ANNALS OF MATHEMATICS
Vol. 78, No. 2, September, 1963
Printed in Japan

APPROXIMATING STABLE HOMEOMORPHISMS BY PIECEWISE LINEAR ONES

BY E. H. CONNELL

(Received August 21, 1962)

1. Introduction

A homeomorphism h of E_n or S_n onto itself is stable if $\exists$ homeomorphisms $h_1, h_2, \cdots, h_m$ and non-void open sets $U_1, U_2, \cdots, U_m$ such that $h = h_m h_{m-1} \cdots h_1$ and $h_i \mid U_i = I$ for $i = 1, 2, \cdots, m$. All orientation preserving homeomorphisms on E_n or S_n are stable provided $n = 1, 2$, or 3. There is no example known in any dimension of an orientation preserving homeomorphism which is not stable. In fact, the conjecture that all orientation preserving homeomorphisms are stable is equivalent to the annulus conjecture (see [3]).

It is known that any homeomorphism of E_3 onto itself can be approximated by a piecewise linear one (see [2] or [6]). The purpose of this paper is to show that, if h is an orientation preserving homeomorphism of E_n onto E_n ($n \geqq 7$), then h is stable if and only if h can be approximated by a piecewise linear homeomorphism (Theorem 3 and Theorem 5). Also, h is stable if and only if h can be approximated by a diffeomorphism (Theorem 4 and Theorem 5). In addition, it is shown that a stable homeomorphism on S_n can be approximated by a piecewise linear one (Theorem 2). The author thanks John Stallings for counsel.

Notation

E_n is eulidean n-space, S_{n-1} is the unit sphere in E_n, and O_n is the open unit ball in E_n. Thus $O_n \cup S_{n-1} = \bar{O}_n$. For a given integer n, O_n and S_{n-1} will usually be denoted by O and S respectively. If $U \subset E_n$ and $a > 0$, $aU = \{x \in E_n : \exists y \in U \text{ with } x = ay\}$. Furthermore, Ua will denote $C(aU)$, the complement of aU. Thus for a given n, aO will be the canonical open ball in E_n of radius a, and Oa will be its complement. If $x, y \in E_n$, $|x - y|$ will be the usual distance from x to y. The distance from x to the origin is $\|x\|$. If o is the origin and $x \neq o \neq y$, then $\theta\{x, y\}$ will represent the angle in radians between the two line intervals, one joining o to x and the other joining o to y. Thus $0 \leqq \theta\{x, y\} \leqq \pi$. A piecewise linear structure (p.w.l. structure) or combinatorial structure on an open subset of E_n or S_n is a triangulation such that the star of each vertex is a combinatorial cell (see [10, § 3]). The identity function will be denoted by I.

2. Lemmas

LEMMA 0. *Hypothesis*: K_1 *is an* $m+1$ *dimensional finite complex, and* $K_1 \times [0, 1]$ *has a triangulation* V *such that if* S *is a subcomplex of* $K_1 \times [0, 1]$, *then* $\pi(S) \times [0, 1]$ *is a subcomplex of* $K_1 \times [0, 1]$, *where* $\pi(S)$ *is the projection of* S *on* $K_1 \times 0$. E_n *has a* p.w.l. *structure* T_1, *and* $f: K_1 \times [0, 1] \rightarrow E_n$ *is a map; for each simplex* Δ *of* V *there is a simplex* σ *of* T_1 *such that* $f(\Delta) \subset \sigma$ *and* f *is linear on* Δ *with respect to the linear structures of* Δ *and* σ. *Also* $f \mid K_1 \times 0$ *is a homeomorphism, and* $S \subset K_1 \times [0, 1]$ *denotes the singularities of* f *and* $\dim S \leqq m - 1$. $U \subset E_n$ *is open,* $U \supset f(K_1 \times 1) \cup f(L \times [0, 1])$ *where* L *is the* $m-1$ *skeleton of* $K_1 \times 0$. *Thus* $U \supset f(\pi(S) \times [0, 1])$ *also. Finally, the interiors of the* m *dimensional simplexes of* $K_1 \times 0$ *are* $s_1, s_2, \cdots s_v$.

Conclusion: *There exist disjoint closed sets* $B_1, B_2, \cdots, B_v$, $B_i \subset f[s_i \times [0, 1)] \subset E_n$ *such that, if* $O_1, O_2, \cdots, O_v$ *are disjoint open sets in* E_n, $O_i \supset B_i$, *then* $\exists$ p.w.l. *homeomorphisms* $h_1, h_2, \cdots, h_v$ *with* $h_i: E_n \rightarrow E_n$, $h_i \mid C(O_i) = I$ *and the composition* $g_1 = h_v h_{v-1} \cdots h_1$ *satisfying* $g_1(U) \supset f(L_1 \times [0, 1]) \cup f(K_1 \times 1)$ *where* L_1 *is the* m-*skeleton of* $K_1 \times 0$.

OUTLINE OF PROOF. Let B_i, $i = 1, 2, \cdots, v$, be combinatorial $m + 1$ cells, $B_i \subset f[s_i \times [0, 1)]$. Let $C_i = B_i \cap f[s_i \times 0]$ and $D_i =$ boundary B_i minus interior C_i. Clearly the B_i may be chosen so that the C_i and D_i are combinatorial m-cells. Each B_i collapses onto C_i (see [12, Th. 6] and [10, 3.4]). It now follows from the proof of [15, Lemma 1] that if $O_1, O_2, \cdots, O_v$ are disjoint open sets, $B_i \subset O_i \subset E_n$, there exist p.w.l. homeomorphisms $h_i: E_n \rightarrow E_n$ such that $h_i \mid C(O_i) = I$ and $h_v h_{v-1} \cdots h_1(U) \supset f(L_1 \times [0, 1]) \cup f(K_1 \times 1)$.

The results of this paper are based primarily upon Lemma 1 below, a modification of the *engulfing lemma* (see [10, 3.4]). The modification consists of adding to the conclusion the requirement $\theta\{h(x), x\} < \varepsilon$. If E_n is given the usual p.w.l. structure, it is possible to expand radially in a p.w.l. manner. For instance, define $f(x) = 2x$. According to Lemma 1, and its consequence Lemma 3, regardless of the p.w.l. structure on E_n, it is possible to expand almost radially in a p.w.l. manner.

LEMMA 1. *Suppose* $E_n (n \geqq 4)$ *has an arbitrary* p.w.l. *structure* T, K *is a finite subcomplex of* T, $\dim K \leqq n - 4$, a, b, *and* ε *are numbers with* $0 < a < b$, $\varepsilon > 0$, *and* $K \subset bO = bO_n$. *Then* $\exists$ *a homeomorphism* $h: E_n \rightarrow E_n$ *such that* h *is* p.w.l. *relative to* T, $h \mid (a - \varepsilon)O = I$, $h \mid Ob = I$, $h(aO) \supset K$ *and* $\theta\{h(x), x\} < \varepsilon$ *for* $x \in E_n$.

PROOF. Let n be fixed, $n \geqq 4$. The proof will be by induction on $\dim K$. The lemma is immediate for $\dim K = 0$. Assume the lemma is true for

$\dim K \leqq m$ where $m \leqq n-5$. Now let K, T, a, b, and ε be given, $\dim K = m+1$, $0 < a < b$, $\varepsilon > 0$, and let $\varepsilon < a$. (Note: For simplicity in defining f_1 below, assume K does not contain the origin o. If $K \ni o$, in a fine enough subdivision of T, the open star w of o will be in $(a-\varepsilon)O$. Thus obtaining the conclusion for $K - w$ would obtain the conclusion for K.)

Let T_1 be a subdivision of T such that each closed simplex r of T_1 which intersects bO has diameter $<\varepsilon/5$ and whenever r also intersects $O(a-\varepsilon)$ and $x, y \in r$, then $\theta\{x, y\} < \varepsilon/5$. Let K_1 be K under the new triangulation T_1. Let $K_1 \times [0, 1]$ be the cross product complex, and identify $K_1 \times 0$ with K_1. Define f_1: $K_1 \times [0, 1] \rightarrow bO \subset E_n$ as follows: $f_1(k, 0) = k$, $f_1(k, 1) = \{(a-\varepsilon/2)/\|k\|\}k$ and $f_1(k, t) = tf_1(k, 1) + (1-t)f_1(k, 0)$.

Let f: $K_1 \times [0, 1] \rightarrow E_n$ be a p.w.l. approximation to f_1 such that f is in general position in some triangulation V_1 (see [10, 3.4D]), and $f(k, 0) = f_1(k, 0) = k$ for $k \in K_1$. Let the approximation be so close that f has the following properties:

(1) If r is a closed simplex of K_1 (in T_1) and $(x, t) \in r \times [0, 1] \subset K_1 \times [0, 1]$ and $(y, u) \in r \times [0, 1] \subset K_1 \times [0, 1]$ then $\theta\{f(x, t), f(y, u)\} < \varepsilon/5$.

(2) $f(K_1 \times [0, 1]) \subset bO$ and $f(K_1 \times 1) \subset (a - 2\varepsilon/5)O \cap \bar{O}(a - 3\varepsilon/5)$.

(3) If r is a closed simplex of K_1 (in T_1), $r \subset (a-\varepsilon/5)O$, then $f(r \times [0, 1]) \subset (a - \varepsilon/5)O$.

(4) If r is a closed simplex of K_1 (in T_1), $r \subset \bar{O}(a - 3\varepsilon/5)$, then $f(r \times [0, 1]) \subset \bar{O}(a - 3\varepsilon/5)$.

(5) If r is a closed simplex of K_1 (in T_1), $r \subset \bar{O}(a-\varepsilon)$, then $f(r \times [0, 1]) \subset \bar{O}(a - \varepsilon)$.

This is possible because f_1 satisfies these five conditions with some room to spare.

Let V be a subdivision of V_1 such that:

(a) If S is a subcomplex of V, and $\pi(S)$ is its projection on $K_1 \times 0$, then $\pi(S) \times [0, 1]$ is a subcomplex of V.

(b) For each simplex Δ of V, there is a simplex σ of T_1 such that $f(\Delta) \subset \sigma$ and f is linear on Δ with respect to the linear structures of Δ and σ.

(c) The set of singularities of f is contained in a subcomplex of V.

To obtain V, one triangulates the projection π: $K_1 \times [0, 1] \rightarrow K_1$ by a triangulation V which is fine enough to satisfy (b) and (c).

Denote by S the set of singularities of f. $S \subset K_1 \times [0, 1]$ will be a subcomplex of V. Since f is in general position, $\dim S \leqq 2\dim(K_1 \times [0, 1]) - n \leqq 2(m+2) - n \leqq 2(m+2) - (m+5) = m-1$. Let L be the $m-1$ skeleton of $K_1 \times 0$ in the triangulation V. Then $L \times [0, 1] \supset \pi(S) \times [0, 1]$.

Since $\dim f(L \times [0, 1]) \leqq m$, by induction $\exists$ a p.w.l. homeomorphism $g: E_n \rightarrow E_n$ such that:

(A) $g \mid (a - \varepsilon/5)O = I$ and $g \mid Ob = I$

(B) $g(aO) \supset f(L \times [0, 1])$

(C) $\theta\{g(z), z\} < \varepsilon/5$ for $z \in E_n$.

Now the procedure stated in Lemma 0 will be used to finish the proof. Let U of Lemma 0 be $g(aO)$. Now $U = g(aO) \supset g([a - \varepsilon/5]O)$ which by (A) is $=[a - \varepsilon/5]O$. By (2), $f(K_1 \times 1) \subset [a - 2\varepsilon/5]O$ and thus $U \supset f(K_1 \times 1)$ and the hypothesis of Lemma 0 is satisfied.

Let $s_1, s_2, \cdots, s_v$ be the open m simplexes of $K_1 \times 0$ in V and let $B_1, B_2, \cdots, B_v, O_1, O_2, \cdots, O_v$, and $h_1, h_2, \cdots, h_v$ be as in Lemma 0. Because V is a subdivision of the cross product triangulation of $K_1 \times [0, 1]$, properties (1), (2), (3), (4) and (5) above apply to the V simplexes of $K_1 \times [0, 1]$ as well as the T_1 simplexes.

REMARK (1). According to (1), the "θ diameter" of $f(s_i \times [0, 1])$ is $<\varepsilon/5$. Thus each O_i can be chosen so that $\theta\{h_i(z), z\} < \varepsilon/5$ for $z \in E_n$.

REMARK (2). Each O_i can be chosen so that $O_i \subset bO$. This follows from (2).

REMARK (3). If $\bar{s}_i \subset (a - \varepsilon/5)O$, then by (3), h_i can be chosen as the identity.

REMARK (4). If $\bar{s}_i \not\subset (a - \varepsilon/5)O$, then by (4), O_i can be chosen so that $O_i \subset \bar{O}(a - 3\varepsilon/5)$.

The conclusion is that $g_1 = h_v h_{v-1} \cdots h_1$ satisfies $\theta\{g(z), z\} < \varepsilon/5$, $g_1 | Ob = I$, $g_1 \mid (a - 3\varepsilon/5)O = I$ and $g_1(U) = g_1 g(aO) \supset f(L_1 \times [0, 1])$ where L_1 is the m skeleton of $K_1 \times 0$ in V.

Now the same procedure will be applied again, with $U = g_1(U)$ and $s_1, s_2, \cdots, s_w$ the open $m + 1$ simplexes of $K_1 \times 0$ in V. When $\bar{s}_i \subset (a - 3\varepsilon/5)O$, h_i can be chosen as the identity, since $g_1 g(aO) \supset \bar{s}_i$. Otherwise, when $\bar{s}_i \not\subset (a - 3\varepsilon/5)O$, it follows from (5) that O_i can be chosen so that $O_i \subset \bar{O}_i(a - \varepsilon)$. The conclusion is $\exists$ $g_2: E_n \rightarrow E_n$, a p.w.l. homeomorphism with $\theta\{g_2(z), z\} < \varepsilon/5$, $g_2 \mid Ob = I$, $g_2 \mid (a - \varepsilon)O = I$, and $g_2 g_1(U) = g_2 g_1 g(aO) \supset K_1 \times 0 = K_1$. Set $h = g_2 g_1 g$, and note that $\theta\{h(z), z\} < 3\varepsilon/5 < \varepsilon$, $h \mid Ob = I$, $h \mid (a - \varepsilon)O = I$, and $h(aO) \supset K$. This completes the proof.

LEMMA 2. *Suppose $E_n (n \geqq 4)$ has an arbitrary* p.w.l. *structure T, K is a finite subcomplex of T,* dim $K \leqq n - 4$, *a, b, and ε are numbers with $0 < a < b$, $\varepsilon > 0$, and $K \subset \bar{O}a$. Then $\exists$ a homeomorphism $h: E_n \rightarrow E_n$ such that h is* p.w.l. *relative to T, $h \mid O(b + \varepsilon) = I$, $h \mid aO = I$, $h(\bar{O}b) \supset K$ and $\theta\{h(x), x\} < \varepsilon$ for $x \in E_n$.*

PROOF. This lemma is the same as Lemma 1 except that the expansion is in toward the origin instead of away from the origin. The proof is essentially identical to the proof of Lemma 1.

The only purpose of Lemma 1 and Lemma 2 is to prove Lemma 3 below, which can be used to approximate stable homeomorphisms by p.w.l. ones.

LEMMA 3. *Suppose* $E_n (n \geqq 7)$ *has an arbitrary* p.w.l. *structure* T, *and* a, b, *and* ε *are numbers with* $0 < a < b$ *and* $\varepsilon > 0$. *Then* $\exists$ *a homeomorphism* $h: E_n \to E_n$ *such that* h *is* p.w.l. *relative to* T, $h \mid (a - \varepsilon)O = I$, $h \mid O(b + \varepsilon) = I$, $h(aO) \supset bO$, *and* $\theta\{h(x), x\} < \varepsilon$ *for* $x \in E_n$.

PROOF. The proof is a trivial modification of [10, § 4] and [11, § 8.1]. Suppose $0 < \varepsilon < a$. Let T_1 be a subdivision of T such that if v is a simplex of T_1 which intersects $(b + \varepsilon)O$ and $x, y \in v \subset E_n$, then $|x - y| < \varepsilon/3$ and $\theta\{x, y\} < \varepsilon/3$. Let J be the star (in T_1) of $[O(a - 2\varepsilon/3)] \cap [(b + 2\varepsilon/3)\bar{O}]$ and K its 3-skeleton.

Now $n - \dim K \geqq 7 - 3 = 4$, and thus by Lemma 1, $\exists$ a p.w.l. homeomorphism $h_1: E_n \to E_n$ such that $h_1 \mid (a - \varepsilon/3)O = I$, $h_1 \mid O(b + \varepsilon) = I$, $h_1(aO) \supset K$, and $\theta\{h_1(x), x\} < \varepsilon/3$ for all $x \in E_n$.

Let L be the subcomplex of the barycentric subdivision of J which is maximal with respect to the property of not intersecting K, $L = J \div K$. Now $\dim L = n - (\dim K + 1) = n - 4$ and, by Lemma 2, $\exists$ a p.w.l. homeomorphism $g: E_n \to E_n$ such that $g \mid (a - \varepsilon)O = I$, $g \mid O(b + \varepsilon/3) = I$, $g(\bar{O}b) \supset L$, and $\theta\{g(x), x\} < \varepsilon/3$ for all $x \in E_n$.

If A is an n-simplex of J, A is the join of $A \cap K$ and $A \cap L$, and $A \cap K \subset h_1(aO)$ and $A \cap L \subset g(\bar{O}b)$. By a trivial extension of [11, Lemma 8.1], $\exists$ a p.w.l. homeomorphism $g_1: E_n \to E_n$ such that $g_1 \mid (a - \varepsilon)O = I$, $g_1 \mid O(b + \varepsilon) = I$, for each n-simplex u of J (in T_1), $h_1(aO) \cup g_1 g(\bar{O}b) \supset u$, and for each simplex v of E_n (in T_1), $g_1(v) = v$. The triangulation T_1 was chosen fine enough so that $g_1(v) = v$ implies $\theta\{g(x), x\} < \varepsilon/3$ for all $x \in E_n$.

Let $h_2 = g_1 g$, and show $h_1(aO) \cup h_2(Ob) \supset E_n$. Now g_1 was chosen so that $h_1(aO) \cup h_2(Ob) \supset J$ and, since $h_1 \mid (a - \varepsilon/3)O = I$, it clearly contains $(a - \varepsilon/3)O \cup J \supset (b + 2\varepsilon/3)O$. It remains to show that $g_1 g(Ob) \supset C[(b + 2\varepsilon/3)O]$. Let v be a simplex of T_1 which intersects $C[(b + 2\varepsilon/3)O]$. Then $v \subset O(b + \varepsilon/3)$ and thus $g \mid v = I$. Since $g_1(v) = v$, $g_1 g(v) = v$ and therefore $g_1 g(Ob) \supset v$. This shows that $h_1(aO) \cup h_2(Ob) = E_n$.

This gives $h_1(aO) \supset h_2(C[Ob])$ which gives $h_2^{-1}h_1(aO) \supset C[Ob] = bO$. Let $h = h_2^{-1}h_1$, and note that $h \mid (a - \varepsilon)O = I$, $h \mid O(b + \varepsilon) = I$ and $\theta\{h(x), x\} < \varepsilon$, and that h is a p.w.l. homeomorphism. Thus h satisfies the conclusion of the lemma.

DEFINITION. A homeomorphism $h: S_n \to S_n$ is said to have property P if for any p.w.l. structure T on S_n and any $\varepsilon > 0$, $\exists$ a homeomorphism $f: S_n \to S_n$ such that f is p.w.l. relative to T, any $|h(x) - f(x)| < \varepsilon$ for

$x \in S_n$.

The set of all homeomorphisms on S_n forms a group under composition. Let G_n be the set of homeomorphisms on S_n which possess property P.

Observation Q. G_n is a normal subgroup of the group of all homeomorphisms.

PROOF. The proof that it is a subgroup is immediate. It will be shown that G_n is normal. Suppose $h \in G_n$, and $g: S_n \to S_n$ is any homeomorphism. Show that $g^{-1}hg \in G_n$. Let T and ε be given.

$\exists$ a $\delta > 0$ such that if $|x - y| < \delta$, then $|g^{-1}(x) - g^{-1}(y)| < \varepsilon$. Let T_1 be the p.w.l. structure of S_n which is the g image of T, $T_1 = g(T)$. Thus if v is a simplex of S_n in the triangulation T, $g(v)$ is a simplex of S_n in the triangulation T_1. Since $h \in G_n$, $\exists$ a homeomorphism $f: S_n \to S_n$ which is p.w.l. relative to T_1, and with $|h(x) - f(x)| < \delta$ for $x \in S_n$. Thus $|g^{-1}hg(x) - g^{-1}fg(x)| < \varepsilon$ for $x \in S_n$. Note that $g^{-1}fg$ is p.w.l. relative to T because g is p.w.l. from T to T_1, f is p.w.l. from T_1 to T_1, and g^{-1} is p.w.l. from T_1 to T. This completes *Observation* Q.

3. The main theorems

THEOREM 1. *Let T be an arbitrary* p.w.l. *structure of $S_n (n \geqq 7)$, and let $h: S_n \to S_n$ be stable homeomorphism. If $\varepsilon > 0$, $\exists$ a homeomorphism $f: S_n \to S_n$ such that f is* p.w.l. *relative to T and $|h(x) - f(x)| < \varepsilon$ for $x \in S_n$.*

PROOF. The set of all stable homeomorphisms of S_n is a simple, normal subgroup of the group of all homeomorphisms. The fact that it is a normal subgroup is immediate and the fact that it is simple follows from [1] and is even stated explicity in [4, Th. 14]. Therefore, using *Observation* Q, it will follow that G_n contains the stable group if G_n contains some stable homeomorphism distinct from the identity. This will now be shown.

Let h be a symmetric radial expansion, i.e., let $h: E_n \to E_n$ be a homeomorphism such that $h(x) = x$ for $\|x\| \geqq 1$, $h(o) = o$, $\theta\{h(x), x\} = 0$ for all x, and if $0 < r < 1$, $\exists$ a number $u(r)$, $r < u(r) < 1$ such that $h[r(\bar{O} - O)] = u(r)(\bar{O} - O)$. Let T be any p.w.l. structure on E_n and $\varepsilon > 0$. It will be shown that $\exists f: E_n \to E_n$ which is a p.w.l. homeomorphism relative to T, and with $f(x) = x$ for $\|x\| \geqq 1$ and $|h(x) - f(x)| < \varepsilon$ for $x \in E_n$. Since h determines a homeomorphism from S_n to itself by defining $h(\infty) = \infty$, this will show that G_n is non-trivial and will complete the proof of Theorem 1.

Let $0 = r_0 < r_1 < r_2 \cdots < r_{m+1} = 1$ be numbers such that $(u(r_{i+2}) - u(r_i)) < \varepsilon/2$ for $i = 0, 1, 2, \cdots, (m - 1)$. By Lemma 3, $\exists$ p.w.l. homeomorphisms

$f_1, f_2, \cdots, f_m$ such that $f_i \mid r_{i-1}O = I$, $f_i \mid Ou(r_{i+1}) = I$, $\theta\{f_i(x), x\} < \varepsilon/4$ for $x \in E_n$, and $f_i(r_iO) \supset u(r_i)O$ for $i = 1, 2, \cdots, m$. Let $f = f_1 f_2 \cdots f_m$. Now f is a homeomorphism of E_n onto E_n that is p.w.l. relative to T, and $f \mid C(O) = I$. It will be shown that $|f(x) - h(x)| < \varepsilon$ for $x \in O$. Let $x \in r_{k+1}O \cap Or_k = r_{k+1}O - r_kO$, $0 \leqq k \leqq m$. Then $f(x) = f_1 f_2 \cdots f_{k+1}$ because $f_t \mid r_{k+1}O = I$ for $t > k + 1$. In fact, $f(x) = f_k f_{k+1}(x)$ because $f_k f_{k+1}(x) \in Ou(r_k)$ and $f_t \mid Ou(r_k) = I$ for $t < k$. (In the special case $k = 0$, $f(x) = f_1(x)$.) Now since $f(x)$ and $h(x) \in u(r_{k+2})O \cap Ou(r_k)$, $\|f(x)\|$ and $\|h(x)\|$ differ by $<\varepsilon/2$. Since $\theta\{h(x), f(x)\} < \varepsilon/2$ is measured in radians and any radius under consideration is <1, it follows that $|h(x) - f(x)| < \varepsilon$. This completes the proof.

The following theorem is a restatement of Lemma 3. It could be called a "controlled expanding theorem."

THEOREM 2. *Suppose $E_n (n \geqq 7)$ has an arbitrary* p.w.l. *structure T, and a, b, and ε are numbers with $0 < a < b$ and $\varepsilon > 0$. If f is any homeomorphism from E_n onto E_n, then $\exists$ a homeomorphism g: $E_n \to E_n$ such that g is* p.w.l. *relative to T, $g \mid f[(a - \varepsilon)O] = I$, $g \mid f[O(b + \varepsilon)] = I$, $g[f(aO)] \supset f(bO)$, and $\theta\{f^{-1}[g(y)], f^{-1}(y)\} < \varepsilon$ for all $y \in E_n$.*

PROOF. Let T_1 be the p.w.l. structure of E_n which is the image of T under f^{-1}, $T_1 = f^{-1}(T)$. By Lemma 3, $\exists$ a homeomorphism h: $E_n \to E_n$ such that h is p.w.l. relative to T_1, $h \mid (a - \varepsilon)O = I$, $h \mid O(b + \varepsilon) = I$, $h(aO) \supset bO$, and $\theta\{h(x), x\} < \varepsilon$ for $x \in E_n$. Now $g = fhf^{-1}$ is a homeomorphism of E_n which is p.w.l. relative to T and satisfies the conclusion of the theorem.

Theorem 2 will be used to approximate stable homeomorphisms of E_n in somewhat the same manner that Lemma 3 was used to approximate stable homeomorphisms of S_n. The heart of the matter of approximating stable homeomorphisms of E_n is contained in Lemma 4 below.

LEMMA 4. *Let T be an arbitrary* p.w.l. *structure on $E_n (n \geqq 7)$. Let $O = O_n$ as before, and h: $O \to E_n$ be a homeomorphism such that $h(o) = o$, $\theta\{h(x), x\} = 0$ for $x \in O$, and if $0 < r < 1$, $\exists$ a number $u(r) > r$ such that $h[r(\bar{O} - O)] = u(r)(\bar{O} - O)$. Then if $\varepsilon(x)$: $O \to (0, \infty)$ is continuous, $\exists$ a homeomorphism f: $O \to E_n$ which is* p.w.l. *relative to T, and such that $|f(x) - h(x)| < \varepsilon(x)$ for $x \in O$.*

PROOF. The proof calls for a p.w.l. expansion of O onto E_n which is nearly radial. This expansion will be obtained through a sequence of steps, each step using Theorem 2. A difficulty appears here that did not appear in the proof of Theorem 1. This difficulty is that, after a sequence of expansions, the "angle error" may accumulate. This is overcome in parts D, E, and F.

Let $\delta(w)$: $E_n \to (0, \infty)$ be a continuous function such that if $v, w \in E_n$, $-\delta(w) < ||v|| - ||w|| < \delta(w)$, and $\theta\{v, w\} < \delta(w)$, then $|v - w| < \varepsilon(h^{-1}[w])$. Let $0 = r_0 < r_1 < r_2 \cdots$ be an increasing sequence of numbers such that $r_n \to 1$ as $n \to \infty$, and $u(r_{i+2}) - u(r_i) < \max \delta(w)$ for $w \in u(r_{i+2})\bar{O}$, and denote this max by δ_i.

It follows from Lemma 3 that $\exists$ a homeomorphism f_1: $E_n \to E_n$ such that f_1 is p.w.l. relative to T, $f_1(o) = o$, $f_1(r_1O) \supset u(r_1)O$, $f_1 | Ou(r_2) = I$, and $\theta\{f_1(x), x\} < \delta_1/2$. Now applying Theorem 2, $\exists$ a p.w.l. homeomorphism f_2: $E_n \to E_n$ such that $f_2 | f_1(r_1O) = I$, $f_2 | f_1(Ou[r_3]) = I$, $f_2[f_1(r_2O)] \supset f_1(u[r_2]O)$, and $\theta\{f_1^{-1}[f_2(y)], f_1^{-1}(y)\} < \delta_2/2$.

In general, suppose $f_1, f_2, \cdots, f_{k-1}$ have been defined. Let the "f" of Theorem 2 be $f_{k-1}f_{k-2} \cdots f_1$. Then $\exists$ a p.w.l. homeomorphism f_k: $E_n \to E_n$ such that

(1) $f_k | f_{k-1}f_{k-2} \cdots f_1(r_{k-1}O) = I$,

(2) $f_k | f_{k-1}f_{k-2} \cdots f_1(Ou[r_{k+1}]) = I$,

(3) $f_k[f_{k-1}f_{k-2} \cdots f_1(r_kO)] \supset f_{k-1}f_{k-2} \cdots f_1(u[r_k]O)$,

(4) $\theta\{(f_{k-1}f_{k-2} \cdots f_1)^{-1}f_k(z), (f_{k-1}f_{k-2} \cdots f_1)^{-1}(z)\} < \delta_k/2$ for $z \in E_n$.

From (2) follows

(2′) $f_kf_{k-1} \cdots f_1 | Ou(r_{k+1}) = I$.

Suppose (2′) is true for $k - 1$, i.e., suppose $f_{k-1} \cdots f_1 | Ou(r_k) = I$. Then since $Ou(r_{k+1}) \subset Ou(r_k)$, $f_{k-1}f_{k-2} \cdots f_1 | Ou(r_{k+1}) = I$, and thus by (2), $f_kf_{k-1} \cdots f_1 | Ou(r_{k+1}) = I$. Therefore (2′) follows from (2) by induction.

Define f: $O \to E_n$ by $f = \cdots f_3f_2f_1 | O$. This f will satisfy the conclusion of the lemma. First it will be shown that f is well defined. Let $x \in O$. Then $\exists$ an integer k such that $x \in r_kO$. From (1), $f_s | f_{s-1} \cdots f_1(r_{s-1}O) = I$ for $s = 1, 2, \cdots$, and since $x \in r_sO$ when $s > k$, $f(x) = f_k \cdots f_1(x)$ and thus f is well defined. Since each f_i is p.w.l. relative to T and, on any compact subset of O, f is defined by a finite number of the f_i, it is clear that f is a p.w.l. homeomorphism. It remains to be shown that $|f(x) - h(x)| < \varepsilon(x)$ for $x \in O$.

Observation A. $f(r_kO) \supset u(r_k)O$ for $k = 1, 2, \cdots$. To see this, note that $f(r_kO) = f_kf_{k-1} \cdots f_1(r_kO)$ which, by (3), $\supset f_{k-1}f_{k-2} \cdots f_1(u(r_k)O)$, and it follows from (2′) that this is equal to $u(r_k)O$.

Observation B. $f(r_{k+1}O) \subset u(a_{k+2})O$ for $k = 0, 1, 2, \cdots$. To see this, note that $f(r_{k+1}O) = f_{k+1}f_k \cdots f_1(r_{k+1}O) \subset f_{k+1}f_k \cdots f_1(u[r_{k+2}]O)$ and this is equal to $u[r_{k+2}]O$ because $f_{k+1}f_k \cdots f_1$ is the identity on $Ou[r_{k+2}]$.

Observation C. If $x \in r_{k+1}O - r_kO$, then $-\delta_k < ||f(x)|| - ||h(x)|| < \delta_k$. It follows from A and B that $f(x) \in Ou(r_k) \cap u(r_{k+2})O$, and thus $u(r_k) \leqq ||f(x)|| < u(r_{k+2})$. Also $u(r_k) \leqq ||h(x)|| < u(r_{k+1})$. Thus the absolute value of $||f(x)|| - ||h(x)||$ is $< u(r_{k+2}) - u(r_k)$ which is $<\delta_k$, and the result follows.

Observation D. If $y \in Or_k$, then $\theta\{f_k f_{k-1} \cdots f_1(y), y\} < \delta_k/2$. According to A, $f_k f_{k-1} \cdots f_1(y) \in Ou(r_k)$. By (2′), $(f_{k-1} \cdots f_1)^{-1} | Ou(r_k) = I$. Therefore

$$\begin{aligned}\theta\{f_k \cdots f_1(y), y\} &= \theta\{(f_{k-1} \cdots f_1)^{-1}[f_k \cdots f_1(y)], y\} \\ &= \theta\{(f_{k-1} \cdots f_1)^{-1}[f_k \cdots f_1(y)], \\ &\qquad (f_{k-1} \cdots f_1)^{-1}[f_{k-1} \cdots f_1(y)]\}\end{aligned}$$

which is $<\delta_k/2$ by (4), setting z of (4) to be $f_{k-1} \cdots f_1(y)$. This shows D.

Observation E. $(f_k \cdots f_1)^{-1}(f_{k+1} \cdots f_1)(Or_k) \subset Or_k$. This will be true if $(f_{k+1} \cdots f_1)(Or_k) \subset (f_k \cdots f_1)(Or_k)$ which follows easily from (1), $f_{k+1} | (f_k \cdots f_1)(r_k O) = I$.

Observation F. If $x \in r_{k+1}O - r_k O$, then $\theta\{f(x), x\} < \delta_k$. Let $y = (f_k \cdots f_1)^{-1}(f_{k+1} \cdots f_1(x))$. By E, $y \in Or_k$. Substitute y in the inequality of D and obtain $\theta\{f(x), y\} = \theta\{f_{k+1} \cdots f_1(x), y\} < \delta_k/2$. Now $\theta\{y, x\} = \theta\{(f_k \cdots f_1)^{-1}f_{k+1}[f_k \cdots f_1(x)], (f_k \cdots f_1)^{-1}[f_k \cdots f_1(x)]\}$ which is $<\delta_{k+1}/2$ by (4). Now by the triangle inequality, $\theta\{f(x), x\} < \delta_k/2 + \delta_{k+1}/2 \leqq \delta_k$, and F follows.

Conclusion. If $x \in O$, then $\exists$ an integer k such that $x \in r_{k+1}O - r_k O$. By C, $-\delta_k < ||f(x)|| - ||h(x)|| < \delta_k$ and by F, $\theta\{f(x), x\} = \theta\{f(x), h(x)\} < \delta_k$. In the definition of $\delta(x)$ at the beginning of the proof, let $v = f(x)$ and $w = h(x)$, and note that $\delta(h(x)) \leqq \delta_k$ because $||h(x)|| < u(r_{k+1})$. Thus from the definition of $\delta(w)$, $|v - w| < \varepsilon(h^{-1}(w))$ or $|f(x) - h(x)| < \varepsilon(h^{-1}[h(x)]) = \varepsilon(x)$. This completes the proof.

THEOREM 3. *Let T be an arbitrary* p.w.l. *structure on E_n ($n \geqq 7$). If g: $E_n \to E_n$ is a stable homeomorphism and $\varepsilon(x)$: $E_n \to (0, \infty)$ is a continuous function, then $\exists$ a homeomorphism f: $E_n \to E_n$ which is* p.w.l. *relative to T, and such that $|f(x) - g(x)| < \varepsilon(x)$ for $x \in E_n$.*

PROOF. Since g is stable, $\exists$ homeomorphisms $g_1, g_2, \cdots, g_m$ and non-void open sets $U_1, U_2, \cdots, U_m$ such that $g_i | U_i = I$ for $i = 1, 2, \cdots, m$ and $g = g_m g_{m-1} \cdots g_1$. If each g_i can be approximated by a p.w.l. homeomorphism, then clearly g can also. Thus it may be assumed that $\exists$ a non-void open set U such $g | U = I$. For convenience, suppose $U \supset O_n = O$.

Let $\delta(z)$: $E_n \to (0, \infty)$ be a continuous function such that, if $z, b, c \in E_n$, $|b - z| < \delta(z)$, $|c - z| < \delta(z)$, then $|g(b) - g(c)| < \varepsilon(c)$.

Let h: $O \to E_n$ be a homeomorphism as in the statement of Lemma 4, $h(x) = u(||x||)x$. Then $\exists$ a homeomorphism f_1: $O \to E_n$ which is p.w.l. relative to T and such that $|f_1(y) - h(y)| < \delta[h(y)]$ for $y \in O$.

Let T_1 be the p.w.l. structure on E_n which is the image of T under g^{-1}, $T_1 = g^{-1}(T)$. Since $g | O = I$, T_1 and T agree on O. Thus f_1^{-1}: $E_n \to O$ is p.w.l. from T to T_1. Now using Lemma 4 again, $\exists$ a homeomorphism f_2: $O \to E_n$ which is p.w.l. from T_1 to T_1 and such that $|f_2(y) - h(y)| < \delta[h(y)]$.

The homeomorphism $gf_2f_1^{-1}$: $E_n \to E_n$ will satisfy the conclusion of the theorem. Since $f_2f_1^{-1}$ is p.w.l. from T to T_1 and g is p.w.l. from T_1 to T, $gf_2f_1^{-1}$ is p.w.l. from T to T.

It remains to show that $|gf_2f_1^{-1}(x) - g(x)| < \varepsilon(x)$ for $x \in E_n$. In the definition of $\delta(z)$, let $z = h[f_1^{-1}(x)]$, $b = f_2f_1^{-1}(x)$, and $c = x$. Then $|b - z| = |f_2f_1^{-1}(x) - hf_1^{-1}(x)|$ which is $< \delta(h[f_1^{-1}(x)]) = \delta(z)$. Also $|c - z| = |x - h[f_1^{-1}(x)]| = |f_1f_1^{-1}(x) - h[f_1^{-1}(x)]| < \delta(h[f_1^{-1}(x)]) = \delta(z)$. The conclusion is that $|g(b) - g(c)| < \varepsilon(c)$, which is to say $|gf_2f_1^{-1}(x) - g(x)| < \varepsilon(x)$. This proves the theorem.

THEOREM 4. *Suppose D is any C^2 differentiable structure on E_n $(n \geqq 7)$. If g: $E_n \to E_n$ is a stable homeomorphism and $\varepsilon(x)$: $E_n \to (0, \infty)$ is a continuous function, then $\exists$ a homeomorphism f: $E_n \to E_n$ which is a C^2 diffeomorphism relative to D and such that $|f(x) - g(x)| < \varepsilon(x)$ for $x \in E_n$.*

PROOF. Let T be a C^2 triangulation of E_n which is compatible with D (see [5] or [13]). By THEOREM 3, g may be approximated by a homeomorphism f_1 which is p.w.l. relative to T. Now by [7, Theorems 5.7 and 6.2], f_1 may be approximated by a diffeomorphism f. This completes the proof. The theorem also holds if C^2 is replaced by C^∞.

LEMMA 5. *Bounded homeomorphisms on euclidean space are stable, i.e., if f: $E_n \to E_n$ $(n \geqq 1)$ is a homeomorphism and $\exists$ an $M > 0$ such that $|f(x) - x| < M$ for $x \in E_n$, then f is stable.*

PROOF. Let $0 < a < b < c$ be numbers such that $f(aO) \subset bO$. Let h: $cO \to E_n$ be a homeomorphism with $h \mid bO = I$ and $\theta\{h(x), x\} = 0$ for $x \in cO$. Then $h^{-1}fh$: $cO \to cO$ can be extended to a homeomorphism g: $E_n \to E_n$ by defining $g(x) = x$ for $x \in Oc$. The lemma now follows from the fact that $f = g^{-1}gf$ where $g^{-1} \mid Oc = I$ and $gf \mid aO = I$.

THEOREM 5. *Suppose g and h are homeomorphisms from E_n onto E_n $(n \geqq 1)$, and $\exists$ an $M > 0$ such that $|g(x) - h(x)| < M$ for $x \in E_n$. Then if g is stable, h is also stable.*

PROOF. Let $f = hg^{-1}$. By hypothesis $|x - f(x)| < M$ for $x \in E_n$, and thus by Lemma 5, f is stable. Since $h = fg$, h is the product of stable homeomorphisms and is thus stable.

4. Two auxiliary theorems

THEOREM 6. *Let g: $E_n \to E_n$ $(n \geqq 7)$ be a stable homeomorphism, $O =$*

O_n, and $\varepsilon(x): O \to (0, \infty)$ be continuous. If T is an arbitrary p.w.l. *structure on E_n, $\exists$ a homeomorphism $f: E_n \to E_n$ such that $f \mid C(O) = g \mid C(O)$, $f \mid O$ is* p.w.l. *relative to T, and $|f(x) - g(x)| < \varepsilon(x)$ for $x \in O$. If D is an arbitary C^2 differentiable structure on E_n, $\exists$ a homeomorphism $f: E_n \to E_n$ such that $f \mid C(O) = g \mid C(O)$, $f \mid O$ is a diffeomorphism relative to D, and $|f(x) - g(x)| < \varepsilon(x)$ for $x \in O$.*

OUTLINE OF PROOF. Let $h_1: E_n \to O$ be a p.w.l. homeomorphism which is the identity on some non-void open set in O. Let $h_2: g(O) \to E_n$ be a p.w.l. homeomorphism which is the identity on some non-void open set in $g(O)$. Such homeomorphisms h_1 and h_2 can be constructed by using Theorem 2 in the same manner as it was used in the proof of Lemma 4. Now $h_2 g h_1: E_n \to E_n$ is stable (see [3]) and, according to Theorem 3, can be approximated by an $h: E_n \to E_n$ which is p.w.l. relative to T. Now define $f: O \to g(O)$ by $f = h_2^{-1} h h_1^{-1}$. This f will be a p.w.l. approximation to $g \mid O$ and may be extended to all of E_n by letting $f \mid C(O) = g \mid C(O)$.

The second part of the theorem, the differentiable case, follows from the first part of the theorem in the same manner that Theorem 4 follows from Theorem 3.

THEOREM 7. (A) *Let D be an arbitrary C^2 differentiable structure on E_n ($n \geqq 7$) and T an arbitrary* p.w.l. *structure on E_n. Thus T may or may not be compatible with D. If $g: E_n \to E_n$ is an orientation preserving C^2 diffeomorphism relative to D and $\varepsilon(x): E_n \to (0, \infty)$ is continuous, then $\exists$ a homeomorphism $f: E_n \to E_n$ such that f is* p.w.l. *relative to T, and $| g(x) - f(x) | < \varepsilon(x)$ for $x \in E_n$.*

(B) *Let T_1 and T_2 be two arbitrary* p.w.l. *structures on E_n ($n \geqq 7$). If $g: E_n \to E_n$ is an orientation preserving homeomorphism* p.w.l. *relative to T_1 and $\varepsilon(x): E_n \to (0, \infty)$ is continuous, then $\exists$ a homeomorphism $f: E_n \to E_n$ such that f is* p.w.l. *relative to T_2, and $| g(x) - f(x) | < \varepsilon(x)$ for $x \in E_n$.*

(C) *The same as* (A) *with E_n replaced by S_n.*

(D) *The same as* (B) *with E_n replaced by S_n.*

(E) *Let D_1 and D_2 be two arbitray C^2 differentiable structures on E_n ($n \geqq 7$). If $g: E_n \to E_n$ is an orientation preserving diffeomorphism relative to D_1 and $\varepsilon(x): E_n \to (0, \infty)$ is continuous, then $\exists$ a homeomorphism $f: E_n \to E_n$ such that f is a diffeomorphism relative to D_2, and $| g(x) - f(x) | < \varepsilon(x)$ for $x \in E_n$.*

OUTLINE OF PROOF. Since orientation preserving diffeomorphisms and p.w.l. homeomorphisms are stable, each of (A) and (B) is a corollary to Theorem 3, and each of (C) and (D) is a corollary to Theorem 1. Part (E) follows from (A) in the same manner that Theorem 4 follows from Theorem 3. Theorem 6 is true if C^2 is replaced by C^∞.

5. Remarks

The *engulfing lemma* holds for codimension 3 (see [10]). Lemma 1 of this paper, a modification of the *engulfing lemma*, is stated for codimension 4, i.e., the dimension of K is 4 less than the dimension of E_n. It is certainly possible that Lemma 1 also holds for codimension 3. If this is true, the results of this paper hold for $n \geqq 5$ instead of merely for $n \geqq 7$. However, this is of limited interest since dimension 4 would still remain unsolved.

Let M be a topological manifold with or without boundary. A homeomorphism $g: M \to M$ is said to have *property* Q if $\exists$ closed n-cells A and B with $A \subset$ Interior $B \subset M$ such that $g \mid C(A) = I$. Define $G^0(M)$ to be all homeomorphisms $h: M \to M$ which can be expressed as $h = h_v h_{v-1} \cdots h_1$ where each h_i has *property* Q. The group $G^0(M)$ is studied in [4]. If M has a p.w.l. structure, then any $h \in G^0(M)$ can be approximated by a p.w.l. homeomorphism. This may be proved by modifying Theorem 1 or Theorem 3.

Suppose T_1 and T_2 are two arbitrary p.w.l. structures on E_n. It is known (except for $n = 4$) that $\exists$ a homeomorphism $h: E_n \to E_n$ which is p.w.l. from T_1 to T_2 (see [10] and [6]). If h could be chosen as a bounded homeomorphism, then by Lemma 5, h would be stable, and it would follow immediately that all orientation preserving homeomorphisms are stable. Thus the annulus conjecture in dimension n would be true (see [3]). Conversely, if the annulus conjecture were true in all dimensions, it would follow from the procedures of this paper that (for $n \geqq 7$) h could be chosen as bounded. Thus the annulus conjecture is roughly equivalent to this strong form of the Hauptvermutung for euclidean space where h is to be chosen as bounded.

INSTITUTE FOR ADVANCED STUDY

REFERENCES

1. R. D. ANDERSON, *The algebraic simplicity of certain groups of homeomorphisms,* Amer. J. Math., 80 (1958), 955-963.
2. R. H. BING, *An alternate proof that* 3-*manifolds can be triangulated,* Ann. of Math., 69 (1959), 37-65.
3. M. BROWN and H. GLUCK, *Stable structures on manifolds,* I. *Homeomorphisms of* S_n, to be published in Ann. of Math.
4. G. M. FISHER, *On the group of all homeomorphisms of a manifold,* Trans. Amer. Math. Soc., 97 (1960), 193-212.
5. J. MILNOR, On the relationship between differentiable manifolds and combinatorial manifolds, mimeographed notes, Princeton Univ., 1956.

6. E. E. MOISE, *Affine structures in 3-manifolds*, IV. *Piecewise linear approximations of homeomorphisms*, Ann. of Math., 55 (1952), 215-222.
7. J. MUNKRES, *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*, Ann. of Math., 72 (1960) 521-554.
8. R. PENROSE, J. H. C. WHITEHEAD, and E. C. ZEEMAN, *Imbedding of manifolds in euclidean space*, Ann. of Math., 73 (1961), 613-623.
9. J. STALLINGS, *Polyhedral homotopy-spheres*, Bull. Amer. Math. Soc., 66 (1960), 485-488.
10. ———, *The piecewise-linear structure of euclidean space*, Proc. Camb. Philos. Soc., 58 (1962), 481-488.
11. ———, *On topologically unknotted spheres*, Ann. of Math., 77 (1963), 490-503.
12. J. H. C. WHITEHEAD, *Simplicial spaces, nuclei and m-groups*, Proc. London Math. Soc., 45 (1939), 243-327.
13. ———, *On C^1-complexes*, Ann. of Math., 41 (1940), 809-824.
14. E. C. ZEEMAN, *The Poincaré conjecture for $n \geqq 5$*, Topology of 3-Manifolds by M. K. Fort, Jr., Prentice-Hall, 1962, 198-204.
15. ——— and D. R. MCMILLAN, *On contractible open manifolds*, Proc. Camb. Phil. Soc., 58 (1962) 221-224.