Nil GROUPS IN K-THEORY AND SURGERY THEORY

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Abstract: We study Cappell's *UNil* group, $UNil^h_{2n}(R;\mathcal{B}_1,\mathcal{B}_{-1})$, for any ring and pair of bimodules with involution $(R;\mathcal{B}_1,\mathcal{B}_{-1})$. We show that, in the geometrically significant cases, this group is isomorphic to the Wall-Ranicki L -group, $L_\varepsilon(\mathbb{A}_\alpha[t])$, for a certain additive polynomial extension category $\mathbb{A}_\alpha[t]$. We then introduce an Arf invariant for $UNil^h_{2n}(R;R,R)$ when the involution is trivial. We use this to compute $UNil^h_{2n}(R;R,R)$ when R is a Dedekind domain in which 2 is prime. We also show that for a suitable choice of $(\mathbb{A},\alpha)$, the *Nil* group of $(\mathbb{A},\alpha)$ coincides with the *Nil* group of Bass-Farrell and with the *Nil* group of Waldhausen.
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§ 1. Introduction

1.0 The primary goal of this paper is to analyze Cappell's group $UNil^h_n(R;\mathcal{B}_1,\mathcal{B}_{-1})$, $n \geq 0$ as defined in [C1], (See also [R1] Chapter 7). In particular, we calculate $UNil^h_{2n}(R;R,R)$ for any semisimple ring, and any Dedekind domain in which 2 is prime. Our analysis inevitably also draws in the *Nil* groups defined by Waldhausen [W1], [W2], [W3], (also see [C4] p.125), by Farrell [F1], and by Bass [B]. It leads to interesting new interpretations of these as well. The main results are 2.11, 3.9, 4.9, 6.1, and 6.2. We summarize our results in 1.2 below. But first we want to explain why a better understanding of *UNil* seems so important to us.

1.1 Motivation.

Suppose $f : M \to X$ is a homotopy equivalence of compact closed manifolds, and X is the union of two other manifolds, $X = X_1 \cup X_{-1}$, with $X_0 = \partial X_1 = \partial X_{-1} = X_1 \cap X_{-1}$. Assume that X and X_0 are connected, and that the fundamental groups of X_1, X_{-1}, X, and X_0 are denoted G_1, G_{-1}, Γ, and H respectively. We want to know if f, or a map "close" to f, is *split* along X_0, in the sense that f is transverse to X_0 and the three maps, $f^{-1}(X_i) \to X_i, i = -1,0,1$ are again

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homotopy equivalences. Therefore, as a preliminary assumption, we assume that the Whitehead torsion of f is in the image of $Wh(G_1) \oplus Wh(G_{-1}) \to Wh(\Gamma)$. Let $R = \mathbb{Z}H$, $\mathcal{B}_i = \mathbb{Z}(G_i \setminus H)$. There is then defined an obstruction in an abelian group $UNil_n^h(R; \mathcal{B}_1, \mathcal{B}_{-1})$, where $n = dim(X) + 1$. When $n \geq 5$ this obstruction vanishes if and only if there is an h-cobordism of (M, f) to a second homotopy equivalence, $f' : M' \to X$ which is split along X_0.

$UNil_n^h(R; \mathcal{B}_1, \mathcal{B}_{-1})$ is a direct summand of the surgery obstruction group $L_n^h(\Gamma)$, and acts freely on the structure set, $S^h(X)$. Similarly Waldhausen's Nil group, $\widetilde{Nil}(R; \mathcal{B}_1, \mathcal{B}_{-1})$ is a direct summand of $Wh(\Gamma)$. These Nil and $UNil$ groups obstruct the exactness of Mayer-Vietoris type sequences in the corresponding K and L groups. For, if we write $\hat{L}_n^h(\Gamma)$ for the complementary summand in $L_n^h(\Gamma)$, and write $\hat{W}h(\Gamma)$ for the complementary summand in $Wh(\Gamma)$, we get exact sequences:

$$\cdots \to L_n^h(H) \to L_n^h(G_1) \oplus L_n^h(G_{-1}) \to \hat{L}_n^h(\Gamma) \to L_{n-1}^p(H) \to \ldots .$$

$$Wh(H) \to Wh(G_1) \oplus Wh(G_{-1}) \to \hat{W}h(\Gamma) \to \tilde{K}_0(\mathbb{Z}H) \to \tilde{K}_0(\mathbb{Z}G_1) \oplus \tilde{K}_0(\mathbb{Z}G_{-1}) \to \ldots .$$

See [C1], [C2], [C4], [W1],[W2], for all of this (and more).

This should suffice to establish the importance of these Nil groups, but we want to mention a second point of view which also attractively displays their significance.

A compact aspherical manifold has the form $\tilde{X}/\Gamma = B\Gamma$ where $\tilde{X}$ is a contractible manifold and Γ acts on $\tilde{X}$ with compact quotient, properly discontinuously and freely. It is known that the structure set, $\mathcal{S}(B\Gamma)$ or $\mathcal{S}^h(B\Gamma)$, for various compact aspherical manifolds $B\Gamma$, consists of one element (see, inter alia, [FH2], [FH3], [FJ]). When the action of Γ is no longer required to be free, but instead $\tilde{X}^H$ is required to be a contractible submanifold for each finite subgroup H in Γ, we no longer have a manifold as orbit space. But we still can discuss a structure set, $\mathcal{S}(\Gamma)$ or $S^h(\Gamma)$ (see [CK3] for a definition), and there is evidence here too of rich rigidity phenomena (see [CK1], [CK2]).

In this case too, when Γ is an amalgamated product of subgroups G_1 and G_{-1}, Cappell's Nil group, $UNil_n^h(R; \mathcal{B}_1, \mathcal{B}_{-1})$, forms a kind of obstruction to any such rigidity results. To be precise in at least one case, suppose the manifolds X, X_0, X_{-1}, X_1, mentioned in the first paragraph of 1.1 are all aspherical. Then, as mentioned above, $UNil_n^h(R; \mathcal{B}_1, \mathcal{B}_{-1})$ acts freely on $\mathcal{S}^h(B\Gamma)$ Therefore rigidity of Γ requires the vanishing of the corresponding $UNil$ group. This phenomenon persists when the Γ action on $\tilde{X}$ is no longer free.

But very little is known of $UNil_n^h(R;\mathcal{B}_1,\mathcal{B}_{-1})$. Even its definition is rather involved (see[C1] and [R1]). It is known to be zero in the so called "square root closed" case [C1]. More generally, F.T. Farrell has proved that:

$$4 \cdot UNil_{2n}^h(\mathbb{Z}H;\mathbb{Z}(G_1 \setminus H),\mathbb{Z}(G_{-1} \setminus H)) = 0, \quad \text{[F2]}.$$

Even when H, $G_1 \setminus H$, $G_{-1} \setminus H$ consist of a single element (so that $\Gamma = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$), a complete set of invariants for this group has never been previously discovered. (Cappell in [C1] asserts it has exponent 2 in this case, and, when n is odd, proves in [C3] that it is countably infinite). There is also evidence that Nil groups, when they are nonzero, must be very large ([F2], [F3]).

1.2 Main Results. We concentrate on $UNil_{2n}^h(R;\mathcal{B}_1,\mathcal{B}_{-1})$.

a. We first express the $UNil$ groups in terms of the surgery obstruction groups, $L_\varepsilon(\mathbb{A}_\alpha[t])$, of an additive category $\mathbb{A}_\alpha[t]$ with involution. Here $L_\varepsilon(\ )$ denotes the Wall - Ranicki surgery group, as defined in [R2], and $\mathbb{A}_\alpha[t]$ is a "twisted" polynomial extension category, defined in 2.1 below, for an additive category $\mathbb{A}$ and a suitable additive functor α. It turns out that, in purely categorical terms, one can construct Nil groups and unitary Nil groups for such pairs $(\mathbb{A},\alpha)$. We denote them $Nil(\mathbb{A},\alpha)$ and $UNil_\varepsilon(\mathbb{A},\alpha)$ where $\varepsilon = \pm 1$. (See 2.3 and 3.2). We prove that for an appropriate choice of $\mathbb{A}$ and α, in terms of $(R;\mathcal{B}_1,\mathcal{B}_{-1})$, one has isomorphisms:

$$Nil(\mathbb{A},\alpha) \cong \widetilde{Nil}(R;\mathcal{B}_1,\mathcal{B}_{-1}),$$

$$UNil_\varepsilon(\mathbb{A},\alpha) \cong UNil_{2n}^h(R;\mathcal{B}_1,\mathcal{B}_{-1})\ ,\ \varepsilon = (-1)^n.$$

See 2.6 and 3.4.b. Here $\widetilde{Nil}(R;\mathcal{B}_1,\mathcal{B}_{-1})$ is the Nil group constucted by Waldhausen in [W1], [W2]. For Farrell's group, $\widetilde{Nil}(R,a)$ of [F1], one also has, for a suitable choice of $\mathbb{A}$ and α,

$$Nil(\mathbb{A},\alpha) \cong \widetilde{Nil}(R,a).$$

We next use a "quadratic" analogue of Higman's trick (see [B] or [BHS] for the linear version) to construct exact sequences:

$$Nil(\mathbb{A},\alpha) \to K_1(\mathbb{A}_\alpha[t]) \to K_1(\mathbb{A}) \to 0$$

$$UNil_\varepsilon(\mathbb{A},\alpha) \to L_\varepsilon(\mathbb{A}_\alpha[t]) \to L_\varepsilon(\mathbb{A}) \to 0, \quad \varepsilon = \pm 1$$

(see 2.9 and 3.6.b). For this it is necessary that $\mathbb{A}_\alpha[t]$ be linearizable (see 2.8 for a definition). The polynomial extension defined by any $(R; \mathcal{B}_1, \mathcal{B}_{-1})$ is linearizable (2.10).

The most geometrically significant case of these ideas is when H is a subgroup of two groups G_1 and G_{-1}. One sets $R = kH$ where k is a subring of $\mathbb{Q}$. One sets $\mathcal{B}_i = k(G_i \setminus H)$, an R bimodule with involution. For this special case we obtain (2.11, 3.9) an exact sequence and an isomorphism:

$$0 \to \widetilde{Nil}(R; \mathcal{B}_1, \mathcal{B}_{-1}) \to K_1(\mathbb{A}_\alpha[t]) \to K_1(\mathbb{A}) \to 0$$

$$UNil^h_{2n}(R; \mathcal{B}_1, \mathcal{B}_{-1}) \cong L_\varepsilon(\mathbb{A}_\alpha[t]), \qquad \varepsilon = (-1)^n.$$

b. We write $UNil^h_{2n}(R)$ for the group $UNil^h_{2n}(R; R, R)$. The importance of this "universal case" has been shown by Farrell [F2]. Using the categorical descriptions above, we construct endomorphisms (natural in R) F_s and V_s, of $UNil^h_{2n}(R)$ for each odd positive integer s. We prove (see 4.8):

$$V_s V_{s'} = V_{ss'} \ ; \ F_s F_{s'} = F_{ss'} \ ; \ V_1 = F_1 = \text{identity. Also:}$$

$$\forall x \in UNil^h_{2n}(R), \text{there is an } N > 0, \text{ so that } F_s(x) = 0 \text{ if } s \geq N.$$

Then we specialize to the case of a perfect field F of characteristic 2, and trivial involution. We construct an Arf invariant for $UNil^h_{2n}(F)$, denoted A, and establish an exact sequence:

$$UNil^h_{2n}(F) \xrightarrow{A} coker(\psi_2 - 1) \to F/(\psi_2 - 1)F \to 0.$$

Here ψ_2 denotes the Frobenius monomorphism, $\psi_2 : F[t] \to F[t]$ sending p to p^2 for all $p \in F[t]$. See 5.6, 5.7. The group $coker(\psi_2 - 1)$ is huge (5.8.a).

c. Finally, we completely calculate $UNil^h_{2n}(R)$ in two cases (6.1, 6.2):
- when R is a Dedekind domain with involution for which $R/2R$ is a perfect field.
-when R is a semi-simple algebra with involution over a perfect field F.

For the second case, the task boils down to computing $UNil^h_{2n}(F)$ when F is a perfect field of characteristic two, with trivial involution. For this case the Arf invariant, and the operators F_s mentioned above, combine to provide an isomorphism:

$$UNil^h_{2n}(F) \approx \sum_0^\infty (F)$$

The first case, that of a Dedekind domain R, is reduced to the second, by showing that, when $UNil^h_{2n}(R)$ is nonzero, the reduction map $UNil^h_{2n}(R) \to UNil^h_{2n}(R/2R)$ is an isomorphism.

We close with some unsolved problems raised by this paper.

1.3 Notation.

Throughout this paper $\mathcal{F}_R$ (resp. $\mathcal{P}_R$) denotes the category of free (resp. projective), finitely generated right R modules. Also, if $\mathcal{A}$ is a right R module and $\mathcal{B}$ is a left R module we write $\mathcal{AB}$ for $\mathcal{A} \otimes_R \mathcal{B}$. For any left R-module $\mathcal{B}$, over a ring with involution R, we write $\mathcal{B}^t$ for the right R module structure on the abelian group $\mathcal{B}$, given by the rule: $br = \bar{r}b, \ \forall r \in R, \ \forall b \in \mathcal{B}$.

If $\mathcal{A}$ is a right R module, then $\mathcal{A}^*$ denotes the right R-module $(Hom_R(\mathcal{A}, R))^t$. We always write $< \ , \ >: \mathcal{A} \times \mathcal{A}^* \to R$ for the cannonical sesquilinear pairing: $< x, f >= \overline{f(x)}$. When $\mathcal{B}$ is a bimodule with involution, we write $S^\varepsilon(\mathcal{B})$ for $\{r \in \mathcal{B} : r = -\varepsilon\bar{r}\}$ and $S_\varepsilon(\mathcal{B})$ for $\{r - \varepsilon\bar{r} : r \in \mathcal{B}\}$.

§2. Twisted Polynomial Extension Categories and *Nil* Groups

For any additive functor $\alpha : C \to C$ on an additive category and full subcategory $\mathbb{A} \subset C$, we construct $\mathbb{A}_\alpha[t]$, its twisted polynomial extension category. This generalizes a construction of Ranicki, in [R1], when $\alpha = 1$. These give rise to a Nil group $Nil(\mathbb{A}, \alpha)$ and an exact sequence $Nil(\mathbb{A}, \alpha) \to K_1(\mathbb{A}_\alpha[t]) \to K_1(\mathbb{A}) \to 0$.

The Nil groups of Farrell [F1] and Waldhausen [W1], [W2], are then described in these terms. The maps from these to their associated Whitehead groups both factor through the K_1 - group of the relevant (twisted) polynomial extension. See 2.3, 2.6, 2.9, 2.10.

2.1 Construction: Let $\alpha : C \longrightarrow C$ be an additive functor on an additive category (or even on an Ab-category, in the sense of MacLane [M]). We define the twisted polynomial extension category $C_\alpha[t]$ as follows.

Objects of $C_\alpha[t]$: $|C_\alpha[t]| = |C|$ (here $| \ |$ means "objects of").

Maps of $C_\alpha[t]$: Let $u, v \in |C|$. The group of morphisms between these is:

$$C_\alpha[t](u, v) = \sum_{i=0}^{\infty} C(u, \alpha^i v),$$

a graded group. Here $C(u, v)$ means the group of C-maps from u to v.

If $\varphi_i \in C(u, \alpha^i v)$ we write $t_v^{(i)}\varphi_i$ (or just $t^{(i)}\varphi_i$) for the corresponding degree i morphism in $C_\alpha[t](u,v)$. When $\varphi_i = 1 : \alpha^i v \to \alpha^i v$ we write $t_v^{(i)}$ instead of $t_v^{(i)}(1_{\alpha^i v})$. Each element φ of $C_\alpha[t](u,v)$ has a unique expression: $\varphi = \sum_{i=o}^{\infty} t_v^{(i)}\varphi_i$ where $\varphi_i \in C(u, \alpha^i v)$, and almost all of these are zero. The morphisms of degree i from u to v form a subgroup denoted $P_i(u,v)$ for brevity.

Composition Law : Given u, v, w, in $|C|$ and maps
$\varphi = \sum_{i=0}^{\infty} t_v^{(i)}\varphi_i \in C_\alpha[t](u,v)$, $\psi = \sum_{j=o}^{\infty} t_w^{(j)}\psi_j \in C_\alpha[t](v,w)$,
we define $\psi\varphi \in C_\alpha[t](u,w)$ by:

$$\psi\varphi = \sum_{k=0}^{\infty} t_w^{(k)}\chi_k, \quad \text{where} \quad \chi_k = \sum_{i+j=k} \alpha^i(\psi_j)\varphi_i.$$

Note $t_v^{(k)} = t_v^{(1)} t_{\alpha(v)}^{(k-1)}$ if $k \geq 1$. We write t for $t^{(1)}$.

Subcategories : Let $\mathbb{A}$ be a full subcategory of C. We define the (twisted) polynomial extension category $\mathbb{A}_\alpha[t]$, as the full subcategory of $C_\alpha[t]$ for which $|\mathbb{A}_\alpha[t]| = |\mathbb{A}|$. We do not require α to send $\mathbb{A}$ to $\mathbb{A}$. When $\mathbb{A}$ and C are additive categories, so is $\mathbb{A}_\alpha[t]$.

Augmentation and Inclusion : There is an additive functor $\eta : \mathbb{A}_\alpha[t] \longrightarrow \mathbb{A}$ which is a right inverse of the inclusion $\iota : \mathbb{A} \to \mathbb{A}_\alpha[t]$. η sends a morphism $\varphi = \sum t^{(i)}\varphi^i$ to $\eta(\varphi) = \varphi_0$. The functor ι sends a morphism $\varphi : u \to v$ to $\iota(\varphi) = t_v^{(0)}\varphi$. Both η and ι send each object to itself.

2.2.a. Example : Let $a : R \longrightarrow R$ be an endomorphism of a ring with unit. R is the morphism set of an Ab- category with one object; write C_R for the opposite of this category. a specifies an additive covariant functor $\alpha : C_R \to C_R$. The twisted polynomial ring $R_a[t]$ is the free left R module on $\{t^0, t^1, t^2, \dots\}$ with the multiplication:

$$(\sum_{i=0}^{\infty} r_i t^i)(\sum_{j=0}^{\infty} s_j t^j) = \sum_{k=0}^{\infty}(\sum_{i+j=k} r_i a^i(s_j))t^k.$$

One therefore concludes : $(C_R)_\alpha[t] \approx C_{R_a[t]}$.

2.2.b. Example: Again let $a : R \longrightarrow R$ be an endomorphism of a ring with unit. Let $\mathcal{M}_R$ be the category of all right R modules. Now the endomorphism $a : R \to R$ induces an additive functor $\alpha : \mathcal{M}_R \to \mathcal{M}_R$: one defines $\alpha(M)$ as a^*M for any module M. One then gets an additive functor:

$$l : (\mathcal{M}_R)_\alpha[t] \longrightarrow \mathcal{M}_{R_a[t]}$$

sending an R module M to $M \otimes_R R_a[t]$. One defines l on morphisms as the following composite map :

$$(\mathcal{M}_R)_\alpha[t](M,N) = \sum_{i=0}^{\infty} Hom_R(M,(a^i)^*N) \approx \sum_{i=0}^{\infty} Hom_R(M, N \otimes_R Rt^i)$$
$$\subset Hom_R(M, N \otimes_R R_a[t]) \approx Hom_{R_a[t]}(l(M), l(N)) \ .$$

When M is finitely generated this composite is an isomorphism.

But now, let $\mathcal{F}_R$ be the full subcategory of $\mathcal{M}_R$ consisting of finitely generated free right R modules. Then $l : (\mathcal{F}_R)_\alpha[t] \longrightarrow \mathcal{F}_{R_a[t]}$ is a weak equivalence of categories (i.e. it is a bijection on the set of isomorphism classes of objects and is an isomorphism on each morphism set).

Note 1 : For $F \in |\mathcal{F}_R|$, $\alpha^i(F)$ is usually neither free nor finitely generated, unless the map $a : R \to R$ is bijective. Hence α does not stabilize $\mathcal{F}_R$, yet we can still form the twisted polynomial extension category, $(\mathcal{F}_R)_\alpha[t]$.

Note 2 : l would be an equivalence of categories if we defined $\mathcal{F}_R$ so that $|\mathcal{F}_R|$ were a set.

2.3 Definition of $Nil(\mathbb{A},\alpha)$: Let $\mathbb{A}$ be a full additive subcategory of the additive category C with additive functor $\alpha : C \to C$. $\mathcal{N}il\ (\mathbb{A},\alpha)$ is the category consisting of those objects (u, ν), $u \in |\mathbb{A}|$, for which $t\nu$ is a nilpotent element of degree one in the ring $\mathbb{A}_\alpha[t](u,u)$.This implies that ν is in $C(u,\alpha(u))$. The morphisms in $Nil(\mathbb{A},\alpha)$ from (u,ν) to (u',ν') are $\{\phi \in \mathbb{C}(u,u') |\ \phi t\nu = t\nu'\phi\}$. (The functor $(u,\nu) \mapsto (u,t\nu)$ embeds $Nil(\mathbb{A},\alpha)$, as a full subcategory, into $End(\mathbb{A}_\alpha[t])$).

$\mathcal{N}il(\mathbb{A},\alpha)$ is an exact category; a sequence:

$$0 \to (u',\nu') \longrightarrow (u,\nu) \longrightarrow (u'',\nu'') \to 0$$

is declared to be short exact if the underlying sequence in $\mathbb{C}$ is split exact. The obvious functor $\mathbb{A} \to \mathcal{N}il(\mathbb{A},\alpha)$ sending an object $u \in |\mathbb{A}|$ to (u,0) induces a homomorphism $j : K_0(\mathbb{A}) \longrightarrow K_0(\mathcal{N}il(\mathbb{A},\alpha))$.

$Nil(\mathbb{A},\alpha)$ is defined to be $coker(j)$.

There is a homomorphism $\sigma : Nil(\mathbb{A},\alpha) \to K_1(\mathbb{A}_\alpha[t])$ sending the class [u, ν] in $Nil(\mathbb{A},\alpha)$ to the class $[u, 1-t\nu]$ in $K_1(\mathbb{A}_\alpha[t])$. For the augmentation $\eta : \mathbb{A}[t] \to \mathbb{A}$, it is clear that $\eta_* \circ \sigma = 0$.

Example : The twisted Nil group of Farrell (see [F1]), defined for any ring R with endomorphism $a : R \to R$, is just $Nil(\mathcal{F}_R,\alpha)$ where $\alpha : \mathcal{M}_R \longrightarrow \mathcal{M}_R$ is the functor $M \longmapsto a^*M$. We will write this Nil group $\widetilde{Nil}(R,a)$.

2.4 The Fundamental Example : Let R be a ring with unit. Let $\mathcal{B}_1, \mathcal{B}_{-1}$ be R bimodules which are free as left R modules. We construct a twisted polynomial extension category whose Nil group is Waldhausen's Nil group $\widetilde{Nil}(R; \mathcal{B}_1, \mathcal{B}_{-1})$. Let $C = \mathcal{M}_R \times \mathcal{M}_R$, $\mathbb{A} = \mathcal{F}_R \times \mathcal{F}_R$. Let $\alpha : C \to C$ be the functor defined by:

$$\alpha(P_1, P_{-1}) = (P_{-1}\mathcal{B}_1, P_1\mathcal{B}_{-1}) \text{ for objects}$$

$$\alpha(f_1, f_{-1}) = (f_{-1} \otimes 1, f_1 \otimes 1) \text{ for morphisms.}$$

$(\mathbb{A}, \alpha)$ is called the *additive category and functor defined by* $(R; \mathcal{B}_1, \mathcal{B}_{-1})$; $\mathbb{A}_\alpha[t]$ is called the *polynomial extension defined by* $(R; \mathcal{B}_1, \mathcal{B}_{-1})$.

2.4.a Let k denote a commutative ring with unit now, and suppose H is a group which is a subgroup of two other groups, G_1, G_{-1}. Form $\Gamma = G_1 *_H G_{-1}$. The group ring $k\Gamma$ will be written Λ. This data yields a ring R with two bimodules $\mathcal{B}_1$, $\mathcal{B}_{-1}$ as follows.

Let $R = kH$ and, for $\varepsilon = \pm 1$, let $\mathcal{B}_\varepsilon = k(G_\varepsilon \setminus H)$, the k-submodule of kG_ε generated by $G_\varepsilon \setminus H$. The left and right multiplications of H on G_ε make each $\mathcal{B}_\varepsilon$ into an R bimodule. The R bimodule Λ can then be expressed a sum of R bimodules:

$$\begin{aligned}\Lambda &= R \oplus (\mathcal{B}_1 + \mathcal{B}_{-1}) \oplus (\mathcal{B}_{-1}\mathcal{B}_1 + \mathcal{B}_1\mathcal{B}_{-1}) \oplus (\mathcal{B}_1\mathcal{B}_{-1}\mathcal{B}_1 + \mathcal{B}_{-1}\mathcal{B}_1\mathcal{B}_{-1}) \oplus \dots \\ &= \mathcal{A}_0 \oplus \sum_{i=1}^{\infty}(\mathcal{A}_i + \mathcal{A}_{-i}) = \sum_{-\infty}^{\infty} \mathcal{A}_i\end{aligned}$$

where $\mathcal{A}_0 = R$ and $\mathcal{A}_i$ (resp $\mathcal{A}_{-i}$) denotes the i-letter alternating word in $\mathcal{B}_1$ and $\mathcal{B}_{-1}$ which terminates in $\mathcal{B}_1$ (resp. $\mathcal{B}_{-1}$). (compare Stallings [S] or Cappell [C4] p.84). The triple $(R; \mathcal{B}_1, \mathcal{B}_{-1})$ then defines the twisted polynomial extension category $\mathbb{A}_\alpha[t]$ as in 2.4.

There is an additive functor :

$$r : \mathbb{A}_\alpha[t] \longrightarrow \mathcal{F}_\Lambda$$

sending an object (P_1, P_{-1}) to $r(P_1, P_{-1}) = (P_1 \oplus P_{-1}) \otimes_R \Lambda$; if $u = (P_1, P_{-1})$, $v = (Q_1, Q_{-1})$ are objects of $\mathbb{A}_\alpha[t]$, the map $r : \mathbb{A}_\alpha[t](u, v) \longrightarrow \mathcal{F}_\Lambda(u, v)$ is the following composite homomorphism:

$$\begin{aligned}
\mathbb{A}_\alpha[t](u,v) &= \sum_{i=0}^{\infty} Hom_R(P_1, Q_{(-1)^i}\mathcal{A}_i) \times Hom_R(P_{-1}, Q_{-(-1)^i}\mathcal{A}_{-i}) \\
&\to \sum_{i=0}^{\infty} Hom_R(P_1 \oplus P_{-1}, Q_{(-1)^i}\mathcal{A}_i \oplus Q_{-(-1)^i}\mathcal{A}_{-i}) \\
&\to Hom_R(P_1 \oplus P_{-1}, (Q_1 \oplus Q_{-1})(\sum_{i=-\infty}^{\infty} \mathcal{A}_i)) \\
&\approx Hom_\Lambda((P_1 \oplus P_{-1})\Lambda, (Q_1 \oplus Q_{-1})\Lambda) \\
&= \mathcal{F}_\Lambda(ru, rv).
\end{aligned}$$

We will only lightly sketch the routine details of the proof that $r(\psi)r(\varphi) = r(\psi\varphi)$ for morphisms $\varphi : u \to v$, $\psi : v \to w$. One may as well assume φ has degree k, ψ has degree j, so that $\varphi = t^{(k)}\varphi_k \in P_k(u,v)$, $\psi = t^{(j)}\psi_j \in P_j(v,w)$. Then $\psi_j = (g_1, g_{-1})$, $\varphi_k = (f_1, f_{-1})$ where g_ε, f_ε are morphisms in $\mathcal{M}_R$ for $\varepsilon = \pm 1$. We compute $\psi\varphi = t^{(j+k)}(h_1, h_{-1})$ where $h_\varepsilon = (g_\delta \otimes 1)f_\varepsilon$, $\delta = (-1)^k\varepsilon$, and where 1 is the identity on $\mathcal{A}_{k\varepsilon}$. One must verify that, in $(P_1 \oplus P_{-1})\Lambda$,

$$r(\psi\varphi)|_{P_\varepsilon} = (r(\psi))|_{Q_\delta \mathcal{A}_{k\varepsilon}} \circ (r(\varphi))|_{P_\varepsilon}$$

This means one must establish that : $(g_\delta \otimes 1)f_\varepsilon = ((r(\psi))|_{Q_\delta \mathcal{A}_{k\varepsilon}})f_\varepsilon$. In turn this amounts to proving : $(r(\psi))|_{Q_\delta \mathcal{A}_i} = g_\delta \otimes 1_{\mathcal{A}_i}$; but this is clear.

We will return to this example 2.4 and 2.4.a frequently.

2.5 Waldhausen, in [W1] (see also [W2], p.166, or [C4], p.125) defines a group $\widetilde{Nil}\ (R; \mathcal{B}_1, \mathcal{B}_{-1})$ for any ring R with two bimodules $\mathcal{B}_1$, $\mathcal{B}_{-1}$, which are both left and right free over R. In the special case discussed in 2.4.a above, when $R = kH$, $\mathcal{B}_\varepsilon = k(G_\varepsilon \setminus H)$ he provides a homomorphism $s : \widetilde{Nil}(R; B_1, B_{-1}) \to K_1(\Lambda)$, $\Lambda = k(G_1 *_H G_2)$, which we will review below. We will then construct an isomorphism $\Phi : \widetilde{Nil}(R; \mathcal{B}_1, \mathcal{B}_{-1}) \to Nil(\mathbb{A}, \alpha)$ where $\mathbb{A}_\alpha[t]$ is the polynomial extension defined by $(R; \mathcal{B}_1, \mathcal{B}_{-1})$ in 2.4 .

We prove the following easy result, which is the paradigm on which our "quadratic" results are modeled:

2.6 Proposition. *a) The map $\Phi : \widetilde{Nil}(R; \mathcal{B}_1, \mathcal{B}_{-1}) \to Nil(\mathbb{A}, \alpha)$ (constructed in the proof below) is an isomorphism for any ring R and pair of bimodules $\mathcal{B}_1$, $\mathcal{B}_{-1}$, which are left and right R-free.*

b) In case $R = kH$, $\mathcal{B}_\varepsilon = k(G_\varepsilon \setminus H)$, $\Lambda = k(G_1 *_H G_2)$ *as in 2.4.a , the diagram below commutes:*

$$\begin{array}{ccc} Nil(\mathbb{A},\alpha) & \xrightarrow{\sigma} & K_1(\mathbb{A}_\alpha[t]) \\ \cong \uparrow \Phi & & \downarrow r_* \\ \widetilde{Nil}(R;\mathcal{B}_1\mathcal{B}_{-1}) & \xrightarrow{s} & K_1(\Lambda) \end{array}$$

2.7 Proof of Proposition 2.6.

a) We begin with the definition of $\widetilde{Nil}(R;\mathcal{B}_1,\mathcal{B}_{-1})$ and Φ. Waldhausen studies pairs (u,ν) where $u = (P_1, P_{-1}) \in \mathcal{F}_R \times \mathcal{F}_R$, and $\nu : (P_1, P_{-1}) \longrightarrow (P_{-1}\mathcal{B}_1, P_1\mathcal{B}_{-1})$ is a pair of R maps, (p_1, p_{-1}), in $C = \mathcal{M}_R \times \mathcal{M}_R$. He says (u,$\nu$) has an assailable filtration if there are submodules: $P_1 = M_n \supset M_{n-1} \supset \cdots \supset M_0 = 0$; $P_{-1} = N_n \supset N_{n-1} \supset \cdots \supset N_0 = 0$, so that $p_1(M_i) \subset N_{i-1}\mathcal{B}_1$, $p_{-1}(N_i) \subset M_{i-1}\mathcal{B}_{-1}$ $\forall i$. Letting $\mathbb{A} = \mathcal{F}_R \times \mathcal{F}_R$, and $\alpha : C \to C$ be as in 2.4, we see that $t\nu$ is a nilpotent element of degree one in the ring $\mathbb{A}_\alpha[t](u,u)$ when (u,ν) has an assailable filtration. Conversely suppose $\nu \in C(u, \alpha u)$ and $t\nu$ is nilpotent of exponent n in $\mathbb{A}_\alpha[t](u,u)$. Then, because each $\mathcal{A}_i$ (as defined in 2.4.a) is left R-free, $\alpha^i : C \to C$ is an exact functor on the abelian category C, and $\alpha^i(Ker\ \nu) = ker\ \alpha^i(\nu)$. If we set $ker\ \nu = (M_1, N_1)$, then ν induces a nilpotent map $t\nu' : (u/ker(\nu)) \longrightarrow (u/ker(\nu))$ in $C_\alpha[t]$, of exponent $n-1$. By induction we argue that $(u/ker\ \nu,\ \nu')$ has a finite assailable filtration (of length $n-1$). Since $u/ker\ \nu = (P_1/M_1, P_{-1}/N_1)$, we conclude (u, ν) has an assailable filtration (of length n). Therefore, we conclude that for any $u \in |\mathbb{A}|$, $\nu \in C(u, \alpha(u))$, the pair (u, ν) has an assailable filtration if and only if $t\nu$ is nilpotent.

Waldhausen considers the category $\mathcal{N}il(\mathbb{A},\alpha)$ as defined in 2.3 when $\mathbb{A}$, C, $\alpha : C \to C$ are as in 2.4. In effect, using the discussion above, we see his definition of $\widetilde{Nil}(R;\mathcal{B}_1,\mathcal{B}_{-1})$ amounts to $Nil(\mathbb{A},\alpha)$, and that will be our approach: Φ is the identity map.

b) The map $s : \widetilde{Nil}(R;\mathcal{B}_1,\mathcal{B}_{-1}) \to K_1(\Lambda)$ of Waldhausen (see [W1]) sends the element $[(u,\nu)]$ of $\widetilde{Nil}(R;\mathcal{B}_1\ \mathcal{B}_{-1})$, when $u = (P_1, P_{-1})$, and $\nu = (p_1, p_{-1})$, to the element

$$[(P_1\Lambda \oplus P_{-1}\Lambda), \begin{pmatrix} 1 & -p'_{-1} \\ -p'_1 & 1 \end{pmatrix}]$$

where $p'_\varepsilon : P_\varepsilon\Lambda \to P_{-\varepsilon}\Lambda$ is the unique Λ map extending p_ε. To prove 2.6 b), we note:

$$r_*\sigma[u,\nu] = r_*[(P_1,P_{-1}), 1 - t_u(p_1,p_1)] = [(P_1\Lambda \oplus P_{-1}\Lambda),\ \begin{pmatrix} 1 & -p'_{-1} \\ -p'_1 & 1 \end{pmatrix}] \text{ by 2.4.a}$$

$= s[u,\nu]$, as required. This completes the proof of 2.6.

2.8 Higman's trick on linearization of nonsingular matrices over polynomial rings fits into our setting to yield the exact sequence of 2.9 below, when suitable conditions hold on the category. The argument will be similar to that in Bass [B] p. 643-645, or [BHS].

Consider a twisted polynomial extension $\mathbb{A}_\alpha[t]$, when $\mathbb{A}$ is an additive full subcategory of C and $\alpha : C \longrightarrow C$ is an additive functor. $\mathbb{A}_\alpha[t]$ is said to be *linearizable* if, for each $u \in \mathbb{A}$ one can find $v \in \mathbb{A}$ such that $\mathbb{A}_\alpha[t](u \oplus v, u \oplus v)$ is generated as a graded ring, by elements of degree 0 and degree 1.

2.9 Proposition. *Suppose the twisted polynomial extension category $\mathbb{A}_\alpha[t]$ is linearizable. Then the following sequence is exact:*

$$Nil(\mathbb{A}, \alpha) \xrightarrow{\sigma} K_1(\mathbb{A}_\alpha[t]) \xrightarrow{\eta_*} K_1(\mathbb{A}) \rightarrow 0.$$

Proof. : The functor η admits a right inverse, so η_* is obviously a split epimorphism. It is also clear that $\eta_*\sigma = 0$ since $\eta_*\sigma[u,\nu] = [u, \eta(1-t\nu)] = [u,1] = 0$.

Now we prove $ker\ \eta_* \subset Im\ \sigma$. Let $x = [u,\varphi] \in ker\ \eta_*$. Since $\eta \circ \iota = id_{\mathbb{A}}$, we can assume that the isomorphism $\varphi = \sum_{i=0}^{n} t_u^{(i)} \varphi_i$ satisfies : $\eta(\varphi) = \varphi_0 = 1_u$. If n = 0, then $x = 0$ so we may assume $n \geq 1$. Stablizing, if necessary we can also assume $\varphi_n \in C(u, \alpha^n u)$ is expressible in the form $\sum_{j=1}^{r} \alpha(c_j)d_j$ where $d_j \in C(u, \alpha u)$, $c_j \in C(u, \alpha^{n-1}u)$. This is where we use that $\mathbb{A}_\alpha[t]$ is linearizable. Replace (u,φ) now by the equivalent representatives :

$$(u \oplus u, \begin{pmatrix} \varphi & 0 \\ 0 & 1 \end{pmatrix}) \sim (u \oplus u, \begin{pmatrix} \varphi & t^{(n-1)}c_1 \\ 0 & 1 \end{pmatrix}) \sim (u \oplus u, \begin{pmatrix} \varphi - t^{(n)}\alpha(c_1)d_1 & t^{(n-1)}c_1 \\ td_1 & 1 \end{pmatrix})$$

The map $\varphi - t^{(n)}\alpha(c_1)d_1$ is of degree n still; its degree n summand is $\sum_{j=2}^{r} \alpha(c_j)d_j$. Repeat this process $r-1$ more times and one gets : $x = [u', \varphi']$ for some φ' whose degree is max $(1, n-1)$. By induction we can get : $x = [v, \psi]$ when $\psi = 1 - t\psi_1$. Since ψ is invertible in the graded ring $\mathbb{A}_\alpha[t](v,v)$ it follows that ψ_1 is nilpotent so $x = \sigma[v, \psi_1]$. Hence $ker\ \eta_* \subset Im\ \sigma$. This proves 2.9.

We will use 2.9 for the Nil groups given by the Fundamental Example. So we need:

2.10 Lemma. *Let $\mathbb{A}_\alpha[t]$ be the polynomial extension defined by a ring with two bimodules, $(R; \mathcal{B}_1, \mathcal{B}_{-1})$. Then $\mathbb{A}_\alpha[t]$ is linearizable.*

Proof. Let $u \in |\mathbb{A}|$. We have $\mathbb{A} \subset C$, $\alpha : C \rightarrow C$. We only have to show that if $n \geq 1$, then the composition law induces an epimorphism:

$$C(u, \alpha^{n-1}u) \otimes C(u, \alpha u) \xrightarrow{K} C(u, \alpha^n u)$$

Here K is defined by $K(\phi \otimes \psi) = \alpha(\varphi) \circ \psi$.

But since $\mathbb{A} = \mathcal{F}_R \times \mathcal{F}_R$, $C = \mathcal{M}_R \times \mathcal{M}_R$ this amounts to showing that the map:

$$Hom_R(P', P''\mathcal{A}) \otimes Hom_R(P, P'\mathcal{B}) \xrightarrow{K} Hom_R(P, P''\mathcal{AB}), (\varphi \otimes \psi \longmapsto (\varphi \otimes 1_{\mathcal{B}})\psi)$$

is an epimorphism, when P, P', P'' are in $|\mathcal{F}_R|$ and $\mathcal{A}$, $\mathcal{B}$ are R bimodules and $P' \neq 0$. By distributivity it is enough to show this when P, P', P'' are all free of rank 1. But in this case the result is clear.

We can now conclude:

2.11 Theorem. *Let H be a subgroup of two groups G_1 and G_{-1}. Let k be a subring of $\mathbb{Q}$. Let $R = kH$, $\mathcal{B}_i = k(G_i \smallsetminus H)$. Let $(\mathbb{A}, \alpha)$ be the additive category and functor defined by $(R; \mathcal{B}_1, B_{-1})$. Then the following sequence is split exact:*

$$0 \longrightarrow \widetilde{Nil}(R; \mathcal{B}_1, \mathcal{B}_{-1}) \xrightarrow{\sigma} K_1(\mathbb{A}_\alpha[t]) \xrightarrow{\eta} K_1(\mathbb{A}) \to 0$$

Proof. Waldhausen's map $s : \widetilde{Nil}(R; \mathcal{B}_1, \mathcal{B}_{-1}) \longrightarrow K_1(k[G_1 *_H G_{-1}])$ (explained in 2.7.b above) is a split monomorphism (see [W1], Ch. 5). By 2.6.b) then, σ is also a split monomorphism. Since $\mathbb{A}_\alpha[t]$ is linearizable by 2.10, the result follows from 2.9.

Note : If $a : R \to R$ is a ring endomorphism and $\alpha : \mathcal{M}_R \to \mathcal{M}_R$ is as in 2.2, then $(\mathcal{F}_R)_\alpha[t]$ is linearizable. The argument is simpler than, and similar to the argument above. So from 2.9 one gets the exact sequence:

$$Nil(R, a) \xrightarrow{\sigma} K_1(R_a[t]) \xrightarrow{\eta_*} K_1(R) \to 0$$

first established by Farrell and Hsiang [FH] when a is an automorphism. In their case σ was shown to be injective. (Even when a is only an endomorphism we can show σ is injective, but we will not prove or use this fact). In general twisted polynomial extension categories, σ does not seem to be a monomorphism.

§3. *UNil* groups of polynomial categories with involution.

In this chapter we show that Cappell's *UNil* groups are isomorphic to the L groups of a polynomial extension category, at least in the geometrically significant cases. For any polynomial extension category $\mathbb{A}_\alpha[t]$, with involution, we define an abelian group $UNil_\varepsilon(\mathbb{A},\alpha)$. Cappell's group $UNil^h_{2n}(R;\mathcal{B}_1,\mathcal{B}_{-1})$ is shown to be a special case of this construction. In the case when $(R;\mathcal{B}_1,\mathcal{B}_{-1})$ come from group rings as in 2.4.a, we show:

$$UNil^h_{2n}(R;\mathcal{B}_1,\mathcal{B}_{-1}) \cong L_\varepsilon(\mathbb{A}_\alpha[t]), \varepsilon = (-1)^n,$$

where $\mathbb{A}_\alpha[t]$ is the category defined in 2.4. We relate this isomorphism to Cappell's map $\rho : UNil^h_{2n}(R;\mathcal{B}_1,\mathcal{B}_{-1}) \to L^h_{2n}(k(G_1 *_H G_2))$. See 3.2, 3.5., 3.6.b, 3.9.

3.1 We briefly recall Ranicki's notion (in [R2], and also [R3]) of the surgery groups of an additive category.

Let $\mathbb{A}$ be an additive category and let $* : \mathbb{A} \to \mathbb{A}$ be an involution. This means that $*$ is a contravariant functor, together with a natural equivalence $e : id \to **$ which satisfies $e(u)^* e(u^*) = id_{u^*}$ for all u in $|\mathbb{A}|$. For every u one then has an involution T on the abelian group $\mathbb{A}(u,u^*)$ given by $\varphi \mapsto \varphi^T = \varphi^* \circ e(u)$. For $\varepsilon = (-1)^n$, let $N_\varepsilon = id + \varepsilon T$. An *$\varepsilon$-quadratic form* in $\mathbb{A}$ is a pair (u,ψ) where $\psi \in coker(N_{-\varepsilon} : \mathbb{A}(u,u^*) \to \mathbb{A}(u,u^*))$. (u,ψ) is *nonsingular* if $N_\varepsilon(\psi)$ is an isomorphism. A morphism $i \in \mathbb{A}(y,u)$ is a *Lagrangian* for (u,ψ) if $i^*\psi i = 0 \in \text{coker} N_{-\varepsilon}$, and the sequence $0 \to y \xrightarrow{i} u \xrightarrow{i^*(\psi+\varepsilon\psi^T)} y^* \to 0$ is split exact. The *Wall - Ranicki surgery group,* $L_\varepsilon(\mathbb{A})$, is defined as the Witt group of nonsingular, ε-quadratic forms in $\mathbb{A}$, modulo those admitting Lagrangians. If $\mathbb{A} = \mathcal{F}_R$ (resp.$\mathcal{P}_R$) then $L_\varepsilon(\mathbb{A}) \cong L^h_{2n}(R)$ (resp. $L^p_{2n}(R)$).

3.2 Definition of $UNil_\varepsilon(\mathbb{A},\alpha)$ and $\sigma_\varepsilon : UNil_\varepsilon(\mathbb{A},\alpha) \to L_\varepsilon(\mathbb{A}_\alpha[t])$.

Let $\mathbb{A}_\alpha[t]$ be a polynomial extension category and let $*$ be an involution on $\mathbb{A}_\alpha[t]$ which satisfies the following properties for all objects u, v of $\mathbb{A}$:

(i) $* : \mathbb{A}_\alpha[t](u,v) \to \mathbb{A}_\alpha[t](v^*,u^*)$ is a degree preserving map;

(ii) The isomorphism $e(u) \in \mathbb{A}_\alpha[t](u,u^{**})$ has degree 0 .

This implies that the maps $T : \mathbb{A}_\alpha[t](u,u^*) \to \mathbb{A}_\alpha[t](u,u^*)$ and $N_\varepsilon = id + \varepsilon T$ are degree-preserving. $\mathbb{A}$ is then also a category with involution, and the functor $\eta : \mathbb{A}_\alpha[t] \to \mathbb{A}$ induces a homomorphism $\eta_* : L_\varepsilon(\mathbb{A}_\alpha[t]) \to L_\varepsilon(\mathbb{A})$.

For any object u of $\mathbb{A}$ write $Q_\varepsilon(u) = coker(N_{-\varepsilon} : \mathbb{A}_\alpha[t](u,u^*) \to \mathbb{A}_\alpha[t](u,u^*))$. $Q_\varepsilon(u)$ is a graded group.

An ε-UNil form in $\mathbb{A}_\alpha[t]$ is a nonsingular ε-quadratic form (u,ψ) in $\mathbb{A}_\alpha[t]$ such that:

(a) $\psi \in Q_\varepsilon(u)$ has filtration ≤ 1; that is to say, $\psi = \psi_0 + t\psi_1$;

(b) The ε-quadratic form(u,ψ_0) in $\mathbb{A}$ has a Lagrangian in $\mathbb{A}$.

A UNil Lagrangian for an ε-*UNil* form (u,ψ) is a Lagrangian $i \in \mathbb{A}_\alpha[t](y,u)$ for the ε-quadratic form (u,ψ) in $\mathbb{A}_\alpha[t]$, with the additional property that i has degree 0. Two ε-*UNil* forms (u,ψ), (u',ψ') are *isomorphic* if there exists an isomorphism $f \in \mathbb{A}(u,u')$ such that $f^*\psi' f = \psi \in Q_\varepsilon(u)$.

We now define $UNil_\varepsilon(\mathbb{A},\alpha)$ as the abelian group with one generator for each isomorphism class of ε-*UNil* forms in $\mathbb{A}_\alpha[t]$ and relations:

$$(1)\ (u,\psi) + (u',\psi') = (u \oplus u', \psi \oplus \psi'),$$

$$(2)\ (u,\psi) = 0 \text{ if } (u,\psi) \text{ has a } UNil \text{ Lagrangian.}$$

We define the homomorphism $\sigma_\varepsilon : UNil_\varepsilon(\mathbb{A},\alpha) \to L_\varepsilon(\mathbb{A}_\alpha[t])$ by mapping $[u,\psi]$ to $[u,\psi]$. Clearly $im(\sigma_\varepsilon) \subset ker(\eta_*)$.

3.3 Construction:. Let R be a ring with involution. Let $\mathcal{B}_1, \mathcal{B}_{-1}$ be R bimodules with involution and let $\mathbb{A} = \mathcal{F}_R \times \mathcal{F}_R$, $\alpha(P_1, P_{-1}) = (P_{-1}\mathcal{B}_1, P_1\mathcal{B}_{-1})$ as in 2.4. We will show that the involutions on R, $\mathcal{B}_1, \mathcal{B}_{-1}$ induce an involution on the polynomial extension category $\mathbb{A}_\alpha[t]$.

The involution on R induces an involution $*$ on $\mathcal{F}_R$ (see [R2] p.168). This makes $\mathbb{A} = \mathcal{F}_R \times \mathcal{F}_R$ into a category with involution by: $(P_1,P_{-1})^* = (P_{-1}^*, P_1^*)$ and $(f_1,f_{-1})^* = (f_{-1}^*, f_1^*)$. For $u = (P_1,P_{-1})$, $e(u)$ is $(e(P_1), e(P_{-1}))$.

We now extend this involution on $\mathbb{A}$ to one on $\mathbb{A}_\alpha[t]$, satisfying 3.2(i) and (ii).

Suppose M,N,P are free right R modules of finite rank, and let A be an R bimodule which is a "word" in $\mathcal{B}_1$ and $\mathcal{B}_{-1}$: $A = \mathcal{B}_{j_1}\mathcal{B}_{j_2}\ldots\mathcal{B}_{j_r}$. Set $A' = \mathcal{B}_{j_r}\ldots\mathcal{B}_{j_2}\mathcal{B}_{j_1}$. So, for example, for $\mathcal{A}_k$ of 2.4.a, one has: $\mathcal{A}'_k = \mathcal{A}_{(-1)^{k+1}k}$. One has sesquilinear pairings:

$$< , >:\ N^* \times NA \longrightarrow A\ :< \lambda, na >= \lambda(n)a$$

$$(,):\ M^*A' \times M \longrightarrow A:\ (\lambda \otimes a, m) = \bar{a}\lambda(m),$$

where, if $a = b_{j_r}\ldots b_{j_2}b_{j_1}$, then $\bar{a} = \bar{b}_{j_1}\bar{b}_{j_2}\ldots\bar{b}_{j_r}$. $< , >$ and $(,)$ are right and left nonsingular respectively. They allow one to define an isomorphism:

$$Hom_R(M,NA) \longrightarrow Hom_R(N^*, M^*A'):\ f \to f^!$$

by the rule: $(f^!(\lambda), m) = <\lambda, f(m)> \forall\lambda \in N^*, m \in M, f \in Hom(M, NA)$.

For this isomorphism, the following diagram commutes:

$$\begin{array}{ccc} Hom(M, NA_1) \times Hom(N, PA_2) & \rightarrow Hom(M, PA_2A_1) : (f,g) \mapsto (g \otimes 1) \circ f \\ \downarrow ! \times ! & ! \downarrow \\ Hom(N^*, M^*A'_1) \times Hom(P^*, N^*A'_2) & \rightarrow Hom(P^*, M^*A'_1A'_2) : (f,g) \mapsto (f \otimes 1) \circ g \end{array}$$

We next define a homomorphism, $* : \mathbb{A}_\alpha[t](u,v) \rightarrow \mathbb{A}_\alpha[t](v^*, u^*)$ for any objects $u = (P_1, P_{-1}), v = (Q_1, Q_{-1})$. Suppose $\phi = t^{(k)}(f_k, f_{-k}) \in P_k(u,v)$, for $k \geq 0$. Let $\delta = (-1)^k$. Then $f_k \in Hom(P_1, Q_\delta \mathcal{A}_k)$, so $f_k^! \in Hom(Q_\delta^*, P_1^* \mathcal{A}_{-\delta k})$. Similarly, $f_{-k}^! \in Hom(Q_{-\delta}^*, P_{-1}^* \mathcal{A}_{\delta k})$.

Define $\phi^* = t^{(k)}(f_{\delta k}^!, f_{-\delta k}^!) \in P_k(v^*, u^*)$. For $k = 0$, this is the involution on $\mathbb{A}$. The commutativity of the above square shows that $*$ is a contravariant functor. Moreover it is easy to see that $\phi^{**}e(u) = e(v)\phi$. It is obvious that 3.2(i) and (ii) hold. This completes the construction of the involution on $\mathbb{A}_\alpha[t]$.

The above involution on $\mathbb{A}_\alpha[t]$ allows us to define the group $UNil_\varepsilon(\mathbb{A}, \alpha)$. In fact this is Cappell's $UNil$ group, as we show below.

3.4 Cappell's $UNil$ group.

We briefly review Cappell's definition.

Let R be a ring with involution. For a free, finitely generated right R module P and an R bimodule with involution $\mathcal{B}$ one has the involution T on the abelian group $Hom_R(P, P^*\mathcal{B})$ given by $T(\varphi) = \varphi^! \circ e(P)$ where $e(P) : P \rightarrow P^{**}$ is the natural isomorphism. It is well known ([Wa1] p.260, [Wa2] p.246) that an ε-Hermitian form (P, λ, μ) over $\mathcal{B}$ specifies and is specified by a form $\hat{\mu} \in$ coker $\{N_{-\varepsilon} : Hom_R(P, P^*\mathcal{B}) \rightarrow Hom_R(P, P^*\mathcal{B})\}$ where $N_\varepsilon = id + \varepsilon T$.

Let $\mathcal{B}_1, \mathcal{B}_{-1}$ be R bimodules with involution which are free as right R modules and let $\varepsilon = (-1)^n$. Cappell in [C1] defines an ε-$UNil$ form over $(\mathcal{B}_1, \mathcal{B}_{-1})$ as a sextuple $z = (P_1, \lambda_1, \mu_1, P_{-1}, \lambda_{-1}, \mu_{-1})$ where,(P_i, λ_i, μ_i) are ε-Hermitian forms over $\mathcal{B}_i$ and:

(i) $P_{-1} = P_1^*$, with natural sesquilinear pairing $< , >: P_1 \times P_{-1} \rightarrow R$. This pairing defines R maps $p_i : P_i \rightarrow P_{-i}\mathcal{B}_i$, $p_i = Ad\lambda_i$, for $i = \pm 1$.

(ii) If $u = (P_1, P_{-1})$, $\nu = (p_1, p_{-1})$, the pair (u, ν) is in $\widetilde{\mathcal{N}il}(R; \mathcal{B}_1, \mathcal{B}_{-1})$ (defined in 2.5). A Lagrangian for the ε-$UNil$ form z over $(\mathcal{B}_1, \mathcal{B}_{-1})$ is defined to be a pair V_1, V_{-1} of free submodules of P_1, P_{-1}, respectively, so that $V_{-1} = V_1^\perp$ under $< , >$, and P_j/V_j is free, $p_j(V_j) \subset V_{-j}\mathcal{B}_j$, $\mu_j(V_j) = 0$ for $j = \pm 1$. For $\varepsilon = (-1)^n$, Cappell then defines $UNil_{2n}^h(R; \mathcal{B}_1, \mathcal{B}_{-1})$ as the Witt group of ε-$UNil$ forms over $(\mathcal{B}_1, \mathcal{B}_{-1})$ modulo those admitting Lagrangians.

Let $\mathbb{A}_\alpha[t]$ be the polynomial extension category defined by $(R; \mathcal{B}_1, \mathcal{B}_{-1})$, with involution constructed in 3.3. By the above remarks, if $z = (P_1, \lambda_1, \mu_1, P_{-1}, \lambda_{-1}, \mu_{-1})$ is a sextuple which satisfies (i), and if $u = (P_1, P_{-1})$, then z specifies a quadratic form $\varphi_z \in Q_\varepsilon(u)$ of filtration one; namely, $\varphi_z = [H] + t\hat{\mu}$, where $\hat{\mu} = (\hat{\mu}_1, \hat{\mu}_{-1})$, and $[H]$ is the class of $(0, id_{P_1}) \in \mathbb{A}(u, u^*)$. Notice $N_\varepsilon H = (\varepsilon e(P_1), id_{P_{-1}}) \in \mathbb{A}(u, u^*)$ is an isomorphism. The map $(P_1, 0) \to (P_1, P_{-1})$ is a Lagrangian for [H] in $\mathbb{A}$.

3.5. Theorem. *Let $R, \mathcal{B}_1, \mathcal{B}_{-1}$ be a ring and two bimodules with involution. Assume $\mathcal{B}_1, \mathcal{B}_{-1}$ are free as right R modules. Let $\mathbb{A}, \alpha$ be the additive category and functor defined by $(R, \mathcal{B}_1, \mathcal{B}_{-1})$ (in 2.4), equipped with the involution on $\mathbb{A}_\alpha[t]$ defined in 3.3. Let $\varepsilon = (-1)^n$. Then the rule, $z \mapsto (u, \varphi_z)$ defines an isomorphism*

$$\Psi : UNil^h_{2n}(R; \mathcal{B}_1, \mathcal{B}_{-1}) \to UNil_\varepsilon(\mathbb{A}, \alpha).$$

Proof. Let $z = (P_1, \lambda_1, \mu_1, P_{-1}, \lambda_{-1}, \mu_{-1})$ be a sextuple which satisfies (i). Let $\nu = (p_1, p_{-1}) \in \mathbb{A}(u, \alpha u)$, where $p_i = Ad\lambda_i$. Then $1 + t\nu = (N_\varepsilon H)^{-1}(N_\varepsilon \varphi_z)$. Therefore:
z is an ε-*UNil* form over $(\mathcal{B}_1, \mathcal{B}_{-1}) \Leftrightarrow (u, \nu)$ is in $\mathcal{N}il(R; \mathcal{B}_1, \mathcal{B}_{-1}) \Leftrightarrow t\nu$ is nilpotent in $\mathbb{A}_\alpha[t](u, u)$ (see 2.7) $\Leftrightarrow 1 + t\nu$ is an isomorphism $\Leftrightarrow N_\varepsilon \varphi_z$ is an isomorphism $\Leftrightarrow (u, \varphi_z)$ is an ε-*UNil* form in $\mathbb{A}_\alpha[t]$. So each ε-*UNil* form in $\mathbb{A}_\alpha[t]$ specifies an ε-*UNil* form over $(\mathcal{B}_1, \mathcal{B}_{-1})$ (in Cappell's sense) and vice versa.

Let $v = (V_1, V_{-1})$ be a pair of submodules of (P_1, P_{-1}). The inclusions $V_j \subset P_j$ define a degree zero map, $i : v \to u$, and it is surely clear that the pair (V_1, V_{-1}) is a Lagrangian for z, in Cappell's sense, if and only if i is a *UNil* Lagrangian for (u, φ_z). This proves 3.5.

Let $\mathbb{A}_\alpha[t]$ be a polynomial extension category with an involution, as in 3.2. We now show that if the category $\mathbb{A}_\alpha[t]$ is linearizable in the sense of 2.8, then there is an exact sequence relating the *UNil* group and the surgery groups, analogous to the sequence 2.9.

We will write $(u, \varphi) \sim (v, \psi)$ if the nonsingular, ε-quadratic forms $(u, \varphi), (v, \psi)$ are equivalent, i.e. they represent the same element of $L_\varepsilon(\mathbb{A}_\alpha[t])$.

3.6.a Lemma. *Let (w, φ) be a nonsingular ε-quadratic form in $\mathbb{A}_\alpha[t]$ and $\varphi = \sum_{i=0}^{m} t^{(i)}\varphi_i$. Assume that $m \geq 2$. Suppose that*
(i) The ε-quadratic form (w, φ_0) in $\mathbb{A}$ represents 0 in $L_\varepsilon(\mathbb{A})$,

(ii) $t^{(m)}\varphi_m = \beta^*\gamma$ *where* $\beta \in P_1(w,w)$ *and* $\gamma \in P_{m-1}(w,w^*)$.
Then (w,φ) *is equivalent to a form* (u,ψ) *with* $\psi = \sum_{i=0}^{m-1} t^{(i)}\psi_i$

Proof. Let $(u,\theta) = (w,\varphi) \oplus (w,\varphi_0) \oplus (w \oplus w^*, H)$, where $H = \begin{pmatrix} 0 & id_{w^*} \\ 0 & 0 \end{pmatrix}$ Since $(w,\varphi_0) \sim 0$ and $(w \oplus w^*, H) \sim 0$ we have $(w,\varphi) \sim (u,\theta)$. Now let $x \in \mathbb{A}_\alpha[t](u,u)$ be the nilpotent morphism given by the matrix of maps: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X & 0 & 0 \end{pmatrix}$ where $X : w \to w \oplus w^*$ is $(\beta, -\gamma)$.

Let $c = id_u + x \in \mathbb{A}_\alpha[t](u,u)$. The isomorphism c provides an isomorphism of forms $(u,\theta) \cong (u,\psi)$ where $\psi = c^*\theta c$. Also, $\psi = \sum_{i=0}^{m-1} t^{(i)}\psi_i$, as required.

3.6.b Proposition. *Suppose that* $\mathbb{A}_\alpha[t]$ *is linearizable. Then the following sequence is exact:*

$$UNil_\varepsilon(\mathbb{A},\alpha) \xrightarrow{\sigma_\varepsilon} L_\varepsilon(\mathbb{A}_\alpha[t]) \xrightarrow{\eta_*} L_\varepsilon(\mathbb{A}) \to 0.$$

Proof. We only need to show that $ker\ \eta_* \subset im\ \sigma_\varepsilon$. Let $x \in ker\ \eta_*$ be represented by a form (u,ψ), $\psi = \sum_{i=0}^{m} t^{(i)}\psi_i$. We have to show that $x \in im(\sigma_\varepsilon)$. The proof will be by induction on m. If $m \le 1$ we are done. Suppose $m \ge 2$. We will show that (u,ψ) is equivalent to a form of degree $\le m-1$. Since u^* is $\mathbb{A}$ isomorphic to u and $\mathbb{A}_\alpha[t]$ is linearizable we can (after stabilizing, if necessary) write $t^{(m)}\psi_m = \sum_{i=1}^{r} \beta_i^*\gamma_i$ where $\beta_i \in P_1(u,u)$ and $\gamma_i \in P_{m-1}(u,u^*)$ for all i. $(u,\psi_0) \sim 0$ because $x \in ker\ \eta_*$. Let $(w,\varphi) = (u,\psi) \oplus (u,\psi_0) \oplus \cdots \oplus (u,\psi_0)$ (r summands in all). Clearly $(u,\psi) \sim (w,\varphi)$. Note that $\varphi = \sum_{i=0}^{m} t^{(i)}\varphi_i$ where $t^{(m)}\varphi_m = \beta^*\gamma$ for $\beta = \begin{pmatrix} \beta_1 & 0 & \\ \vdots & \ddots & \\ \beta_r & & 0 \end{pmatrix} \in P_1(w,w)$ and $\gamma = \begin{pmatrix} \gamma_1 & 0 & \\ \vdots & \ddots & \\ \gamma_r & & 0 \end{pmatrix} \in P_{m-1}(w,w^*)$.

The ε-*UNil* form (w,φ) satifies the condition of 3.6.a, so $(u,\varphi) \sim (z,\theta)$ with $\theta = \sum_{i=0}^{m-1} t^{(i)}\theta_i$. This ends the inductive step and completes the proof.

3.7 Let H be a subgroup of two groups G_1 and G_{-1} and let k be a commutative ring with unit. Set $\Gamma = G_1 *_H G_{-1}$, $R = kH$, $\mathcal{B}_i = k(G \setminus H)$ and $\Lambda = k\Gamma$, as in 2.4. In [$C1$] Cappell constructs a map $\rho : UNil_{2n}^h(R;\mathcal{B}_1,\mathcal{B}_{-1}) \to L_{2n}^h(\Lambda)$ as follows. For $\varepsilon = (-1)^n$ the class of an ε-*UNil* form $(P_1,\lambda_1,\mu_1,P_{-1},\mu_{-1})$ is mapped to the class of the special ε-Hermitian form (P,λ,μ) where: $P = (P_1 \oplus P_{-1}) \otimes_R \Lambda$, and
$\lambda(u,v) = \lambda_i(u,v)$ for $u,v \in P_i$, $i = \pm 1$, $\lambda(u,v) =< u,v >$ for $u \in P_1, v \in P_{-1}$,
$\mu(u) = \mu_i(u)$ for $u \in P_i$, $i = \pm 1$.

For $\mathbb{A} = \mathcal{F}_R \times \mathcal{F}_R$ with $\alpha(P_1,P_{-1}) = (P_{-1}\mathcal{B}_1, P_1\mathcal{B}_{-1})$ as in 2.4, let

$$r_* : L_\varepsilon(\mathbb{A}_\alpha[t]) \to L_{2n}^h(\Lambda)$$

be the homomorphism induced by the functor $r : \mathbb{A}_\alpha[t] \to \mathcal{F}_\Lambda$ defined in 2.4.a.

We now relate Cappell's map ρ to our map $\sigma_\varepsilon : UNil_\varepsilon(\mathbb{A}, \alpha) \to L_\varepsilon(\mathbb{A}_\alpha[t])$.

3.8 Proposition. *The following diagram commutes:*

$$\begin{array}{ccc} UNil_\varepsilon(\mathbb{A},\alpha) & \xrightarrow{\sigma_\varepsilon} & L_\varepsilon(\mathbb{A}_\alpha[t]) \\ \cong \uparrow \Psi & & \downarrow r_* \\ UNil^h_{2n}(R; \mathcal{B}_1, \mathcal{B}_{-1}) & \xrightarrow{\rho} & L^h_{2n}(\Lambda) \end{array}$$

where Ψ is the map defined in 3.5.

Proof. : Let $z = (P_1, \lambda_1, \mu_1, P_{-1}, \lambda_{-1}, \mu_{-1})$ as before. Then we compute: $r_*\sigma_\varepsilon\Psi([z]) = r_*\sigma_\varepsilon[u, \varphi_z] = [P, \lambda, \mu]$, where $P = (P_1 + P_{-1})\Lambda$, and (P, λ, μ) is specified by a quadratic form $[\hat{\mu}]$. In turn, $[\hat{\mu}]$ is computed (using 2.4.a). It is the unique Λ map $P \to Hom_\Lambda(P, \Lambda)$ whose restrictions to P_1 and P_{-1} are the following composite homomorphisms :

$$P_1 \xrightarrow{\hat{\mu}_1} Hom_R(P_1, \mathcal{B}_1) \to Hom_R(P_1 \oplus P_{-1}, \mathcal{B}_1) \to Hom_R(P_1 \oplus P_{-1}, \Lambda) = Hom_\Lambda(P, \Lambda)$$

$$P_{-1} \xrightarrow{(id, \hat{\mu}_{-1})} Hom_R(P_1, R) \times Hom_R(P_{-1}, \mathcal{B}_{-1}) \to Hom_R(P_1 \oplus P_{-1}, \Lambda) = Hom_\Lambda(P, \Lambda)$$

(the unmarked maps are obvious induced maps). But this is exactly Cappell's form $\rho((P_1, \lambda_1, \mu_1, P_{-1}, \lambda_{-1}, \mu_{-1})$ as defined in 3.7. Therefore $r_*\sigma_\varepsilon\Psi = \rho$ as required.

3.9. Theorem. *Let $(\mathbb{A}, \alpha)$ be the additive category and functor defined by some ring R and bimodules $\mathcal{B}_1, \mathcal{B}_{-1}$ with involution. Then the map σ_ε provides an epimorphism:*

$$\sigma_\varepsilon : UNil_\varepsilon(\mathbb{A}, \alpha) \to L_\varepsilon(\mathbb{A}_\alpha[t]).$$

Let H be a finitely presented subgroup of two finitely presented groups G_1 and G_{-1}. Assume G_1 and G_{-1} have orientation characters which agree on H, and let k be a subring of $\mathbb{Q}$. Suppose $R = kH$, $\mathcal{B}_i = k(G \setminus H)$. Then the map $\sigma_\varepsilon \circ \Psi$ provides an isomorphism:

$$\sigma_\varepsilon \circ \Psi : UNil^h_{2n}(R; \mathcal{B}_1, \mathcal{B}_{-1}) \cong L_\varepsilon(\mathbb{A}_\alpha[t]).$$

Proof:. The polynomial extension category, $\mathbb{A}_\alpha[t]$, defined by $(R; \mathcal{B}_1, \mathcal{B}_{-1})$, is shown in 2.10 to be linearizable. Also in this case $L_\varepsilon(\mathbb{A}) = 0$. Indeed: if (u, φ) is a nonsingular ε-quadratic form in $\mathbb{A}$ and $u = (P_1, P_{-1})$ then the inclusion $(P_1, 0) \to (P_1, P_{-1})$ is a Lagrangian for (u, φ). Therefore σ_ε and $\sigma_\varepsilon \circ \Psi$ are epimorphisms by

3.5. and 3.6.b. On the other hand, Cappell proves that the map ρ in 3.7 is a split monomorphism (see [C1]). So the commutativity of the square in 3.8 implies that $\sigma_\varepsilon \circ \Psi$ is also a monomorphism. This completes the proof.

3.10 Remarks on a projective version of *UNil*.

Let $\mathcal{P}_R$ denote the category of finitely generated projective right modules over a ring with involution R. For bimodules $\mathcal{B}_1, \mathcal{B}_{-1}$ with involution which are right R-free, (or projective) we define:

$$UNil^p_{2n}(R; \mathcal{B}_1, \mathcal{B}_{-1}) = UNil_\varepsilon(\mathbb{A}, \alpha)$$

where $\mathbb{A} = \mathcal{P}_R \times \mathcal{P}_R$; α and $*$ are exactly as in 2.4 and 3.3, and $\varepsilon = (-1)^n$.

For a direct product of two triples $(R; \mathcal{B}_1, \mathcal{B}_{-1})$ and $(R'; \mathcal{B}'_1, \mathcal{B}'_{-1})$, each consisting of a ring and two bimodules with involution, one easily obtains:

$UNil^p_{2n}(R \times R'; \mathcal{B}_1 \times \mathcal{B}'_1, \mathcal{B}_{-1} \times \mathcal{B}'_{-1}) = UNil^p_{2n}(R; \mathcal{B}_1, \mathcal{B}_{-1}) \oplus UNil^p_{2n}(R', \mathcal{B}'_1, \mathcal{B}'_{-1})$

But this projective version is only a technical convenience; one has:

3.11 Proposition. *The change-of-decoration map gives an isomorphism:*

$$UNil^h_{2n}(R; \mathcal{B}_1, \mathcal{B}_{-1}) \approx UNil^p_{2n}(R; \mathcal{B}_1, \mathcal{B}_{-1})$$

Proof. The Rothenberg-Ranicki exact sequence relating these two groups has, as its third term $\hat{H}^*(\mathbb{Z}/2\mathbb{Z}; \tilde{K}_0(R) \times \tilde{K}_0(R))$ where $\mathbb{Z}/2\mathbb{Z}$ acts by transposition. But this Tate cohomology group is zero.

4. Operations in *UNil* Groups

Our goal here is to analyze the groups $UNil^h_{2n}(R; R, R)$ for any ring with involution R. Following Farrell, [F2], we denote this group $UNil^h_{2n}(R)$. We show it admits a monoid of "restriction" endomorphisms $\{F_{2k+1}; k = 0, 1, 2 \dots\}$., and (at least when R is a group ring), a second monoid of "induction" endomorphims $\{V_{2k+1}; k = 0, 1, 2 \dots\}$. These are natural in R. We will use the restriction operators F_{2k+1} in Chapter 6 to make calculations. The reader may wish to compare the constructions here with analogous constructions for Nil groups in [CD].

The group $UNil^h_{2n}(R)$ merits special attention because it has the following universal property noticed by Farrell. Let H be a subgroup of groups G_1, G_{-1}, and $\Lambda = k(G_1 *_N G_{-1})$ for some commutative ring k. Set $R = kH$, $B_\varepsilon = k(G_\varepsilon \setminus H)$.

The map $\rho : UNil^h_{2n}(R; \mathcal{B}_1, \mathcal{B}_{-1}) \to L^h_{2n}(\Lambda)$ of Cappell [C1] factors through a map $\rho' : UNil^h_{2n}(\Lambda) \to L^h_{2n}(\Lambda)$.

4.1 Notation, and standing assumptions for this chapter:

$\mathbb{A}$: an additive category with involution $*$.

$\alpha : \mathbb{A} \to \mathbb{A}$; an additive covariant functor satisfying $\alpha * = * \alpha$, and $\alpha^2 = 1$.

$\sigma_\varepsilon : UNil_\varepsilon(\mathbb{A}, \alpha) \to L_\varepsilon(\mathbb{A}_\alpha[t])$: the map constructed in Chapter 3.

(The example to think of is $\mathbb{A} = \mathcal{F}_R \times \mathcal{F}_R$, when R is a ring with involution, $\alpha(P, Q) = (Q, P)$, and $(P, Q)^* = (Q^*, P^*)$ where $P^* = Hom_R(P, R)$,. By 3.5., $UNil_\varepsilon(\mathbb{A}, \alpha) \cong UNil^h_{2n}(R))$, if $\varepsilon = (-1)^n$).

There is exactly one involution on $\mathbb{A}_\alpha[t]$ (also denoted $*$) and one covariant functor $\alpha : \mathbb{A}_\alpha[t] \to \mathbb{A}_\alpha[t]$ extending $*$ and α on $\mathbb{A}$ and satisfying : $\alpha(t_u) = t_{\alpha(u)}$ for all u in $|\mathbb{A}|$. For any morphism in $\mathbb{A}_\alpha[t]$ we then have:

$$(\sum_{i=0}^{n} t^{(i)}\varphi_i)^* = \sum_{i=0}^{n} t^{(i)}\alpha^i(\varphi_i^*); \qquad \alpha(\sum_{i=0}^{\infty} t^{(i)}\varphi_i) = \sum_{i=0}^{\infty} t^{(i)}\alpha^i(\varphi_i).$$

4.2 Construction of $F_{2k+1} : UNil_\varepsilon(\mathbb{A}, \alpha) \to UNil_\varepsilon(\mathbb{A}, \alpha)$.

Let $k \geq 0$ be an integer. Let (u, φ) be any $\varepsilon - UNil$ form for $(\mathbb{A}, \alpha)$. So $\varphi = \varphi_0 + t\varphi_1$. The morphism $t\nu(\varphi) = N_\varepsilon(\varphi_\circ)^{-1}N_\varepsilon(t\varphi_1)$ is nilpotent in the ring $\mathbb{A}_\alpha[t](u, u)$. Write $\nu = \nu(\varphi) \in \mathbb{A}(u, \alpha u)$. Now, for any map $f \in \mathbb{A}(u, \alpha u)$ we shall write $f_{(k)}$ for $\alpha^{k-1}(f)\alpha^{k-2}(f)\ldots\alpha(f)f$; $f_0 = 1_u$. We get therefore: $(tf)^k = t^k f_{(k)}$. We then define:

$$F_{2k+1}(u, \varphi) = (u, \psi), \text{ where } \psi = \varphi_\circ + t\psi_1 \text{ and } t\psi_1 = (\nu_{(k)})^* \alpha^k(t\varphi_1)\nu_{(k)}.$$

4.3 Lemma. *(u, ψ) is an ε-$UNil$ form for $(\mathbb{A}, \alpha)$*

Proof: Let $D = N_\varepsilon(\varphi_0)$. Set $t\nu(\psi) = D^{-1}N_\varepsilon(t\psi_1)$. We only have to prove that ψ is nonsingular. This amounts to showing that $1 - t\nu(\psi)$ is an isomorphism. So we need to show $t\nu(\psi)$ is nilpotent. In fact we will show:

(4.4) $\nu(\psi) = (\nu(\varphi))_{(2k+1)}$; $\quad t^{(2k+1)}\nu(\psi) = (t\nu(\varphi))^{2k+1}$.

The second equation proves $t^{(2k+1)}\nu(\psi)$ is nilpotent and therefore $t\nu(\psi)$ is nilpotent (since $t^{(2)}$ is a central non-zero-divisor in the ring $\mathbb{A}_\alpha[t](u, u)$). The second equation is immediate from the first one (cf. 4.2), so we prove only the first equation of (4.4). Write $\nu = \nu(\varphi)$. We first note that:

$$t\alpha(D)\nu = Dt\nu = N_\varepsilon(t\varphi_1) = (N_\varepsilon(t\varphi_1))^T = (Dt\nu)^T = (t\nu)^* D^T = t\alpha(\nu^*)D.$$

Comparing first and last term, we see: $\alpha(D^{-1})\alpha(\nu^*) = \nu D^{-1}$; $D^{-1}\nu^* = \alpha(\nu)\alpha(D^{-1})$. Therefore $D^{-1}(\nu_{(k)})^* = \alpha^k(\nu_{(k)}D^{-1})$ and we compute: $t\nu(\psi)=$

$$D^{-1}N_\varepsilon(\nu^*_{(k)}\alpha^k(t\varphi_1)\nu_{(k)}) = D^{-1}(\nu_{(k)})^*\alpha^k(N_\varepsilon(t\varphi_1))\nu_{(k)} = D^{-1}(\nu_{(k)})^*\alpha^k(Dt\nu)\nu_{(k)} =$$

$$D^{-1}(\nu_{(k)})^*\alpha^k(D)t\nu_{(k+1)} = \alpha^k(\nu_{(k)}D^{-1}D)t\nu_{(k+1)} = t\nu_{(2k+1)}.$$

We conclude that $\nu(\psi) = \nu_{(2k+1)}$ as required. This proves 4.4 and therefore 4.3 .

It is easy to see that:

4.5 a) $F_{2k+1}((u,\varphi) \perp (u',\varphi')) = F_{2k+1}(u,\varphi) \perp F_{2k+1}(u',\varphi')$

b) If (y,i) is a Lagrangian for the ε- $UNil$ form (u,φ), then (y,i) is also a Lagrangian for $F_{2k+1}(u,\varphi)$.

It follows that the rule $(u,\varphi) \to F_{2k+1}(u,\varphi)$ defines a homomorphism $F_{2k+1} : UNil_\varepsilon(\mathbb{A},\alpha) \longrightarrow UNil_\varepsilon(\mathbb{A},\alpha)$.

4.6 Lemma. : *For any $x \in UNil_\varepsilon(\mathbb{A},\alpha)$, there is an integer $N > 0$ such that $F_a x = 0$ for every $a \geq N$.*

Proof. : If $x = [u,\varphi]$, then $t\nu(\varphi)$ is nilpotent of exponent n, for some n. Therefore $\nu_{(k)} = 0$ if $k \geq n$. Therefore $F_a x = 0$ if $a \geq 2n+1$.

4.7 Lemma. : *$F_aF_b = F_{ab}$ for all odd integers $a,b > 0$; $F_1 =$ identity .*

Proof. Let $a = 2i+1, b = 2j+1, ab = 2k+1, k = bi+j$. Take an ε-$UNil$ form (u,φ); abbreviate $\nu(\varphi)$ to ν. We get $F_aF_b(u,\varphi) = F_a(u,\psi)$
where $\psi = \varphi_0 + t\psi_1$ and, $t\psi_1 = (\nu_{(j)})^*\alpha^j(t\varphi_1)\nu_j$ and $\nu(\psi) = \nu_{(b)}$ (by 4.4). So $F_aF_b(u,\varphi) = (u,\varphi_0 + t\chi)$ where

$t\chi = ((\nu_{(b)})_i)^*\alpha^i\{(\nu_{(j)})^*\alpha^j(t\varphi_1)(\nu_{(j)})\}((\nu_{(b)})_{(i)}) = (\nu_{(k)})^*\alpha^k(t\varphi_1)\nu_{(k)}$.

Therefore, $F_{2k+1}(u,\varphi) = (u,\varphi_0 + t\chi)$. So $F_aF_b = F_{ab}$.

The second statement of 4.7 is obvious. This completes the proof.

4.8 Construction of the operators $V_{2k+1} : UNil^h_{2n}(kG) \to UNil^h_{2n}(kG)$:

Let $a \geq 1$ be an odd integer. Define a covariant additive functor $V_a : \mathbb{A}_\alpha[t] \to \mathbb{A}_\alpha[t]$, sending each object to itself by the rule:

$$V_a(\sum_{i=0}^{n} t^{(i)}\varphi_i) = \sum_{i=0}^{n} t^{(ai)}\varphi_i$$

This is a functor because $\alpha^2 = 1$ and $a \equiv 1 \bmod 2$. V_a clearly commutes with $*$ and therefore yields an endomorphism of $L_\varepsilon(\mathbb{A}_\alpha[t])$, also written V_a. It is clear that $V_a V_b = V_{ab}$ and V_1 = identity. Now suppose G is a finitely presented group with orientation character, and k is a subring of $\mathbb{Q}$. The isomorphism $\sigma_\varepsilon \Psi : UNil^h_{2n}(kG) \to L_\varepsilon(\mathbb{A}_\alpha[t])$ of 3.9 allows one to transport this monoid of endomorphisms to a monoid of endomorphisms $\{V_1, V_3, V_5, \dots\}$ on $UNil^h_{2n}(kG)$.

4.9 Summary Theorem. *For each ring with involution, R, and each positve odd integer a, we have defined an endomorphism F_a of $UNil^h_{2n}(R)$. Moreover F_1 = identity ; $F_a F_b = F_{ab}$. These endomorphisms are natural for maps of rings with involution. Moreover, if $x \in UNil^h_{2n}(R)$, then $F_a(x) = 0$ for all but finitely many integers a. In case k is a subring of $\mathbb{Q}$, and G is a finitely presented group with orientation character, then we have also constructed endomorphisms V_a of $UNil^h_{2n}(kG)$ for each odd integer $a > 0$. Moreover V_1 = identity; $V_a V_b = V_{ab}$. These are natural in k and in G.*

4.10 Remark on Morita Theory.

Our goal here is to establish the isomorphism:

$$UNil^p_{2n}(R) \approx UNil^p_{2n}(M_k(R))$$

for any ring with involution R and any n and any k.

For a ring R, write Λ for $M_k(R)$, and V_r for the projective right Λ-module of $1 \times k$ row vectors from R. Set $V_l = Hom_\Lambda(V_r, \Lambda)$, a projective left Λ-module. V_r and V_l are left and right R-modules respectively. We can, and shall, identify V_l with the space of $k \times 1$ column vectors from R, thereby obtaining an obvious isomorphism, $V_r^t \cong V_l$ of (Λ, R) modules (namely, the isomorphism sends $(r_1, r_2, \dots r_k)$ to $(\bar{r}_1, \bar{r}_2, \dots \bar{r}_k)^T$). Morita theory (see Reiner [Re]) provides an equivalence of categories:

$\mathcal{P}_R \xrightarrow{m} \mathcal{P}_\Lambda \quad (P \mapsto P\Lambda_r = P \otimes_R \Lambda_r \ ; f \mapsto f \otimes 1)$

Now if R has an involution, then Λ inherits that involution $((a_{ij}) \mapsto (\bar{a}_{ji}))$ and the duality functors $*_R$ and $*_\Lambda$ on $\mathcal{P}_R$ and $\mathcal{P}_\Lambda$ (see 1.3) respect the equivalence m, in the sense that there is a natural equivalence of functors

$$n : m \circ *_R \xrightarrow{\approx} *_\Lambda \circ m$$

More precisely, for P in $|\mathcal{P}_R|$, $n_p : P^* V_r \to (PV_r)^*$ is the composite of the following three obvious isomorphisms: $P^* V_r \approx (V_l Hom_R(P, R))^t \approx Hom_R(P, V_l)^t = (Hom_R(P, Hom_\Lambda(V_r, \Lambda)))^t \approx Hom_\Lambda(PV_r, \Lambda)^t = (PV_r)^*$.

Let $\mathbb{A}_R = \mathcal{P}_R \times \mathcal{P}_R, \mathbb{A}_\Lambda = \mathcal{P}_\Lambda \times \mathcal{P}_\Lambda$ and let α_R (resp. α_Λ) be the automorphism of $\mathbb{A}_R$ (resp. $\mathbb{A}_\Lambda$) that sends a morphism (f,g) to (g,f), as discussed in 4.1. Set $\mu = m \times m : \mathbb{A}_R \to \mathbb{A}_\Lambda$. Clearly $\mu\alpha_R = \alpha_\Lambda\mu$ so μ extends to an equivalence of categories:

$$\mu : (\mathbb{A}_R)_{\alpha_R}[t] \to (\mathbb{A}_\Lambda)_{\alpha_\Lambda}[t]$$

If $*$ stands for the involution on either of these polynomial categories, then we get a natural equivalence of functors:

$$\nu : \mu \circ * \xrightarrow{\approx} * \circ \mu.$$

For any object $u = (P,Q)$ of $|\mathbb{A}_R|$, ν_u is the isomorphism

$$(n_Q, n_P) : (Q^* V_r, P^* V_r) \to ((QV_r)^*, (PV_r)^*)$$

If $(u, [\psi])$ is an ε-quadratic form in $(\mathbb{A}_R)_{\alpha_R}[t]$ then $(u', \phi') = (u(u), [\nu\mu(\phi)])$ is an ε-quadratic form in $(\mathbb{A}_\Lambda)_{\alpha_\Lambda}[t]$ and this process gives an isomorphism of graded groups (see 3.2):

$$Q_\varepsilon(u) \quad \approx \quad Q_\varepsilon(u')$$

It follows easily that this rule $(u, \varphi) \to (u', \varphi')$ yields an isomorphism:

$$UNil^p_{2n}(R) \approx UNil^p_{2n}(M_k(R))$$

for any n, R, k.

§5. An Arf Invariant for *UNil*

We specialize now to the case of a field F of characteristic two, with trivial involution. Our goal is to construct a kind of Arf invariant:

$$A : UNil^h_0(F) \longrightarrow coker(\psi_2 - 1)$$

where $\psi_2 : F[t] \to F[t]$ is the Frobenius homomorphism. In Chapter 6, we use A to give a complete set of invariants for $UNil^h_0(F)$. A will be defined as the composite of three homomorphisms: $UNil^h_0(F) \xrightarrow{\gamma} L^p_0(P) \xrightarrow{\beta} L^h_0(F[t]) \xrightarrow{\alpha} coker(\psi_2 - 1)$.

5.1 Notation and standing assumptions for Chapter 5.

F: a field of characteristic two with trivial involution;

$F[t]$: its polynomial ring, also with trivial involution;

$\psi_2 : F[t] \to F[t]$: the Frobenius homomorphism: $\psi_2(p) = p^2, \forall p \in F[t]$;

$coker(\psi_2 - 1) := F[t]/(\psi_2 - 1)F[t]$

$a : F \times F \to F$: the automorphism, $a(x,y) = (y,x)$.

$P := (F \times F)_a[t]$, the twisted polynomial ring, as defined in 2.2.a; its involution is: $((x,y)t^i)^- = t^i(y,x) = a^i(y,x)t^i$ for all $(x,y) \in F \times F$ and all $i \geq 0$;

$j : F[t] \to P$: the monomorphism given by $j(xt^i) = (x,x)t^i$, $\forall x \in F$, $\forall i \geq 0$;

$b : P \to F[t]$: the additive map given by $b((x,y)t^i) = (x+y)t^i$, $\forall (x,y) \in F \times F, \forall i \geq 0$.

We note:

5.2 Lemma.

a. *The map j makes P into an $F[t]$ bimodule; it is free of rank 2 as a left or right module.*

b. *The map $P \times P \longrightarrow F[t]$ sending (u,v) to $b(\bar{u}v)$ is a nonsingular, symmetric, $F[t]$-bilinear form on P.*

c. $b(u + \bar{u}) = 0, \forall u \in P$.

5.3 Definition of $\gamma : UNil_0^h(F) \longrightarrow L_0^p(P)$.

If (P_1, P_{-1}) is an object of $\mathcal{F}_F \times \mathcal{F}_F$ then $(P_1 \times P_{-1}) \otimes_{F \times F} (F \times F)_a[t]$ is a projective P-module. This defines a functor $(\mathcal{F}_F \times \mathcal{F}_F)_\alpha[t] \to \mathcal{P}_P$ and a homomorphism

$$L_\varepsilon((\mathcal{F}_F \times \mathcal{F}_F)_\alpha[t]) \xrightarrow{\gamma_1} L_{2n}^p(P),\ \varepsilon = (-1)^n.$$

We define $\gamma = \gamma_1 \circ \sigma_\varepsilon \circ \Psi$, where $\Psi : UNil_{2n}^h(F) \xrightarrow{\approx} UNil_\varepsilon(\mathcal{F}_F \times \mathcal{F}_F, \alpha)$ and $\sigma_\varepsilon : UNil_\varepsilon(\mathcal{F}_F \times \mathcal{F}_F, \alpha) \twoheadrightarrow L_\varepsilon((\mathcal{F}_F \times \mathcal{F}_F)_\alpha[t])$ are as constructed in 3.5 and 3.2.

5.4 Definition of $\beta : L_0^p(P) \longrightarrow L_0^h(F[t])$.

Let (H, λ, μ) be a quadratic form representing an element $[H, \lambda, \mu]$ of $L_0^p(P)$. Define a quadratic form (H', λ', μ') over $F[t]$ as follows:

$H' = j^*H$. H' is finitely generated and free over $F[t]$ since H is finitely generated projective over P, and $F[t]$ is a principal ideal domain. Here we use 5.2.a.

$\lambda' = b \circ \lambda : H' \times H' \to F[t]$. λ' is nonsingular, symmetric and $F[t]$ bilinear, by 5.2.b.

$\mu' = b \circ \mu : H' \to F[t]$. μ' is a well defined function because of 5.2.c.

A projective Lagrangian for (H, λ, μ) is also a free Lagrangian for (H', λ', μ'). Moreover the correspondence $(H, \lambda, \mu) \mapsto (H', \lambda', \mu')$ preserves orthogonal direct sums. Hence we obtain a homomorphism, $L_0^p(P) \xrightarrow{\beta} L_0^h(F[t])$, sending $[H, \lambda, \mu]$ to $[H', \lambda', \mu']$ as defined above.

The definition of $\alpha : L_0^h(F[t]) \longrightarrow coker(\psi_2 - 1)$ will require a lemma:

Let $F(t)$ be the field of fractions of $F[t]$. Let

$$Arf : L_0^h(F(t)) \to F(t)/(\psi_2 - 1)F(t)$$

denote the classical Arf invariant (see [Mh]). Let $k : F[t] \to F(t)$ be the inclusion; k induces a map $\kappa : F[t]/(\psi_2 - 1)F[t] \to F(t)/(\psi_2 - 1)F(t)$.

5.5 Lemma.

(1) *κ is a monomorphism*

(2) $Arf \circ k_*(L_0^h(F[t])) \subset Im(\kappa)$.

Proof of 1. Since $F[t]$ is a principal ideal domain, it is integrally closed. Therefore, for $x \in F(t), (\psi_2 - 1)x = x^2 - x$ is in $F[t]$ if and only if $x \in F[t]$.

This proves κ is a monomorphism.

Proof of 2. Let $[H, \lambda, \mu] \in L_0^h(F[t])$. Since submodules of H are all free, it is routine to see that the symplectic form (H, λ) over $F[t]$, has a symplectic basis $\{e_1 \ldots e_r, f_1 \ldots f_r\}$ over $F[t]$. Therefore $Arf\ k_*[H, \lambda, \mu]$ can be computed as:

$$\sum \mu(e_i)\mu(f_i) mod(\psi_2 - 1)F(t).$$

Hence but $\mu(e_i)\mu(f_i)$ is in $F[t]$. Hence $Arf\ k_*[H, \Lambda, \mu] \in Im(\kappa)$.

5.6 Definition of α. *We define* $\alpha : L_0^h(F[t]) \to coker(\psi_2 - 1)$ *as:*

$$\alpha = \kappa^{-1} \circ Arf \circ k_*$$

The augmentation map, $\varepsilon_\# : F[t] \longrightarrow F$ sending a polynomial $\sum a_i t^i$ to a_0, induces an epimorphism, $\varepsilon : coker(\psi_2 - 1) \longrightarrow F/(\psi_2 - 1)F$.

5.7 Proposition. *Let F be a perfect field of characteristic 2. Then*

$$UNil_0^h(F) \xrightarrow{A} coker(\psi_2 - 1) \xrightarrow{\varepsilon} F/(\psi_2 - 1)F \to 0$$

is exact.

5.8 Remark: a) When F is perfect, it is easy to see that the inclusion $\sum_{j=0}^{\infty} Ft^{2j+1} \longrightarrow F[t]$ induces a short exact sequence:

$0 \to \sum_{j=0}^{\infty} Ft^{2j+1} \to coker(\psi_2 - 1) \xrightarrow{\varepsilon} F/(\psi_2 - 1)F \to 0.$

b) We conjecture that A is a monomorphism.

Proof of 5.7. First we prove that $\varepsilon \circ A = 0$.

Let $\alpha' : L_0^h(F) \longrightarrow F/(\psi_2 - 1)F$ denote the classical Arf invariant (see e.g. Milnor's book, [Mh]). Let $\beta' : L_0^p(F \times F, -) \longrightarrow L_0^h(F)$ denote the transfer. (Here $-$ denotes the involution on $F \times F$). β' is defined exactly as β was defined in 5.4, above. The following diagram obviously commutes:

$$\begin{array}{ccc} L_0^p((F \times F)_a[t]) & \xrightarrow{\eta_*} & L_0^p(F \times F, -) \\ \downarrow \beta & & \downarrow \beta' \\ L_0^h(F[t]) & \xrightarrow{\varepsilon_*} & L_0^h(F) \\ \downarrow \alpha & & \downarrow \alpha' \\ F[t]/(\psi_2 - 1)F[t] & \xrightarrow{\varepsilon} & F/(\psi_2 - 1)F \end{array}$$

But $UNil_0(F) \xrightarrow{\gamma} L_0^p((F \times F)_\alpha[t]) \xrightarrow{\eta_*} L_0^p(F \times F, -)$ is the zero map by 3.6.b. Therefore $\varepsilon A = \varepsilon\alpha\beta\gamma = 0$.

Next we must prove $ker\ \varepsilon \subset ImA$. Now $im(A) = \alpha\beta(image(\gamma))$. We note $im(\gamma) = im(\gamma_1) \supset im(\gamma')$, where γ' is the "change-of-decoration" map:

$$L_0^h(P) = L_0((\mathcal{F}_{F\times F})_\alpha[t]) \xrightarrow{\gamma'} L_0((\mathcal{P}_P)) = L_0^p(P).$$

So it is enough to show $\alpha\beta(im(\gamma')) \supset Ker(\varepsilon)$. In light of Remark 5.8.a, it is enough to exhibit an element $y_{a,j} \in L_0^h(P)$ for each $a \in F$, and each j ≥ 0, such that $\alpha\beta\gamma'(y_{a,j}) = [at^{2j+1}] \in F[t]/(\psi_2 - 1)F[t]$. We now construct $y_{a,j}$.

Let H be the free right P module on one generator e. Define $\lambda : H \times H \to P$ by the rule: $\lambda(ep, ep') = \bar{p}p'$, for all p, p' in P. Define $S(P) := \{p+\bar{p},\ p \in P\}$, and define $\mu : H \longrightarrow P/S(P)$ by the rule: $\mu(ep) = \bar{p}((0,1) + (a,a)t^{2j+1})p$ for all p in P. One checks that (H, λ, μ) is a nonsingular quadratic form. Set $y_{a,j} = [H, \lambda, \mu] \in L_0^h(P)$. One computes $\beta\gamma'(y_{a,j}) = [H', \lambda', \mu']$ where $H' = j^*H$ is a free $F[t]$ module on a basis: $e_1 = e(1,0)$, $f_1 = e(0,1)$. μ' satisfies $\mu'(e_1) = at^{2j+1} = \mu'(f_1)$. Moreover λ' has a symplectic base given by $\{e_1, f_1\}$. So $\alpha[H', \lambda', \mu'] = [(at^{2j+1})^2] mod\, Im(\psi_2 - 1)$, which gives: $\alpha\beta\gamma'(y_{a,j}) = [at^{2j+1}] mod\, Im(\psi_2 - 1)$, as required. This completes the proof of 5.7.

5.9 Let R be a ring with involution with the property that $F = R/2R$ is a perfect field with trivial involution. The natural epimorphism $R \to F$ induces a homomorphism $r_2 : UNil_{2n}^h(R) \to UNil_{2n}^h(F)$. Proposition 5.7 together with Remark 5.8.a provide an epimorphism: $\bar{A} : UNil_{2n}^h(F) \longrightarrow \sum_{j=0}^{\infty} Ft^{2j+1}$. Let $p_0 : \sum_{j=0}^{\infty} Ft^{2j+1} \longrightarrow F$ be the map sending $\sum a_j t^{2j+1}$ to a_0. We will write

$$A_R : UNil_{2n}^h(R) \longrightarrow F$$

for the composition $p_0 Ar_2$.We will use A_R in the next chapter.

6. Computations of $UNil$ groups

The goal of this chapter is to prove the following two results.

6.1. Theorem. *Let R be a division ring with involution.*

(1) *If the characteristic of R is $\neq 2$, then $UNil^h_{2n}(R) = 0$.*

(2) *If char $(R) = 2$, the involution is nontrivial, R has finite dimension over its center, and its center is a perfect field, then $UNil^h_{2n}(R) = 0$.*

(3) *If R is a perfect field of characteristic 2, with trivial involution, then*

$$\sum_{k=0}^{\infty} A_R \circ F_{2k+1} : UNil^h_{2n}(R) \longrightarrow \sum_{k=0}^{\infty} R$$

is an isomorphism.

Here F_{2k+1} is the operation defined in 4.2 and $A_R : UNil^h_{2n}(R) \to R$ is the part of the generalized Arf invariant defined in 5.8.

The reader might profitably compare this result with the methods of Cappell in [C3] where he uses a sequence of Arf invariants to show that $UNil^h_2(Z)$ must be infinitely generated. It is gratifying to see that, by 6.1 and 6.2, the Arf invariant A_R tells the whole story in this, and other cases.

6.2 Theorem. *Let R be a Dedekind domain of characteristic $\neq 2$ with involution. Assume $R/2R$ is a perfect ring (that is to say, ψ_2 is an isomorphism).*

(1) *If the involution is nontrivial, or n is even, then $UNil^h_{2n}(R) = 0$.*

(2) *If the involution is trivial, and $R/2R$ is a field, then the map*

$$r_2 : UNil^h_2(R) \longrightarrow UNil^h_2(R/2R)$$

is an isomorphism.

Remark: It is easy to see that the calculation of $UNil^h_{2n}(R)$ for any semisimple ring R , reduces to the calculation of $UNil^h_{2n}(D)$ where D is a division ring. To prove this, use 3.10, 3.11, and 6.10 (when $R = S \times S^{op}$) to reduce the calculation to that of a simple ring. Then use 4.10, to reduce from $M_q(D)$ to D. Here one must use the well known fact that all involutions on $M_q(D)$ come, up to conjugacy, from involutions on D. (For a proof of this see [A], Th. 12 p.156, and Th.11, p.154).

6.3 Notation and conventions for Chapter 6.

a. Throughout, R denotes a ring with involution. Set $H_\varepsilon(R) = S^\varepsilon(R)/S_\varepsilon(R)$; we view it as a right R module, where the right action is given by the rule: $[x]r = [\bar{r}xr], \forall x \in S^\varepsilon(R), \forall r \in R$.

b. We denote an ε-$UNil$ form over R as $z = (P_1, P_{-1}, p_1, p_{-1}, \mu_1, \mu_{-1})$. Here we write $p_\delta : P_\delta \to P_{-\delta}$ in place of the Hermitian form $\lambda_\delta : P_\delta \times P_\delta \to R$ in view of the fact that $P_{-1} = P_1^*$. Therefore:

$$\lambda_1(x, y) =< x, p_1(y) > \forall x, y \in P_1;\ \ \lambda_{-1}(x, y) =< p_{-1}x, y > \forall x, y \in P_{-1}.$$

c. For any word w in letters p_i, $i = \pm 1$, we write w^t for the same word written backwards. For $k > 0$, let w_k (resp. w_{-k}) denote the alternating k-letter word in p_1 and p_{-1}, beginning in p_{-1} (resp. p_1). w_0 will be understood to mean an identity map. In this formalism, $F_{2k+1}[z]$ is represented by $(P_1, P_{-1}, p'_1, p'_{-1}, \mu'_1, \mu'_{-1})$ where, for $\delta = (-1)^k$,

$$p'_1 = w^t_{\delta k} p_\delta w_{\delta k};\ \ p'_{-1} = w^t_{-\delta k} p_{-\delta} w_{-\delta k},$$

$$\mu'_1 = \mu_\delta \circ w_{\delta k};\ \ \mu'_{-1} = \mu_{-\delta} \circ w_{-\delta k}.$$

d. The *length* of z is defined by declaring length$(z) < k$ iff both w_k and w_{-k} are the zero maps. The length of the zero form $(0, 0, 0, 0, 0, 0)$ is -1. It is the only form whose length is negative.

e. Assume $Char(R) = 2$. Suppose z has length k; we say z is *reduced* if the two maps $\mu_1|image(w_k)$ and $\mu_{-1}|image(w_{-k})$ are injective. These maps are then R-maps to $H_\varepsilon(R)$ since the Hermitian forms $\lambda_{\pm 1}$ vanish on the modules $image(w_{\pm k})$.

f. A *sub-Lagrangian* for z is a pair (V_1, V_{-1}) of submodules $V_i \subset P_i$ for which $< V_1, V_{-1} >= 0$, P_i/V_i is free, $p_i(V_i) \subset V_{-i}$, and $\mu_i(V_i) = 0$ for $i = \pm 1$. The *"sub-Lagrangian construction"* is a second $UNil$ form $z' = S(z; V_1, V_{-1}) = (\tilde{P}_1, \tilde{P}_{-1}, \tilde{p}_1, \tilde{p}_{-1}, \tilde{\mu}_1, \tilde{\mu}_{-1})$ for which $[z] = [z']$. It is defined as follows: $\tilde{P}_i = V_{-i}^\perp / V_i$. The pairing $<,>$: $P_1 \times P_{-1} \to R$ induces a non-singular sesquilinear pairing $<,>$: $\tilde{P}_1 \times \tilde{P}_{-1} \to R$; the functions p_i, μ_i on P_i induce corresponding functions $\tilde{p}_i, \tilde{\mu}_i$ on $\tilde{P}_i$. One easily sees (as usual) that $z' - z$ has a Lagrangian isomorphic to $(V_{-1}^\perp, V_1^\perp)$.

We make heavy use of the following lemma.

6.4 Lemma:. Let R be a division ring with involution. Every ε-$UNil$ form z over R is equivalent to a reduced form, z', for which length(z') $\leq$length(z).

Proof. : Let $z = (P_1, P_{-1}, p_1, p_{-1}, \mu_1, \mu_{-1})$ have length $k \geq 0$. The modules $V_1 = ker\,(\mu_1|image(w_k)), V_{-1} = ker\,(\mu_{-1}|image(w_{-k}))$, form a sub-Lagrangian for z. The form $z' = S(z; V_1, V_{-1}) = (P'_1, P'_{-1}, p'_1, p'_{-1}, \mu'_1, \mu'_{-1})$ has length $\leq k$. In z', the maps $\mu'_1|image(w_k)$, $\mu'_{-1}|image(w_{-k})$ are injective. So either z' is reduced or its length is $< k$. The result follows at once.

6.5 Proof of 6.1(1) and 6.1(2). : We show that every reduced form is zero. Then Lemma 6.4 completes the proof. Let $z = (P_1, P_{-1}, p_1, p_{-1}, \mu_1, \mu_{-1})$ have length $k \geq 0$. It is enough to show that z cannot be a reduced form.

Proof of 6.1(1): The homomorphisms $\mu_1|image(w_k)$, $\mu_{-1}|image(w_{-k})$ take values in $H_\varepsilon(R)$ which is 0, since 2 is a unit in R. If z were reduced, this would imply that both w_k and w_{-k} were zero, contradicting the length hypothesis.

Proof of 6.1(2): Let F denote the center of R, and let F_0 denote the subfield of the center of F which is fixed under the involution. Then $[F : F_0] \leq 2$. Let $d = [R : F_0]$. The Frobenius map $\psi_2 : F_0 \to F_0$ is an isomorphism since F is perfect. Therefore $dim_{F_0}(H_\varepsilon(R)) = dim_{F_0}(\psi_2^{-1})^* H_\varepsilon(R) = dim_{F_0}(S^\varepsilon(R)) - dim_{F_0}(S_\varepsilon(R)) \leq d-2$ since the involution on R is nontrivial. But one, at least, of the R modules $image(w_{\pm k})$ is nonzero and therefore has F_0-dimension $\geq d$. Therefore either $\mu_1|image(w_k)$ or $\mu_{-1}|image(w_{-k})$ is not injective. We again conclude that z is not reduced. This proves 6.1(2).

We begin to prove 6.1.(3) by constructing reduced $UNil$ forms of length $2k+1$ for each nonegative integer k, provided that R is suitable.

6.6 Construction. Let $\varepsilon = (-1)^n$. Suppose R is a ring with involution. For any element $a \in H_\varepsilon(R)$ and any $k \geq 0$, we construct an ε-$UNil$ form $z_{2k,a}$ over R with the following properties:

(1) length($z_{2k,a}$) $= 2k$. If R has characteristic 2, and the involution is trivial, and a is a unit in $R = H_\varepsilon(R)$, then $z_{2k,a}$ is reduced.

(2) In $UNil^h_{2n}(R)$, $F_{2k+1}([z_{2k,a}]) = [z_{0,a}]$.

(3) In $UNil^h_{2n}(R)$, $F_{2j+1}([z_{2k,a}]) = 0$ if $j > k$.

(4) For any map of rings, $f : R \to R'$, $f_*(z_{2k,a}) = (z_{2k,f(a)})$.

(5) If R is a perfect field of characteristic 2, with trivial involution, then

$$A_R \circ F_{2k+1}([z_{2k,a}]) = a.$$

Here is the construction of $z_{2k,a} = (P_1, P_{-1}, p_1, p_{-1}, \mu_1, \mu_{-1})$:

P_1 has base $e_1, \ldots, e_{2k+1}$; P_{-1}, its dual, has the dual base $f_1, \ldots, f_{2k+1}$

$$\begin{array}{llll} p_1(e_{2i}) = f_{2i-1}; & p_1(e_{2i-1}) = \varepsilon f_{2i} & for\ 1 \leq i \leq k; & p_1(e_{2k+1}) = 0 \\ p_{-1}(f_{2i}) = \varepsilon e_{2i+1}; & p_{-1}(f_{2i+1}) = e_{2i} & for\ 1 \leq i \leq k; & p_{-1}(f_1) = 0 \end{array}$$

$$\begin{array}{ll} \mu_1(e_i) = 0 for\ i \leq 2k; & \mu_1(e_{2k+1}) = a \\ \mu_{-1}(f_i) = 0 for\ i \geq 2; & \mu_{-1}(f_1) = a \end{array}$$

Note that $image(w_{2k}) =< e_{2k+1} >$; $image(w_{-2k}) =< f_1 >$. Property (1) above follows at once. To prove (2), note that $F_{2k+1}(z_{2k,a}) = (P_1, P_{-1}, 0, 0, \mu'_1, \mu'_{-1})$ where $\mu'_1(e_i) = 0 = \mu'_{-1}(f_i)\ \forall i \neq k+1$ and $\mu'_1(e_{k+1}) = \mu'_{-1}(f_{k+1}) = a$. The submodules

$$V_1 =< e_1 \ldots e_k >, V_{-1} =< f_{k+2} \ldots f_{2k+1} >$$

form a sub-Lagrangian for which $S(F_{2k+1}(z_{2k,a}), V_1, V_{-1}) = z_{0,a}$. This proves (2).

To prove(3), note that for $j > k$, a Lagrangian for $F_{2j+1}(z_{2k,a})$ is $(V_1, V_{-1}) =$

$$(< e_1 \ldots e_{j+1} >, < f_{j+2} \ldots f_{2k+1} >) \text{or } (< e_{j+2} \ldots e_{2k+1} >, < f_1 \ldots f_{j+1} >)$$

depending on the parity of k.

Property (4) is obvious.

Now we prove (5). Let γ be the map defined in 5.3. For any form $z_{0,a}$ of 6.6 one has $\gamma[z_{0,a}] = [P, \lambda, \mu]$ where $P = (R \times R)_a[t]$, $\lambda(p, p') = \bar{p}p'$ and $\mu(p) = \bar{p}((0,1) + (a,a)t)p$. So $\beta \circ \gamma[z_{0,a}] = [P', \lambda', \mu']$ where $P' = j^*P$. (P', λ') has a $R[t]$ symplectic base e, f : $e = (1,0), f = (0,1)$, and $\mu'(e) = at$, $\mu'(f) = at$. Therefore $A([z_{0,a}]) = [a^2t^2]$, and $A_R([z_{0,a}]) = a$. This completes the construction.

6.7 Lemma. *Let F be a perfect field of characteristic 2 with trivial involution. Suppose $z \neq 0$ is a reduced UNil form over F. Then z has length $2k \geq 0$, for some k, and $F_{2k+1}([z]) = [z_{0,a}]$ for some $a \neq 0$ in F. In particular, $F_{2k+1}([z]) \neq 0$*

Proof. Let $z = (P_1, P_{-1}, p_1, p_{-1}, \mu_1, \mu_{-1})$. Let q = length(z). First assume that q is odd. Then $w_{\pm q} = w^t_{\pm q}$, and the bilinear forms λ defined by $Ad\,\lambda = w_{\pm q}$ are symplectic. Therefore w_q, w_{-q} have even rank, and at least one of them is nonzero. Hence the ψ_2 semi-linear homomorphisms $\mu_{\pm 1} : image(w_{\pm q}) \to F$ can be injective only if $dim_F(image(w_{\pm q})) = 0$. So q cannot be odd. Therefore $length(z) = 2k \geq 0$ for some k. Now $w_{-2k} = w^t_{2k}$, and $\mu_{\pm 1} : image(w_{\pm 2k}) \to F$ are both monomorphisms. Hence w_{2k} and w_{-2k} both have rank one. Let $\delta = (-1)^k$. Note $F_{2k+1}(z) = (P_1, P_{-1}, 0, 0, \mu'_1, \mu'_{-1})$ since all $2k+1$ letter words vanish on z.

Set $V_1 = im(w_k) \cap ker(w_{\delta k})$, $V_{-1} = (im(w_k))^{\perp} = ker(w_k^t) = ker(w_{-\delta k})$. Evidently (V_1, V_{-1}) is a sub-Lagrangian for $F_{2k+1}(z)$. We now show $S(z; V_1, V_{-1}) = (\tilde{P}_1, \tilde{P}_{-1}, 0, 0, \tilde{\mu}_1, \tilde{\mu}_{-1})$ is reduced of length zero. For, $\tilde{\mu}_1$ and $\tilde{\mu}_{-1}$ are both composites of two isomorphisms as follows:

$$V_{-1}^{\perp}/V_1 = im(w_k)/(im(w_k) \cap ker(w_{\delta k})) \xrightarrow{w_{\delta k}} im(w_{\delta k} w_k) = im(w_{2k}) \xrightarrow{\mu_1} F$$

$$V_1^{\perp}/V_{-1} = (im(w_{-k}) + ker(w_{-\delta k}))/ker(w_{-\delta k}) \xrightarrow{w_{-\delta k}} im(w_{-\delta k} w_{-k}) = im(w_{-2k}) \xrightarrow{\mu_{-1}} F$$

Therefore $S(z; V_1, V_{-1})$ is reduced of length 0. Choose $e_1 \in \tilde{P}_1$ so that $\tilde{\mu}_1(e_1) = 1$. Let $f_1 \in \tilde{P}_{-1}$ be the element such that $< e_1, f_1 >= 1$. Then $\mu_{-1}(f_1) = a^4$ for some $a \neq 0$. Set $e = e_1 a, f = f_1 a^{-1}$. With this base one sees that $S(z; V_1, V_{-1}) = z_{0,a^2}$. The fact that $F_{2k+1}([z]) \neq 0$ is now immediate from 6.6(5). This completes the proof of 6.7.

6.8 Proof of 6.1.(3). By 6.7 and 6.6(5), $\sum_{k=0}^{\infty} A_R \circ F_{2k+1}$ is injective. By 6.6. (2), (3), (5), for each $k \geq 0$ and each $a \in F$ there is an element $x \in UNil_{2n}^h(R)$ for which

$$A_R \circ F_{2k+1}(x) = a, \text{ and } A_R \circ F_{2j+1}(x) = 0 \ \forall j > k .$$

This proves that $\sum_{k=0}^{\infty} A_R \circ F_{2k+1}$ is an epimorphism, and ends the proof of 6.1.(3).

To prove 6.2 (1), we need two lemmas:

6.9 Lemma. *Let R be a Dedekind domain with involution. If $char(R) = 2$, assume that R is a field. Assume further that $R/2R$ is a perfect ring (i.e., ψ_2 is an isomorphism).*

$$\text{Then } H_\varepsilon(R) = \begin{cases} 0 \text{ if the involution is nontrivial or if, in R, } \varepsilon \neq -1, \\ R/2R \text{ if the involution is trivial and, in R, } \varepsilon = -1. \end{cases}$$

Proof. If the involution is trivial it is routine to calculate that $S^\varepsilon(R) = R$ when $\varepsilon = -1$ in R, and is zero otherwise. One then obtains the calculation without trouble in this case.

So suppose that the involution is nontrivial. Let R_0 be the fixed subring. Then $rank_{R_0}(S^\varepsilon(R)) = 1$ because the involution is nontrivial. Therefore for any prime ideal $\mathcal{P}_0$ of R_0 containing $2R_0$, $dim_{R_0/\mathcal{P}_0}(S^\varepsilon(R)/\mathcal{P}_0 S^\varepsilon(R)) = 1$ also.

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots \mathcal{P}_r$ be the primes of R_0 containing $2R_0$; let $F_i = R_0/\mathcal{P}_i$, $H_i = H_\varepsilon(R)/H_\varepsilon(R)P_i$. Since $2H_\varepsilon(R) = 0$, we get $H_\varepsilon(R) = H_1 \times H_2 \times \ldots H_r$. The proof is achieved by establishing two claims:

Claim 1: Suppose $\mathcal{P}_i R$ is prime. Then $H_i = 0$.

Claim 2: Suppose $\mathcal{P}_i R$ is not prime. Then $H_i = 0$.

Proof of Claim 1: Let $F = R/\mathcal{P}_i R$. Then $[F : F_i] = 2$, so $dim_{F_i} H_i = 2dim_F(H_i)$, an even integer. However $H_\varepsilon(R) = \psi_2^*(S^\varepsilon(R)/S_\varepsilon(R))$, as an $R/2R$ module. Since F_i is perfect, $dim_{F_i} H_i = dim_{F_i}(\psi_2^*)^{-1} H_i =$

$$dim_{F_i}(S^\varepsilon(R)/(S_\varepsilon(R) + \mathcal{P}_i S^\varepsilon(R))) \leq dim_{F_i}(S^\varepsilon(R)/\mathcal{P}_i S^\varepsilon(R)) = 1.$$

Therefore $dim_{F_i} H_i = 0$ and $H_i = 0$.

Proof of Claim 2: Since $\mathcal{P}_i R$ is not prime, $\mathcal{P}_i R = \mathcal{P} \cap \bar{\mathcal{P}} = \mathcal{P}\bar{\mathcal{P}}$ for some prime $\mathcal{P} \neq \bar{\mathcal{P}}$. So $R/\mathcal{P}_i R = R/\mathcal{P} \times R/\bar{\mathcal{P}}$. One can write the identity of $R/\mathcal{P}_i R$ as $1 = e + \bar{e}$ where $e \in \mathcal{P}, \bar{e} \in \bar{\mathcal{P}}$, $e\bar{e} = 0$. For any $h \in H_\varepsilon(R), r \in R$, $hr = h\bar{r}$. Therefore: $H_i = H_i(e + \bar{e}) = H_i(e + e) = H_i(0) = 0$.

This completes the proof of 6.9.

6.10 Lemma:. *Suppose R is a hereditary ring with involution for which $H_\varepsilon(R) = 0$, where $\varepsilon = (-1)^n$. Then $UNil^p_{2n}(R) = 0$.*

Proof:. Let $z = (P_1, P_{-1}, p_1, p_{-1}, \mu_1, \mu_{-1})$ be any $UNil^p$ form for $UNil^p_{2n}(R)$. Since R is hereditary, $ker(p_1)$ and $ker(p_{-1})$ are projective summands of P_1 and P_{-1}. Then $(ker(p_1), 0)$ is a sub-Lagrangian, as is $(0, ker(p_{-1}))$. Unless $z = 0$, then, a sub-Lagrangian construction on one of these reduces the rank of P_1 and P_{-1}. By induction then, z is equivalent to the zero form.

6.11 Proof of 6.2 (1). : Since a Dedekind ring is hereditary, the proof is immediate from 6.9, 3.11 and 6.10.

6.12. We now begin the job of proving 6.2 (2). So for the rest of this chapter we will assume that R is a Dedekind ring of characteristic different from 2, with trivial involution, $\varepsilon = (-1)$, n is even, and $R/2R$ is a perfect field.

Suppose K is a finitely generated projective over R, and $\mu : K \to R/2R$ is a non-zero R map. Since $R/2R$ is an irreducible $R/2R$ module, and K is a sum of rank-one projectives, (see [Mi]), it is easy to see that we can write $K = X \oplus Y$, where X has rank one, $\mu(X) = R/2R$, and $\mu(Y) = 0$.

This being said, we shall agree that an ε-$UNil^p$ form z over R will be called *semi-reduced* if, for $\delta = \pm 1$, $Ker(p_\delta)$ contains no nonzero summand X_δ, for which $\mu_\delta(X_\delta) = 0$. The remarks of the last paragraph then imply that rank $Ker(p_\delta) \leq 1$, and $Ker(p_\delta) = 0$ iff $\mu_\delta(Ker(p_\delta)) = 0$.

The proof of 6.2(2) is mainly achieved by the following catch-all lemma.

6.13 Lemma. *Let R, ε be as in 6.12. Let $z = (P_1, P_{-1}, p_1, p_{-1}, \mu_1, \mu_{-1})$ be an ε-$UNil^p$ form of length q over R.*

(A). *z is equivalent to a semi-reduced form of length $\leq q$.*

(B). *Suppose q is odd. Then z is equivalent to a form of length $\leq q-1$.*

(C). *Suppose q is even. Then z is equivalent to a form $z' = (P'_1, P'_{-1}, p'_1, p'_{-1}, \mu'_1, \mu'_{-1})$ of length q for which $Im\ (w'_q) = Ker\ (p'_1)$ and $Ker\ (p'_1) \approx Ker\ (p_1)$.*

(D). *Suppose q is even and $Im\ (w_q)$ is a summand of $Ker(p_1)$ with rank 1.*

Then either $r_2(z)$ is reduced of length q, or z is equivalent to a form of length $\leq q-1$.

Proof of (A). If z is not semi-reduced one can find summands $X_\delta \subset Ker\ p_\delta$, $\delta = \pm 1$ not both zero for which $\mu_\delta(X_\delta) = 0$ and $< X_1, X_{-1} >= 0$. Then $z' = S(z; X_1, X_{-1})$ is an equivalent form of smaller rank with length $\leq q$. By an induction on the rank of P_1, this proves A.

Proof of (B). We may as well assume z is semi-reduced with length $\leq q$. We will show that length $z \leq q-1$. Since q is odd, $w_q^t = -w_q$; therefore rank w_q is even. Similarly, rank w_{-q} is even. But $Im\ w_\delta \subset Ker\ p_\delta$, which has rank ≤ 1. Therefore $Im\ w_\delta = 0$ for $\delta = \pm 1$ and so z has length $\leq q-1$.

Proof of (C). Set $q = 2k$. Let $D_{2k} = Ker\ p_1$, a summand of P_1 containing $Im(w_{2k})$. Let $D_0, D_1, \ldots D_{2k-1}$ be copies of D_{2k}. Let $\iota_j : D_{j-1} \to D_j$ be the identity map, for $j = 1 \ldots 2k-1$. Let $\iota_{2k} : D_{2k-1} \to P_1$ be the identity inclusion onto D_{2k}. Construct the new $UNil^p$-form z' as follows:

$$P'_1 = P_1 \oplus \sum_{j=0}^{k-1} (D^*_{2j+1} \oplus D_{2j}) \qquad P'_{-1} = P_{-1} \oplus \sum_{j=0}^{k-1} (D_{2j+1} \oplus D^*_{2j})$$

Obviously, $P'_{-1} = (P'_1)^*$. We define:

$$\begin{array}{ccc} p'_1|P_1 = p_1 & p'_1|D^*_{2j+1} = -\iota^*_{2j+1} & p'_1|D_{2j} = \iota_{2j+1} \\ p'_{-1}|P_{-1} = p_{-1} - \iota^*_{2k} & p'_{-1}|D_{2j+1} = \iota_{2j+2} & p'_{-1}|D^*_{2j} = -\iota^*_{2j}, \text{for } j > 0;\ p'_{-1}|D^*_0 = 0 \\ \mu'_1|P_1 = \mu_1 & \mu'_1|D^*_{2j+1} = 0 & \mu'_1|D_{2j} = 0 \\ \mu'_1|P_{-1} = \mu_{-1} & \mu'_{-1}|D_{2j+1} = 0 & \mu'_{-1}|D^*_{2j} = 0. \end{array}$$

The length of z' is $2k$, and $z = S(z'; V_1, V_{-1})$ where

$$V_1 = \sum_{j=0}^{k-1} D^*_{2j+1}, \quad V_{-1} = \sum_{j=0}^{k-1} D^*_{2j}.$$

Also, in z', $Im\ w_{2k} = D_{2k} = Ker\ p_1' \approx Ker\ p_1$ as required. This proves C.

Proof of (D). Since q is even, $w_q^t = w_{-q}$. Therefore $Im(w_{-q})$ is also a rank 1 summand in P_{-1}. If both $\mu_1 w_q$ and $\mu_{-1} w_{-q}$ are nonzero in z, they are also nonzero in $r_2(z)$, where $Im(w_{\pm q})$ has dimension 1 over $R/2R$. Since $H_\varepsilon(R) \approx H_\varepsilon(R/2R) \approx R/2R$, as an R module, this implies $r_2(z)$ is reduced. On the other hand, if either $\mu_1 w_q$ or $\mu_{-1} w_{-q}$ is zero - say $\mu_1 w_q = 0$ - then $(Im(w_q), 0)$ is a sub-Lagrangian, and $z' = S(z;\ Im(w_q),\ 0)$ is a form equivalent to z, of length $\leq q-1$. This proves (D) and completes the proof of 6.13.

6.14 Proof of 6.2(2). By 3.11 we can work in $UNil_2^p(R)$. Let $[z] \in UNil_2^p(R)$ be a non zero element, and let z be a form of shortest length in this equivalence class. Let $q =$ length z. By 6.13 (B), q is even. 6.13(A) and (C) show that we can choose z so that $Ker\ p_1$ has rank 1 and is equal to $Im(w_q)$. Therefore $Im(w_q)$ is a summand of P_1. 6.13(D) then shows that $r_2(z)$ is reduced of length $q \geq 0$. Then 6.7 shows that $[r_2(z)] \neq 0$. Therefore $r_2 : UNil_2^p(R) \to UNil_2^p(R/2R)$ is a monomorphism. But $UNil_2^p(R/2R)$ is generated by the elements $z_{2k,a}$ where $k \geq 0$, $a \in R/2R$. These elements are in the image of r_2, by 6.6(4), so r_2 is surjective as well. This completes the proof of 6.2.

Some Unsolved Problems

The present paper leaves the following questions unsolved:

a. Let Γ be a group of type VFP, with no elements of order 2, and any orientation character. Does the inclusion $\mathbb{Z} \to \mathbb{Z}\Gamma$ induce an isomorphism of $UNil_{2n}^h$?

b. Is the epimorphism $UNil_{2n}^h(R; \mathcal{B}_1, \mathcal{B}_{-1}) \to L_\varepsilon(\mathbb{A}_\alpha[t])$, of chapter 3, always a monomorphism?

c. Is the Arf invariant $A :\ UNil_0^h(F) \to Coker(\psi_2 - 1)$ an isomorphism? How can one construct operators F_{2k+1} on $Coker(\psi_2 - 1)$ so that A commutes with these?

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