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ON CONNECTED SUMS OF MANIFOLDS

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IF W IS homotopy equivalent to a non-trivial connected sum, is W a non-trivial connected sum? For any set of *P.L.* closed manifolds, a positive answer to such problems leads to a homotopy-theoretic characterization of manifolds which are non-trivial connected sums [5].

Write $P \# Q$ to denote the connected sum of two closed manifolds, P and Q, of the same dimension. We say that the closed manifold Y is a non-trivial connected sum if $Y = P \# Q$, with P and Q not homotopy spheres.

In dimension 3, the Kneser conjecture, proved in [14], implies that a *P.L.* manifold W, homotopy equivalent to $P \# Q$, is itself a connected sum of manifolds homotopy equivalent to P and Q. The same situation exists in dimensions greater than 5 if P and Q are simply connected [1] or even just P simply connected [15] or in odd dimensions greater than 5 if the fundamental groups of P and Q have no elements of order 2 [11]. In fact the same situation exists in all dimensions greater than 4 if the fundamental groups of P and Q have no elements of order 2 [4, 5]. This also extends to all orientable $4k+3$ dimensional manifolds, and to all manifolds W^{2k+1} for which each element g of order 2 in $\pi_1(W)$ satisfies $[g] \cap \omega_1(W) = 0$ for k odd, 1 for k even, $\omega_1(W)$ the first Stiefel–Whitney class of W and $[g]$ the class in $H_1(W; Z_2)$ represented by g [5, 7].

This leaves, in dimension not 4, only some cases when $\pi_1(W)$ has elements of order 2. However, in this remaining case this note constructs an *oriented manifold in each dimension* $4k+1$, $k \geq 1$, *which is homotopy equivalent to, but is not itself a non-trivial connected sum.* Precisely, we prove the following result which was announced in [6] and which shows the *necessity of a restriction on fundamental groups in splitting theorems* [7].

THEOREM 1. *There is a closed differentiable* $4k+1$ *dimensional manifold W, simple homotopy equivalent to* $RP^{4k+1} \# RP^{4k+1}$, $k \geq 1$, *which is not as a differentiable, piecewise-linear or even as a topological manifold a non-trivial connected sum.*

The construction of W shows that it is tangentially homotopy equivalent, and even normally cobordant [2] [15] to $RP^{4k+1} \# RP^{4k+1}$.

Remark. For orientable manifolds P and Q, the definition of $P \# Q$ usually requires a

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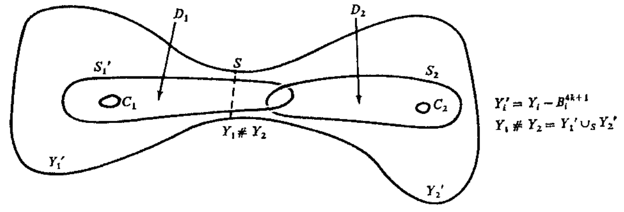
choice of orientations for P and Q [9]. However, as RP^{4k+1} has an orientation reversing diffeomorphism to itself, see Lemma 1, this is not needed to define $RP^{4k+1} \# RP^{4k+1}$.

W will be constructed essentially by the following procedure. Let $Y_1 \cong RP^{4k+1}$ and $Y_2 \cong RP^{4k+1}$ and let g_i denote the non-trivial element of $\pi_1(Y_i) \subset \pi_1(Y_1 \# Y_2)$, $i = 1, 2$. By van Kampen's theorem $\pi_1(Y_1 \# Y_2) = \pi_1(Y_1) * \pi_1(Y_2) = Z_2 * Z_2$. Construct $2k$-dimensional embedded spheres S_1 and S_2 in $Y_1 \# Y_2$ with S_i bounding an immersed disc D_i of dimension $k+1$ with the double points of D_i being a single circle C_i representing g_i and with S_1 and S_2 having linking number 1. Now perform surgery on both S_1 and S_2 to obtain W.



In the proof of Theorem 1, we will describe the linking and self-linking of S_1 and S_2 and the construction of W in terms of a Hermitian form (M_1, λ_1, μ_1) over the ring $Z[Z_2 * Z_2]$. The stable indecomposability proved below of this Hermitian form into forms defined over $Z[Z_2]$, will imply the corresponding indecomposability of W.

Set $S^n = \{(x_1, \ldots, x_{n+1}) \in R^{n+1} \mid x_1{}^2 + x_2{}^2 + \cdots + x_{n+1}^2 = 1\}$; the antipodal map α_n of S^n is given by $\alpha_n(x_1, x_2, \ldots, x_{n+1}) = (-x_1, -x_2, \ldots, -x_{n+1})$. RP^n is the quotient of S^n by the Z_2 action given by α_n. Let β_n denote the map defined on S^n by $\beta_n(x_1, x_2, x_3, \ldots, x_{n+1}) = (-x_1, x_2, x_3, \ldots, x_{n+1})$. We start with two easy lemmas.

LEMMA 1. *RP^{4k+1} has an orientation-reversing diffeomorphism.*

Proof. β_{4k+1} induces an orientation-reversing diffeomorphism on $RP^{4k+1} = S^{4k+1}/\alpha_{4k+1}$.

LEMMA 2. *Let V be a manifold homotopy equivalent to $RP^{4k+1} \# RP^{4k+1}$. If $V = P \# Q$, for some closed manifolds P and Q, with P and Q not homotopy spheres, then P and Q are homotopy equivalent to RP^{4k+1}.*

Proof. First observe that the universal cover of $RP^{4k+1} \# RP^{4k+1}$ is $S^{4k} \times R$. Thus, the universal cover of V is $4k-1$ connected, and hence $\tilde{P}$ and $\tilde{Q}$ are $4k-1$ connected. But as $Z_2 * Z_2 = \pi_1 V = \pi_1 P * \pi_1 Q$, either $\pi_1 P = Z_2$ and $\pi_1 Q = Z_2$ or one of these groups, say $\pi_1 P$, is zero and the other is $Z_2 * Z_2$[10]. But if $\pi_1 P = 0$, $P = \tilde{P}$ is $4k-1$ connected and hence is a homotopy sphere. As we assumed that P and Q were not homotopy spheres we get $\pi_1 P = Z_2$, $\pi_1 Q = Z_2$.

Since there groups are finite, $\tilde{P}$ and $\tilde{Q}$ are closed manifolds, and hence are homotopy spheres. Thus, P and Q are the quotients of free Z_2 actions on homotopy spheres of dimension $4k+1$, and are therefore by an easy argument [12] [15] homotopy equivalent to RP^{4k+1}.

Let S as above denote the $4k$-dimensional sphere joining Y_1 and Y_2, so that $Y = Y_1 \# Y_2 = Y_1' \bigcup_S Y_2'$.

LEMMA 3. *Every homotopy equivalence* $\gamma : RP^{4k+1} \# RP^{4k+1} \to RP^{4k+1} \# RP^{4k+1}$ *is homotopic to a map, which we continue to denote by* γ, *with* γ *transverse to* S *and with* $\gamma^{-1}(S) = S$.

Proof. Let Aut denote the group under composition of homotopy classes of auto-homotopy equivalences of $RP^{4k+1} \# RP^{4k+1}$. Clearly it suffices to check Lemma 3 for a set of generators of Aut.

Let γ_1 denote the orientation-preserving map of $Y = Y_1 \# Y_2 = RP^{4k+1} \# RP^{4k+1}$ which switches both copies of RP^{4k+1}; precisely, γ_1 is induced from the map $\tilde{\gamma}_1$ of the universal cover of $RP^{4k+1} \# RP^{4k+1}$, $\tilde{\gamma}_1 : S^{4k} \times R \to S^{4k} \times R$, $\tilde{\gamma}_1(x, t) = (x, t+1)$. Let γ_2 be the map induced on $RP^{4k+1} \# RP^{4k+1}$ by $\tilde{\gamma}_2 : S^{4k} \times R \to S^{4k} \times R$, $\tilde{\gamma}_2(x, t) = (\beta_{4k}(x), t)$. Lastly, to define γ_3, let $\tau : S^1 \to SO_{4k+2}$ denote the non-trivial element of $\pi_1(SO_{4k+2})$, $S^1 = \{z \in C \mid |z| = 1\}$ with $\tau(1)$ the identity matrix. Let γ_3 be be the map which is the identity outside a neighborhood $S \times I$, $I = [0, 1]$, of S and which restricts to $(\gamma_3 | S \times I) : S \times I \to S \times I$, $\gamma_3(x, t) = (\tau(e^{2\pi i t})(x), t)$, $x \in S$, $t \in I$. Clearly γ_1, γ_2 and γ_3 satisfy the conclusion of Lemma 3, and the proof of Lemma 3 is completed by showing that they generate Aut.

Every automorphism of $\pi_1(Y) = Z_2 * Z_2$ is easily seen to be either an inner automorphism, or the composite of an inner automorphism with γ_{1*}, the automorphism of $Z_2 * Z_2$ which switches both copies of Z_2. Therefore, it suffices to show that $\{\gamma_2, \gamma_3\}$ generate Aut_+, the group of base-point preserving auto-homotopy equivalences $\gamma : Y \to Y$, satisfying $\gamma_* = 1_{\pi_1}(Y) : \pi_1(Y) \to \pi_1(Y)$, classified up to base-point preserving homotopy.

For a basepointed space X, let $[X, Y]$ denote the set of basepoint preserving maps of X to $Y = RP^{4k+1} \# RP^{4k+1}$, classified up to basepoint preserving homotopy. The cofibration sequence

$$S^{4k} \longrightarrow RP^{4k} \vee RP^{4k} \xrightarrow{\ j\ } RP^{4k+1} \# RP^{4k+1} \longrightarrow \Sigma S^{4k} \longrightarrow \cdots$$

gives an induced "exact sequence"

$$[RP^{4k} \vee RP^{4k}, Y] \xleftarrow{\ j_*\ } [Y, Y] \longleftarrow \pi_{4k+1}(Y).$$

Here, "exactness" means that the cosets of this action of $\pi_{4k+1}(Y)$ on $[Y, Y]$ go injectively into $[RP^{4k} \vee RP^{4k}, Y]$. It is easy to see that the orbit of the action of $\pi_{4k+1}(Y) = \pi_{4k+1}(S^{4k} \times R) = Z_2$ on $1_Y \in [Y, Y]$ is $\{1_Y, \gamma_3\}$. Routine obstruction theory then shows that $j_*(\mathrm{Aut}_+) = \{j, \gamma_2 j\}$ and hence Aut_+ is easily seen to be generated by γ_2 and γ_3.

We now precisely describe the construction of W and of a homotopy equivalence of W to Y. Let $H_i \cong Z_2$, $i = 1, 2$, and let g_i denote the non-trivial element of H_i. Let u_1 be the element of the Wall [15] surgery group† $L_{4k+2}(H_1 * H_2)$, represented by the Hermitian. form (M_1, λ_1, μ_1) where

† In our notation for surgery groups, as we are always studying only orientable manifolds, we omit the orientation homorphisms to Z_2.

(i) M_1 is a free $Z[H_1 * H_2]$ module on two generators $\{e_1, f_1\}$;

(ii) $\lambda(e_1, e_1) = \lambda(f_1, f_1) = 0$, $\lambda(e_1, f_1) = 1$;

(iii) $\mu(e_1) = g_1$, $\mu(f_1) = g_2$.

Realize [15] the element u_1 by a $4k + 2$ dimensional normal cobordism (T, F)

$$F: T^{4k+2} \to Y, \partial T = Y \cup W, (F|Y) = 1_Y,$$

$f = (F|W)$ a simple homotopy equivalence.

Covering bundle maps, not recorded in our notation, are of course part of the structure of this normal map [2].

LEMMA 4. *The simple homotopy equivalence $f: W \to Y$ is not homotopic to a map transverse regular to $S \subset Y$ with $f^{-1}(S) \to S$ a homotopy equivalence.*

We defer the proof of Lemma 4.

Proof of Theorem. If $W = P \# Q$, P, Q not homotopy spheres, by Lemma 2 there are homotopy equivalences $g_1 : P \to RP^{4k+1}$, $g_2 : Q \to RP^{4k+1}$. Clearly, g_1 and g_2 induce a homotopy equivalence $g: P \# Q \to RP^{4k+1} \# RP^{4k+1}$ with $g^{-1}(S) \to S$ a homeomorphism. But f is, up to homotopy, $(fg^{-1})g$ and fg^{-1} is by Lemma 3 homotopic to a map γ with $\gamma^{-1}(S) = S$. Hence, varying f by a homotopy to get $f = \gamma g$ we get $f^{-1}(S) = g^{-1}(\gamma^{-1}(S)) = g^{-1}(S)$, which is homotopy equivalent to S. This contradicts Lemma 4.

Proof of Lemma 4. Assume, contrary to the conclusion of Lemma 4, that f is homotopic to a map with $f^{-1}(S) \to S$ a homotopy eqiuvalence. Keeping it fixed on ∂T, make F transverse to $S \subset Y$. Let $V = F^{-1}(S)$, a normal cobordism of $(S, 1_S)$ to $(f^{-1}(S), f|f^{-1}(S))$. As $L_{4k+1}(0) = 0$ the normal map $V \to S$ is normally cobordant, relative to the boundary, to a simple homotopy equivalence. Hence, by the normal cobordism extension lemma [2], (T, F) is normally cobordant relative to the boundary to a normal cobordism (T', F') with $F'^{-1}(S) \to S$ a homotopy equivalence. Splitting T' along $F'^{-1}(S)$, the normal cobordism (T', F') is seen to be produced by pasting together normal maps, restricting to homotopy equivalences on the boundary, to manifolds with fundamental group H_1 and H_2. But then the surgery obstruction of (T', F') which equals u_1, the surgery obstruction of (T, F), is in Image $(L_{4k+2}(H_1) \oplus L_{4k+2}(H_2) \to L_{4k+2}(H_1 * H_2))$. We complete the proof by showing that in fact

$$u_1 \notin \text{Image}(L_{4k+2}(H_1) \oplus L_{4k+2}(H_2) \to L_{4k+2}(H_1 * H_2))$$

and hence f is not homotopic to a map with $f^{-1}(S) \to S$ a homotopy equivalence.

Recall [15] that $Z_2 \cong L_{4k+2}(0) \to L_{4k+2}(H_i)$, $i = 1, 2$ is an isomorphism and hence it suffices to show that $u_1 \notin \text{Image}(L_{4k+2}(0) \to L_{4k+2}(H_1 * H_2))$. Under the obvious retraction $L_{4k+2}(H_1 * H_2) \to L_{4k+2}(0)$, (M_1, λ_1, μ_1) goes to a Hermitian form with non-zero Arf invariant and hence $u_1 \neq 0$. Thus it suffices to show $u_1 \neq u_0$ where u_0 is the non-trivial element of $\text{Image}(L_{4k+2}(0) \to L_{4k+2}(H_1 * H_2))$. Recall [9] [2] that u_0 is represented by the Hermitian form (M_0, λ_0, μ_0) where

(i) M_0 is the free $Z[H_1 * H_2]$ module on 2 generators $\{e_0, f_0\}$;

(ii) $\lambda_0(e_0, e_0) = \lambda_0(f_0, f_0) = 0$, $\lambda_0(e_0, f_0) = 1$;

(iii) $\mu(e_0) = 1$, $\mu(f_0) = 1$.

To show $u_1 \neq u_0$, we define a homomorphism which has value 0 on u_1 but is non-zero on u_0. The definition of this homomorphism involves the transfer construction and transfer homorphism for surgery theory [15; p. 242]. The following definition is convenient for the purpose of explicit computation. Let G be a group, H a subgroup of finite index in G, $\omega_G : G \to Z_2$ a homomorphism restricting to $\omega_H : H \to Z_2$. As a $Z[H]$ bimodule, $Z[G] \cong Z[H] \oplus \widehat{Z[G]}$ where $\widehat{Z[G]}$ is additively generated by $g \in \{G - H\}$. Given a $(-1)^k$ Hermitian form (M, λ, μ) defined over $Z[G]$, corresponding to the decomposition of $Z[G]$ we may write $\lambda = \lambda^H \oplus \lambda^{\widehat{Z[G]}}$, $\mu = \mu^H \oplus \mu^{\widehat{Z[G]}}$. Now let the transfer of (M, λ, μ) be the $(-1)^k$ Hermitian over $Z[H]$, $\operatorname{tr}(M, \lambda, \mu) = (M, \lambda^H, \mu^H)$. This induces a homomorphism $\operatorname{tr}_H : L_{2k}{}^h(G, \omega_G) \to L_{2k}{}^h(H, \omega_H)$.

We apply this to the case where G is the non-commutative group of order 6, H any subgroup of order 2 in G, ω_G and hence also ω_H trivial. Let $\varphi : H_1 * H_2 \to G$ be a surjective homomorphism i.e. let $\varphi(g_1)$ and $\varphi(g_2)$ be different elements of order 2 in G. We complete the argument by showing that $\operatorname{tr}_H \varphi_*(u_1) = 0$ and $\operatorname{tr}_H \varphi_*(u_0) \neq 0$ in $L^h_{4k+2}(H)$ by explicitly computing the Arf-ivariants of $\operatorname{tr}_H \varphi_*(M_0, \lambda_0, \mu_0)$ and of $\operatorname{tr}_H \varphi_*(M_1, \lambda_1, \mu_1)$.

We continue to write λ_i and μ_i for the $Z[G]$ Hermitian form induced by λ_i and μ_i on $M_i' = M_i \oplus_{Z[H_1 * H_2]} Z[G]$, $i = 0, 1$. Clearly $\operatorname{tr}_H \varphi_*(M_i, \lambda_i, \mu_i) = (M_i', \lambda_i^H, \mu_i^H)$ for $i = 0$ or $i = 1$ where:

(i) As a $Z[H]$ module, M_i' is generated by the six elements $\{e_i \otimes V^j, f_i \otimes V^j\}$, $0 \le j \le 2$ where V is a fixed choice of an element of order 3 in G.

(ii) As $\lambda_i(e_i \otimes V^j, e_i \otimes V^m) = 0$, $\lambda_i^H(e_i \otimes V^j, e_i \otimes V^m) = 0$. Similarly

$$\lambda_i^H(f_i \otimes V^j, f_i \otimes V^m) = 0.$$

Moreover, as $\lambda_i(e_i \otimes V^j, f_i \otimes V^m) = V^{m-j}$ and for $0 \le m \le 2$, $0 \le j \le 2$, V^{m-j} is an element of H only for $m = j$, we get $\lambda_i^H(e_i \otimes V^j, f_i \otimes V^m) = 0$, $m \neq j$, $0 \le j \le 2$, $0 \le m \le 2$, and $\lambda_i^H(e_i \otimes V^j, f_i \otimes V^j) = 1$, $0 \le j \le 2$.

(iii) *Case* $i = 0$: First observe that $\mu_0(e_0 \otimes V^j) = V^{-j}\mu_0(e_0)V^j = V^{-j}V^j$ and

$$\mu_0(f_0 \otimes V^j) = V^{-j}\mu_0(f_0)V^j = V^{-j}V^j = 1.$$

Hence for each j, $0 \le j \le 2$, $\mu_0{}^H(e_0 \otimes V^j) = 1$ and $\mu_0{}^H(f_0 \otimes V^j) = 1$.

Case $i = 1$. First observe that $\mu_1(e_1 \otimes V^j) = V^{-j}\mu_1(e_1)V^j = V^{-j}\varphi(g_1)V^j$ and similarly $\mu_1(f_1 \otimes V^j) = V^{-1}\mu_1(f_1)V^j = V^{-j}\varphi(g_2)V^j$. But as $\varphi(g_1) \neq \varphi(g_2)$ and

$$V^{-j}\varphi(g_1)V^j \neq V^{-j}\varphi(g_2)V^j$$

and as they both have order 2, $V^{-j}\varphi(g_1)V^j$ and $V^{-j}\varphi(g_2)V^j$ cannot both be in $H \cong Z_2$. Hence for each j, $0 \le j \le 2$, $\mu_1{}^H(e_1 \otimes V^j) = 0$ or $\mu_1{}^H(f_1 \otimes V^j) = 0$.

Letting $P : Z[H] \to Z_2$ denote the unique ring homomorphism, from (ii) the Arf-invariant of $(M_i', \lambda_i^H, \mu_i^H)$ is given by $\sum_{j=0}^{2} P(\mu_i{}^H(e_i \otimes V^j))P(\mu_i{}^H(f_i \otimes V^j))$ [9]. By (iii) for $i = 0$, this sum is 1 and for $i = 1$, this sum is 0. Thus, $\operatorname{tr}_H \varphi_*(u_0) \neq \operatorname{tr}_H \varphi_*(u_1)$ and $u_0 \neq u_1$.

Geometrically, the above proof could be interpreted as showing that u_1 is represented by the normal cobordism T with the Kervaire invariant of T being 1 but with the Kervaire-invariant of a 3-fold covering space of T being 0.

The following result could be used to construct infinitely many different smooth manifolds, homotopy equivalent to $RP^{4k+1} \# RP^{4k+1}$, which are not non-trivial connected sums.

THEOREM 2. *$\bigotimes_{\infty} Z_2 \subset L_2(Z_2 * Z_2)$ and $\bigotimes_{\infty} Z_2 \subset L_2(Z * Z_2)$.*

Outline of Proof. Let $u_p \in L_2(Z_2 * Z_2)$, p an odd prime, be the element represented by (M_p, λ_p, μ_p) with M_p generated by $\{e_p, f_p\}$ and $\lambda_p(e_p, e_p) = \lambda_p(f_p, f_p) = 0$, $\lambda_p(e_p, f_p) = 1$, $\mu_p(e_p) = g_1$, $\mu_p(f_p) = (g_2 g_1)^p g_1 (g_2 g_1)^{-p}$. Define $\theta_q : Z_2 * Z_2 \to Z_2 * Z_2/\langle (g_2 g_1)^q \rangle$, q an odd prime, and let $H_q = \{\theta_q(1), \theta_q(g_1)\}$. Then the Arf invariant of $\mathrm{tr}_{H_q} \theta_{q*}(u_p)$ is 0 if $p \neq q$ and 1 if $p = q$. Hence $\{u_p \,|\, p \text{ an odd prime}\}$ are linearly independent over Z_2. Moreover, clearly $u_p \in \mathrm{Image}(L_2(Z_2 * Z) \xrightarrow{f_*} L_2(Z_2 * Z_2))$, $f: Z_2 * Z \to Z_2 * Z_2$ the obvious surjection.

Note that Theorem 2 provides counterexamples to a splitting theorem of Miščenko [13, p. 676].

Further results on surgery groups of free products will be presented in [8].

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