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The asymmetry of an anti-automorphism

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Abstract

The asymmetry of a nonsingular pairing on a vector space is an endomorphism of the space on which the classification of arbitrary pairings (not necessarily symmetric or skew-symmetric) is based. A general notion of asymmetry is defined for arbitrary anti-automorphisms on a central simple algebra, and conditions are given to characterize the elements which are the asymmetries of some anti-automorphism. The asymmetry is used to define the determinant of an anti-automorphism. © 2002 Elsevier Science B.V. All rights reserved.

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0. Introduction

The asymmetry of an arbitrary nonsingular pairing (not necessarily symmetric or skew-symmetric) on a finite-dimensional vector space V is an invertible endomorphism of V which is an important invariant of the pairing. It is 1 if and only if the pairing is symmetric and -1 if and only if it is skew-symmetric. This invariant was first considered by Williamson [9], and more recently by Riehm [6].

In the present paper, we determine under which conditions a linear map $a \in \text{GL}(V)$ is the asymmetry of some nonsingular pairing on V : the map a must be conjugate to its inverse and satisfy some conditions on the generalized eigenspaces of eigenvalues $+1$ and -1 , see Theorem 1. As pointed out by Ranicki, the property that a is an

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asymmetry could be rephrased by saying that a certain asymmetric Poincaré complex of dimension 1 is round simple null-cobordant. (See [5, Chapter 20] for background information on Poincaré complexes.)

In Section 2, we define the asymmetry of an anti-automorphism σ on a central simple algebra A : it is an element $a_\sigma \in A^\times$ which is mapped, under scalar extension to a splitting field of A , to the asymmetry of any nonsingular pairing to which σ is adjoint. It is defined up to sign by the properties that $\sigma^2(x) = a_\sigma x a_\sigma^{-1}$ for all $x \in A$ and that $\sigma(a_\sigma) = a_\sigma^{-1}$. This element was incidentally used by Saltman [7, Lemma 3.3, Theorem 4.4] to show that if an Azumaya algebra A carries an anti-automorphism, then the ring of 2×2 matrices $M_2(A)$ carries an involution, and that Azumaya algebras over connected semilocal rings which are isomorphic to their opposite have an involution. We show that in a central simple algebra of exponent 2, an invertible element is the asymmetry of some anti-automorphism if and only if it is conjugate to its inverse (Theorem 2). Albert's theorem that every central simple algebra of exponent 2 has an involution is an immediate consequence, since involutions are the anti-automorphisms of asymmetry ± 1 . In the final section, the asymmetry is used to define the determinant of an anti-automorphism.

1. The asymmetry of a nonsingular pairing

Throughout this section, V denotes a finite-dimensional vector space over an arbitrary field F . We define the asymmetry and the adjoint anti-automorphism of a nonsingular pairing on V , and determine which linear transformations of V are asymmetries.

1.1. Definitions

Let $V^* = \text{Hom}_F(V, F)$ be the dual of V . Every pairing (or bilinear form) $b: V \times V \rightarrow F$ induces a linear map $\hat{b}: V \rightarrow V^*$ which carries $x \in V$ to $b(x, \bullet) \in V^*$. The transpose map $\hat{b}^t: V = V^{**} \rightarrow V^*$ carries $x \in V$ to $b(\bullet, x) \in V^*$.

Proposition 1. *For a pairing b on V , the following conditions are equivalent:*

- (a) *if $x \in V$ is such that $b(x, y) = 0$ for all $y \in V$, then $x = 0$;*
- (b) *if $y \in V$ is such that $b(x, y) = 0$ for all $x \in V$, then $y = 0$;*
- (c) *the map \hat{b} is bijective.*

If these conditions hold, the pairing b is called *nonsingular*.

Proof. Condition (a) is equivalent to injectivity of \hat{b} , and (b) to injectivity of \hat{b}^t , hence also to surjectivity of \hat{b} . Since $\dim V = \dim V^*$, each of these conditions implies that \hat{b} is bijective. \square

All the pairings considered in the sequel are nonsingular. To every nonsingular pairing b on V we attach an anti-automorphism σ_b of $\text{End}_F V$ and a linear transformation $a_b \in \text{GL}(V)$ as follows:

Proposition 2. *Let b be a nonsingular pairing on V . There is a unique map $\sigma_b: \text{End}_F V \rightarrow \text{End}_F V$ and a unique map $a_b: V \rightarrow V$ such that*

$$b(f(x), y) = b(x, \sigma_b(f)(y)) \quad \text{for all } x, y \in V, f \in \text{End}_F V \quad (1)$$

and

$$b(x, y) = b(y, a_b(x)) \quad \text{for all } x, y \in V. \quad (2)$$

The map σ_b is an F -linear anti-automorphism of $\text{End}_F V$ and the map a_b is linear and invertible. These maps satisfy the following properties:

- (i) $\sigma_b^2(f) = a_b \circ f \circ a_b^{-1}$ for all $f \in \text{End}_F V$;
- (ii) $\sigma_b(a_b) = a_b^{-1}$.

Proof. For $f \in \text{End}_F V$, let $\sigma_b(f) = (\hat{b} \circ f \circ \hat{b}^{-1})^t$. Equality (1) is easily checked, and the fact that σ_b is an F -linear anti-automorphism of $\text{End}_F V$ follows. Uniqueness of σ_b follows from the hypothesis that b is nonsingular.

On the other hand, let $a_b = (\hat{b}^t)^{-1} \circ \hat{b}$. This map is clearly linear and invertible, and it satisfies (2). Uniqueness of a_b is clear. To check the additional properties, observe that for $f \in \text{End}_F V$

$$\sigma_b^2(f) = (\hat{b} \circ (\hat{b} \circ f \circ \hat{b}^{-1})^t \circ \hat{b}^{-1})^t = ((\hat{b}^t)^{-1} \circ \hat{b}) \circ f \circ ((\hat{b}^t)^{-1} \circ \hat{b})^{-1}$$

and

$$\sigma_b((\hat{b}^t)^{-1} \circ \hat{b}) = (\hat{b} \circ ((\hat{b}^t)^{-1} \circ \hat{b}) \circ \hat{b}^{-1})^t = ((\hat{b}^t)^{-1} \circ \hat{b})^{-1}. \quad \square$$

We call σ_b the anti-automorphism *adjoint* to b . Using the Skolem–Noether theorem, it is easily seen that every F -linear anti-automorphism of $\text{End}_F V$ is adjoint to some nonsingular pairing, see [4, p. 1]. The map a_b is called the *asymmetry* of b . From the definition, it is clear that the adjoint anti-automorphism and the asymmetry of any scalar multiple of b are the same as those of b . Moreover, the map a_b is determined up to sign by properties (i) and (ii).

We combine a_b and σ_b into a linear involution of $\text{End}_F V$ as follows:

Proposition 3. *Let b be a nonsingular pairing on V . There is a unique linear map $\gamma_b: \text{End}_F V \rightarrow \text{End}_F V$ such that*

$$b(x, f(y)) = b(y, \gamma_b(f)(x)) \quad \text{for all } x, y \in V, f \in \text{End}_F V. \quad (3)$$

This map satisfies the following additional properties:

- (i) $\gamma_b(f \circ g \circ h) = \sigma_b(h) \circ \gamma_b(g) \circ \sigma_b^{-1}(f)$ for $f, g, h \in \text{End}_F V$;
- (ii) $\gamma_b^2 = \text{Id}_{\text{End}_F V}$;
- (iii) $\gamma_b(\text{Id}_V) = a_b$.

Proof. Set $\gamma_b(f) = \sigma_b(f) \circ a_b (= a_b \circ \sigma_b^{-1}(f))$ for $f \in \text{End}_F V$; then (iii) is clear and (3), (i), (ii) follow from the properties of σ_b and a_b . \square

We call γ_b the *linear involution* of $\text{End}_F V$ associated to b . As for the adjoint anti-automorphism σ_b and the asymmetry a_b , it is clear that γ_b is also the linear involution associated to any scalar multiple of b .

Remark. There are corresponding notions for pairings on faithfully projective modules with values in invertible modules (over an arbitrary commutative ring R): see [3, Chapter III, (8.2)].

1.2. Characterization of asymmetries

The goal of this subsection is to answer the following question: Under which conditions on a map $a \in \text{GL}(V)$ does there exist a nonsingular pairing b on V whose asymmetry is a , i.e., such that $a_b = a$? Identifying $\text{End}_F V$ with a matrix algebra $M_n(F)$ through the choice of a basis of V , this amounts to asking for which invertible matrices $a \in \text{GL}_n(F)$ the equation $a = (x^t)^{-1}x$ has a solution $x \in \text{GL}_n(F)$, in view of the definition of a in terms of \hat{b} in the proof of Proposition 2.

The conditions involve the following vector spaces: for an arbitrary integer $m \geq 1$ and $\varepsilon = \pm 1$, we let

$$V_m^\varepsilon = \frac{\ker(a - \varepsilon \text{Id}_V)^m}{\ker(a - \varepsilon \text{Id}_V)^{m-1} + (a - \varepsilon \text{Id}_V)(\ker(a - \varepsilon \text{Id}_V)^{m+1})}.$$

Theorem 1. *Suppose $\text{char } F \neq 2$. A map $a \in \text{GL}(V)$ is the asymmetry of some nonsingular pairing on V if and only if the following conditions hold:*

- (1) *a is conjugate to a^{-1} in $\text{GL}(V)$;*
- (2) *for every even integer m , $\dim V_m^{+1}$ is even;*
- (3) *for every odd integer m , $\dim V_m^{-1}$ is even.*

If $\text{char } F = 2$, a map $a \in \text{GL}(V)$ is the asymmetry of some nonsingular pairing on V if and only if conditions (1) and (2) hold.

Proof. We first show that the conditions are necessary. Suppose b is a nonsingular pairing on V such that $a_b = a$. Proposition 2 shows that $\sigma_b(a) = a^{-1}$. To see how this equality implies condition (1), we argue in terms of matrices. Using a basis of V , we identify $\text{End}_F V$ with the matrix algebra $M_n(F)$. Since the transpose map t is an anti-automorphism, $\sigma_b \circ t$ is a linear automorphism of $M_n(F)$, hence the Skolem–Noether theorem yields an invertible matrix u such that $\sigma_b \circ t$ is the conjugation by u . Then $\sigma_b(x) = ux^t u^{-1}$ for all $x \in M_n(F)$. In particular, since $\sigma_b(a) = a^{-1}$ it follows that a^{-1} is conjugate to a^t . But it is well known that every matrix is conjugate to its transpose, hence condition (1) is proved.

To show that conditions (2) and (3) are necessary if $\text{char } F \neq 2$, we show that the nonsingular pairing b induces a nonsingular skew-symmetric pairing on V_m^{+1} if m is even and on V_m^{-1} if m is odd. Conditions (2) and (3) follow because only even-dimensional vector spaces carry nonsingular skew-symmetric pairings if the characteristic of the base field is different from 2.

Fix some integer m and $\varepsilon = \pm 1$. For the convenience of notation, we let

$$U_m^\varepsilon = \ker(a - \varepsilon \text{Id}_V)^m,$$

so $V_m^\varepsilon = U_m^\varepsilon / (U_{m-1}^\varepsilon + (a - \varepsilon \text{Id}_V)(U_{m+1}^\varepsilon))$. For $x, y \in U_m^\varepsilon$, define

$$b_m^\varepsilon(x, y) = b(x, (a - \varepsilon \text{Id}_V)^{m-1}(y)).$$

Since $y \in U_m^\varepsilon$, we have

$$a \circ (a - \varepsilon \text{Id}_V)^{m-1}(y) = \varepsilon(a - \varepsilon \text{Id}_V)^{m-1}(y), \quad (4)$$

hence

$$\begin{aligned} b(y, (a - \varepsilon \text{Id}_V)^{m-1}(x)) &= \varepsilon b(y, a \circ (a - \varepsilon \text{Id}_V)^{m-1}(x)) \\ &= \varepsilon b((a - \varepsilon \text{Id}_V)^{m-1}(x), y). \end{aligned} \quad (5)$$

On the other hand, equality (4) yields

$$\begin{aligned} (a - \varepsilon \text{Id}_V)^{m-1}(y) &= (\varepsilon a^{-1})^{m-1}(a - \varepsilon \text{Id}_V)^{m-1}(y) \\ &= (-1)^{m-1} \sigma_b(a - \varepsilon \text{Id}_V)^{m-1}(y), \end{aligned}$$

hence

$$b((a - \varepsilon \text{Id}_V)^{m-1}(x), y) = (-1)^{m-1} b(x, (a - \varepsilon \text{Id}_V)^{m-1}(y)). \quad (6)$$

Comparing (5) and (6), we obtain

$$b_m^\varepsilon(y, x) = (-1)^{m-1} \varepsilon b_m^\varepsilon(x, y).$$

Therefore, b_m^ε is a skew-symmetric bilinear form on U_m^ε if $\varepsilon = +1$ and m is even, and also if $\varepsilon = -1$ and m is odd.

To see that b_m^ε induces a nonsingular pairing on V_m^ε , we consider the radical of b_m^ε , which is

$$\text{rad } b_m^\varepsilon = \{x \in U_m^\varepsilon \mid b(x, z) = 0 \text{ for all } z \in (a - \varepsilon \text{Id}_V)^{m-1}(U_m^\varepsilon)\}.$$

Thus, $\text{rad } b_m^\varepsilon$ is the intersection of U_m^ε with the orthogonal² complement for the form b of

$$(a - \varepsilon \text{Id}_V)^{m-1}(U_m^\varepsilon) = \text{im } (a - \varepsilon \text{Id}_V)^{m-1} \cap \ker(a - \varepsilon \text{Id}_V),$$

² If b is not symmetric nor skew-symmetric, one has to distinguish orthogonality on the left and on the right; the orthogonal complements of a -invariant subspaces coincide, however.

which is $\ker \sigma_b(a - \varepsilon \text{Id}_V)^{m-1} + \text{im } \sigma_b(a - \varepsilon \text{Id}_V)$. Since $\sigma_b(a) = a^{-1}$, we have $\ker \sigma_b(a - \varepsilon \text{Id}_V)^{m-1} = \ker(a - \varepsilon \text{Id}_V)^{m-1}$ and $\text{im } \sigma_b(a - \varepsilon \text{Id}_V) = \text{im}(a - \varepsilon \text{Id}_V)$, hence

$$\begin{aligned} \text{rad } b_m^\varepsilon &= (U_{m-1}^\varepsilon + \text{im}(a - \varepsilon \text{Id}_V)) \cap U_m^\varepsilon \\ &= U_{m-1}^\varepsilon + (\text{im}(a - \varepsilon \text{Id}_V) \cap U_m^\varepsilon) \\ &= U_{m-1}^\varepsilon + (a - \varepsilon \text{Id}_V)(U_{m+1}^\varepsilon). \end{aligned}$$

Therefore, b_m^ε induces a nonsingular pairing on $U_m^\varepsilon / (U_{m-1}^\varepsilon + (a - \varepsilon \text{Id}_V)(U_{m+1}^\varepsilon)) = V_m^\varepsilon$.

Suppose now $\text{char } F = 2$. The arguments above still show that b_m^ε induces a nonsingular bilinear pairing on V_m^ε , but in characteristic 2 skew-symmetric pairings are symmetric, hence we cannot conclude that $\dim V_m^\varepsilon$ is even. To show that $\dim V_m^{+1}$ is even if m is even, we show that b_m^{+1} is in fact alternating if m is even. For $x \in U_m^{+1}$ we have

$$(a - \text{Id}_V)^{m-2}(x) \in \ker(a - \text{Id}_V)^2 = \ker(a^2 - \text{Id}_V),$$

hence $a^2 \circ (a - \text{Id}_V)^{m-2}(x) = (a - \text{Id}_V)^{m-2}(x)$. Since m is even, we obtain by induction

$$a^{m-2} \circ (a - \text{Id}_V)^{m-2}(x) = (a - \text{Id}_V)^{m-2}(x),$$

hence

$$(a - \text{Id}_V)^{m-2}(x) = a^{2-m} \circ (a - \text{Id}_V)^{m-2}(x) = \sigma(a - \text{Id}_V)^{m-2}(x).$$

Therefore,

$$b(x, (a - \text{Id}_V)^{m-2}(x)) = b((a - \text{Id}_V)^{m-2}(x), x) = b(x, a \circ (a - \text{Id}_V)^{m-2}(x)).$$

It follows that $b(x, (a - \text{Id}_V)^{m-1}(x)) = 0$, hence b_m^{+1} is alternating. This completes the proof that the conditions are necessary.

To prove that the conditions are sufficient, we shall make V into a module over the ring $F[X, X^{-1}]$ of Laurent polynomials in one indeterminate X . As a preparation, we make some observations on the prime ideals of this principal ideal domain.

Let J be the automorphism of $F[X, X^{-1}]$ which maps X to X^{-1} . We also denote by J the extension of this automorphism to the field of fractions $F(X)$ and to the factor module $E = F(X)/F[X, X^{-1}]$. Every prime ideal $P \subset F[X, X^{-1}]$ is generated by an irreducible polynomial of the form

$$\pi = a_0 + a_1X + \cdots + a_dX^d \in F[X]$$

such that $a_0, a_d \neq 0$. If $P^J = P$, the Laurent polynomials π, π^J differ by a factor which is invertible in $F[X, X^{-1}]$, hence $\pi = \alpha X^d \pi^J$ for some $\alpha \in F^\times$. Comparing coefficients, we have

$$a_i = \alpha a_{d-i} \quad \text{for } i = 0, \dots, d,$$

hence $a_d = \alpha a_0 = \alpha^2 a_d$ and therefore $\alpha = \pm 1$. If d is odd, then

$$\pi = \sum_{i=0}^{(d-1)/2} a_i (X^i + \alpha X^{d-i}),$$

hence π is divisible by $1 + \alpha X$. As π is irreducible, we may then choose $\pi = X + 1$ if $\alpha = 1$, and $\pi = X - 1$ if $\alpha = -1$. Suppose next d is even. If $\alpha = -1$ and $\text{char } F \neq 2$, then $a_{d/2} = -a_{d/2}$ implies $a_{d/2} = 0$. In that case, we have

$$\pi = \sum_{i=0}^{d/2-1} a_i(X^i - X^{d-i}),$$

hence π is divisible by $1 - X$. This is a contradiction, since π is assumed to be irreducible. Therefore, $\alpha = 1$ and $(X^{d/2}\pi^{-1})^J = X^{d/2}\pi^{-1}$. We may then choose π of the form

$$\pi = 1 + a_1X + a_2X^2 + \cdots + a_2X^{d-2} + a_1X^{d-1} + X^d.$$

Let \mathcal{R}_1 be the set of irreducible polynomials of this form.

For each pair of prime ideals $\{P, P^J\}$ with $P^J \neq P$, we arbitrarily choose a generator $\pi \in F[X]$ of one of P, P^J and denote by \mathcal{R}_2 the set of irreducible polynomials thus chosen. Thus, the set of prime ideals of $F[X, X^{-1}]$ is $\{\pi F[X, X^{-1}]\}$ where π runs over the set $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_2^J \cup \{X - 1, X + 1\}$, and we have $\pi^J F[X, X^{-1}] \neq \pi F[X, X^{-1}]$ if and only if $\pi \in \mathcal{R}_2 \cup \mathcal{R}_2^J$.

Returning to the proof of Theorem 1, we define a structure of $F[X, X^{-1}]$ -module on V by letting

$$X \cdot v = a(v) \quad \text{for all } v \in V.$$

Since $F[X, X^{-1}]$ is a principal ideal domain, the $F[X, X^{-1}]$ -module V decomposes as a (finite) direct sum of quotients of $F[X, X^{-1}]$, as follows:

$$V \simeq \bigoplus_{\pi, m} (F[X, X^{-1}]/\pi^m)^{\mu(\pi, m)}$$

for some integers $\mu(\pi, m)$ which all vanish except a finite number, where π runs over $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_2^J \cup \{X - 1, X + 1\}$, and m over the positive integers.

Condition (1) shows that the elementary divisors of a are the same as those of a^{-1} , hence

$$V \simeq \bigoplus_{\pi, m} (F[X, X^{-1}]/(\pi^J)^m)^{\mu(\pi, m)}.$$

Therefore, we have $\mu(\pi, m) = \mu(\pi^J, m)$ for all m if $\pi \in \mathcal{R}_2$.

For all integers m and for $\varepsilon = \pm 1$ we have

$$\dim V_m^\varepsilon = \mu(X - \varepsilon, m).$$

Therefore, condition (2) says that $\mu(X - 1, m)$ is even for all m even, and condition (3) says that $\mu(X + 1, m)$ is even for all m odd. Assuming $\text{char } F \neq 2$ and conditions (1), (2) and (3) hold, we may decompose V into a direct sum of six $F[X, X^{-1}]$ -submodules

$$V = V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5 \oplus V_6,$$

where

$$\begin{aligned}
 V_1 &\simeq \bigoplus_{\pi \in \mathcal{R}_1} \bigoplus_m (F[X, X^{-1}]/\pi^m)^{\mu(\pi, m)}, \\
 V_2 &\simeq \bigoplus_{\pi \in \mathcal{R}_2} \bigoplus_m (F[X, X^{-1}]/\pi^m \oplus F[X, X^{-1}]/(\pi^J)^m)^{\mu(\pi, m)}, \\
 V_3 &\simeq \bigoplus_{m \text{ odd}} (F[X, X^{-1}]/(X-1)^m)^{\mu(X-1, m)}, \\
 V_4 &\simeq \bigoplus_{m \text{ even}} (F[X, X^{-1}]/(X-1)^m \oplus F[X, X^{-1}]/(X-1)^m)^{\mu(X-1, m)/2}, \\
 V_5 &\simeq \bigoplus_{m \text{ even}} (F[X, X^{-1}]/(X+1)^m)^{\mu(X+1, m)}, \\
 V_6 &\simeq \bigoplus_{m \text{ odd}} (F[X, X^{-1}]/(X+1)^m \oplus F[X, X^{-1}]/(X+1)^m)^{\mu(X+1, m)/2}.
 \end{aligned}$$

If $\text{char } F = 2$ and conditions (1), (2) hold, there is a similar decomposition

$$V = V_1 \oplus V_2 \oplus V_3 \oplus V_4,$$

where V_1, \dots, V_4 are as above. We shall show below (see Lemma 1) that there are nonsingular $(-X)$ -hermitian forms with values in E (with respect to J) on

$$\begin{aligned}
 &F[X, X^{-1}]/\pi^m \quad \text{if } \pi \in \mathcal{R}_1, \\
 &F[X, X^{-1}]/\pi^m \oplus F[X, X^{-1}]/(\pi^J)^m \quad \text{if } \pi \in \mathcal{R}_2, \\
 &F[X, X^{-1}]/(X-1)^m \quad \text{if } m \text{ is odd,} \\
 &(F[X, X^{-1}]/(X-1)^m)^2 \quad \text{if } m \text{ is even,} \\
 &F[X, X^{-1}]/(X+1)^m \quad \text{if } m \text{ is even and } \text{char } F \neq 2, \\
 &(F[X, X^{-1}]/(X+1)^m)^2 \quad \text{if } m \text{ is odd and } \text{char } F \neq 2.
 \end{aligned} \tag{7}$$

The orthogonal sum of these forms yields a nonsingular $(-X)$ -hermitian form

$$h: V \times V \rightarrow E$$

with respect to J . As Ischebeck-Scharlau [2] or Waterhouse [8], define an F -linear map $T: E \rightarrow F$ by observing that every element in E is represented by a unique rational fraction f which has a zero at ∞ and does not have a pole at 0, and letting

$$T(f + F[X, X^{-1}]) = f(0).$$

It is easily verified that $T(r^J) = -T(r)$ for all $r \in E$. Moreover, for every nonzero $r \in E$ there exists an integer k such that $T(X^{-k}r) \neq 0$, hence T does not vanish on any nonzero $F[X, X^{-1}]$ -submodule of E .

Let $T_*(h): V \times V \rightarrow F$ be the transfer bilinear map, defined by

$$T_*(h)(x, y) = T(h(x, y)) \quad \text{for } x, y \in V.$$

If $x \in V$ is such that $T_*(h)(x, y) = 0$ for all $y \in V$, then T vanishes on the $F[X, X^{-1}]$ -submodule $h(x, V)$, hence $h(x, V) = \{0\}$ and therefore $x = 0$ since h is nonsingular. This shows that $T_*(h)$ is nonsingular.

Moreover, since h is $(-X)$ -hermitian we have

$$\begin{aligned} T_*(h)(y, x) &= T((-X)h(x, y)^J) = -T(Xh(x, y)^J) \\ &= T(X^J h(x, y)) = T(h(x, Xy)) = T_*(h)(x, a(y)) \end{aligned}$$

for all $x, y \in V$. Therefore, a is the asymmetry of $T_*(h)$.

To complete the proof, we prove the existence of nonsingular $(-X)$ -hermitian forms as asserted above.

Lemma 1. *There are nonsingular $(-X)$ -hermitian forms with values in E (with respect to J) on the modules listed in (7).*

Proof. Suppose first $\pi \in \mathcal{R}_1$, hence $(X^{d/2}\pi^{-1})^J = X^{d/2}\pi^{-1}$, where d is the degree of π . For $u, v \in F[X, X^{-1}]$, let

$$h(u, v) = (X - 1)(X^{d/2}\pi^{-1})^m u^J v + F[X, X^{-1}] \in E.$$

This map induces a sesquilinear form on $F[X, X^{-1}]/\pi^m$. The induced form is $(-X)$ -hermitian since $(X - 1)^J = -X^{-1}(X - 1)$; it is nonsingular since $h(1, v) = 0$ implies π^m divides $(X - 1)v$ in $F[X, X^{-1}]$, hence $v = 0$ in $F[X, X^{-1}]/\pi^m$ since π is prime to $X - 1$.

Next, suppose $\pi \in \mathcal{R}_2$. For $u_1, u_2, v_1, v_2 \in F[X, X^{-1}]$, we let

$$h((u_1, u_2), (v_1, v_2)) = \pi^{-m} u_1^J v_2 - X(\pi^J)^{-m} u_2^J v_1 + F[X, X^{-1}] \in E.$$

Computation shows that this map induces a nonsingular $(-X)$ -hermitian form on $(F[X, X^{-1}]/\pi^m) \times (F[X, X^{-1}]/(\pi^J)^m)$.

Similarly, the following maps induce nonsingular $(-X)$ -hermitian forms on the corresponding modules (where e is an arbitrary nonnegative integer):

$$h(u, v) = X^{e-1}(X - 1)^{-2e-1} u^J v + F[X, X^{-1}] \in E \quad \text{on } F[X, X^{-1}]/(X - 1)^{2e+1},$$

$$h((u_1, u_2), (v_1, v_2)) = X^e(X - 1)^{-2e} (u_1^J v_2 - X u_2^J v_1) + F[X, X^{-1}] \in E$$

$$\text{on } (F[X, X^{-1}]/(X - 1)^{2e})^2$$

and if $\text{char } F \neq 2$,

$$h(u, v) = (X - 1)X^e(X + 1)^{-2e}u^Jv + F[X, X^{-1}] \in E \quad \text{on } F[X, X^{-1}]/(X + 1)^{2e},$$

$$h((u_1, u_2), (v_1, v_2)) = (X - 1)^{2e+1}(X + 1)^{-2e-1}(u_1^Jv_2 + Xu_2^Jv_1) + F[X, X^{-1}] \in E \\ \text{on } (F[X, X^{-1}]/(X + 1)^{2e+1})^2.$$

We omit the straightforward verifications. \square

Remark. The theory of hermitian forms over principal ideal domains can also be used to show that the conditions in Theorem 1 are necessary.

2. The asymmetry of an anti-automorphism

2.1. Definition

Let A be a (finite-dimensional) central simple algebra over an arbitrary field F , and let $\sigma: A \rightarrow A$ be an F -linear anti-automorphism of A . Our goal is to attach to σ a unit $a_\sigma \in A^\times$ which plays the same rôle as the asymmetry a_b of a nonsingular pairing b with respect to the adjoint anti-automorphism σ_b . The key to the definition is an analogue of the linear involution γ_b , which we now define.

Proposition 4. *There is a unique linear map $\gamma_\sigma: A \rightarrow A$ which satisfies the following property: for any splitting field K of A , any isomorphism*

$$\theta: A_K = A \otimes_F K \rightarrow \text{End}_K V$$

and any nonsingular pairing b on V such that $\sigma_b = \theta \circ (\sigma \otimes \text{Id}_K) \circ \theta^{-1}$,

$$\theta \circ (\gamma_\sigma \otimes \text{Id}_K) \circ \theta^{-1} = \gamma_b.$$

This map satisfies the following additional properties:

- (i) $\gamma_\sigma(xyz) = \sigma(z)\gamma_\sigma(y)\sigma^{-1}(x)$ for $x, y, z \in A$;
- (ii) $\gamma_\sigma^2 = \text{Id}_A$.

Proof. It suffices to prove the existence of γ_σ . Uniqueness is then clear, and the additional properties follow from those of γ_b in Proposition 3.

Let $T_\sigma: A \times A \rightarrow F$ be the nonsingular pairing defined by

$$T_\sigma(x, y) = \text{Trd}_A(\sigma(x)y) \quad \text{for } x, y \in A,$$

where Trd_A is the reduced trace. Let $(e_i)_{i \in I}$ be a basis of A and let $(e_i^\#)_{i \in I}$ be the dual basis with respect to the pairing T_σ , so that

$$T_\sigma(e_i^\#, e_j) = \delta_{ij} \quad \text{for } i, j \in I.$$

We let

$$\gamma_\sigma(x) = \sum_{i \in I} e_i x e_i^\# \quad \text{for } x \in A.$$

In other words, γ_σ is the image of $\sum_{i \in I} e_i \otimes e_i^\# \in A \otimes_F A$ under the “sandwich” map $\text{Sand}: A \otimes_F A \rightarrow \text{End}_F A$ defined by $\text{Sand}(x \otimes y)(z) = xzy$. Observe that γ_σ does not depend on the choice of the basis $(e_i)_{i \in I}$ since $\sum_{i \in I} e_i \otimes e_i^\#$ is the element which corresponds to Id_A under the bijection $\text{Id}_A \otimes \hat{T}_\sigma: A \otimes_F A \rightarrow A \otimes_F A^* = \text{End}_F A$.

As a consequence, for every field extension K/F , the map $\gamma_{\sigma \otimes \text{Id}_K}: A \otimes K \rightarrow A \otimes K$ satisfies

$$\gamma_{\sigma \otimes \text{Id}_K} = \gamma_\sigma \otimes \text{Id}_K$$

since for $x \in A \otimes K$,

$$\gamma_{\sigma \otimes \text{Id}_K}(x) = \sum_{i \in I} (e_i \otimes 1)x(e_i^\# \otimes 1) = (\gamma_\sigma \otimes \text{Id}_K)(x).$$

To show that γ_σ is as required, assume that A is split: let $A = \text{End}_F V$ and let b be a nonsingular pairing on V such that $\sigma = \sigma_b$. We have to show that $\gamma_\sigma = \gamma_b$. To prove this equality, we use the identification $V \otimes_F V = \text{End}_F V$ defined by the linear isomorphism $\text{Id}_V \otimes \hat{b}: V \otimes_F V \rightarrow V \otimes_F V^* = \text{End}_F V$. Then $(v \otimes w)(x) = vb(w, x)$ for $v, w, x \in V$ and moreover

$$f \circ (v \otimes w) = f(v) \otimes w, \quad \sigma(v \otimes w) = a_b(w) \otimes v \quad \text{and} \quad \text{Trd}(v \otimes w) = b(w, v)$$

for $v, w \in V$ and $f \in \text{End}_F V$. Let $(v_i)_{1 \leq i \leq n}$ be a basis of V and let $(v'_i)_{1 \leq i \leq n}$ be the dual basis for the pairing b , so that

$$b(v'_i, v_j) = \delta_{ij} \quad \text{for } i, j = 1, \dots, n. \quad (8)$$

Then $(v_i \otimes v_j)_{1 \leq i, j \leq n}$ is a basis of $\text{End}_F V$, and the dual basis with respect to T_σ is given by

$$(v_i \otimes v_j)^\# = v'_i \otimes v'_j.$$

Therefore, we have for $f \in \text{End}_F V$

$$\begin{aligned} \gamma_\sigma(f) &= \sum_{i,j=1}^n (v_i \otimes v_j) \circ f \circ (v'_i \otimes v'_j) \\ &= \sum_{i,j=1}^n v_i \otimes v'_j b(v_j, f(v'_i)) \\ &= \sum_{i,j=1}^n v_i \otimes v'_j b(v'_i, \gamma_b(f)(v_j)). \end{aligned}$$

For all $x \in V$ we have $x = \sum_{i=1}^n v_i b(v'_i, x)$, hence $\sum_{i=1}^n v_i b(v'_i, \gamma_b(f)(v_j)) = \gamma_b(f)(v_j)$ for all j , and the last equality above simplifies to

$$\gamma_\sigma(f) = \sum_{j=1}^n \gamma_b(f)(v_j) \otimes v'_j = \gamma_b(f) \circ \left(\sum_{j=1}^n v_j \otimes v'_j \right).$$

Since $\sum_{j=1}^n v_j \otimes v'_j = \text{Id}_V$, it follows that $\gamma_\sigma(f) = \gamma_b(f)$. \square

In view of property (i), we have

$$\gamma_\sigma(x) = \sigma(x) \gamma_\sigma(1) = \gamma_\sigma(1) \sigma^{-1}(x) \quad \text{for all } x \in A. \quad (9)$$

Therefore, γ_σ is completely determined by the element $\gamma_\sigma(1) \in A^\times$.

Definition. The *asymmetry* of the anti-automorphism σ is the element $a_\sigma = \gamma_\sigma(1) \in A^\times$, where γ_σ is the linear involution defined in Proposition 4.

If $A = \text{End}_F V$ and $\sigma = \sigma_b$ is the anti-automorphism adjoint to some nonsingular pairing b on V , it follows from Proposition 4 and property (iii) of Proposition 3 that a_σ is the asymmetry of the nonsingular form b , i.e.,

$$a_\sigma = a_b.$$

In the general case, Eq. (9) shows that

$$\sigma^2(x) = a_\sigma x a_\sigma^{-1} \quad \text{for all } x \in A. \quad (10)$$

Moreover, since $\gamma_\sigma^2 = \text{Id}_A$ we have

$$1 = \gamma_\sigma(a_\sigma) = \sigma(a_\sigma) a_\sigma. \quad (11)$$

The element a_σ is uniquely determined up to sign by (10) and (11).

Recall that an anti-automorphism σ is called an *involution* if $\sigma^2 = \text{Id}_A$.

Proposition 5. A linear anti-automorphism is an involution if and only if its asymmetry is $+1$ or -1 .

Proof. If $a_\sigma = \pm 1$, Eq. (10) shows that $\sigma^2 = \text{Id}_A$. Conversely, if σ is an involution, (10) shows that $a_\sigma \in F^\times$. It then follows from (11) that $a_\sigma^2 = 1$, hence $a_\sigma = \pm 1$. \square

If $\text{char } F \neq 2$, a linear involution σ is called *orthogonal* (resp. *symplectic*) if after scalar extension to a splitting field it is adjoint to a symmetric (resp. skew-symmetric) bilinear pairing. Therefore, orthogonal involutions are exactly the linear anti-automorphisms with asymmetry $+1$, and symplectic involutions are those with asymmetry -1 . Therefore, Eqs. (10) and (11) are not sufficient to determine the type of the involution.

The following proposition yields an alternative definition of the asymmetry a_σ , without reference to the linear involution γ_σ and without scalar extension to a splitting field.

Let $\sigma_* : A \otimes_F A \rightarrow \text{End}_F A$ be the F -algebra homomorphism defined by

$$\sigma_*(a \otimes b)(x) = ax\sigma(b) \quad \text{for } a, b, x \in A$$

and recall (from [4, (3.5)], for instance) the *Goldman element* of A : this is the element $g \in A \otimes_F A$ such that $\text{Sand}(g)(x) = \text{Trd}_A(x)$ for all $x \in A$. Thus, there is a well-defined linear endomorphism $\sigma_*(g) : A \rightarrow A$.

Proposition 6. *The asymmetry of σ is the unique element $a_\sigma \in A^\times$ such that*

$$\sigma(\sigma_*(g)(f)) = a_\sigma f$$

for all $f \in A$.

Proof. It suffices to prove that a_σ satisfies the property above, since uniqueness is clear. To do this, we may extend scalars to a splitting field. Therefore, we may assume $A = \text{End}_F V$ for some F -vector space V , and $\sigma = \sigma_b$ is the anti-automorphism adjoint to some nonsingular pairing b on V .

For all $f \in A$ and all $x, y \in V$ we have

$$b(f(x), y) = b(y, a_\sigma \circ f(x)),$$

by definition of the asymmetry (see (2)), hence we have to show

$$b(f(x), y) = b(y, \sigma(\sigma_*(g)(f))(x))$$

or, equivalently (by definition of $\sigma = \sigma_b$),

$$b(f(x), y) = b(\sigma_*(g)(f)(y), x) \tag{12}$$

for all $f \in A$ and all $x, y \in V$.

In order to compute the right-hand side, we identify $A = \text{End}_F V$ to $V \otimes_F V$ via the linear isomorphism $\text{Id}_V \otimes \hat{b} : V \otimes_F V \rightarrow V \otimes_F V^* = \text{End}_F V$, as in the proof of Proposition 4. If $(v_i)_{1 \leq i \leq n}$ is a basis of V and $(v'_i)_{1 \leq i \leq n}$ is the dual basis for the pairing b (see (8)), then the Goldman element is

$$g = \sum_{i,j} (v_i \otimes v'_j) \otimes (v_j \otimes v'_i)$$

since it is easily computed that for all $u, w \in V$

$$\begin{aligned} \text{Sand}(g)(u \otimes w) &= \sum_{i,j} (v_i \otimes v'_j) \circ (u \otimes w) \circ (v_j \otimes v'_i) \\ &= \left(\sum_i v_i \otimes v'_i \right) \left(\sum_j b(v'_j, u) b(w, v_j) \right) \\ &= b(w, u) \sum_i v_i \otimes v'_i = \text{Trd}(u \otimes w) \text{Id}_V. \end{aligned}$$

Now, for $u, w \in V$,

$$\sigma_*(g)(u \otimes w) = \sum_{i,j} (v_i \otimes v'_j) \circ (u \otimes w) \circ \sigma(v_j \otimes v'_i).$$

Since $(u \otimes w) \circ \sigma(f) = u \otimes f(w)$ for $f \in \text{End}_F V$, the right-hand side of the last equality simplifies to

$$\sum_{i,j} ((v_i \otimes v'_j(u)) \otimes ((v_j \otimes v'_i)(w))) = \sum_{i,j} v_i \otimes v_j b(v'_j, u) b(v'_i, w),$$

hence

$$\sigma_*(g)(u \otimes w) = w \otimes u.$$

Therefore, for $u, w, x, y \in V$,

$$b(\sigma_*(g)(u \otimes w)(y), x) = b((w \otimes u)(y), x) = b(w, x) b(u, y).$$

Since we also have $b((u \otimes w)(x), y) = b(u, y) b(w, x)$, Eq. (12) holds for $f = u \otimes w$. Since $\text{End}_F V = V \otimes_F V$, it follows that (12) holds for all $f \in A$, and the proof is complete. \square

Remark. Asymmetries can be defined on the same model for anti-automorphisms of Azumaya algebras; one may avoid the use of a basis of A in Proposition 4 by defining $\gamma_\sigma = \text{Sand}(\xi_\sigma)$ where $\xi_\sigma \in A \otimes A$ is the element mapped to Id_A by $\text{Id}_A \otimes \hat{T}_\sigma$. Alternatively, we may set $\xi_\sigma = (\text{Id}_A \otimes \sigma^{-1})(g)$ where $g \in A \otimes A$ is the Goldman element. This is the approach taken by Saltman in [7] (see also [3, Chapter III, Section 8]).

2.2. Characterization of asymmetries

In this subsection, we show that in a central simple algebra of exponent 2, every unit which is conjugate to its inverse is the asymmetry of some anti-automorphism.

We first compare the asymmetries of two anti-automorphisms σ, τ on a central simple algebra A . The Skolem–Noether theorem shows that the automorphism $\tau \circ \sigma^{-1}$ is the conjugation by some unit $u \in A^\times$, i.e.,

$$\tau(x) = u\sigma(x)u^{-1} \quad \text{for all } x \in A. \quad (13)$$

Proposition 7. *Let σ, τ be anti-automorphisms of a central simple algebra A , and let $u \in A^\times$ be such that (13) holds. The asymmetries a_σ, a_τ of σ and τ are related by*

$$a_\tau = u\sigma(u)^{-1}a_\sigma.$$

Proof. We use the definition of asymmetry provided by Proposition 6. For $a, b, x \in A$, we have

$$\tau_*(a \otimes b)(x) = ax\tau(b) = axu\sigma(b)u^{-1},$$

hence

$$\tau_*(a \otimes b)(x) = \sigma_*(a \otimes b)(xu)u^{-1}.$$

Therefore, denoting by $r_u : A \rightarrow A$ the linear map of multiplication on the right by u , we have

$$\tau_*(a \otimes b) = (r_u)^{-1} \circ \sigma_*(a \otimes b) \circ r_u$$

for all $a, b \in A$, hence also

$$\tau_*(g) = (r_u)^{-1} \circ \sigma_*(g) \circ r_u$$

for g the Goldman element of A . It follows that for all $f \in A$,

$$\tau_*(g)(f) = \sigma_*(fu)u^{-1}. \quad (14)$$

By Proposition 6, the asymmetry a_τ satisfies

$$a_\tau f = \tau(\tau_*(g)(f)) \quad \text{for all } f \in A.$$

Using (14), we obtain

$$a_\tau f = \tau(\sigma_*(g)(fu)u^{-1}) = u\sigma(\sigma_*(g)(fu)u^{-1})u^{-1} = u\sigma(u)^{-1}\sigma(\sigma_*(g)(fu))u^{-1}.$$

Proposition 6 also yields $\sigma(\sigma_*(g)(fu)) = a_\sigma fu$, hence

$$a_\tau f = u\sigma(u)^{-1}a_\sigma f \quad \text{for all } f \in A.$$

The proposition follows. \square

Theorem 2. *Let A be a central simple algebra of exponent 2 over an arbitrary field F . A unit is the asymmetry of some anti-automorphism of A if and only if it is conjugate to its inverse.*

Proof. Suppose $a \in A^\times$ is the asymmetry of some anti-automorphism σ . We have to show that the F -vector space

$$U = \{x \in A \mid xa = a^{-1}x\}$$

contains an invertible element. This amounts to proving that the restriction of the reduced norm polynomial Nrd_A does not vanish on U . Theorem 1 shows that this polynomial does not vanish on $U \otimes K$, for any splitting field K of A , since a is the asymmetry of $\sigma \otimes \text{Id}_K$. Therefore, the reduced norm does not vanish on U , since F is an infinite field. (Note that every central simple algebra over a finite field is split, hence of exponent 1.)

For the converse, suppose $a \in A^\times$ is conjugate to a^{-1} . Let K be a splitting field of A ; identify $A \otimes K = \text{End}_K V$ for some K -vector space V . We first show, by using Theorem 1, that $a (=a \otimes 1)$ is the asymmetry of some anti-automorphism of $\text{End}_K V$. With the same notation as in Theorem 1, we have to prove that $\dim_K V_m^{+1}$ is even if

m is even, and moreover that $\dim_K V_m^{-1}$ is even if m is odd and $\text{char } F \neq 2$. For every integer $m \geq 1$ and $\varepsilon = \pm 1$, we have an exact sequence of K -vector spaces

$$0 \rightarrow \frac{\ker(a - \varepsilon \text{Id}_V)^{m+1}}{\ker(a - \varepsilon \text{Id}_V)^m} \xrightarrow{a - \varepsilon \text{Id}_V} \frac{\ker(a - \varepsilon \text{Id}_V)^m}{\ker(a - \varepsilon \text{Id}_V)^{m-1}} \rightarrow V_m^\varepsilon \rightarrow 0,$$

hence

$$\dim V_m^\varepsilon = \text{rk}(a - \varepsilon \text{Id}_V)^{m-1} - 2 \text{rk}(a - \varepsilon \text{Id}_V)^m + \text{rk}(a - \varepsilon \text{Id}_V)^{m+1}, \quad (15)$$

where rk denotes the rank.

For all $b \in A$ we have

$$\text{rk } b = \frac{\dim_K b(A \otimes K)}{\deg(A \otimes K)} = \frac{\dim_F bA}{\deg A},$$

hence $\text{rk } b$ is divisible by the Schur index $\text{ind } A$ (see [4, (1.9)]). Since A has exponent 2, $\text{ind } A$ is even, by Albert [1, Theorem 5.17]. Therefore, $\text{rk } b$ is even for all $b \in A$, and Eq. (15) shows that $\dim V_m^\varepsilon$ is even for every integer m and for $\varepsilon = \pm 1$. By Theorem 1, it follows that a is the asymmetry of some anti-automorphism θ of $A \otimes K$.

Now, fix some anti-automorphism σ of A . Let a_σ be its asymmetry and consider the F -vector space

$$W = \{x \in A \mid xa = \sigma(x)a_\sigma\}.$$

If $u \in (A \otimes K)^\times$ is such that $\theta(x) = u(\sigma \otimes \text{Id}_K)(x)u^{-1}$ for all $x \in A \otimes K$, then $u^{-1} \in W \otimes K$, by Proposition 7. Therefore, the same arguments as in the first part of the proof show that W contains an invertible element w . Using Proposition 7 again, we see that a is the asymmetry of the anti-automorphism $x \mapsto w^{-1}\sigma(x)w$. \square

Corollary 1 (Albert). *Every central simple algebra of exponent 2 carries an involution. Moreover, if the characteristic of the base field is different from 2, every central simple algebra of exponent 2 carries involutions of both orthogonal and symplectic types.*

Proof. It readily follows from Theorem 2 that $+1$ and -1 are asymmetries of some anti-automorphisms. These anti-automorphisms are involutions, by Proposition 5. \square

2.3. The determinant of an anti-automorphism

Let σ be a linear anti-automorphism of a central simple algebra A over an arbitrary field F . Let $a_\sigma \in A^\times$ be the asymmetry of A and γ_σ the linear involution of Proposition 4. Consider the vector spaces

$$\text{Alt}(A, \sigma) = \{x - \sigma(x)a_\sigma \mid x \in A\} = \{x - \gamma_\sigma(x) \mid x \in A\}$$

and

$$\text{Sk}(A, \sigma) = \{x \in A \mid \sigma(x) + xa_\sigma^{-1} = 0\} = \{x \in A \mid \gamma_\sigma(x) = -x\}.$$

From Eqs. (10) and (11), it follows that $\text{Alt}(A, \sigma) \subset \text{Sk}(A, \sigma)$. Moreover, we have $x - \gamma_\sigma(x) = 2x$ for all $x \in \text{Sk}(A, \sigma)$, hence $\text{Alt}(A, \sigma) = \text{Sk}(A, \sigma)$ if $\text{char } F \neq 2$.

Lemma 2. Suppose σ, τ are anti-automorphisms of A , and let $u \in A^\times$ be such that

$$\tau(x) = u\sigma(x)u^{-1} \quad \text{for all } x \in A.$$

Then

$$\text{Alt}(A, \tau) = u \text{Alt}(A, \sigma) \quad \text{and} \quad \text{Sk}(A, \tau) = u \text{Sk}(A, \sigma).$$

Proof. Proposition 7 yields $a_\tau = u\sigma(u)^{-1}a_\sigma$ and $a_\sigma = u^{-1}\tau(u)a_\tau$. Therefore, for all $x \in A$ we have

$$x - \tau(x)a_\tau = u(u^{-1}x - \sigma(u^{-1}x)a_\sigma) \quad \text{and} \quad u(x - \sigma(x)a_\sigma) = ux - \tau(ux)a_\tau,$$

proving that $\text{Alt}(A, \tau) = u \text{Alt}(A, \sigma)$. The proof that $\text{Sk}(A, \tau) = u \text{Sk}(A, \sigma)$ is along the same lines. \square

Lemma 3. If $\deg A$ is even, $\text{Alt}(A, \sigma)$ contains invertible elements. Moreover, the square class $\text{Nrd}_A(x)F^{\times 2} \in F^\times/F^{\times 2}$ does not depend on the choice of $x \in A^\times \cap \text{Alt}(A, \sigma)$.

Proof. Let τ be an anti-automorphism of A with asymmetry $+1$ and let $u \in A^\times$ be such that

$$\tau(x) = u\sigma(x)u^{-1} \quad \text{for all } x \in A.$$

By Lemma 2, we have

$$\text{Alt}(A, \sigma) = u^{-1} \text{Alt}(A, \tau). \tag{16}$$

Since τ is an involution, Corollary (2.8) of [4] shows that $\text{Alt}(A, \tau)$ contains invertible elements if $\deg A$ is even, hence $\text{Alt}(A, \sigma)$ also contains invertible elements. Moreover, from [4, (7.1)], it follows that all the invertible elements have the same reduced norm up to a square of F ; therefore, if $v \in A^\times \cap \text{Alt}(A, \tau)$ it follows from (16) that $\text{Nrd}_A(x) \in \text{Nrd}_A(u^{-1}v)F^{\times 2}$ for all $x \in A^\times \cap \text{Alt}(A, \sigma)$. \square

This last lemma allows us to define the *determinant* of an anti-automorphism σ of a central simple algebra A of even degree, as follows:

$$\det \sigma = \text{Nrd}_A(x)F^{\times 2} \in F^\times/F^{\times 2}$$

for any $x \in A^\times \cap \text{Alt}(A, \sigma)$.

This definition is consistent with [4, (7.2)], where the determinant of an orthogonal involution is defined.

Example 1. Since clearly $1 - a_\sigma \in \text{Alt}(A, \sigma)$, we have

$$\det \sigma = \text{Nrd}_A(1 - a_\sigma) F^{\times 2}$$

if $1 - a_\sigma$ is invertible. Therefore, the determinant of σ is entirely determined by its asymmetry in this particular case.

Example 2. The transpose involution on a matrix algebra $M_n(F)$ (with n even) has trivial determinant. Indeed, the matrix

$$\begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_{n/2} \end{pmatrix} \quad \text{where } m_1 = \cdots = m_{n/2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is in $\text{Alt}(M_n(F), t)$ and has determinant 1.

Proposition 8. Let σ, τ be anti-automorphisms of a central simple algebra A of even degree, and let $u \in A^\times$ be such that

$$\tau(x) = u\sigma(x)u^{-1} \quad \text{for all } x \in A.$$

Then

$$\det \tau = \text{Nrd}_A(u) \det \sigma.$$

Proof. This readily follows from Lemma 2. \square

Proposition 9. Let V be an even-dimensional vector space over an arbitrary field F and let b be a nonsingular pairing on V . For every basis $(v_i)_{1 \leq i \leq n}$ of V ,

$$\det \sigma_b = \det(b(v_i, v_j))_{1 \leq i, j \leq n} F^{\times 2}.$$

Proof. Identify $\text{End}_F V$ with the matrix algebra $M_n(F)$ by means of the basis $(v_i)_{1 \leq i \leq n}$. The anti-automorphism σ_b is then given by

$$\sigma_b(m) = u^{-1} m^t u \quad \text{for all } m \in M_n(F),$$

where $u = (b(v_i, v_j))_{1 \leq i, j \leq n} \in M_n(F)$. Therefore, Proposition 8 yields

$$\det \sigma_b = \det u^{-1} \det t.$$

Since it was observed in Example 2 above that $\det t$ is trivial, the proposition follows. \square

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References

- [1] A.A. Albert, *Structure of Algebras*, Coll. Pub., vol. 24, Amer. Math. Soc., Providence, RI, 1939.
- [2] F. Ischebeck, W. Scharlau, Hermiteische und orthogonale Operatoren über kommutativen Ringen, *Math. Ann.* 200 (1973) 327–334.
- [3] M.-A. Knus, *Quadratic and Hermitian Forms over Rings*, Grundlehren der Mathematischen Wissenschaften, vol. 294, Springer, Berlin, 1991.
- [4] M.-A. Knus, A.S. Merkurjev, M. Rost, J.-P. Tignol, *The Book of Involutions*, Coll. Pub., vol. 44, Amer. Math. Soc., Providence, RI, 1998.
- [5] A. Ranicki, *High-dimensional Knot Theory*, Springer Monographs in Mathematics, Springer, Berlin, 1998.
- [6] C. Riehm, The equivalence of bilinear forms, *J. Algebra* 31 (1974) 45–66.
- [7] D.J. Saltman, Azumaya algebras with involution, *J. Algebra* 52 (1978) 526–539.
- [8] W.C. Waterhouse, A nonsymmetric Hasse–Minkowski theorem, *Amer. J. Math.* 99 (1977) 755–759.
- [9] J. Williamson, On the algebraic problem concerning the normal form of linear dynamical systems, *Amer. J. Math.* 58 (1936) 141–163.