JACOBI IDENTITIES IN LOW-DIMENSIONAL TOPOLOGY

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Abstract. The Jacobi identity is the key relation in the definition of a Lie algebra. In the last decade, it also appeared at the heart of the theory of finite type invariants of knots, links and 3-manifolds (and is there called the IHX-relation). In addition, this relation was recently found to arise naturally in a theory of embedding obstructions for 2-spheres in 4-manifolds [20]. We expose the underlying topological unity between the 3- and 4-dimensional IHX-relations, deriving from a picture, Figure 3, of the Borromean rings embedded on the boundary of an unknotted genus 3 handlebody in 3-space. This is most naturally related to knot and 3-manifold invariants via the theory of grope cobordisms [4].

Contents

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1. Introduction

The only axiom in the definition of a Lie algebra, in addition to the bilinearity and skew-symmetry of the Lie bracket, is the *Jacobi identity*

$$[[a, b], c] - [a, [b, c]] + [[c, a], b] = 0.$$

If the Lie algebra arises as the tangent space at the identity element of a Lie group, the Jacobi identity follows from the associativity of the group multiplication. Picturing the Lie bracket as a rooted Y-tree with two inputs (the leaves) and one output (the root), the Jacobi identity can be encoded by the following figure:

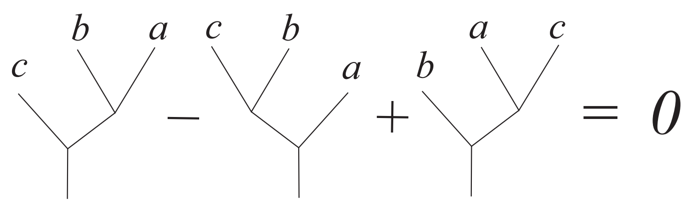


Figure 1. The Jacobi identity

One should read this tree from top to bottom, and note that the planarity of the tree (together with the counter-clockwise orientation of the plane) induces an ordering of each trivalent vertex which can thus be used as the Lie bracket. A change of this ordering just introduces a sign due to the skew-symmetry of the bracket. This will later correspond to the antisymmetry relation for diagrams.

Changing the input letters a, b, c to $1, 2, 3$ and labeling the root 4, Figure 1 may be redrawn with the position of the labeled univalent vertices fixed as follows:

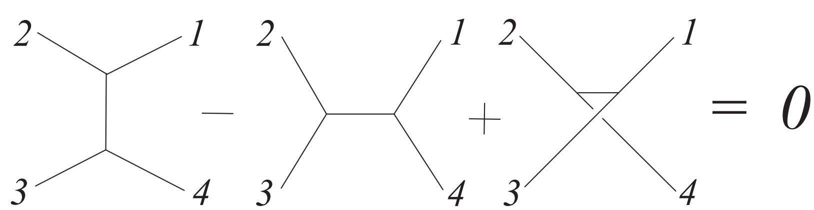


Figure 2. The IHX-relation

This (local) relation is an unrooted version of the Jacobi identity, and is well known in the theory of finite type (or Vassiliev) invariants of knots, links and 3-manifolds. Because of its appearance it is called the IHX-relation. The precise connection between finite type invariants and Lie algebras is very well explained in many references, see e.g. [2].

1.1. **2-spheres in 4-manifolds.** In Section 2 of this paper we will rediscover the Jacobi or IHX relation in the context of intersection invariants for Whitney towers in 4-manifolds. It is actually a direct consequence of a beautifully symmetric picture, Figure 3. The expert will see three standard Whitney disks whose Whitney arcs are drawn in an unconventional way (to be explained in 2.3 below). Ultimately, the freedom of choosing the Whitney arcs in this way forces the IHX-relation upon us.

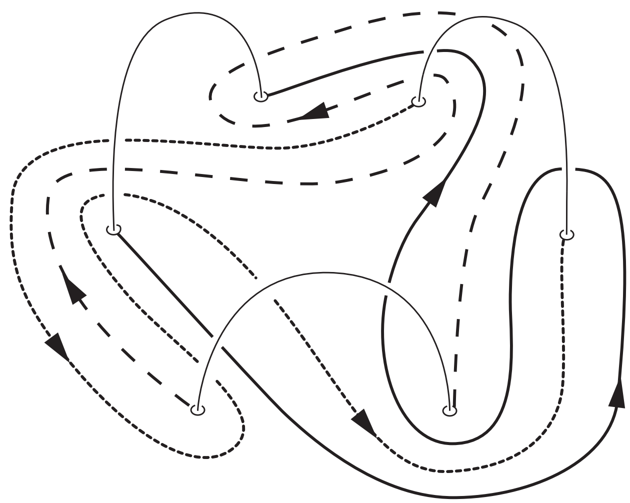


FIGURE 3. The geometric origin of the Jacobi identity in Dimension 4.

The reader will recognize the 3-component link in the figure as the Borromean rings. Each component consists of a semicircle and a planar arc (solid, dashed, dotted respectively), exhibiting the Borromean rings as embedded on the boundary of an unknotted genus 3 handlebody in 3-space.

The IHX relation for Whitney towers plays a key role in the obstruction theory for embedding 2-spheres into 4-manifolds developed in [20]. However, here no background is required of the reader beyond a willingness to try to visualize surfaces in 4–space and our elementary construction can also serve as an introduction to Whitney towers.

Roughly speaking, a Whitney tower is a 2-complex in a 4-manifold, formed inductively by attaching layers of Whitney disks to pairs of intersection points of previous surface stages, see Section 2. A Whitney tower has an *order* which measures how many layers were used. Moreover, for any unpaired intersection point p in a Whitney tower T of order $n-1$, one can associate a tree $t(p) \subset T$, see Figure 7. This is a trivalent tree of *degree* n, i.e. $t(p)$ has $n-1$ trivalent vertices. Orientations of the surface stages in T give vertex-orientations of $t(p)$, i.e. cyclic orderings of the trivalent vertices, and they also give a sign ϵ_p. We define the *geometric intersection tree* $t_n(T)$ as the disjoint union of signed vertex-oriented trees $t_n(T) := \amalg_p \epsilon_p \cdot t(p)$. Properly interpreted, $t_n(T)$ contains information on the homotopy classes of the A_i and represents a possible obstruction to the existence of an order n Whitney tower.

In the easiest case $n = 1$, a Whitney tower of order 0 is just a union of immersed 2-spheres $A_1, \dots, A_\ell : S^2 \to M^4$, and its geometric intersection tree $t_1(\cup_i A_i)$ is a disjoint union of signed arcs, one for each intersection point among the A_i, including self-intersections. The endpoints of the arcs are labeled by the 2-spheres, or better by the set $\{1, \dots, \ell\}$, organizing the information as to which A_i are involved in the intersection.

In this case we know how to proceed, namely by actually summing all these intersections to get exactly the *intersection numbers* among the A_i. Actually, if M is not simply connected, these "numbers" should be evaluated in the group ring of $\pi_1 M$, rather than in $\mathbb{Z}$, leading to Wall's intersection invariants [21]. This corresponds to putting group elements on the edges of each $t(p)$ and has been worked out in [20] for higher order Whitney towers. In the present paper our constructions are local so that we may safely ignore these group elements.

If $t_1(\cup_i A_i) = 0$ then all the intersections can be paired up by Whitney disks, i.e. there is a Whitney tower T of order 1 with the A_i as bottom stages. Then $t_2(T)$ is a signed sum of oriented Y-trees, and again the univalent vertices have labels from $\{1, \dots, \ell\}$. It was shown in [19] (and in [16], [23] for simply connected 4-manifolds) that a summation as above leads to an invariant $\tau_2(T)$ which vanishes if and only if there is a Whitney tower T of order 2 with the A_i as bottom stages. In fact, we showed that if defined in the correct target group, $\tau_2(T)$ only depends on the regular homotopy classes of the A_i and hence is a well defined higher obstruction for representing these classes by disjoint embeddings.

For arbitrary n, the summation can be formalized by a map

$$\widehat{\tau}_n : \mathbb{W}_{n-1}(\ell) \longrightarrow \widehat{\mathcal{B}}^t_n(\ell)$$

where $\mathbb{W}_{n-1}(\ell)$ denotes the set of 0-oriented Whitney towers of order $n-1$ on ℓ bottom stages A_i. A 0-orientation is an orientation of these bottom stages alone; it is a consequence of the AS relation explained below that the orientation of all other surface stages in T is irrelevant for $\widehat{\tau}_n(T)$. $\widehat{\mathcal{B}}^t_n(\ell)$ is the free abelian group generated by trivalent vertex-oriented trees of degree n, with univalent vertices labeled by $\{1, \dots, \ell\}$, modulo the antisymmetry relations AS. This should be viewed in analogy to the chains of a (combinatorial) simplicial complex: There one also takes *ordered* simplices σ as a basis, and divides out by the relations

$$(\sigma, -or) = -(\sigma, or).$$

Similarly, AS is the relation which introduces a minus sign whenever the cyclic ordering at one vertex is changed. This is related to the skew-symmetry of the Lie bracket, as explained above.

The map $\widehat{\tau}_n$ takes a Whitney tower T to the sum $\sum_p \epsilon_p \cdot t(p)$. The question arises as to whether this can be made into an obstruction for representing the bottom stages A_i by disjoint embeddings, up to homotopy. The punch-line of the first part of this paper is that this can only be possible if we quotient the groups $\widehat{\mathcal{B}}^t_n(\ell)$ by all IHX-relations, obtaining groups $\mathcal{B}^t_n(\ell)$ (containing elements $\tau_n(T)$), which are more customary in the theory of finite type invariants:

Theorem 1. *There exists an oriented order 2 Whitney tower T on four immersed 2-spheres in the 4–ball such that $t_3(T) = (+I) \amalg (-H) \amalg (+X)$ where I, H and X are the trees shown in Figure 2.*

This result comes from the fact alluded to before, namely that Whitney towers have the indeterminancy of choosing the Whitney arcs!

1.2. **Moving down to Dimension 3.** In sections 3 and 4, we extend the geometric IHX-relation explained above to a 3-dimensional setting via the connection between capped gropes and Whitney towers worked out in [17]. More precisely, we shall explain in full detail the following commutative diagram:

$$\begin{array}{ccc} \mathbb{G}^c_n(\ell) & \xrightarrow{\textbf{push-in}} & \mathbb{W}_{n-1}(\ell) \\ {\scriptstyle\widehat{\Psi}^c_n}\big\downarrow & & {\scriptstyle\widehat{\tau}_n}\big\downarrow \\ \widehat{\mathcal{A}}^t_n(\ell) & \xrightarrow{\textbf{pull-off}} & \widehat{\mathcal{B}}^t_n(\ell) \end{array} \tag{1}$$

Here $\mathbb{G}^c_n(\ell)$ is the set of 0-oriented capped gropes in S^3, with ℓ boundary components and of class n. The map **push-in** takes a capped grope, pushes it slightly into the 4-ball, and then surgers it into a Whitney tower (of order $n-1$). The group $\widehat{\mathcal{A}}^t_n(\ell)$ is just like its $\mathcal{B}$-analogue, except that the univalent vertices of the trees are attached to ℓ numbered strands (which form a trivial string link). The homomorphism **pull-off** takes such a tree and pulls off the strands, just remembering their indices in $\{1, \dots, \ell\}$. Thus the diagram above says exactly what information is lost when one moves from 3 to 4 dimensions. Moreover, the existence of the map $\widehat{\Psi}^c_n$ shows that the 4-dimensional IHX-relation from Theorem 1 can be lifted to a 3-dimensional version. We will prove this by re-interpreting our central picture, Figure 3, in terms of capped gropes in S^3. See Theorem 19 for the precise formulations.

One consequence of this work is the following theorem, phrased in the language of claspers familiar to many in the finite-type community.

Theorem 2. *Suppose three tree claspers C_i differ locally by the three terms in the IHX relation. Given an embedding of C_1 into a 3-manifold, there are embeddings of C_2 and C_3 inside a regular neighborhood of C_1, such that the leaves are parallel copies of the leaves of C_1, and the edges avoid any caps that C_1 may have. Moreover, surgery on $C_1 \cup C_2 \cup C_3$ is diffeomorphic (rel boundary of the handlebody neighborhood) to doing no surgery at all.*

This theorem was stated and utilized in [5], although the fact that the claspers must be tree claspers was accidentally omitted. The theorem was needed in [5] to prove Theorem 24(a), restated and proven here. Our treatment here makes it clear that only tree claspers are needed.

Garoufalidis and Ohtsuki [10] were the first to prove a version of a topological IHX relation. It was needed to show that a map from trivalent diagrams to homology spheres was well-defined. Habegger [12] improved and conceptualized their construction. Moving to the modern techniques of claspers (clovers), Garoufalidis, Goussarov and Polyak [7] sketch a proof of our Theorem 2, a theorem of which Habiro was also aware. Our proof is completely new, and, we believe, more conceptual. Moreover it serves as a bridge between the three- and four-dimensional worlds.

1.3. **Grope cobordism of string links.** In the last section, we shall use the techniques developed in this paper to obtain new information about string links. Let $\mathcal{L}(\ell)$ be the set of isotopy classes of string links in D^3 with ℓ components (which is a monoid with respect to the usual "stacking" operation). Its quotient by the relation of grope cobordism of class n is denoted $\mathcal{L}(\ell)/G_n$, compare Definition 11. The submonoid of $\mathcal{L}(\ell)$, consisting of those string links which cobound a class n grope with the trivial string link, is denoted by $G_n(\ell)$. Using results of Habiro [11] we show that $\mathcal{L}(\ell)/G_{n+1}$ are finitely generated groups and that the iterated quotients

$$G_n(\ell)/G_{n+1}$$

are central subgroups (implying that $\mathcal{L}(\ell)/G_{n+1}$ are nilpotent). We will construct a surjective homomorphism from diagrams to string links modulo grope cobordism:

$$\Phi_n(\ell)\colon \mathcal{B}^g_n(\ell) \to G_n(\ell)/G_{n+1}.$$

where $\mathcal{B}^g_n$ denotes the usual abelian group of trivalent graphs, modulo IHX- and AS-relations, but graded by the *grope degree* (which is the Vassiliev degree plus the first Betti number), compare Section 4.3.

Theorem 3. *$\Phi_n(\ell)\otimes\mathbb{Q}\colon \mathcal{B}^g_n(\ell)\otimes\mathbb{Q} \to G_n(\ell)/G_{n+1}\otimes\mathbb{Q}$ is an isomorphism.*

This extends a result in [5] from knots to string links and it relies on the existence of the Kontsevich integral for string links, which serves as an inverse to the above map.

Acknowledgment: It is a pleasure to thank Tara Brendle and Stavros Garoufalidis for helpful discussions.

2. A Jacobi Identity in Dimension 4

In this section we prove Theorem 1 after first sketching some background material and stating a more general Corollary which is proved and used in [20]. For more details on immersed surfaces in 4–manifolds we refer to [6], for more details on Whitney towers compare [17], [18], [20].

2.1. **Whitney towers.** Using local coordinates $\mathbb{R}^3\times(-\epsilon,+\epsilon)$, Figure 4 shows a pair of disjoint local sheets of surfaces A_1 and A_2 in 4–space. Figure 5 shows the result of applying a (Casson) *finger move* to the sheets of Figure 4, with A_1 having been changed by an isotopy supported near an arc from A_1 to A_2, creating a pair of transverse intersection

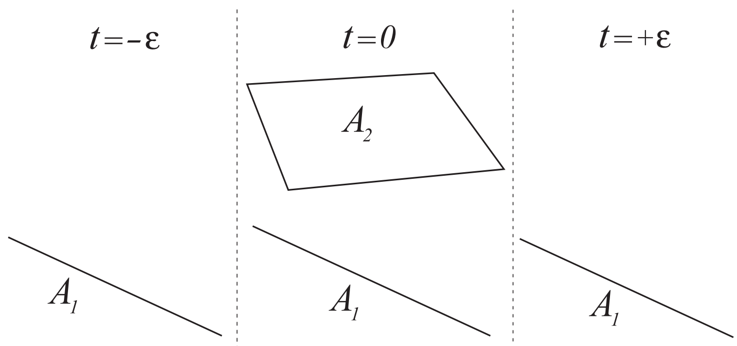


FIGURE 4.

points in $A_1 \cap A_2 \subset \mathbb{R}^3 \times \{0\}$. Such a pair of intersection points is called a *cancelling*

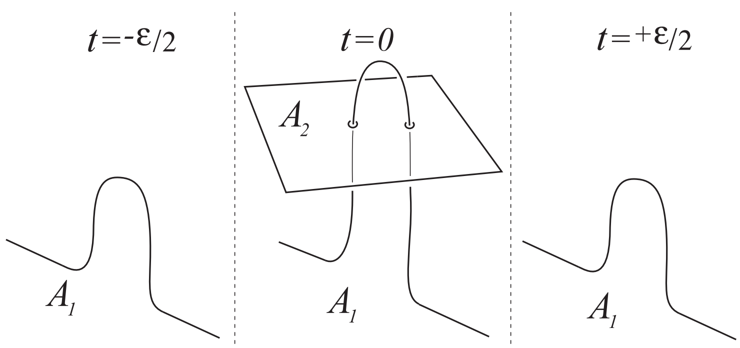


FIGURE 5.

pair since their signs differ and they can be paired by a *Whitney disk* as illustrated in Figure 6. Such a Whitney disk guides a motion (of either sheet) called a *Whitney move*

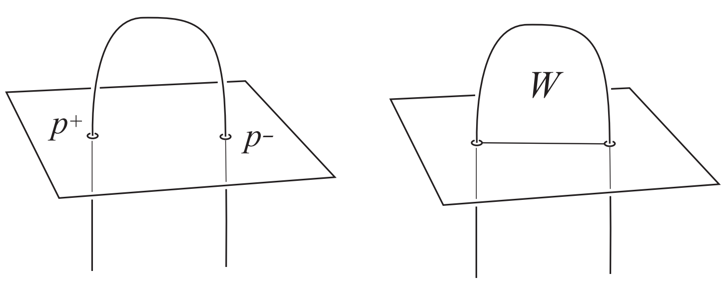


FIGURE 6. Left: A cancelling pair of intersections $p^{\pm}$. Right: A Whitney disk pairing $p^{\pm}$.

that eliminates the pair of intersection points [6]. A Whitney move guided by a Whitney disk whose interior is free of singularities can be thought of as an "inverse" to the finger move since it eliminates a cancelling pair without creating any new intersections. In general, Whitney disks may have interior self-intersections and intersections with other surfaces so that eliminating a cancelling pair via a Whitney move may also create new singularities. Pairing up "higher order" interior intersections in a Whitney disk by "higher order" Whitney disks leads to the notion of a Whitney tower:

Definition 4 (compare [17],[18],[20])**.**

- A *surface of order 0* in a 4–manifold M is a properly immersed surface (boundary embedded in the boundary of M and interior immersed in the interior of M). A *Whitney tower of order 0* in M is a collection of order 0 surfaces.
- The *order of a (transverse) intersection point* between a surface of order n and a surface of order m is $n+m+1$.
- The *order of a Whitney disk* is n if it pairs intersection points of order n.
- For $n \geq 1$, a *Whitney tower T of order n* is a Whitney tower of order $n-1$ together with (framed) Whitney disks pairing all order n intersection points of T, see Figure 7. (These top order disks are allowed to intersect each other as well as lower order surfaces.)

The boundaries of the Whitney disks in a Whitney tower are required to be disjointly embedded.

Framings of Whitney disks will not be discussed here (see e.g. [6]). In the construction of an order 2 Whitney tower, in the proof of Theorem 1, the reader familiar with framings can check that the Whitney disks are correctly framed.

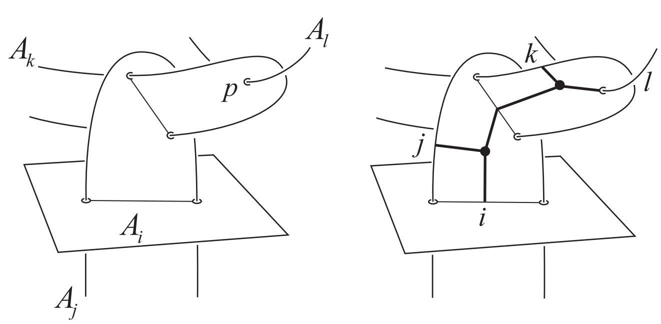


FIGURE 7. Part of an order 2 Whitney tower on order 0 surfaces A_i, A_j, A_k, and A_l and the labeled tree $t(p)$ (of Vassiliev degree 3) associated to the order 3 intersection point p.

2.2. **Intersection trees for Whitney towers.** For each order n intersection point p in a Whitney tower T there is an associated labeled trivalent tree $t(p)$ of Vassiliev degree n (Figure 7). (The Vassiliev degree of a unitrivalent graph is half the number of vertices.) This tree $t(p)$ is most easily described as a subset of T which "branches down" from p to the order 0 surfaces, bifurcating in each Whitney disk: The trivalent vertices of $t(p)$

correspond to Whitney disks in T, the labeled univalent vertices of $t(p)$ correspond to the labeled order 0 surfaces of T and the edges of $t(p)$ correspond to sheet-changing paths between adjacent surfaces in T.

Fixing orientations on all surfaces in T (including Whitney disks) *orients* T and determines a cyclic orientation for each of the trivalent vertices of $t(p)$ via a bracketing convention which will be illustrated explicitly during the proof of Theorem 1 below. The orientation of T also endows each intersection point p with a sign $\epsilon_p \in \{\pm\}$, determined as usual by comparing the orientations of the intersecting sheets at p with that of the ambient manifold. A precise and more elaborate definition of $t(p)$ is given in [20].

The "interesting" intersection points in an order $n-1$ Whitney tower T are the order n intersection points, since these points may represent an obstruction to the existence of an order n Whitney tower on the order 0 surfaces of T.

Definition 5. For an oriented order $(n-1)$ Whitney tower T, define $t_n(T)$, the order n *geometric intersection tree* of T, to be the disjoint union of signed labeled vertex-oriented trivalent trees

$$t_n(T) := \amalg_p \, \epsilon_p \cdot t(p)$$

over all order n intersection points $p \in T$.

We emphasize that $t_n(T)$ is a collection of signed trees of Vassiliev degree n, possibly with repetitions, *without* cancellation of terms.

As mentioned in the introduction, taking $t_n(T)$ as a sum in an appropriate abelian group defines an element $\tau_n(T)$ which gives information on the homotopy classes of the order 0 surfaces. The nature of this group depends in general on the 4–manifold and the order 0 surfaces. However, it turns out that for a fixed Whitney tower T, the AS antisymmetry relations correspond *exactly* to the indeterminacies coming from orientation choices on the Whitney disks in T, so that the element $\widehat{\tau}_n(T) \in \widehat{\mathcal{B}}^t_n$ only depends on the 0-orientation of T, i.e. the orientations of the order 0 surfaces. On the other hand, by fixing the order 0 surfaces and varying the choices of Whitney disks we are led to the IHX relations, as we describe in the next subsection.

2.3. The IHX relation for 2-spheres in 4–space.

The element $\tau_n(T)$ determined by $t_n(T)$ should vanish for any Whitney tower T on immersed 2–spheres into 4–space since all such spheres are null-homotopic. Theorem 1 from the introduction (proven below), and its corollary (Corollary 6) illustrate the necessity of the IHX relation in the target of τ_n. Since Theorem 1 is a local statement (taking place in a 4–ball) it can be used to "geometrically realize" all higher degree IHX relations for Whitney towers in arbitrary 4–manifolds, a key part of the obstruction theory described in [20]. The following corollary of Theorem 1 is proved in [20].

Corollary 6. *Let T be any (oriented) order $(n-1)$ Whitney tower on order 0 surfaces A_i. Then, given any order n trivalent trees t_I, t_H and t_X differing only by the usual local IHX relation, there exists an (oriented) order $(n-1)$ Whitney tower T' on A_i' homotopic* *(rel boundary) to the A_i such that*

$$t_n(T') = t_n(T) \amalg (+t_I) \amalg (-t_H) \amalg (+t_X).$$

□

The idea of the proof of Corollary 6 is that by applying finger moves to surfaces in a Whitney tower one can create clean Whitney disks which are then tubed into the spheres in Theorem 1. This construction can be done without creating extra intersections since finger moves are supported near arcs and the construction of Theorem 1 is contained in a 4–ball (see [20]).

Proof of Theorem 1. The 4–dimensional IHX construction starts with any four disjointly embedded oriented 2-spheres A_1, A_2, A_3, A_4 in 4–space. Perform finger moves on each A_i, for $i = 1, 2, 3$, to create a cancelling pair of order 1 intersection points $p^{\pm}_{(i,4)}$ between each of the first three 2-spheres (still denoted A_i) and A_4 as pictured in the left-hand side of Figure 8 where A_4 appears as the "plane of the paper" with the standard counter-clockwise orientation. Choose disjointly embedded oriented order 1 Whitney disks $W_{(3,4)}$, $W_{(2,4)}$ and $W_{(4,1)}$ for the cancelling pairs $p^{\pm}_{(i,4)}$ as in the right-hand side of Figure 8. Here the bracket sub-script notation corresponds to the following *orientation convention*: The bracket subscript (i,j) on a Whitney disk indicates that the boundary $\partial W_{(i,j)}$ of the Whitney disk is oriented from the negative intersection point to the positive intersection point along A_i and from the positive to the negative intersection point along A_j. This orientation of $\partial W_{(i,j)}$ together with a second "inward pointing" tangent vector induces the orientation of $W_{(i,j)}$.

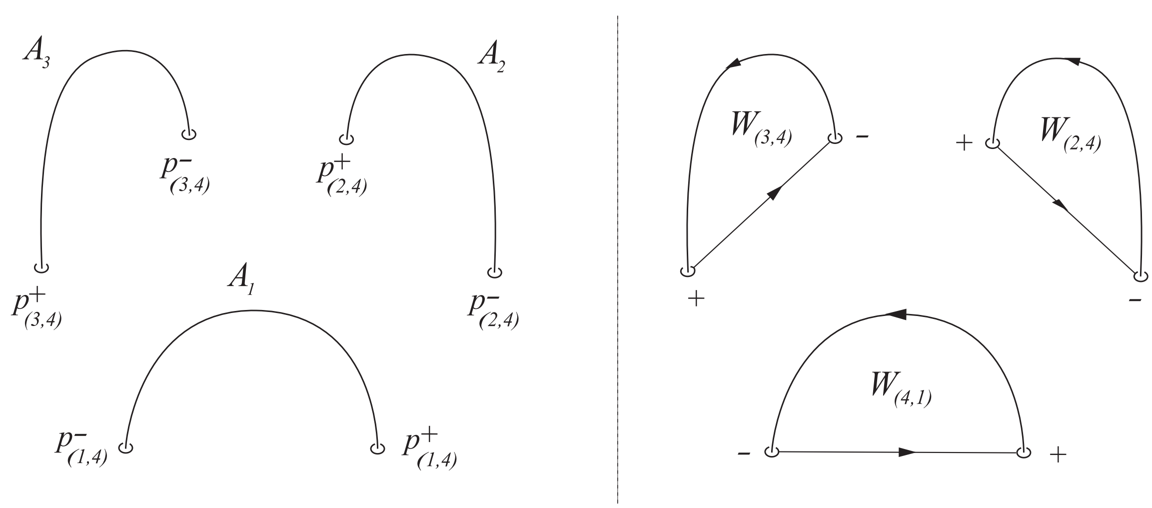


FIGURE 8. Left: The order zero 2-spheres and cancelling pairs of first order intersection points. Right: The clean first order Whitney tower T^0.

We have constructed an order 1 Whitney tower T^0 which is *clean*, meaning that T^0 has no higher order intersection points and hence is in fact a Whitney tower of order n for all n. Now change $W_{(3,4)}$ by isotoping its boundary $\partial W_{(3,4)}$ along A_4 and across $p^+_{(2,4)}$ and

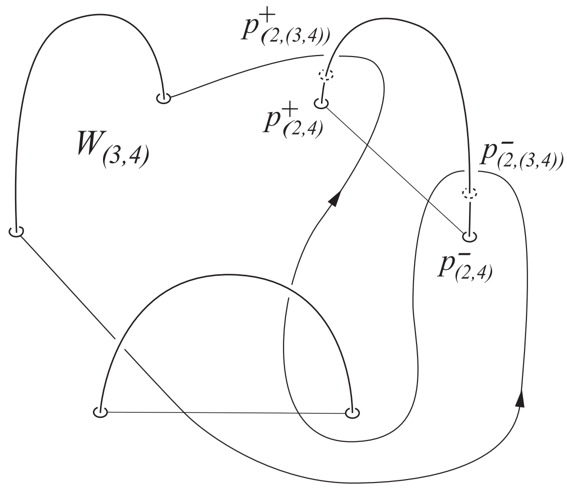


FIGURE 9.

$p^-_{(2,4)}$ as indicated in Figure 9 and extending this isotopy to a collar of $\partial W_{(3,4)}$. Note that a cancelling pair of order 2 intersection points $p^\pm_{(2,(3,4))}$ has been created between A_2 and the interior of the "new" $W_{(3,4)}$ (still denoted by $W_{(3,4)}$). The pair $p^\pm_{(2,(3,4))}$ is indicated in Figure 9 by the small dashed circles near $p^\pm_{(2,4)}$ and, since the orientation of A_4 is the standard counter-clockwise orientation of the plane, the sign of $p^+_{(2,(3,4))}$ (resp. $p^-_{(2,(3,4))}$) agrees with the sign of $p^+_{(2,4)}$ (resp. $p^-_{(2,4)}$). By perturbing the interior of $W_{(3,4)}$ slightly into past or future we may assume that $p^\pm_{(2,(3,4))}$ lie near, but not on, $\partial W_{(2,4)}$. A Whitney disk $W_{(2,(3,4))}$ (of order 2) for the cancelling pair $p^\pm_{(2,(3,4))}$ can be constructed by altering a parallel copy of $W_{(2,4)}$ in a collar of its boundary as indicated in Figure 10a. The part of the boundary of $W_{(2,(3,4))}$ that lies on $W_{(3,4)}$ is indicated by a dashed line in Figure 10a. The other arc (lying in A_2) of $\partial W_{(2,(3,4))}$ is not visible in the figure but would run parallel (perturbed slightly into past or future) to the arc of $\partial W_{(2,4)}$ in A_2.

Take the orientation of $W_{(2,(3,4))}$ that corresponds to its bracket sub-script via the above convention, i.e., that induced by orienting $\partial W_{(2,(3,4))}$ from $p^-_{(2,(3,4))}$ to $p^+_{(2,(3,4))}$ along A_2 and from $p^+_{(2,(3,4))}$ to $p^-_{(2,(3,4))}$ along $W_{(3,4)}$ together with a second inward pointing vector.

Note that $W_{(2,(3,4))}$ has a single *positive* intersection point p_{1234} (of order 3) with A_1. To this point p_{1234} we associate the positively signed labeled I-tree (of Vassiliev degree 3) as illustrated in Figure 10b. This I-tree, $t(p_{1234})$, is embedded in the construction with the trivalent vertices lying in the interiors of the Whitney disks, $W_{(2,4)}$ and $W_{(2,(3,4))}$, and each i-labeled univalent vertex lying on A_i. Each trivalent vertex of $t(p_{1234})$ inherits a

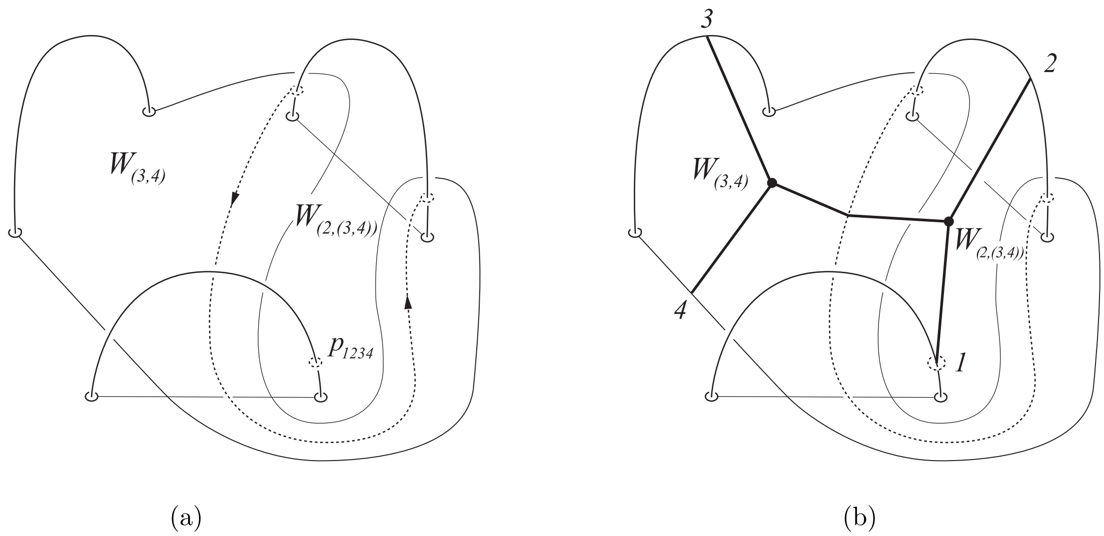


FIGURE 10.

cyclic orientation from the ordering of the components in the bracket associated to the corresponding oriented Whitney disk. Note that the pair of edges which pass from a trivalent vertex down into the lower order surfaces paired by a Whitney disk determine a "corner" of the Whitney disk which does not contain the other edge of the trivalent vertex. If this corner contains the *positive* intersection point paired by the Whitney disk, then the vertex orientation and the Whitney disk orientation agree ([20]). Our figures are drawn to satisfy this convention.

The antisymmetry relation can be seen here by noting that, according to our orientation conventions, switching the cyclic orientation of *one* trivalent vertex changes the orientations of *both* Whitney disks and hence changes the sign of the unpaired order 3 intersection point associated to the tree.

We have described how to construct (from the original $W_{(3,4)}$ of T^0) Whitney disks $W_{(2,(3,4))}$ and $W_{(3,4)}$ such that $W_{(2,(3,4))}$ pairs $A_2 \cap W_{(3,4)}$ and $A_1 \cap W_{(2,(3,4))}$ consists of a single point p_{1234}. In fact, this construction can be carried out symmetrically and simultaneously on all three of the original Whitney disks in T^0 yielding additional order 3 intersection points $p_{2341} \in A_2 \cap W_{(3,(4,1))}$ (with negative sign and associated labeled trivalent tree H) and $p_{3124} \in A_3 \cap W_{(1,(2,4))}$ (with positive sign and associated labeled trivalent tree X). Here $W_{(3,(4,1))}$ pairs $A_3 \cap W_{(4,1)}$ and $W_{(1,(2,4))}$ pairs $A_1 \cap W_{(2,4)}$ and it can be arranged that all the Whitney disks have pairwise disjointly embedded interiors and pairwise disjointly embedded boundaries: One way to see this is to first observe that the boundaries of the first order Whitney disks $W_{(3,4)}$, $W_{(4,1)}$ and $W_{(2,4)}$ can be disjointly embedded, as pictured in Figure 3, then push the collars of two of the first order Whitney disks into past and future respectively and continue with the previously described construction. The resulting order 2 Whitney tower T has exactly three order 3 intersection points with $t_3(T) = (+I) \amalg (-H) \amalg (+X)$. The correspondence between the Whitney disks in this construction and the trivalent vertices in the IHX relation is indicated in Figure 11. □

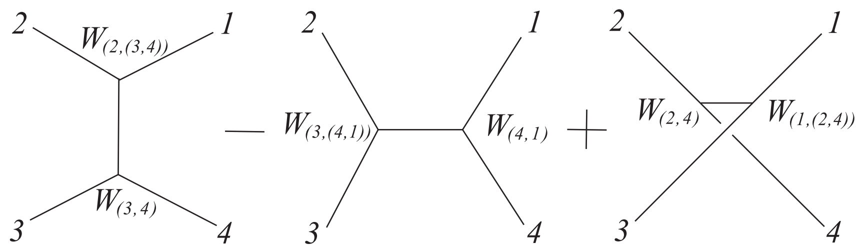


FIGURE 11. The correspondence between the trivalent vertices in the IHX relation and the (oriented) Whitney disks in the construction. (The trivalent orientations are all counter-clockwise.)

3. Connecting Dimensions 3 and 4

In this section we motivate the 3–dimensional IHX construction by describing a connection between 3–dimensional and 4–dimensional tree-valued invariants of string links. Summarizing this section, the ideas are as follows: A 3–dimensional *capped grope cobordism* between string links σ and σ' is a collection of disjointly embedded gropes in the 3–ball with boundary $\sigma \cup \sigma'$. Moreover, the tips of the gropes have to bound disjointly embedded *caps* (2–disks) which are disjoint from all but the bottom stage surfaces of the gropes. Analogously to Whitney towers, such a grope cobordism is assumed to be oriented and then has an associated *geometric intersection tree*, which in this case is a disjoint union of signed vertex-oriented trivalent trees whose univalent vertices are attached to the string link components. A class n 3–dimensional grope cobordism can be pushed into 4–space and surgered to an *order* $(n-1)$ *Whitney concordance*, that is, a collection of properly immersed 2–disks admitting an order $(n-1)$ Whitney tower in the 4–ball with boundary $\sigma \cup \sigma' \subset S^3$. It follows from Theorem 6 in [17] that this pushing and surgering "preserves trees" in the sense that applying summation maps to the 3– and 4–dimensional geometric intersection trees, respectively, yields isomorphic group elements after pulling the 3–dimensional trees off of the string link components.

The rest of this section is devoted to making this discussion precise by explaining in detail the commutative diagram 1 in 1.2 of the introduction.

3.1. (Capped) Gropes and Grope Cobordisms.

Definition 7. A *genus one grope* g is constructed by the following method. Start with a compact orientable connected surface of any genus (the *bottom stage* of g); choose a symplectic basis of circles on this bottom stage surface and attach punctured tori to any number of the basis circles. Next choose hyperbolic pairs of circles on each attached torus and attach punctured tori to any of these circles. Iterating this construction any number of times yields g. The attached tori are the *higher stages* of g. The basis circles in all stages of g that do not have a torus attached to them are called the *tips* of g. Attaching 2–disks along all the tips of g yields a *capped* grope (of genus one), denoted g^c. In the case of an (uncapped) grope, it is often convenient to attach an annulus along one of its boundary components to each tip. These annuli are called *pushing annuli*, and every tame embedding of a grope in a 3–manifold can be extended to include the pushing annuli. In general a grope is allowed to have surfaces of arbitrary genus at all stages, see [4] for the precise general definition.

Let g^c be a capped genus one grope. We define an associated rooted trivalent tree $t(g^c)$ as follows:

Definition 8. Assume first that the bottom stage of g^c is a genus one surface with boundary. Then define $t(g^c)$ to be the rooted trivalent tree which is dual to the 2–complex g^c; specifically, $t(g^c)$ sits as an embedded subset of g^c in the following way: The root univalent vertex of $t(g^c)$ is a point in the boundary of the bottom stage of g^c, each of the other univalent vertices is a point in the interior of a cap of g^c, each higher stage of g^c contains a single trivalent vertex of $t(g^c)$, and each edge of $t(g^c)$ is a sheet-changing path between trivalent vertices in adjacent stages or caps (here "adjacent" means "intersecting in a circle"), see Figure 12b. In the case where the bottom stage of g^c has genus > 1, then $t(g^c)$ is defined by cutting the bottom stage into genus one pieces and taking the disjoint union of the trees just described. In the case of genus zero, $t(g^c)$ is the empty tree.

We can now define the relevant complexity of a grope.

Definition 9. The *class* of g^c is the minimum of the Vassiliev degrees of the connected trees in $t(g^c)$. The underlying uncapped grope g (the *body* of g^c) inherits the same tree, $t(g) = t(g^c)$, and the same notion of class. For a general grope, refer to [4] for the definition of class. If the grope consists of a surface of genus zero, we regard it as a grope of class n for all n.

The non-root univalent vertices of $t(g)$ are called *leaves* and each leaf of $t(g)$ corresponds to a tip of g.

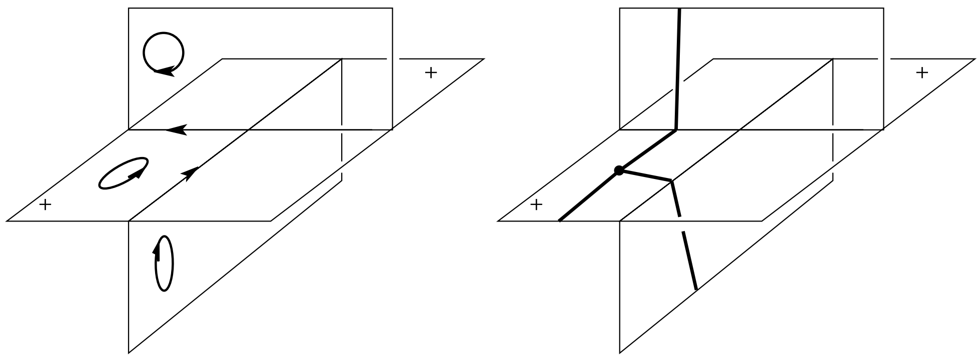
Definition 10. A grope is said to be *minimal* if the deletion of any stage will reduce the class. Any grope can be turned into a minimal grope by deleting superfluous stages, and we will assume throughout the paper that our gropes are minimal.

A 0-*orientation* of a (capped) grope is a choice of orientation for the bottom stage.

Given an embedding of a grope into an oriented three manifold, a 0-orientation determines orientations for each stage and cap of the grope, up to some indeterminacy, in the following way. Each surface stage or cap is attached to a previous stage along a circle, which hits the attaching region for one other surface stage or cap in a point. Near this point, the 2-complex is modeled by the following subset of $\mathbb{R}^3$:

$$\{(x, y, z) : z = 0\} \cup \{(x, y, z) : x = 0, z \geq 0\} \cup \{(x, y, z) : y = 0, z \leq 0\}.$$

Distinguish two of the quadrants as *positive*, namely the quadrants where both $x, y > 0$ and where both $x, y < 0$. See Figure 12a, where the two positive quadrants are indicated. Now suppose that the bottom stage ($z = 0$) has an orientation. Choose one of the two positive quadrants. The orientation of the surface induces an orientation of a small triangle in the positive quadrant which has a vertex at the origin and two edges contained in the axes. This then induces an orientation of the boundaries of the two higher surface stages, and hence induces an orientation of the higher surface stages. If we use the other positive quadrant instead, this has the effect of flipping the orientation of both higher surface stages, and this is the indeterminacy we mentioned previously. This is all pictured in Figure 12a, where an orientation of the bottom stage is pictured inside a positive quadrant, and the induced orientations on the two higher surfaces and their boundaries is indicated.



(a) Positive quadrants and orientation.

(b) A trivalent vertex of $t(g^c)$.

FIGURE 12.

The orientation of a capped grope g^c induces vertex-orientations of the trivalent vertices of $t(g^c)$ by taking each trivalent vertex of $t(g_i^c)$ to lie in a positive quadrant (see Figure 12b). Here also, the pairs of edges that cross into the next stages are required to do so through positive quadrants.

Definition 11. A *class n grope cobordism* G between ℓ-component string links σ and σ' is a collection of disjointly embedded class n gropes g_i ($i = 1, 2, \ldots, \ell$) in the 3–ball, such that the boundary of each g_i is equal to the union of the ith strands σ_i and σ'_i along their boundary points. If all the tips of all the g_i bound embedded caps whose interiors are disjoint from each other and disjoint from all but the bottom stages of the g_i, then G together with these caps forms a (class n) *capped* grope cobordism G^c between σ and σ'.

If G^c is oriented and the orientations of the bottom stages induce the orientations of the strands of σ, then we say that G^c is a capped grope cobordism "of σ", or "from σ to σ'." Denote by $\mathbb{G}_n^c(\ell)$ the set of class n oriented capped grope cobordisms of ℓ-component string links.

3.2. **Intersection trees for capped gropes.** Let $G^c \in \mathbb{G}_n^c(\ell)$ be a capped grope cobordism of a string link σ. It turns out that one can assume that the intersections of the caps with the bottom stages are arcs from σ to σ'. This can be accomplished by finger moves of the caps across the boundary of the bottom stages. Also, by applying Krushkal's splitting technique (as adapted to 3–dimensions in [4]) it can be assumed that each grope component is genus one and that each cap contains just a single intersection arc.

Now let g_i^c be an oriented, genus one, capped grope component of G^c. Assume that each cap of g_i^c contains only a single arc of intersections, which can be with any bottom stage surface in $g_j^c \subset G^c$. Then $t(g_i^c)$ is a disjoint union $\amalg_r t_i^r$ of vertex-oriented trees t_i^r, each of which sits as an embedded subset of g_i^c, with the root of t_i^r lying on the ith strand of σ (in ∂g_i^c) and each leaf of t_i^r lying on a jth strand of σ at an intersection point between a cap of g_i^c and that jth strand (see left hand side of Figure 13). Associate to each leaf of t_i^r the sign of the corresponding intersection point (between the cap and the jth strand) and denote by $\epsilon_i^r \in \{+,-\}$ the product of these signs.

Now the *geometric intersection tree* $t(G^c)$ of G^c is defined to be the disjoint union $\amalg_i \amalg_r \epsilon_i^r \cdot t_i^r$ of all the signed trees associated to all the g_i^c. Note that each tree should avoid the intersections between caps and the bottom stage, and this forces the roots to attach to the ith strand of σ in a specific ordering.

Definition 12. The abelian group $\widehat{\mathcal{A}}_n^t(\ell)$ is additively generated by (isomorphism classes of) vertex-oriented trivalent trees of Vassiliev degree n whose univalent vertices are attached to ℓ directed line segments. The relations are exactly the antisymmetry relations which introduce a minus sign whenever the cyclic ordering at one vertex is changed. Note that we do *not* divide out by the IHX relations. That's why our groups have hats on top of them.

To define the map $\widehat{\Psi}_n^c \colon \mathbb{G}_n^c(\ell) \longrightarrow \widehat{\mathcal{A}}_n^t(\ell)$, interpret each signed tree in the oriented intersection tree $t(G^c) = \amalg_i \amalg_r \epsilon_i^r \cdot t_i^r$ as a generator of $\widehat{\mathcal{A}}_n^t(\ell)$ by choosing the leaves to lie on the strands of σ (and forgetting the embedding of σ). Then sum over i and r to get:

$$\widehat{\Psi}_n^c(G^c) := \sum_i \sum_r \epsilon_i^r \cdot t_i^r \in \widehat{\mathcal{A}}_n^t(\ell).$$

Well-definedness of this map is an issue for a couple of reasons. First one must show that different orientations induced by the same 0-orientation do not affect $\widehat{\Psi}_n^c$. Secondly, one must show that different choices of tips for a surface stage will also leave $\widehat{\Psi}_n^c$ invariant. Proposition 23 addresses this in the uncapped situation. We leave it to the reader to modify the argument for the capped case.

We will relax the condition that G^c is of genus one and further generalize $\widehat{\Psi}_n^c$ in the next section.

Definition 13. A *singular concordance* between string links σ and σ' is a collection of properly immersed 2–disks D_i in the product $B^3 \times I$ of the 3–ball with the unit interval $I = [0,1]$, with ∂D_i equal to the union of the ith strands $\sigma_i \subset B^3 \times \{0\}$ and $\sigma_i' \subset B^3 \times \{1\}$ together with their end points crossed with I. For instance, any generic homotopy between σ and σ' defines such a singular concordance. An order n Whitney tower on a singular concordance is an *order n Whitney concordance.* Denote by $\mathbb{W}_n(\ell)$ the set of oriented order n Whitney concordances of ℓ-component string links, where an oriented singular concordance "of" σ induces the orientation of σ.

Definition 14. The abelian group $\widehat{\mathcal{B}}^t_n(\ell)$ is additively generated by isomorphism classes of vertex-oriented trivalent trees of Vassiliev degree n whose univalent vertices are labeled from $\{1, 2, \ldots, \ell\}$. Again the relations are only the antisymmetry relations and not IHX.

To define the map $\widehat{\tau}_n \colon \mathbb{W}_{n-1}(\ell) \longrightarrow \widehat{\mathcal{B}}^t_n(\ell)$, take a Whitney tower T and apply the summation map to the intersection tree $t_n(T) = \amalg_p \, \epsilon_p \cdot t(p)$ of Definition 5:

$$\widehat{\tau}_n(T) := \sum_p \epsilon_p \cdot t(p) \in \widehat{\mathcal{B}}^t_n(\ell)$$

3.3. **From grope cobordism to Whitney concordance.** Let G^c be a capped grope cobordism (from σ to σ') in $\mathbb{G}^c_n(\ell)$. Think of G^c as sitting in the middle slice $B^3 \times \{1/2\}$ of $B^3 \times I$. Extending $\sigma \subset G^c$ to $B^3 \times \{0\}$, via the product with $[0, 1/2]$, and extending $\sigma' \subset G^c$ to $B^3 \times \{1\}$, via the product with $[1/2, 1]$, yields a collection of class n capped gropes properly embedded in $B^4 = B^3 \times I$, i.e. a *4–dimensional grope cobordism*, or *grope concordance*, from σ to σ'. After perturbing the interiors of the caps slightly, we may assume that all caps are still disjointly embedded and that a cap which intersected the jth string link component in the 3–dimensional grope cobordism now has a single transverse intersection point with the interior of a bottom stage of the jth grope in the 4–dimensional grope concordance. By fixing the appropriate orientation conventions, the construction preserves the signs of these intersection points.

Consider the effect of this construction on the trees $t(g^c_i)$ which were embedded in the original G^c and are now sitting in the class n capped gropes in the 4–ball: Any root vertex that was lying on an ith string link strand is now in the interior of the ith bottom stage, and any leaf that corresponded to an intersection between a cap and a jth strand now corresponds to an intersection between a cap and a jth bottom stage. These are exactly the labeled trees associated to gropes in 4–manifolds as described in [17] and Theorem 6 of [17] describes how to surger such gropes to an order $(n-1)$ Whitney concordance T while preserving trees, meaning that the labeled trees associated to the gropes become the order n intersection tree $t_n(T)$. Although signs and orientations are not discussed in [17], the notation there is chosen to be compatible with the sign conventions of this paper and a basic case of the compatibility is illustrated in Figure 13 which shows a "push and surger" step in the modification of a 3-dimensional grope cobordism to a Whitney concordance applied to a top stage. (The modification in general involves "hybrid" grope-towers but reduces essentially to this case as explained in [17]).

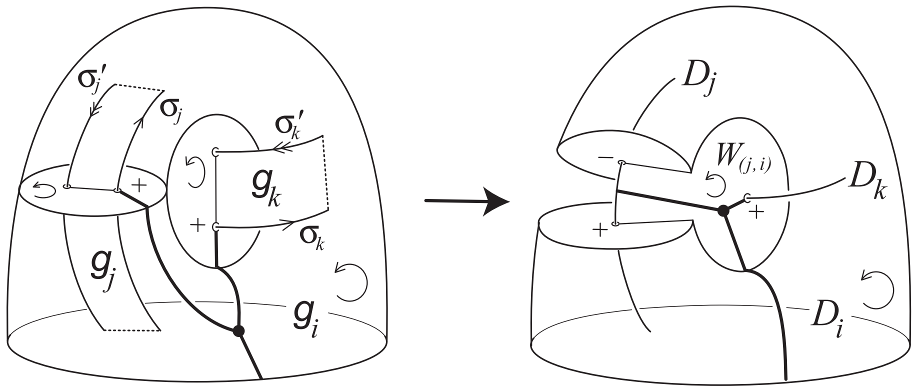


FIGURE 13. Left: A top stage of a capped grope cobordism. Right: The corresponding part of a Whitney concordance after pushing into 4–space and surgering a cap.

Definition 15. The above constructions define the map **push-in** : $\mathbb{G}_n^c(\ell) \longrightarrow \mathbb{W}_{n-1}(\ell)$ which pushes a grope into the 4–ball and surgers it into a Whitney tower. It is used in our main diagram 1 in the introduction.

Remark 16. The only information contained in the original geometric intersection tree $t(G^c)$ that is *lost* by the map (induced by) **push-in** is the ordering in which the univalent vertices of the $t(g_i^c)$ were attached to the string link components. Thus, pushing a class n 3–dimensional grope cobordism into 4–dimensions, surgering to an order $(n-1)$ Whitney concordance and applying the map $\widehat{\tau}_n$ is the same as the composition of the map $\widehat{\Psi}_n$ with the homomorphism

$$\textbf{pull-off} : \widehat{\mathcal{A}}_n^t \longrightarrow \widehat{\mathcal{B}}_n^t$$

that pulls the trees off the string link components and labels the univalent vertices accordingly.

Notice that the map **pull-off** is very different from the rational PBW-type isomorphism $\sigma\colon \mathcal{A} \otimes \mathbb{Q} \to \mathcal{B} \otimes \mathbb{Q}$.

4. Jacobi Identities in Dimension 3

As a consequence of our work so far, IHX relations appear in $\widehat{\mathcal{B}}_n^t$ as the image under $\widehat{\tau}_n$ of Whitney concordances from any string-link to itself (e.g., tube the 2–spheres in Theorem 1 into a product concordance). In this section we show that this phenomenon pulls back to the 3–dimensional world: There are capped grope cobordisms from any string link to itself whose images under $\widehat{\Psi}_n^c$ give all IHX relations. We will also realize all IHX relations in a group generated by unitrivalent *graphs* by defining a more general map $\widehat{\Psi}_n$ on *un*-capped class n grope cobordisms. We want to work in a general setting that includes knots, links and string links, which leads to the definitions below.

4.1. **$\mathcal{X}$-links.** Let $\mathcal{X}$ be a finite union of directed circles and line segments. Define an $\mathcal{X}$-link to be a proper embedding of $\mathcal{X}$ into a 3–ball, where the endpoints of the line segments are fixed as in the case of string links. For example, if $\mathcal{X}$ is a union of ℓ circles, then an $\mathcal{X}$-link is the same as an ℓ-component link, whereas if $\mathcal{X}$ is a union of ℓ line segments, then an $\mathcal{X}$-link is an ℓ-component string link. Let $\{x_i\}$ denote the components of $\mathcal{X}$.

Let $\mathbb{G}_n(\mathcal{X})$ be the set of class n oriented grope cobordisms of $\mathcal{X}$-links, where we allow genus > 1 at all stages and the orientations of all bottom stages are induced by the orientations on the components of $\mathcal{X}$. Let $\mathbb{G}^c_n(\mathcal{X})$ be the same, except that the gropes are capped.

Define the abelian group $\widehat{\mathcal{A}}^t(\mathcal{X})$ to be generated by connected trivalent vertex-oriented trees which have $\mathcal{X}$ as a skeleton, modulo AS relations. That is, each generator is formed by attaching the univalent vertices of a tree to $\mathcal{X}$. Let $\widehat{\mathcal{A}}^t_n(\mathcal{X})$ be the subgroup generated by trees of Vassiliev degree n.

We will define a map $\widehat{\Psi}^c_n(\mathcal{X}) : \mathbb{G}^c_n(\mathcal{X}) \longrightarrow \widehat{\mathcal{A}}^t_n(\mathcal{X})$ which is a straightforward generalization of the map $\widehat{\Psi}^c_n$ in Definition 12 which was only defined for genus one grope cobordisms of string links; here we just sum over "all genus and all cap-intersection components": Let $G^c \in \mathbb{G}_n(\mathcal{X})$ be a capped grope cobordism. It consists of disjointly embedded capped gropes g^c_i, one for each component of $\mathcal{X}$. Choosing contiguous genus one pieces of stages in a g^c_i, from the bottom stage up, determines a genus one *branch* of g^c_i. For each genus one branch, construct a vertex-oriented trivalent tree sitting as a subset of g^c_i whose leaves correspond to caps of g^c_i, as we did previously. Choose an intersection component for each cap; each such component corresponds to a place where the embedding of $\mathcal{X}$ punctures the cap. Glue the leaves of the tree to $\mathcal{X}$ based on where this intersection occurs. The root of the tree is glued to x_i in the boundary of the bottom stage of g^c_i. Together with the product of the signs of the intersections in the caps, this gives a generator of $\widehat{\mathcal{A}}^t_n(\mathcal{X})$. Now $\widehat{\Psi}^c_n(\mathcal{X})(G^c)$ by definition is the sum of these generators, over all choices of branches, over all choices of intersections with each cap, and over all g^c_i in G^c.

Remark 17. This version of $\widehat{\Psi}^c(\mathcal{X})$ can also be defined by applying Krushkal's grope-splitting procedure (as adapted to 3–dimensions in [4]) and then using the genus one definition of the previous section.

When $\mathcal{X}$ is understood, the notation $\widehat{\Psi}^c(G^c)$ may be used.

4.2. **The IHX relation for string links.** The geometric degree 3 IHX construction for string links contained in Theorem 19 below will play a key role in all subsequent IHX constructions. At the heart of the proof of Theorem 19 is a 3–dimensional interpretation of Figure 3 which leads to the following construction of a (slightly) singular capped grope cobordism.

Construction 18. Consider a trivial three-component string link in the 3–ball. We will construct a singular capped grope $\bar{g}^c$ of class three with an unknotted boundary component on the surface of the ball. (So $\mathcal{X}$ is the union of three line segments with a circle.) Its bottom stage is of genus three and embedded. The second stage surfaces of $\bar{g}^c$ are of genus one and are each embedded. The interiors of the second stage surfaces intersect each other but are disjoint from the bottom stage of $\bar{g}^c$. Denote by $\bar{G}^c$ the union of $\bar{g}^c$ together with trivial cobordisms of the strands of the string link (embedded 2–disks traced out by perturbations of the interiors of the strands). Then the key property of $\bar{g}^c$ is that $\widehat{\Psi}^c_3(\mathcal{X})(\bar{G}^c) \in \widehat{\mathcal{A}^t_3}(\mathcal{X})$ is equal to the three terms of the IHX relation in Figure 2. Here the strands of the trivial string link are labeled by 1, 2, 3 and $\bar{g}^c$ is interpreted as a *null* bordism of its unknotted boundary which is labeled 4. Note that $\widehat{\Psi}^c$ still makes sense as a sum of subtrees of $\bar{g}^c$ whose leaves are attached to intersections with caps, even though $\bar{g}^c$ is singular.

To begin the construction of $\bar{g}^c$, consider Figure 3 again. Think of it as taking place inside a 3-ball B, so that the horizontal plane has an unknotted boundary on ∂B. The arcs that each puncture the plane twice are the three strands of a trivial string link. Add tubes around the arcs to turn the plane into a genus three surface Σ. Σ is the bottom stage of our singular grope. We construct a symplectic basis for Σ as follows. Three of the curves are meridians to the tubes. To get the other three basis curves, connect the endpoints of each of the three pictured arcs in the plane (formerly Whitney arcs) by an untwisted arc that travels once over a tube. (Exercise: these three curves form a Borromean rings.) We fix surfaces bounding these latter three basis curves in the following way. Consider Figure 9, where a Whitney disk pushed slightly in the future is pictured. Think of the Whitney disk, instead, as being entirely in the present. It will have two intersections with an arc as pictured. Add a tube along the arc to make a surface s_1. This surface has two dual caps, one of which hits the upper right strand 2, and one of which hits the bottom strand 1. The curve dual to the attaching curve of s_1 is a meridian to the strand 3 and so bounds a cap hitting strand 3 once. Symmetrically, we can construct s_2 and s_3. Adding these three capped surfaces s_1, s_2, s_3 to the surface Σ we get the desired singular capped grope $\bar{g}^c$. The tree structure for the stage s_1 is $[[1,2],3]$, and for s_2 and s_3 we get $[1,[2,3]]$ and $[[3,1],2]$ which, if we add strand 4 as the root, are exactly the terms of the IHX relation. Keeping track of orientations, the signs work out correctly.

Theorem 19. *Suppose $l \geq 4$ and $t_I - t_H + t_X$ is any IHX relation in $\widehat{\mathcal{A}^t_3}(\ell)$. Then there is a class three capped grope cobordism G^c, which takes the ℓ-component trivial string link to itself, such that $\widehat{\Psi}^c_3(G^c) = t_I - t_H + t_X$.*

Remark 20. As will be clear from the proof, we emphasize that there are no hidden cancelling terms here and the sum $t_I - t_H + t_X$ of basis elements is also equal to the geometric intersection tree $t(G^c)$ as in section 3.2.

Proof. First consider the case where $\ell = 4$ and $t_I - t_H + t_X$ is as in Figure 2. We will construct G^c as a connected cobordism of strand 4 together with trivial cobordisms (disks) of the other three strands. Take the 3–ball B from the above Construction 18 and remove regular neighborhoods of the three strands of the trivial string link in B to get a handlebody M which contains the uncapped body $\bar{g}$ of the singular capped grope $\bar{g}^c$. Let m_i be a meridian to the ith strand on the surface of M. Now in the complement of a trivial 4-component string link, embed M so that m_i is a meridian to strand i. Connect a parallel copy of the fourth strand by a band to the unknot $\partial\bar{g}$ on the boundary surface of M calling the resulting strand $4'$. The embedding of M extends (by attaching disks to the m_i) to an embedding of B into the 3–ball containing the 4-component string link. Thus, 4 and $4'$ cobound the singular capped grope $\bar{g}^c$ from Construction 18 which sits inside B, where, by abuse of terminology, we let $\bar{g}^c$ also denote the grope that has 4 and $4'$ as its boundary.

Pick arcs α and β contained in the bottom stage of $\bar{g}^c$ and sharing endpoints with 4 and $4'$ such that $\alpha \cup \beta$ splits $\bar{g}^c$ into three genus one capped grope cobordisms g_1^c, g_2^c and g_3^c. If we number them appropriately, g_1^c modifies strand 4 to the strand α, g_2^c modifies α to β and g_3^c modifies β to $4'$. Note that each of these three grope cobordisms is nonsingular.

Examining the way in which the caps hit the strands, we see that $\sum \widehat{\Psi}_3^c(G_j^c) = t_I - t_H + t_X$, where each G_j^c is just g_j^c together with trivial cobordisms on the first three strands.

In order to get the desired G^c, we wish to glue these cobordisms G_i^c back together so that the resulting grope is *embedded*. To do this, we use the transitivity argument from [4], which is easily adapted to the current situation of arcs rel boundary (as opposed to knots). In that argument the individual gropes that are being glued together are homotoped inside the ambient 3–manifold until they match up. However, the homotopies are always isotopies when restricted to individual gropes. (Except in the framing correction move where some twists are introduced, which will not affect $\widehat{\Psi}^c(G^c)$.) Thus $\widehat{\Psi}_3^c(G^c) = t_I - t_H + t_X$ is not changed during this procedure.

Now consider the case where $\ell = 4$ but the univalent vertices of the trees in the IHX relation are attached to strands j_1, j_2, j_3 and j_4 which are not necessarily distinct. Then the only modification needed in the above proof is to embed M so that the m_i are meridians to the j_ith strand arranged in the correct ordering ($i = 1, 2, 3$), and make sure that the band from $\partial\bar{g}$ attaches to the j_4th strand in the right place.

Finally, if there are more than four strands, add the rest of the strands to the picture away from the above construction. □

More generally, let us consider grope cobordisms of higher class. We begin by realizing IHX relations for trees whose univalent vertices lie on distinct components. The non-distinct case will be covered in Theorem 24 below.

Theorem 21. *Let t_I, t_H and t_X be three trees which differ by the terms in an IHX relation in $\widehat{\mathcal{A}}_n^t(\ell)$, where $\ell \geq n+1$. Assume that no two univalent vertices of any one* *tree are attached to the same component. Then there is a class n capped grope cobordism G^c, from the ℓ-component trivial string link to itself, such that $\widehat{\Psi}^c_n(G^c) = t_I - t_H + t_X$.*

Proof. As in the previous Theorem, we will construct a cobordism of one of the strands, extending the others by disks. Also, it will be sufficient to consider the case $l = n+1$ since extra strands can be added away from the construction.

Decompose t_I into rooted trees I, A, B, C, R, where I represents the "I" in the IHX relation, a chosen root of I is connected to R, and the leaves of I connect to the roots of the trees A, B and C. Let the rooted tree given by I union A, B and C be called t as illustrated in Figure 14. Think of the ball containing $n+1$ strands as a boundary-

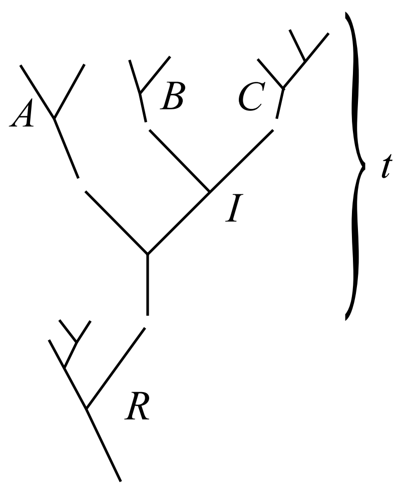


Figure 14.

connected-sum $B_t \# B_R$, where B_t is a ball with strands which inherit the (distinct) labels of t and B_R is ball with strands labeled distinctly from the rest of $\{1, 2, \ldots, n+1\}$.

Consider a capped grope g^c_t with one boundary component having geometric intersection tree equal to the tree t and contained in B_t. (To see that such a grope exists note that a regular neighborhood of a grope is a handlebody, which can be thought of as a ball with a tubular neighborhood of some arcs removed. The tips are part of a spine for the handlebody, so that there is a bijection between tips and arcs, with each arc going through a single tip once. Thus, the tips bound disks that are punctured by distinct arcs. Now there is an embedding of this ball-with-arcs to B_t that takes the arcs to strands in B_t according to any bijection.)

Pruning the "I" part g^c_I of g^c_t, we get three capped gropes realizing the trees A, B, C, denoted g^c_A, g^c_B, g^c_C respectively. As in Theorem 19, consider the genus three handlebody M which is the complement of a trivial 3–strand string link with m_i meridians to the strands on ∂M. Taking M to be a regular neighborhood of g_I^c, there is an embedding of M into B_t such that the m_i map to ∂g_A^c, ∂g_B^c, ∂g_C^c. Now, by Construction 18, there is a singular grope $\bar{g}$ of class three inside M which bounds an unknot on the boundary of M such that the tips of $\bar{g}$ bound parallel copies of g_A^c, g_B^c and g_C^c. (Note that these parallel copies intersect each other because the second stages of $\bar{g}$ intersect each other, and because parallel gropes in dimension three intersect.)

Let g_R^c be a capped grope realizing the tree R inside B_R, such that the tip T_0 of g_R^c corresponding to the leaf of t that connects to the root of I bounds a cap that does not intersect any strands. Note that g_R^c can be surgered into a disk, so that its boundary is unknotted.

Tube the cap on the (unknotted) tip T_0 of g_R^c to the (unknotted) $\partial\bar{g}$ on the boundary of M. Connect-sum the (unknotted) ∂g_R^c to (a push-off of) the strand in B_R corresponding to the root of R.

We get a singular capped grope cobordism $\bar{G}^c$ taking the trivial $(n+1)$-component string link to itself such that $\widehat{\Psi}_n^c(\bar{G}^c) = t_I - t_H + t_X$. The connected grope cobordism of the strand corresponding to the root of R is genus three at one stage and is embedded at that and all lower stages (the "R part"). Higher stages (the "A, B, and C parts") that lie above different genus one subsurfaces of the genus three stage (in the "I part") may intersect. Splitting the grope via Proposition 16 of [4], we get three grope cobordisms, each separately embedded, which can then be reglued by transitivity, as in the proof of Theorem 19. □

The previous theorem can be rephrased in the language of claspers as Theorem 2 of the introduction.

A picture of three claspers of degree three as in Theorem 2 is given in Figure 9 of [5]. This was derived from Theorem 19 using a mixture of claspers and gropes in the following way. First, (using the notation in the proof of Theorem 19) the clasper representing g_1^c was drawn. Next, we modified strand 1 by g_1^c to the new position α. We then drew in the clasper representing g_2^c. This clasper intersects the grope g_1^c, but using the usual pushing-down argument we pushed all the intersections down to the bottom stage. We then pushed them off the strand 0 boundary component of the grope, which is an isotopy in the complement of α. This gave rise to two disjoint claspers, surgery on which moves strand 0 to the arc β. The process was repeated for the clasper representing g_3^c: it was pushed out of the trace of the first two grope cobordisms/clasper surgeries. We double-checked the result by performing surgery along these three claspers and verified the result was isotopic to the original trivial 4-component string link.

4.3. **General IHX relations and the map $\widehat{\Psi}(\mathcal{X})$.** Next, we extend the realization of IHX relations from trees to arbitrary diagrams. Extending the map $\widehat{\Psi}^c$ to *un*-capped grope cobordisms involves two new wrinkles. First of all, in the absence of caps bounding the grope tips, it will not be possible to attach the leaves of the grope-trees to $\mathcal{X}$ with a meaningful ordering; however leaves will still be associated to components of $\mathcal{X}$ according to the linking between the components and the corresponding tips. Secondly, non-trivial linking between certain tips will lead to the construction of graphs with non-zero Betti number which result from gluing together the corresponding leaves.

Define $\widehat{\mathcal{B}}(\mathcal{X})$ to be the abelian group generated by connected *diagrams* whose univalent vertices are labeled by the components x_i of $\mathcal{X}$, modulo the AS antisymmetry relations. Here a *diagram* is a vertex-oriented unitrivalent graph having at least one univalent vertex. Let $\widehat{\mathcal{B}}^g_n(\mathcal{X})$ be the subgroup generated by diagrams of grope degree n, where the *grope degree* is equal to the Vassiliev degree plus the first Betti number.

Now we define $\widehat{\Psi}_n(\mathcal{X})\colon \mathbb{G}_n(\mathcal{X}) \longrightarrow \widehat{\mathcal{B}}^g_n(\mathcal{X})$. Let G be a grope cobordism of class n. First, choose a grope component $g \subset G$. As before, each branch of g has an associated vertex-oriented trivalent rooted tree t whose leaves L_i correspond to tips T_i of g. For each such T_i, choose either a component x_j of $\mathcal{X}$, or another tip T_j of t, and label the corresponding leaf L_i of t by (L_i, x_j), or (L_i, T_j) respectively. The root of t is labeled by the component of $\mathcal{X}$ that the boundary of g meets. Now sum over all choices to get a formal sum of labeled trees denoted $((G))$.

Now we proceed to glue together some of the leaves on each of these labeled trees, based on the geometric information of how the tips link each other and $\mathcal{X}$. Let t be a labeled tree. It has leaves L_i labeled (L_i, T_j) or labeled (L_i, x_j), where each leaf L_k corresponds to the tip T_k. A *matching* of such a labeled tree t is a partition of the set of all the leaves of t labeled by tips (and not $\mathcal{X}$ components) into pairs, such that the labels on each pair are of the form $(L_i, T_j), (L_j, T_i)$. A matching determines a labeled connected graph Γ, gotten by gluing together matched leaves of t, where each edge resulting from such a gluing assumes the coefficient $\mathrm{lk}(T_i, T_j) = \mathrm{lk}(T_j, T_i)$. Each of the remaining univalent vertices L_i is labeled by some component x_j, and assumes the coefficient $\mathrm{lk}(L_i, x_j)$. Each such Γ determines a multiple of a generating diagram of $\widehat{\mathcal{B}}^g_n(\mathcal{X})$, where the coefficient of the diagram is the product of the coefficients on the leaves and edges of Γ. Define $< t >$ as the sum of these elements in $\widehat{\mathcal{B}}^g_n(\mathcal{X})$ over all matchings of t. If there are no matchings, then $< t > = 0$ by definition. Extend $< \cdot >$ to linear combinations of trees linearly. Now define $\widehat{\Psi}_n(\mathcal{X})(G)$ to be $< ((G)) >$.

Remark 22. If G extends to a capped grope cobordism $G \subset G^c$, then $\widehat{\Psi}_n(G)$ is just the image of $\widehat{\Psi}^c_n(G^c)$ under the map **pull-off**$\colon \widehat{\mathcal{A}}^t_n(\mathcal{X}) \longrightarrow \widehat{\mathcal{B}}^g_n(\mathcal{X})$ that pulls the trees off the components of $\mathcal{X}$ and labels their univalent vertices accordingly.

Proposition 23. *$\widehat{\Psi}_n$ is well-defined.*

Proof. The first ambiguity is the orientation. The same cyclic orientation is induced at a trivalent vertex independently of which positive quadrant is chosen. What changes is that the orientation of the next two dual stages S_1, S_2 is switched. If S_i is a pushing annulus, this will reverse the orientation of a tip. The effect of switching a tip's orientation is to reverse the overall sign of $\widehat{\Psi}_n$. If S_i is a surface stage, this will reverse the cyclic orientation of the corresponding trivalent vertex, and by the AS antisymmetry relation, this also introduces a sign. Thus two factors of -1 are introduced in all cases, and the overall sign remains unchanged.

The second ambiguity arises from choosing different tips for a grope component $g \subset G$. Notice that $\widehat{\Psi}_n$ never sees the linking of tips on the same stage of g: Either they belong to different branches and hence will be part of different tree summands, or they are dual to each other in which case a graph with a loop at a vertex (which is zero by the AS relations) would result. Thus on a single surface stage, the linking number with objects c_i is all that matters, where c_i is either a component of $\mathcal{X}$ or another tip of g on a different stage.

Suppose we are not at a top stage. Then at least one curve in every hyperbolic pair bounds a higher surface stage. Removing a regular neighborhood of the higher surface stages, we get a planar surface. The tips become arcs joining some pairs of boundary components. Different choices of tips are related by Dehn twists on curves in the planar surface. Note that the boundary components of the planar surface are all null-homologous in the complement of $\cup c_i$. (They bound surfaces, and if the surfaces are slightly perturbed, they avoid c_i.) Hence choices of tips differ by multiples of curves which link the c_i trivially and hence do not change the contribution of g to $\widehat{\Psi}_n(G)$.

Now suppose we are at a top stage of genus m. Any two choices of tips = symplectic bases $(\alpha_1, \beta_1, \ldots, \alpha_m, \beta_m)$ are related by an element of $Sp(2m, \mathbb{Z})$, which is generated by the following automorphisms:

- for some i, $\alpha_i \mapsto \alpha_i + \beta_i$ and everything else is fixed
- for some i, $\beta_i \mapsto \alpha_i + \beta_i$ and everything else is fixed
- for some $i \neq j$, $\begin{cases} \alpha_i \mapsto \alpha_i + \alpha_j \\ \beta_j \mapsto -\beta_i + \beta_j \end{cases}$ and everything else is fixed
- for some $i \neq j$ $\begin{cases} \alpha_i \mapsto \alpha_i + \beta_j \\ \alpha_j \mapsto \beta_i + \alpha_j \end{cases}$ and everything else is fixed
- for some $i \neq j$ $\begin{cases} \beta_i \mapsto \beta_i + \alpha_j \\ \beta_j \mapsto \alpha_i + \beta_j \end{cases}$ and everything else is fixed
- for some $i \neq j$ $\begin{cases} \beta_i \mapsto \beta_i + \beta_j \\ \beta_j \mapsto -\alpha_i + \alpha_j \end{cases}$ and everything else is fixed

Let us adopt the following notation for expressing the contribution $((g))$ of g to $((G))$. Compute the disjoint union of trees where the leaves correspond to the tips of g, and label each leaf L_i by a linear combination $\sum_r n_r \mathbf{c_r}$ where the labels $\mathbf{c_r}$ correspond to components of $\mathcal{X}$ and tips T_j of g with $j \neq i$ (and n_r is the corresponding linking number with T_i). This represents $((g))$ by expanding the trees linearly in the labels. Note that if any labeled trees in $((g))$ represent zero modulo AS relations, then these relations will still be present upon gluing, so that the corresponding contribution to $\widehat{\Psi}_n(G) =< ((G)) >$ will also be zero.

The trees in $((g))$ before and after applying the first automorphism above only differ in a subtree isomorphic to a "Y", which we can represent by a bracket $[\,,\,]$. The difference is then represented by

$$\left[\sum_r \mathrm{lk}(\alpha_i, c_r)\mathbf{c_r}, \sum_r \mathrm{lk}(\beta_i, c_r)\mathbf{c_r}\right] - \left[\sum_r \mathrm{lk}(\alpha_i+\beta_i, c_r)\mathbf{c_r}, \sum_r \mathrm{lk}(\beta_i, c_r)\mathbf{c_r}\right].$$

Breaking the second summand into two terms, and using the fact that

$$\left[\sum_r \mathrm{lk}(\beta_i, c_r)\mathbf{c_r}, \sum_r \mathrm{lk}(\beta_i, c_r)\mathbf{c_r}\right] = 0$$

by the AS relations, we see that $((g))$, and hence $\widehat{\Psi}_n(G)$, remains unchanged. The case of the second automorphism is handled in the same way.

Let's consider the third automorphism. Abbreviate the notations $\sum_r \mathrm{lk}(\alpha, c_r)\mathbf{c_r}$ by $\mathrm{lk}(\alpha, c)$. Then notice that the difference in $((g))$ only occurs in the i and j trees, and this difference is

$$[\mathrm{lk}(\alpha_i, c), \mathrm{lk}(\beta_i, c)] + [\mathrm{lk}(\alpha_j, c), \mathrm{lk}(\beta_j, c)] - [\mathrm{lk}(\alpha_i+\alpha_j, c), \mathrm{lk}(\beta_i, c)] - [\mathrm{lk}(\alpha_j, c), \mathrm{lk}(-\beta_i+\beta_j, c)],$$

which is easily seen to be zero. The cases of the last three automorphisms are handled identically. □

Theorem 24.

(a) *Let $D_I, D_H, D_X \in \widehat{\mathcal{B}}^g_n(\mathcal{X})$ be diagrams differing by the terms in an IHX relation. Then there is a grope cobordism G, from the trivial $\mathcal{X}$-link to itself, such that $\widehat{\Psi}_n(G) = D_I - D_H + D_X$.*

(b) *Let $t_I, t_H, t_X \in \widehat{\mathcal{A}}^t_n(\mathcal{X})$ be trees differing by the terms in an IHX relation. Then there is a capped grope cobordism G^c, from the trivial $\mathcal{X}$-link to itself, such that $\widehat{\Psi}^c_n(G^c) = t_I - t_H + t_X$.*

Proof. (a) First, cut some edges (not contained in the "I" part) of D_I to make a tree D_I^t. Pick a univalent vertex that did not come from a cut as the root. Let ℓ be the number of leaves. As before, think of the complement of a trivial ℓ string link as a handlebody, M, with special curves $\{m_i\}_{i=1}^{\ell}$ on its boundary. Let the leaves of D_I^t be placed in correspondence with the curves m_i. Embed M in the complement of a trivial $\mathcal{X}$-link, such that if a leaf L_i of D_I^t is labeled by a component x of $\mathcal{X}$, then the corresponding m_i links x exactly once. Also, leaves resulting from cuts of D_I should have the corresponding m_i linking exactly once. Take a trivial subarc of the component of $\mathcal{X}$ corresponding to the root of D_I^t and perform a finger move so that it goes through M as a trivial subarc η. Now the proof of Theorem 21 yields a "weak" capped grope cobordism $g^{\bar{c}}$ (with $g \subset M$) which modifies η, where the weakness comes from the fact that here the linking pairs of leaves have intersecting caps. Ignoring this defect, $g^{\bar{c}}$ extends (as in the proof of Theorem 21)

to a (weak) capped grope cobordism $G^{\bar{c}}$ of $\mathcal{X}$ such that $\widehat{\Psi}^{\bar{c}}_n(G^{\bar{c}})$ equals three terms in an IHX relation which looks locally like $D_I - D_H + D_X$, where $\widehat{\Psi}^{\bar{c}}$ is the obvious extension of $\widehat{\Psi}^c$ which identifies leaves corresponding to Hopf-linked tips. When we throw away the caps and apply $< \cdot >$ to glue these trees into graphs, note that the tips of G are parallel to the curves m_i, so that this has exactly the effect of gluing together the broken edges, and labeling the univalent vertices appropriately.
(b) The proof of this part is similar to that of part (a). When embedding the handlebody M, make sure all m_i bound disjoint disks that intersect $\mathcal{X}$ in the same pattern that the univalent vertices of t_I hit $\mathcal{X}$. When picking the finger move of the "root" component to M, have it start where the root of t_I hits $\mathcal{X}$. Then find a grope cobordism inside M. □

5. Mapping diagrams to string links

Let $\mathcal{L}(\ell)$ be the set of isotopy classes of string links in D^3 with ℓ components (which is a monoid with respect to the usual "stacking" operation). Its quotient by the relation of grope cobordism (respectively capped grope cobordism) of class n is denoted $\mathcal{L}(\ell)/G_n$ (respectively $\mathcal{L}(\ell)/G^c_n$), compare Definition 11. The submonoid of $\mathcal{L}(\ell)$, consisting of those string links which cobound a class n grope (respectively capped grope) with the trivial string link, is denoted by $G_n(\ell)$ (respectively $G^c_n(\ell)$).

Proposition 25. *$\mathcal{L}(\ell)/G_{n+1}$ and $\mathcal{L}(\ell)/G^c_{n+1}$ are finitely generated groups and the iterated quotients*

$$G_n(\ell)/G_{n+1} \quad \textit{respectively} \quad G^c_n(\ell)/G^c_{n+1}$$

are central subgroups. As a consequence, $\mathcal{L}(\ell)/G_{n+1}$ and $\mathcal{L}(\ell)/G^c_{n+1}$ are nilpotent.

Proof. Let us begin with the statements for the capped case. Then $\mathcal{L}(\ell)/G^c_n$ can be identified with the quotient of $\mathcal{L}(\ell)$ modulo the relation of simple clasper surgery of class n. This translation works just like for knots where it was explained in [4]. All the results then follow from [11, Thm.5.4]. For example, the fact that the iterated quotients are central is proven by showing that $ab = ba$, modulo simple clasper surgery of class $(n+1)$, if a is a string link that is simple clasper n-equivalent to the trivial string link. This follows by sliding the claspers (that turn the trivial string link into a) past another string link b.

In the absence of caps one has to translate into rooted clasper surgery of grope degree n instead, as explained in [4]. Just as above, all results follow from the techniques of Habiro [11]. □

This result makes it possible to try to compute the *abelian* iterated quotients in terms of diagrams, which we proceed to do. We shall first define the map from diagrams to string links modulo grope cobordism:

$$\Phi_n(\ell)\colon \mathcal{B}^g_n(\ell) \to G_n(\ell)/G_{n+1}.$$

Indeed, we defined this for $\ell = 1$ in [5] in the following way. Given a diagram $D \in \widehat{\mathcal{B}}^g_n(\ell)$, find a grope cobordism g of class n, *corresponding to a simple clasper*, such that

$\widehat{\Psi}_n(\ell)(g) = D$. Then define

$$\widehat{\Phi}_n(\ell)(D) = \partial_1 g(\partial_0 g)^{-1},$$

where $\partial g = \partial_0 g \cup \partial_1 g$. One must show that the map is well-defined, i.e. that the choice of the simple clasper (and its embedding) does not matter. The argument given in [5] works with little modification for all $\ell \geq 1$.

The next proposition implies that we can take any grope g satisfying $\widehat{\Psi}_n(g) = D$ in the above definition, not having to restrict to those corresponding to simple claspers. We shall write Φ (respectively Ψ) instead of $\Phi_n(\ell)$ (respectively $\Psi_n(\ell)$) if the indices are clear from the context.

Proposition 26. *Given any grope g of class n, $\partial_1 g(\partial_0 g)^{-1} = \widehat{\Phi} \circ \widehat{\Psi}(g) \in G_n(\ell)/G_{n+1}$*

Proof. Any grope cobordism can be refined to a sequence of genus one grope cobordisms by Proposition 16 of [4] and this refinement evidently commutes with $\widehat{\Psi}$. Then, using Theorem 35 of [4], each of these cobordisms can be refined into a sequence of simple clasper surgeries and clasper surgeries of higher degree, and this refinement commutes with $\widehat{\Psi}$. (To see this it suffices to notice that the "zip construction" commutes with $\widehat{\Psi}$.) Thus

$$\partial_1 g(\partial_0 g)^{-1} = (\partial_1 g)(L_k)^{-1}(L_k)(L_{k-1})^{-1} \cdots ...(L_1)(\partial_0 g)^{-1},$$

where the L_i are string links modified by successive simple clasper surgeries. Note that we can omit any pairs $(L_i)(L_{i-1})^{-1}$ corresponding to clasper surgeries of higher degree, since this product is trivial in $\mathcal{L}(\ell)/G_{n+1}$. On the other hand, we know that for pairs corresponding to simple claspers C_i of degree n, $(L_i)(L_{i-1})^{-1} = \widehat{\Phi}(\widehat{\Psi}(C_i))$, by definition of $\widehat{\Phi}$. Thus

$$\begin{aligned} \partial_1 g(\partial_0 g)^{-1} &= \#_i \widehat{\Phi}(\widehat{\Psi}(C_i)) \\ &= \widehat{\Phi}(\widehat{\Psi}(\sum C_i)) \\ &= \widehat{\Phi}(\widehat{\Psi}(g)) \end{aligned}$$

which completes the proof. □

We next show that $\widehat{\Phi}$ vanishes on all IHX relations and hence descends to a well-defined map Φ.

Theorem 27. $\Phi_n(\ell)\colon \mathcal{B}^g_n(\ell) \to G_n(\ell)/G_{n+1}$ *is a well-defined surjective homomorphism.*

Proof of Theorem 27. By Theorem 24, any IHX relation, R_{IHX}, is the image under $\widehat{\Psi}$ of a cobordism, g, from a trivial string link to another trivial string link, denoted 1_ℓ. So by

Proposition 26,

$$\begin{aligned}\widehat{\Phi}(R_{IHX}) &= \widehat{\Phi}(\widehat{\Psi}(g)) \\ &= (\partial_1 g)(\partial_0 g)^{-1} \\ &= 1_\ell \# 1_\ell^{-1} \\ &= 1_\ell\end{aligned}$$

Next we consider surjectivity of $\phi_n(\ell)$. The elements of $G_n(\ell)$ are by definition of the form $\partial_1 g$ where g is a class n grope cobordism with $\partial_0 g = 1_\ell$. By Proposition 26, $\partial_1 g = \Phi_n(\ell) \circ \widehat{\Psi}_n(\ell)(g)$.

□

Using the Kontsevich integral as a rational inverse, we are now able to prove Theorem 3 which says that $\Phi_n(\ell)$ turns into an isomorphism after tensoring with $\mathbb{Q}$.

Sketch of proof of Theorem 3. This was proven in full detail in [5] for the case when $\ell = 1$. One sets up the (logarithm of the) Kontsevich integral as an inverse. Using the Aarhus integral [1], it is easy to show that the bottom degree term of the Kontsevich integral coincides with our map $\widehat{\Psi}_n(\ell)$. More precisely, if g is a grope cobordism, then Aarhus surgery formulae show that

$$(\log Z)_n(\partial_1 g(\partial_0 g)^{-1}) = \widehat{\Psi}_n(g),$$

where $(\log Z)_n$ is of grope degree n. Thus $\Phi_n((\log Z)_n(\partial_1 g(\partial_0 g)^{-1})) = \partial_1 g(\partial_0 g)^{-1}$, or $\Phi_n \circ (\log Z)_n = id$. On the other hand $(\log Z)_n(\Phi_n(D)) = (\log Z)_n(\partial_1 g(\partial_0 g)^{-1})$ for a grope g satisfying $\widehat{\Psi}_n(g) = D$. But then, by the above highlighted formula we can conclude that $(\log Z)_n \circ \Phi_n = id$.

Also, the Kontsevich integral of grope cobordisms of class $n+1$ will lie in degree $n+1$, so that the Kontsevich integral indeed factors through $G_n(\ell)/G_{n+1} \otimes \mathbb{Q}$. (Here we use the fact that the Kontsevich integral of string links preserves the loop (and hence grope) degree.) The fact that $\log Z_n$ is a homomorphism is straightforward using the Aarhus formula. (In [5] we used the Wheeling isomorphism to show this for knots, but that was unnecessary. The lowest degree part of the Wheeling isomorphism is just the identity.) □

It is unknown whether the analogous statements for the relation of *capped* grope cobordism of string links are true. There are two difficulties, one is the question whether one can realize the STU-relations in $\mathcal{A}_n(\ell)$ by capped grope cobordisms. The other is the question whether Habiro's main theorem [11] generalizes from knots to string links: Does the Vassiliev filtration of string links agree with the relation generated by simple clasper surgery? It follows from the techniques of [4] that the latter agrees with capped grope cobordism.

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