When is $cl[f(\mathcal{S}(E))] = cl[\mathcal{S}(f(E))]$?" Since $f(\mathcal{S}(E)) \subseteq \mathcal{S}(f(E))$, $cl[f(\mathcal{S}(E))] \subseteq cl[\mathcal{S}(f(E))]$. In general this inclusion is proper.

It turns out that to answer this question, it suffices to only consider the case that f is completely non-constant. For example, if λ is an isolated point of f(E) and T is any operator with $\sigma(T) = \{\lambda\}$, then $T \in cl[\mathcal{S}(f(E))]$. But, by (2.1 (f)), the only way that T can belong to $cl[f(\mathcal{S}(E))]$ is for T to be a multiple of the identity.

From the result of Apostol and Morrel (see Corollary 2.6 above), to say that T satisfies conditions (a) and (b) of Theorem 2.1 is equivalent to saying that $T \in cl[\mathcal{S}(f(E))]$. Thus $cl[f(\mathcal{S}(E))] = cl[\mathcal{S}(f(E))]$ if and only if the set E and the function f are such that conditions (c) through (g) can be deduced from (a) and (b) or they do not apply.

2.15 Theorem If f, E, and E_0 are as in Theorem 2.1, then the following statements are equivalent.

(a) cl[f($\mathcal{S}$(E))] = cl[$\mathcal{S}$(f(E))] .

(b) f(E_0) $\subseteq$ cl[f(E \ E_0)], for each analytic Cauchy region Ω with cl Ω $\subseteq$ int [f(E \ E_0)] and ∂Ω ∩ f(Z(f')) = ∅, there is an analytic Cauchy region G with cl G $\subseteq$ int E such that f is a one-to-one map of G onto Ω and for each isolated point λ of f(E), $f^{-1}(\lambda)$ ∩ E is not contained in Z(f').

Proof. First assume that (a) holds. The proof that f(E_0) $\subseteq$ cl[f(E \ E_0)] is left to the reader. Note that this says that cl[f($\mathcal{S}$(E))] = cl[f($\mathcal{S}$(E \ E_0))]. Thus it suffices to assume that f is completely non-constant.

Let Ω be an analytic Cauchy region such that cl Ω $\subseteq$ int [f(E)] and ∂Ω ∩ f(Z(f')) = ∅. Let Ω_1 be a second analytic Cauchy region such that cl Ω $\subseteq$ Ω_1 $\subseteq$ cl Ω_1 $\subseteq$ int f(E). Since A(Ω_1) ∈ $\mathcal{S}$(f(E)), the assumption of (a) implies that A(Ω_1) ∈ cl[f($\mathcal{S}$(E))]. By Theorem 2.1 and the fact that ind (λ - A(Ω_1)) = -1 and nul (λ - A(Ω_1)) = 0 for all λ in cl Ω, there is an analytic Cauchy region G with cl G $\subseteq$ int E such that f is a one-to-one map of G onto Ω.

Suppose that λ is an isolated point of f(E) and f'(a) = 0 for all a in E with f(a) = λ. Then, again using Theorem 2.1 (and, in particular, part (e)) it follows that $T = \lambda + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \notin$ cl[f($\mathcal{S}$(E))]. But clearly T ∈ $\mathcal{S}$(f(E)). This implies (b).

Now assume that (b) is true. Once again, there is no loss of generality in assuming that f is completely non-constant. Since it is always true that f($\mathcal{S}$(E)) $\subseteq$ $\mathcal{S}$(f(E)), it remains to show that $\mathcal{S}$(f(E)) $\subseteq$ cl[f($\mathcal{S}$(E))]. Condition (b) implies that for an analytic Cauchy region Ω with cl Ω $\subseteq$ int [f(E)] and ∂Ω ∩ f(Z(f')) = ∅, the operators A(Ω) and C(Ω) belong to f($\mathcal{S}$(E)). Now assume that T is an operator with σ(T) = {λ} and λ ∈ f(E) ; let a ∈ E such that f(a) = λ. If λ ∉ f(Z(f')), then there is a neighborhood G of a such that f is one-to-one on G. Let g be the inverse of the restriction of f to G ; so g is defined in a neighborhood of λ. If A = g(T), then A ∈ $\mathcal{S}$(E) and T = f(A). If λ ∈ f(Z(f')) but not an isolated point of f(E), there is a λ_1 in f(E) with λ_1 arbitrarily close to λ and λ_1 ∉ f(Z(f')). Thus T + (λ_1 - λ) ∈ f($\mathcal{S}$(E)) and so T ∈ cl[f($\mathcal{S}$(E))]. If λ is isolated, then the second part of condition (b) implies that a can be chosen with f'(a) ≠ 0. Thus the preceding argument can be used to show that T ∈ f($\mathcal{S}$(E)).

Now if T is any operator in $\mathcal{S}$(f(E)), the Similarity Orbit Theorem or the approximation theorem of Apostol and Morrel ([6]; also see Corollary 6.2 of [20]) implies that T can be approximated by an operator that is similar to the finite direct sum of operators of the form $A(\Omega_i)^{(n_i)}$, $C(\Phi_j)^{(m_j)}$, D_k, where Ω_i and Φ_j are analytic Cauchy regions with boundaries missing f(Z(f')) and whose closures are pairwise disjoint subsets of int [f(E)], n_i and m_j are extended positive integers, and each D_k has spectrum equal to a single point of f(E). From the preceding paragraph, each such direct sum belongs to cl[f($\mathcal{S}$(E))] and hence T ∈ cl[f($\mathcal{S}$(E))]. ♦

2.16 Corollary If f and E are as in Theorem 2.1 and, in addition, f(E) has neither interior nor isolated points, then cl[f($\mathcal{S}$(E))] = cl[$\mathcal{S}$(f(E))].

Let f(z) = z^2 and E = [0, 1]. By Corollary 2.16, cl[f($\mathcal{S}$(E))] = cl[$\mathcal{S}$(f(E))]. It is not true, however, that f($\mathcal{S}$(E)) = $\mathcal{S}$(f(E)). In fact, if T is a non-trivial nilpotent acting on a 2 dimensional space, then T ∈ $\mathcal{S}$(f(E)) but T ∉ f($\mathcal{S}$(E)). It is somewhat surprising that for an open set E and a completely non-constant analytic function the two statements are equivalent.

2.17 Corollary If E is open and f is completely non-constant, then the following statements are equivalent.

(a) f($\mathcal{S}$(E)) = $\mathcal{S}$(f(E)).

(b) cl[f($\mathcal{S}$(E))] = cl[$\mathcal{S}$(f(E))].

(c) for each analytic Cauchy region Ω with cl Ω $\subseteq$ f(E), there is an analytic Cauchy region G with cl G $\subseteq$ E such that f is a one-to-one map of G onto Ω.

Proof. It is trivial that (a) implies (b) and (b) implies (c) by Theorem 2.15. To see that (c) implies (a) it suffices to show that $f(\mathcal{S}(E)) \supseteq \mathcal{S}(f(E))$. So let $T \in \mathcal{S}(f(E))$ and construct an analytic Cauchy domain Ω with $\sigma(T) \subseteq \Omega \subseteq \mathrm{cl}\ \Omega \subseteq f(E)$. Let $\Omega_1, \ldots, \Omega_m$ be the components of Ω. Since $f(\mathcal{S}(E))$ and $\mathcal{S}(f(E))$ are similarity invariant, it may be assumed that $T = T_1 \oplus \cdots \oplus T_m$ where $\sigma(T_j) \subseteq \Omega_j$. By (c) there is an analytic Cauchy region G_j with $\mathrm{cl}\ G_j \subseteq E$ such that f is a one-to-one map of G_j onto Ω_j. Let $g_j : \Omega_j \to G_j$ be the inverse of the restriction of f to G_j and put $A_j = g_j(T_j)$. It is immediate that $A = A_1 \oplus \cdots \oplus A_m \in \mathcal{S}(E)$ and $T = f(A)$. ♦

In the case that E is assumed to be open but f is allowed to be constant on some components, in contrast to Corollary 2.17, the equivalence of the equalities $\mathrm{cl}[\ f(\mathcal{S}(E))\ ] = \mathrm{cl}[\ \mathcal{S}(f(E))\ ]$ and $f(\mathcal{S}(E)) = \mathcal{S}(f(E))$ does not hold. The reader is invited to manufacture an example.

When is $\mathrm{cl}[\ f(\mathcal{S}(E))\ ] = \mathcal{B}(\mathcal{H})$? In [9] Arlen Brown gave a necessary and sufficient condition on an analytic function f such that $f(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H})$. It turns out that this is also equivalent to the answer to the question just posed.

Clearly a necessary condition for $\mathrm{cl}[\ f(\mathcal{S}(E))\ ] = \mathcal{B}(\mathcal{H})$ is that f(E) be at least dense in $\mathbb{C}$ (actually it must be that $f(E) = \mathbb{C}$) and a trivial sufficient condition is that $E = \mathbb{C}$ and $f(z) = az + b$ for some $a \neq 0$. However this is not the complete story as the theorem below demonstrates. Also see Examples 3.1, 3.2, 3.3, and 3.4 in the next section.

2.18 Theorem If f is analytic in a neighborhood of E, then the following statements are equivalent.

(a) $f(\mathcal{S}(E)) = \mathcal{B}(\mathcal{H})$.

(b) $\mathrm{cl}[\ f(\mathcal{S}(E))\ ] = \mathcal{B}(\mathcal{H})$.

(c) There are simply connected regions $G_1, G_2, \ldots$ in E such that f is one-to-one on each G_n, $f(G_n) \subseteq f(G_{n+1})$ for all n, and $\mathbb{C}$ is the union of the sets $f(G_n)$.

Proof. It will be assumed throughout this proof that f is completely non-constant. The fact that (c) implies (a) is an easy application of the Riesz functional calculus while it is trivially true that (a) implies (b).

Assume that (b) holds. If S is the unilateral shift of multiplicity 1 and n is any natural number, then (b) implies that $nS \in f(\mathcal{S}(E))$. It follows (for example, by Proposition 1.7) that there is an analytic Cauchy region G_n such that f is a univalent mapping of G_n onto $\{ z : |z| < n \}$. Condition (c) is now immediate. ♦

Although initially the question of when $f(\mathcal{S}(E))$ is closed is only accidentally related to the question of when $f(\mathcal{S}(E)) = \mathcal{B}(\mathcal{H})$, the next result shows that the two questions are almost equivalent.

2.19 Theorem If f is an analytic function defined in a neighborhood of the non-empty set E, then $f(\mathcal{S}(E))$ is closed if and only if either

(a) $E \subseteq \mathrm{int}\ Z(f')$ and f(E) has no limit points in $\mathbb{C}$, or

(b) $f(\mathcal{S}(E)) = \mathcal{B}(\mathcal{H})$.

Proof. Suppose that either (a) or (b) hold. If (b) is true, then clearly $f(\mathcal{S}(E))$ is closed. So assume that (a) is true. Note that $f(\mathcal{S}(E))$ is closed if and only if $f(\mathcal{S}(E)) \cap \{ T \in \mathcal{B}(\mathcal{H}) : r(T) \leq \rho \}$ is closed for every $\rho > 0$. Put $E_\rho = \{ z \in E : |f(z)| \leq \rho \}$; by (a), $f(E_\rho)$ is a finite set for every ρ. Since $E \subseteq \mathrm{int}\ Z(f')$, it is easy to see that

$$\begin{aligned} f(\mathcal{S}(E_\rho)) &= f(\mathcal{S}(E)) \cap \{ T \in \mathcal{B}(\mathcal{H}) : r(T) \leq \rho \} \\ &= \{ T : T \text{ is similar to a normal operator with spectrum included in } f(E_\rho) \}. \end{aligned}$$

Since $f(E_\rho)$ is finite, a simple argument using the Riesz-Dunford functional

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The basis problem for modular forms on $\Gamma_0(N)$

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