CHAPTER XII

The equivariant stable homotopy category

1. An introductory overview

Let us start nonequivariantly. As the home of stable phenomena, the subject of stable homotopy theory includes all of homology and cohomology theory. Over thirty years ago, it became apparent that very significant benefits would accrue if one could work in an additive triangulated category whose objects were "stable spaces", or "spectra", a central point being that the translation from topology to algebra through such tools as the Adams spectral sequence would become far smoother and more structured. Here "triangulated" means that one has a suspension functor that is an equivalence of categories, together with cofibration sequences that satisfy all of the standard properties.

The essential point is to have a smash product that is associative, commutative, and unital up to coherent natural isomorphisms, with unit the sphere spectrum S. A category with such a product is said to be "symmetric monoidal". This structure allows one to transport algebraic notions such as ring and module into stable homotopy theory. Thus, in the stable homotopy category of spectra — which we shall denote by $\bar{h}\mathscr{S}$ — a ring is just a spectrum R together with a product $\phi : R \wedge R \longrightarrow R$ and unit $\eta : S \longrightarrow R$ such that the following diagrams commute in $\bar{h}\mathscr{S}$:

$$\begin{tikzcd} S \wedge R \arrow[r, "\eta\wedge 1"] \arrow[dr, "\simeq"'] & R \wedge R \arrow[d, "\phi"] & R \wedge S \arrow[l, "1\wedge\eta"'] \arrow[dl, "\simeq"] \\ & R & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R \wedge R \wedge R \arrow[r, "1\wedge\phi"] \arrow[d, "\phi\wedge 1"'] & R \wedge R \arrow[d, "\phi"] \\ R \wedge R \arrow[r, "\phi"] & R. \end{tikzcd}$$

The unlabelled isomorphisms are canonical isomorphisms giving the unital property, and we have suppressed associativity isomorphisms in the second diagram. Similarly, there is a transposition isomorphism $\tau : E \wedge F \longrightarrow F \wedge E$ in $\bar{h}\mathscr{S}$, and R is said to be commutative if the following diagram commutes in $\bar{h}\mathscr{S}$:

$$\begin{tikzcd} R \wedge R \arrow[rr, "\tau"] \arrow[dr, "\phi"'] & & R \wedge R \arrow[dl, "\phi"] \\ & R. & \end{tikzcd}$$

A left R-module is a spectrum M together with a map $\mu : R \wedge M \longrightarrow M$ such that the following diagrams commute in $\bar{h}\mathscr{S}$:

$$\begin{tikzcd} S \wedge M \arrow[r, "\eta\wedge 1"] \arrow[dr, "\simeq"'] & R \wedge M \arrow[d, "\mu"] \\ & M \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} R \wedge R \wedge M \arrow[r, "1\wedge\mu"] \arrow[d, "\phi\wedge 1"'] & R \wedge M \arrow[d, "\mu"] \\ R \wedge M \arrow[r, "\mu"] & M. \end{tikzcd}$$

Over twenty years ago, it became apparent that it would be of great value to have more precisely structured notions of ring and module, with good properties before passage to homotopy. For example, when one is working in $\bar{h}\mathscr{S}$ it is not even true that the cofiber of a map of R-modules is an R-module, so that one does not have a triangulated category of R-modules. More deeply, when R is commutative, one would like to be able to construct a smash product $M \wedge_R N$ of R-modules. Quinn, Ray, and I defined such structured ring spectra in 1972. Elmendorf and I, and independently Robinson, defined such structured module spectra around 1983. However, the problem just posed was not fully solved until after the Alaska conference, in work of Elmendorf, Kriz, Mandell, and myself. We shall return to this later.

For now, let us just say that the technical problems focus on the construction of an associative and commutative smash product of spectra. Before June of 1993, I would have said that it was not possible to construct such a product on a category that has all colimits and limits and whose associated homotopy category is equivalent to the stable homotopy category. We now have such a construction, and it actually gives a point-set level symmetric monoidal category.

However, it is not a totally new construction. Rather, it is a natural extension of the approach to the stable category $\bar{h}\mathscr{S}$ that Lewis and I developed in the early 1980's. Even from the viewpoint of classical nonequivariant stable homotopy theory, this approach has very significant advantages over any of its predecessors. What is especially relevant to us is that it is the only approach that extends effortlessly to the equivariant context, giving a good stable homotopy category of G-spectra for any compact Lie group G. Moreover, for a great deal of the homotopical theory, the new point-set level construction offers no advantages over the original Lewis-May theory: the latter is by no means rendered obsolete by the new theory.

From an expository point of view this raises a conundrum. The only real defect of the Lewis-May approach is that the only published account is in the general equivariant context, with emphasis on those details that are special to that setting. Therefore, despite the theme of this book, I will first outline some features of the theory that are nearly identical in the nonequivariant and equivariant contexts, returning later to a discussion of significant equivariant points. I will follow in part an unpublished exposition of the Lewis-May category due to Jim McClure. A comparison with earlier approaches and full details of definitions and proofs may be found in the encyclopedic first reference below. The second reference contains important technical refinements of the theory, as well as the new theory of highly structured ring and module spectra. The third reference gives a brief general overview of the theory that the reader may find helpful. We shall often refer to these as [LMS], [EKMM], and [EKMM′].

General References

[LMS] L. G. Lewis, Jr., J. P. May, and M. Steinberger (with contributions by J. E. McClure). Equivariant stable homotopy theory. Springer Lecture Notes in Mathematics. Vol 1213. 1986.
[EKMM] A. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. Rings, modules, and algebras in stable homotopy theory. Amer. Math. Soc. Surveys and Monographs. To appear.
[EKMM′] A. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. Modern foundations for stable homotopy theory. In "Handbook of Algebraic Topology", edited by I.M. James. North Holland, 1995, pp 213-254.

2. Prespectra and spectra

The simplest relevant notion is that of a *prespectrum* E. The naive version is a sequence of based spaces E_n, $n \geq 0$, and based maps

$$\sigma_n : \Sigma E_n \longrightarrow E_{n+1}.$$

A map $D \longrightarrow E$ of prespectra is a sequence of maps $D_n \longrightarrow E_n$ that commute with the structure maps σ_n. The structure maps have adjoints

$$\tilde{\sigma}_n : E_n \longrightarrow \Omega E_{n+1},$$

and it is customary to say that E is an Ω-spectrum if these maps are equivalences. While this is the right kind of spectrum for representing cohomology theories on spaces, we shall make little use of this concept. By a *spectrum*, we mean a prespectrum for which the adjoints $\tilde{\sigma}_n$ are *homeomorphisms.* (The insistence on homeomorphisms goes back to a 1969 paper of mine that initiated the present approach to stable homotopy theory.) In particular, for us, an "Ω-spectrum" need not be a spectrum: henceforward, we use the more accurate term Ω-prespectrum for this notion.

One advantage of our definition of a spectrum is that the obvious forgetful functor from spectra to prespectra — call it ℓ — has a left adjoint spectrification functor L such that the canonical map $L\ell E \longrightarrow E$ is an isomorphism. Since we are usually concerned only with formal consequences of the adjunction, we will not give the precise construction of L. Its existence follows directly from a categorical result called the Freyd adjoint functor theorem. (We will say a little bit more about the construction in Section 9.) There is a formal analogy between the passage from prespectra to spectra and the passage from presheaves to sheaves, which is the reason for the term "prespectrum". The category of spectra has limits, which are formed in the obvious way by taking the limit for each n separately. It also has colimits. These are formed on the prespectrum level by taking the colimit for each n separately; the spectrum level colimit is then obtained by applying L.

The central technical issue that must be faced in any version of the category of spectra is how to define the smash product of two prespectra $\{D_n\}$ and $\{E_n\}$. Any such construction must begin with the naive bi-indexed smash product $\{D_m \wedge E_n\}$. The problem arises of how to convert it back into a singly indexed object in some good way. It is an instructive exercise to attempt to do this directly. One quickly gets entangled in permutations of suspension coordinates. Let us think of a circle as the one-point compactification of $\mathbb{R}$ and the sphere

S^n as the one-point compactification of $\mathbb{R}^n$. Then the iterated structure maps $\Sigma^n E_m = E_m \wedge S^n \longrightarrow E_{m+n}$ seem to involve $\mathbb{R}^n$ as the last n coordinates in $\mathbb{R}^{m+n}$. This is literally true if we consider the sphere prespectrum $\{S^n\}$ with identity structural maps. This suggests that our entanglement really concerns changes of basis. If so, then we all know the solution: do our linear algebra in a coordinate-free setting, choosing bases only when it is convenient and avoiding doing so when it is inconvenient.

Let $\mathbb{R}^\infty$ denote the union of the $\mathbb{R}^n$, $n \geq 0$. This is a space whose elements are sequences of real numbers, all but finitely many of which are zero. We give it the evident inner product. By a universe U, we mean an inner product space isomorphic to $\mathbb{R}^\infty$. If V is a finite dimensional subspace of U, we refer to V as an indexing space in U, and we write S^V for the one-point compactification of V, which is a based sphere. We write $\Sigma^V X$ for $X \wedge S^V$ and $\Omega^V X$ for $F(S^V, X)$.

By a prespectrum indexed on U, we mean a family of based spaces EV, one for each indexing space V in U, together with structure maps

$$\sigma_{V,W} : \Sigma^{W-V} EV \longrightarrow EW$$

whenever $V \subset W$, where $W - V$ denotes the orthogonal complement of V in W. We require $\sigma_{V,V} = \mathrm{id}$, and we require the evident transitivity diagram to commute for $V \subset W \subset Z$:

$$\begin{tikzcd}
\Sigma^{Z-W}\Sigma^{W-V}EV \arrow[r] \arrow[d, "\cong"'] & \Sigma^{Z-W}EW \arrow[d] \\
\Sigma^{Z-V}EV \arrow[r] & EZ.
\end{tikzcd}$$

We call E a spectrum indexed on U if the adjoints

$$\tilde{\sigma} : EV \longrightarrow \Omega^{W-V} EW$$

of the structural maps are homeomorphisms. As before, the forgetful functor ℓ from spectra to prespectra has a left adjoint spectrification functor L that leaves spectra unchanged. We denote the categories of prespectra and spectra indexed on U by $\mathscr{P}U$ and $\mathscr{S}U$. When U is fixed and understood, we abbreviate notation to $\mathscr{P}$ and $\mathscr{S}$.

If $U = \mathbb{R}^\infty$ and E is a spectrum indexed on U, we obtain a spectrum in our original sense by setting $E_n = E\mathbb{R}^n$. Conversely, if $\{E_n\}$ is a spectrum in our original sense, we obtain a spectrum indexed on U by setting $EV = \Omega^{\mathbb{R}^n - V} E_n$, where n is minimal such that $V \subset \mathbb{R}^n$. It is easy to work out what the structural maps must be. This gives an equivalence between our new category of spectra indexed on U and our original category of sequentially indexed spectra.

More generally, it often happens that a spectrum or prespectrum is naturally indexed on some other cofinal set $\mathscr{A}$ of indexing spaces in U. Here cofinality means that every indexing space V is contained in some $A \in \mathscr{A}$; it is convenient to also require that $\{0\} \in \mathscr{A}$. We write $\mathscr{P}\mathscr{A}$ and $\mathscr{S}\mathscr{A}$ for the categories of prespectra and spectra indexed on $\mathscr{A}$. On the spectrum level, all of the categories $\mathscr{S}\mathscr{A}$ are equivalent since we can extend a spectrum indexed on $\mathscr{A}$ to a spectrum indexed on all indexing spaces V in U by the method that we just described for the case $\mathscr{A} = \{\mathbb{R}^n\}$.

J. P. May. Categories of spectra and infinite loop spaces. Springer Lecture Notes in Mathematics Vol. 99. 1969, 448–479.

3. Smash products

We can now define a smash product. Given prespectra E and E' indexed on universes U and U', we form the collection $\{EV \wedge E'V'\}$, where V and V' run through the indexing spaces in U and U', respectively. With the evident structure maps, this is a prespectrum indexed on the set of indexing spaces in $U \oplus U'$ that are of the form $V \oplus V'$. If we start with spectra E and E', we can apply the functor L to get to a spectrum indexed on this set, and we can then extend the result to a spectrum indexed on all indexing spaces in $U \oplus U'$. We thereby obtain the "external smash product" of E and E',

$$E \wedge E' \in \mathscr{S}(U \oplus U').$$

Thus, if $U = U'$, then two-fold smash products are indexed on U^2, three-fold smash products are indexed on U^3, and so on.

This external smash product is associative up to isomorphism,

$$(E \wedge E') \wedge E'' \cong E \wedge (E' \wedge E'').$$

This is evident on the prespectrum level. It follows on the spectrum level by a formal argument of a sort that pervades the theory. One need only show that, for prespectra D and D',

$$L(\ell L(D) \wedge D') \cong L(D \wedge D') \cong L(D \wedge \ell L(D')).$$

Conceptually, these are commutation relations between functors that are left adjoints, and, by the uniqueness of adjoints, they will hold if and only if the corresponding commutation relations are valid for the right adjoints. We shall soon write down the right adjoint function spectra functors. They turn out to be so simple and explicit that it is altogether trivial to check the required commutation relations relating them and the right adjoint ℓ.

The external smash product is very nearly commutative, but to see this we need another observation. If $f : U \longrightarrow U'$ is a linear isometric isomorphism, then we obtain an isomorphism of categories $f^* : \mathscr{S}U' \longrightarrow \mathscr{S}U$ via

$$(f^*E')(V) = E'(fV).$$

Its inverse is $f_* = (f^{-1})^*$. If $\tau : U \oplus U' \longrightarrow U' \oplus U$ is the transposition, then the commutativity isomorphism of the smash product is

$$E' \wedge E \cong \tau_*(E \wedge E').$$

Analogously, the associativity isomorphism implicitly used the obvious isomorphism of universes $(U \oplus U') \oplus U'' \cong U \oplus (U' \oplus U'')$.

What about unity? We would like $E \wedge S$ to be isomorphic to E, but this doesn't make sense on the face of it since these spectra are indexed on different universes. However, for a based space X and a prespectrum E, we have a prespectrum $E \wedge X$ with

$$(E \wedge X)(V) = EV \wedge X.$$

If we start with a spectrum E and apply L, we obtain a spectrum $E \wedge X$. It is quite often useful to think of based spaces as spectra indexed on the universe $\{0\}$. This makes good sense on the face of our definitions, and we have $E \wedge S^0 \cong E$, where S^0 means the space S^0.

Of course, this is not adequate, and we have still not addressed our original problem about bi-indexed smash products: we have only given it a bit more formal structure. To solve these problems, we go back to our "change of universe functors" $f^* : \mathscr{S}U' \longrightarrow \mathscr{S}U$. Clearly, to define f^*, the map $f : U \longrightarrow U'$ need only be a linear isometry, not necessarily an isomorphism. While a general linear isometry f need not be an isomorphism, it is a monomorphism. For a prespectrum $E \in \mathscr{P}U$, we can define a prespectrum $f_*E \in \mathscr{P}U'$ by

$$(3.1) \qquad (f_*E)(V') = EV \wedge S^{V'-fV}, \quad \text{where } V = f^{-1}(V' \cap f(U)).$$

Its structure maps are induced from those of E via the isomorphisms

$$(3.2) \qquad EV \wedge S^{V'-fV} \wedge S^{W'-V'} \cong EV \wedge S^{W-V} \wedge S^{W'-fW}.$$

As usual, we use the functor L to extend to a functor $f_* : \mathscr{S}U \longrightarrow \mathscr{S}U'$. As is easily verified on the prespectrum level and follows formally on the spectrum level, the inverse isomorphisms that we had in the case of isomorphisms generalize to adjunctions in the case of isometries:

$$(3.3) \qquad \mathscr{S}U'(f_*E, E') \cong \mathscr{S}U(E, f^*E').$$

How does this help us? Let $\mathscr{I}(U, U')$ denote the set of linear isometries $U \longrightarrow U'$. If V is an indexing space in U, then $\mathscr{I}(V, U')$ has an evident metric topology, and we give $\mathscr{I}(U, U')$ the topology of the union. It is vital — and not hard to prove — that $\mathscr{I}(U, U')$ is in fact a contractible space. As we shall explain later, this can be used to prove a version of the following result (which is slightly misstated for clarity in this sketch of ideas).

THEOREM 3.4. *Any two linear isometries $U \longrightarrow U'$ induce canonically and coherently weakly equivalent functors $\mathscr{S}U \longrightarrow \mathscr{S}U'$.*

We have not yet defined weak equivalences, nor have we defined the stable category. A map $f : D \longrightarrow E$ of spectra is said to be a weak equivalence if each of its component maps $DV \longrightarrow EV$ is a weak equivalence. Since the smash product of a spectrum and a space is defined, we have cylinders $E \wedge I_+$ and thus a notion of homotopy in $\mathscr{S}U$. We let $h\mathscr{S}U$ be the resulting homotopy category, and we let $\bar{h}\mathscr{S}U$ be the category that is obtained from $h\mathscr{S}U$ by adjoining formal inverses to the weak equivalences. We shall be more explicit later.

This is our stable category, and we proceed to define its smash product. We choose a linear isometry $f : U^2 \longrightarrow U$. For spectra E and E' indexed on U, we define an internal smash product $f_*(E \wedge E') \in \mathscr{S}U$. Up to canonical isomorphism in the stable category $\bar{h}\mathscr{S}U$, $f_*(E \wedge E')$ is independent of the choice of f. For associativity, we have

$$\begin{aligned} f_*(E \wedge f_*(E' \wedge E'')) &\cong (f(1 \oplus f))_*(E \wedge E' \wedge E'') \\ &\simeq (f(f \oplus 1))_*(E \wedge E' \wedge E'') \cong f_*(f_*(E \wedge E') \wedge E''). \end{aligned}$$

Here we write $\cong$ for isomorphisms that hold on the point-set level and $\simeq$ for isomorphisms in the category $\bar{h}\mathscr{S}U$. For commutativity,

$$f_*(E' \wedge E) \cong f_*\tau_*(E \wedge E') \cong (f\tau)_*(E \wedge E') \simeq f_*(E \wedge E').$$

For a space X, we have a suspension prespectrum $\{\Sigma^V X\}$ whose structure maps are identity maps. We let $\Sigma^\infty X = L\{\Sigma^V X\}$. In this case, the construction of L is quite concrete, and we find that

$$\Sigma^\infty X = \{Q\Sigma^V X\}, \quad \text{where} \quad QY = \bigcup \Omega^W \Sigma^W Y. \tag{3.5}$$

This gives the suspension spectrum functor $\Sigma^\infty : \mathscr{T} \longrightarrow \mathscr{S}U$. It has a right adjoint Ω^∞ which sends a spectrum E to the space $E_0 = E\{0\}$:

$$\mathscr{S}U(\Sigma^\infty X, E) \cong \mathscr{T}(X, \Omega^\infty E). \tag{3.6}$$

The functor Q is the same as $\Omega^\infty\Sigma^\infty$. For a linear isometry $f : U \longrightarrow U'$, we have

$$f_*\Sigma^\infty X \cong \Sigma^\infty X \tag{3.7}$$

since, trivially, $\Omega^\infty f^* E' = E'_0 = \Omega^\infty E'$. A space equivalent to E_0 for some spectrum E is called an infinite loop space.

Remember that we can think of the category $\mathscr{T}$ of based spaces as the category $\mathscr{S}\{0\}$ of spectra indexed on the universe $\{0\}$. With this interpretation, Ω^∞ coincides with i^*, where $i : \{0\} \longrightarrow U$ is the inclusion. Therefore, by the uniqueness of adjoints, $\Sigma^\infty X$ is isomorphic to $i_* X$. Let $i_1 : U \longrightarrow U^2$ be the inclusion of U as the first summand in $U \oplus U$. The unity isomorphism of the smash product is the case $X = S^0$ of the following isomorphism in $\bar{h}\mathscr{S}U$:

(3.8)
$$f_*(E \wedge \Sigma^\infty X) \cong f_*(i_1)_*(E \wedge X) \cong (f \circ i_1)_*(E \wedge X) \simeq 1_*(E \wedge X) = E \wedge X.$$

We conclude that, up to natural isomorphisms that are implied by Theorem 3.4 and elementary inspections, the stable category $\bar{h}\mathscr{S}U$ is symmetric monoidal with respect to the internal smash product $f_*(E \wedge E')$ for any choice of linear isometry $f : U^2 \longrightarrow U$. It is customary, once this has been proven, to write $E \wedge E'$ to mean this internal smash product, relying on context to distinguish it from the external product.

4. Function spectra

We must define the function spectra that give the right adjoints of our various kinds of smash products. For a space X and a spectrum E, the function spectrum $F(X, E)$ is given by

$$F(X, E)(V) = F(X, EV).$$

Note that this is a spectrum as it stands, without use of the functor L. We have the isomorphism

$$F(E \wedge X, E') \cong F(E, F(X, E'))$$

and the adjunction

$$\mathscr{S}U(E \wedge X, E') \cong \mathscr{T}(X, \mathscr{S}U(E, E')) \cong \mathscr{S}U(E, F(X, E')), \tag{4.1}$$

where the set of maps $E \longrightarrow E'$ is topologized as a subspace of the product over all indexing spaces V of the spaces $F(EV, E'V)$. As an example of the use of right adjoints to obtain information about left adjoints, we have isomorphisms

$$(4.2) \qquad (\Sigma^\infty X) \wedge Y \cong \Sigma^\infty(X \wedge Y) \cong X \wedge (\Sigma^\infty Y).$$

For the first, the two displayed functors of X both have right adjoint

$$F(Y, E)_0 = F(Y, E_0).$$

More generally, for universes U and U' and for spectra $E' \in \mathscr{S}U'$ and $E'' \in \mathscr{S}(U \oplus U')$, we define an external function spectrum

$$F(E', E'') \in \mathscr{S}U$$

as follows. For an indexing space V in U, define $E''[V] \in \mathscr{S}U'$ by

$$E''[V](V') = E''(V \oplus V').$$

The structural homeomorphisms are induced by some of those of E'', and others give a system of isomorphisms $E''[V] \longrightarrow \Omega^{W-V}E''[W]$. Define

$$F(E', E'')(V) = \mathscr{S}U'(E', E''[V]).$$

We have the adjunction

$$(4.3) \qquad \mathscr{S}(U \oplus U')(E \wedge E', E'') \cong \mathscr{S}U(E, F(E', E'')).$$

When $E' = \Sigma^\infty Y$, $\mathscr{S}U'(E', E''[V]) \cong \mathscr{T}(Y, E''(V))$. Thus, if $i_1 : U \longrightarrow U \oplus U'$ is the inclusion, then

$$F(\Sigma^\infty Y, E'') \cong F(Y, (i_1)^* E'').$$

By adjunction, this implies the first of the following two isomorphisms:

$$(4.4) \qquad (i_1)_*((\Sigma^\infty X) \wedge Y) \cong \Sigma^\infty X \wedge \Sigma^\infty Y \cong (i_2)_*(X \wedge (\Sigma^\infty Y)).$$

When $U = U'$ and $f : U^2 \longrightarrow U$ is a linear isometry, we obtain the internal function spectrum $F(E', f^*E) \in \mathscr{S}U$ for spectra $E, E' \in \mathscr{S}U$. Up to canonical isomorphism in $\bar{h}\mathscr{S}U$, it is independent of the choice of f. For spectra all indexed on U, we have the composite adjunction

$$(4.5) \qquad \mathscr{S}U(f_*(E \wedge E'), E'') \cong \mathscr{S}U(E, F(E', f^*E'')).$$

Again, it is customary to abuse notation by also writing $F(E', E)$ for the internal function spectrum, relying on the context for clarity. By combining the three isomorphisms (3.7), (4.2), and (4.4) — all of which were proven by trivial inspections of right adjoints — we obtain the following non-obvious isomorphism for internal smash products.

$$(4.6) \qquad \Sigma^\infty(X \wedge Y) \cong (\Sigma^\infty X) \wedge (\Sigma^\infty Y).$$

Generalized a bit, this will be seen to determine the structure of smash products of CW spectra.

suitable left adjoint functors from spaces to spectra. For $n \geq 0$, there is no problem: we take $\underline{S}^n = \Sigma^\infty S^n$. We shall later write S^n ambiguously for both the sphere space and the sphere spectrum, relying on context for clarity, but we had better be pedantic at first.

We also need negative dimensional spheres. We will define them in terms of shift desuspension functors, and these functors will also serve to clarify the relationship between spectra and their component spaces. Generalizing Ω^∞, define a functor

$$\Omega_V^\infty : G\mathscr{S} \longrightarrow G\mathscr{T}$$

by $\Omega_V^\infty = EV$ for an indexing space V in U. The functor Ω_V^∞ has a left adjoint shift desuspension functor

$$\Sigma_V^\infty : G\mathscr{T} \longrightarrow G\mathscr{S}.$$

The spectrum $\Sigma_V^\infty X$ is $L\{\Sigma^{W-V}X\}$. Here the prespectrum $\{\Sigma^{W-V}X\}$ has Wth space $\Sigma^{W-V}X$ if $V \subset W$ and a point otherwise; if $V \subset W \subset Z$, then the corresponding structure map is the evident identification

$$\Sigma^{Z-W}\Sigma^{W-V}X \cong \Sigma^{Z-V}X.$$

The Vth space of $\Sigma_V^\infty X$ is the zeroth space QX of $\Sigma^\infty X$. It is easy to check the prespectrum level version of the claimed adjunction, and the spectrum level adjunction follows:

$$G\mathscr{S}(\Sigma_V^\infty X, E) \cong G\mathscr{T}(X, \Omega_V^\infty E). \tag{6.1}$$

Exactly as in (4.2) and (4.6), we have natural isomorphisms

$$(\Sigma_V^\infty X) \wedge Y \cong \Sigma_V^\infty (X \wedge Y) \cong X \wedge (\Sigma_V^\infty Y) \tag{6.2}$$

and, for the internal smash product,

$$\Sigma_{V+W}^\infty (X \wedge Y) \cong \Sigma_V^\infty X \wedge \Sigma_W^\infty Y \quad \text{if} \quad V \cap W = \{0\}. \tag{6.3}$$

Another check of right adjoints gives the relation

$$\Sigma_V^\infty X \cong \Sigma_W^\infty \Sigma^{W-V} X \quad \text{if} \quad V \subset W. \tag{6.4}$$

It is not hard to see that any spectrum E can be written as the colimit of the shift desuspensions of its component spaces. That is,

$$E \cong \operatorname{colim} \Sigma_V^\infty EV, \tag{6.5}$$

where the colimit is taken over the maps

$$\Sigma_W^\infty \sigma : \Sigma_V^\infty EV \cong \Sigma_W^\infty (\Sigma^{W-V} EV) \longrightarrow \Sigma_W^\infty EW.$$

Let us write U in the form $U = U^G \oplus U'$ and fix an identification of U^G with $\mathbb{R}^\infty$. We abbreviate notation by writing Ω_n^∞ and Σ_n^∞ when $V = \mathbb{R}^n$. Now define $\underline{S}^{-n} = \Sigma_n^\infty S^0$ for $n > 0$. The reader will notice that we can generalize our definitions to obtain sphere spectra $\underline{S}^V$ and $\underline{S}^{-V}$ for any indexing space V. We can even define spheres $\underline{S}^{V-W} = \Sigma_W^\infty S^V$. We shall need such generality later. However, in developing G-CW theory, it turns out to be appropriate to restrict attention to the spheres $\underline{S}^n$ for integers n. Theorem 6.8 will explain why.

In view of our slogan that orbits are the equivariant analogues of points, we also consider all spectra

(6.6) $$\underline{S}^n_H \equiv G/H_+ \wedge \underline{S}^n,\ H \subset G \text{ and } n \in \mathbb{Z},$$

as spheres. By (6.2), $\underline{S}^n_H \cong \Sigma^\infty(G/H_+ \wedge S^n)$ if $n \geq 0$ and $\underline{S}^n_H \cong \Sigma^\infty_n G/H_+$ if $n < 0$. We shall be more systematic about change of groups later, but we prefer to minimize such equivariant considerations in this section. We define the homotopy group systems of G-spectra by setting

(6.7) $$\pi^H_n(E) = \underline{\pi}_n(E)(G/H) = hG\mathscr{S}(\underline{S}^n_H, E).$$

Let $\mathscr{B}_G U$ be the homotopy category of orbit spectra $\underline{S}^0_H = \Sigma^\infty G/H_+$; we generally abbreviate the names of its objects to G/H. This is an additive category, as will become clear shortly, and $\underline{\pi}_n(E)$ is an additive contravariant functor $\mathscr{B}_G U \longrightarrow \mathscr{A}b$. Recall from IX§4 that such functors are called Mackey functors when the universe U is complete. They play a fundamentally important role in equivariant theory, both in algebra and topology, and we shall return to them later. For now, however, we shall concentrate on the individual homotopy groups $\pi^H_n(E)$. We shall later reinterpret these as homotopy groups $\pi_n(E^H)$ of fixed point spectra, but that too can wait.

The following theorem should be viewed as saying that a weak equivalence of G-spectra really is a weak equivalence of G-spectra. Recall that we defined a weak equivalence $f : D \longrightarrow E$ to be a G-map such that each space level G-map $fV : DV \longrightarrow EV$ is a weak equivalence. In setting up CW-theory, which logically should precede the following theorem, one must mean a weak equivalence to be a map that induces an isomorphism on all of the homotopy groups π^H_n of (6.7).

THEOREM 6.8. *Let $f : E \longrightarrow E'$ be a map of G-spectra. Then each component map $fV : EV \longrightarrow E'V$ is a weak equivalence of G-spaces if and only if $f_* : \pi^H_n E \longrightarrow \pi^H_n E'$ is an isomorphism for all $H \subset G$ and all integers n.*

By our adjunctions, we have

(6.9) $$\pi^H_n(E) \cong \pi_n((E_0)^H) \text{ if } n \geq 0 \text{ and } \pi^H_n(E) \cong \pi_0((E\mathbb{R}^n)^H) \text{ if } n < 0.$$

Therefore, nonequivariantly, the theorem is a tautological triviality. Equivariantly, the forward implication is trivial but the backward implication says that if each $E\mathbb{R}^n \longrightarrow E'\mathbb{R}^n$ is a weak equivalence, then each $EV \longrightarrow E'V$ is also a weak equivalence. Thus it says that information at the trivial representations in U is somehow capturing information at all other representations in U. Its validity justifies the development of G-CW theory in terms of just the sphere spectra of integral dimensions.

We sketch the proof, which goes by induction. We want to prove that each map $f_* : \pi_*(EV)^H \longrightarrow \pi_*(E'V)^H$ is an isomorphism. Since G contains no infinite descending chains of (closed) subgroups, we may assume that f_* is an isomorphism for all proper subgroups of H. An auxiliary argument shows that we may assume that $V^H = \{0\}$. We then use the cofiber sequence

$$S(V)_+ \longrightarrow D(V)_+ \longrightarrow S^V,$$

where $S(V)$ and $D(V)$ are the unit sphere and unit ball in V and thus $D(V)_+ \simeq S^0$. Applying $f : F(-,EV)^H \longrightarrow F(-,E'V)^H$ to this cofiber sequence, we obtain a comparison of fibration sequences. On one end, this is

$$f_0 : (\Omega^V EV)^H = (E_0)^H \longrightarrow (E_0')^H = (\Omega^V E'V)^H,$$

which is given to be a weak equivalence. On the other end, we can triangulate $S(V)$ as an H-CW complex with cells of orbit type H/K, where K is a proper subgroup of H. We can then use change of groups and the inductive hypothesis to deduce that f induces a weak equivalence on this end too. Modulo an extra argument to handle π_0, we conclude that the middle map $f : (EV)^H \longrightarrow (E'V)^H$ is a weak equivalence.

7. G-CW spectra

Before getting to CW theory, we must say something about compactness, which plays an important role. A compact spectrum is one of the form $\Sigma_V^\infty X$ for some indexing space V and compact space X. Since a map of spectra with domain $\Sigma_V^\infty X$ is determined by a map of spaces with domain X, facts about maps out of compact spaces imply the corresponding facts about maps out of compact spectra. For example, if E is the union of an expanding sequence of subspectra E_i, then

(7.1) $$G\mathscr{S}(\Sigma_V^\infty X, E) \cong \operatorname{colim} G\mathscr{S}(\Sigma_V^\infty X, E_i).$$

The following lemma clarifies the relationship between space level and spectrum level maps. Recall the isomorphisms of (6.4).

LEMMA 7.2. *Let $f : \Sigma_V^\infty X \longrightarrow \Sigma_W^\infty Y$ be a map of G-spectra, where X is compact. Then, for a large enough indexing space Z, there is a map $g : \Sigma^{Z-V} X \longrightarrow \Sigma^{Z-W} Y$ of G-spaces such that f coincides with*

$$\Sigma_Z^\infty g : \Sigma_V^\infty X \cong \Sigma_Z^\infty(\Sigma^{Z-V} X) \longrightarrow \Sigma_Z^\infty(\Sigma^{Z-W} Y) \cong \Sigma_W^\infty Y.$$

This result shows how to calculate the full subcategory of the stable category consisting of those G-spectra of the form $\Sigma_V^\infty X$ for some indexing space V and finite G-CW complex X in space level terms. It can be viewed as giving an equivariant reformulation of the Spanier-Whitehead S-category. In particular, we have the following consistency statement with the definitions of IX§2.

PROPOSITION 7.3. *If X is a finite based G-CW complex and Y is a based G-space, then*

$$\{X, Y\}_G \cong [\Sigma^\infty X, \Sigma^\infty Y]_G.$$

From here, the development of CW theory is essentially the same equivariantly as nonequivariantly, and essentially the same on the spectrum level as on the space level. The only novelty is that, because we have homotopy groups in negative degrees, we must use two filtrations. Older readers may see more novelty. In contrast with earlier treatments, our CW theory is developed on the spectrum level and has nothing whatever to do with any possible cell structures on the component spaces of spectra. I view the use of space level cell structures in this context as an obsolete historical detour that serves no useful mathematical purpose.

Let $CE = E \wedge I$ denote the cone on a G-spectrum E.

DEFINITION 7.4. A G-cell spectrum is a spectrum $E \in G\mathscr{S}$ that is the union of an expanding sequence of subspectra E_n, $n \geq 0$, such that E_0 is the trivial spectrum (each of its component spaces is a point) and E_{n+1} is obtained from E_n by attaching G-cells $C\underline{S}^q_H \cong G/H_+ \wedge C\underline{S}^q$ along attaching G-maps $\underline{S}^q_H \longrightarrow E_n$. Cell subspectra, or "subcomplexes", are defined in the evident way. A G-CW spectrum is a G-cell spectrum each of whose attaching maps $\underline{S}^q_H \longrightarrow E_n$ factors through a subcomplex that contains only cells of dimension at most q. The n-skeleton E^n is then defined to be the union of the cells of dimension less than or equal to n.

LEMMA 7.5. *A map from a compact spectrum to a cell spectrum factors through a finite subcomplex. Any cell spectrum is the union of its finite subcomplexes.*

The filtration $\{E_n\}$ is called the sequential filtration. It records the order in which cells are attached, and it can be chosen in many different ways. In fact, using the lemma, we see that by changing the sequential filtration on the domain, any map between cell spectra can be arranged to preserve the sequential filtration. Using this filtration, we find that the inductive proofs of the following results that we sketched on the space level work in exactly the same way on the spectrum level. We leave it to the reader to formulate their more precise "dimension ν" versions.

THEOREM 7.6 (HELP). *Let A be a subcomplex of a G-CW spectrum D and let $e : E \longrightarrow E'$ be a weak equivalence. Suppose given maps $g : A \longrightarrow E$, $h : A \wedge I_+ \longrightarrow E'$, and $f : D \longrightarrow E'$ such that $eg = hi_1$ and $fi = hi_0$ in the following diagram:*

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \wedge I_+ \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd, "i"] \\
& E' & & E \arrow[ll, "e"'] & \\
D \arrow[rr, "i_0"'] \arrow[ur, "f"] & & D \wedge I_+ \arrow[ul, dashed, "\tilde{h}"] & & D \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"']
\end{tikzcd}$$

Then there exist maps $\tilde{g}$ and $\tilde{h}$ that make the diagram commute.

THEOREM 7.7 (WHITEHEAD). *Let $e : E \longrightarrow E'$ be a weak equivalence and D be a G-CW spectrum. Then $e_* : hG\mathscr{S}(D, E) \longrightarrow hG\mathscr{S}(D, E')$ is a bijection.*

COROLLARY 7.8. *If $e : E \longrightarrow E'$ is a weak equivalence between G-CW spectra, then e is a G-homotopy equivalence.*

THEOREM 7.9 (CELLULAR APPROXIMATION). *Let (D,A) and (E,B) be relative G-CW spectra, (D',A') be a subcomplex of (D,A), and $f:(D,A)\longrightarrow(E,B)$ be a G-map whose restriction to (D',A') is cellular. Then f is homotopic rel $D'\cup A$ to a cellular map $g:(D,A)\longrightarrow(E,B)$.*

COROLLARY 7.10. *Let D and E be G-CW spectra. Then any G-map $f: D\longrightarrow E$ is homotopic to a cellular map, and any two homotopic cellular maps are cellularly homotopic.*

THEOREM 7.11. *For any G-spectrum E, there is a G-CW spectrum ΓE and a weak equivalence $\gamma:\Gamma E\longrightarrow E$.*

Exactly as on the space level, it follows from the Whitehead theorem that Γ extends to a functor $hG\mathscr{S}\longrightarrow hG\mathscr{C}$, where $G\mathscr{C}$ is here the category of G-CW spectra and cellular maps, and the morphisms of the stable category $\bar{h}G\mathscr{S}$ can be specified by

$$\bar{h}G\mathscr{S}(E,E')=hG\mathscr{S}(\Gamma E,\Gamma E')=hG\mathscr{C}(\Gamma E,\Gamma E'). \tag{7.12}$$

From now on, we shall write $[E,E']_G$ for this set. Again, Γ gives an equivalence of categories $\bar{h}G\mathscr{S}\longrightarrow hG\mathscr{C}$.

We should say something about the transport of functors F on $G\mathscr{S}$ to the category $\bar{h}G\mathscr{S}$. All of our functors preserve homotopies, but not all of them preserve weak equivalences. If F does not preserve weak equivalences, then, on the stable category level, we understand F to mean the functor induced by the composite $F\circ\Gamma$, a functor which preserves weak equivalences by converting them to genuine equivalences.

For this and other reasons, it is quite important to understand when functors preserve CW-homotopy types and when they preserve weak equivalences. These questions are related. In a general categorical context, a left adjoint preserves CW-homotopy types if and only if its right adjoint preserves weak equivalences. When these equivalent conditions hold, the induced functors on the categories obtained by inverting the weak equivalences are again adjoint.

For example, since Ω^∞_V preserves weak equivalences (with the correct logical order, by Theorem 6.8), Σ^∞_V preserves CW homotopy types. Of course, since our left adjoints preserve colimits and smash products with spaces, their behavior on CW spectra is determined by their behavior on spheres. Since Σ^∞_n clearly preserves spheres, it carries G-CW based complexes (with based attaching maps) to G-CW spectra. This focuses attention on a significant difference between the equivariant and nonequivariant contexts. In both, a CW spectrum is the colimit of its finite subcomplexes. Nonequivariantly, Lemma 7.2 implies that any finite CW spectrum is isomorphic to $\Sigma^\infty_n X$ for some n and some finite CW complex X. Equivariantly, this is only true up to homotopy type. It would be true up to isomorphism if we allowed non-trivial representations as the domains of attaching maps in our definitions of G-CW complexes and spectra. We have seen that such a theory of "G-CW(V)-complexes" is convenient and appropriate on the space level, but it seems to serve no useful purpose on the spectrum level.

Along these lines, we point out an important consequence of (6.3). It implies that the smash product of spheres $\underline{S}^m_H$ and $\underline{S}^n_J$ is $(G/H\times G/J)_+\wedge\underline{S}^{m+n}$. When

G is finite, we can use double cosets to describe $G/H \times G/J$ as a disjoint union of orbits G/K. This allows us to deduce that the smash product of G-CW spectra is a G-CW spectrum. For general compact Lie groups G, we can only deduce that the smash product of G-CW spectra has the homotopy type of a G-CW spectrum.

8. Stability of the stable category

The observant reader will object that we have called $\bar{h}G\mathscr{S}$ the "stable category", but that we haven't given a shred of justification. As usual, we write $\Sigma^V E = E \wedge S^V$ and $\Omega^V E = F(S^V, E)$.

THEOREM 8.1. *For all indexing spaces V in U, the natural maps*

$$\eta : E \longrightarrow \Omega^V \Sigma^V E \quad \textit{and} \quad \varepsilon : \Sigma^V \Omega^V E \longrightarrow E$$

are isomorphisms in $\bar{h}G\mathscr{S}$. Therefore Ω^V and Σ^V are inverse self-equivalences of $\bar{h}G\mathscr{S}$.

Thus we can desuspend by any representations that are in U. Once this is proven, it is convenient to write Σ^{-V} for Ω^V. There are several possible proofs, all of which depend on Theorem 6.8: that is the crux of the matter, and this means that the result is trivial in the nonequivariant context. In fact, once we have Theorem 6.8, we have that the functor Σ^∞_V preserves G-CW homotopy types. Using (6.2), (6.4), and the unit equivalence for the smash product, we obtain

$$E \simeq E \wedge \underline{S}^0 \cong E \wedge \Sigma^\infty_V S^V \cong E \wedge (\Sigma^\infty_V S^0 \wedge S^V).$$

This proves that the functor Σ^V is an equivalence of categories. By playing with adjoints, we see that Ω^V must be its inverse. Observe that this proof is independent of the Freudenthal suspension theorem. This argument and (6.2) give the following important consistency relations, where we now drop the underline from our notation for sphere spectra:

$$(8.2) \qquad \Omega^V E \simeq E \wedge S^{-V} \quad \text{and} \quad \Sigma^\infty_V X \cong X \wedge S^{-V}, \text{ where } S^{-V} \equiv \Sigma^\infty_V S^0.$$

Since all universes contain $\mathbb{R}$, all G-spectra are equivalent to suspensions. This implies that $\bar{h}G\mathscr{S}$ is an additive category, and it is now straightforward to prove that $\bar{h}G\mathscr{S}$ is triangulated. In fact, it has two triangulations, by cofibrations and fibrations, that differ only by signs. We have already seen that it is symmetric monoidal under the smash product and that it has well-behaved function spectra. We have established a good framework in which to do equivariant stable homotopy theory, and we shall say more about how to exploit it as we go on.

9. Getting into the stable category

The stable category is an ideal world, and the obvious question that arises next is how one gets from the prespectra that occur "in nature" to objects in this category. Of course, our prespectra are all encompassing, since we assumed nothing about their constituent spaces and structure maps, and we do have the left adjoint $L : G\mathscr{P} \longrightarrow G\mathscr{S}$. However, this is a theoretical tool: its good formal properties come at the price of losing control over homotopical information. We need an alternative way of getting into the stable category, one that retains homotopical information.

We first need to say a little more about the functor L. If the adjoint structure maps $\tilde{\sigma} : EV \longrightarrow \Omega^{W-V}EW$ of a prespectrum E are inclusions, then $(LE)(V)$ is just the union over $W \supset V$ of the spaces $\Omega^{W-V}EW$. Taking $W = V$, we obtain an inclusion $\eta : EV \longrightarrow (LE)(V)$, and these maps specify a map of prespectra. If, further, each $\tilde{\sigma}$ is a cofibration and an equivalence, then each map η is an equivalence.

Thus we seek to transform given prespectra into spacewise equivalent ones whose adjoint structural maps are cofibrations. The spacewise equivalence property will ensure that Ω-prespectra are transported to Ω-prespectra. It is more natural to consider cofibration conditions on the structure maps $\sigma : \Sigma^{W-V}EV \rightarrow EW$, and we say that a prespectrum E is "Σ-cofibrant" if each σ is a cofibration. If E is a Σ-cofibrant prespectrum and if each EV has cofibered diagonal, in the sense that the diagonal map $EV \longrightarrow EV \times EV$ is a cofibration, then each adjoint map $\tilde{\sigma} : EV \longrightarrow \Omega^{W-V}EW$ is a cofibration, as desired.

Observe that no non-trivial spectrum can be Σ-cofibrant as a prespectrum since the structure maps σ of spectra are surjections rather than injections. We say that a spectrum is "tame" if it is homotopy equivalent to LE for some Σ-cofibrant prespectrum E. The importance of this condition was only recognized during the work of Elmendorf, Kriz, Mandell, and myself on structured ring spectra. Its use leads to key technical improvements of [EKMM] over [LMS]. For example, the sharpest versions of Theorems 3.4 and 8.1 read as follows (and are implied by XXII.1.8 below).

THEOREM 9.1. *Let $\mathscr{S}_tU \subset \mathscr{S}U$ be the full subcategory of tame spectra indexed on U. Then any two linear isometries $U \longrightarrow U'$ induce canonically and coherently equivalent functors $h\mathscr{S}_tU \longrightarrow h\mathscr{S}_tU'$. The maps $\eta : E \longrightarrow \Omega\Sigma E$ and $\varepsilon : \Sigma\Omega E \longrightarrow E$ are homotopy equivalences of spectra when E is tame.*

Moreover, analogously to (6.5), but much more usefully, if E is a Σ-cofibrant prespectrum, then

$$LE \cong \operatorname{colim} \Sigma^\infty_V EV, \tag{9.2}$$

where the maps of the colimit system are the cofibrations

$$\Sigma^\infty_W \sigma : \Sigma^\infty_V EV \cong \Sigma^\infty_W(\Sigma^{W-V}EV) \longrightarrow \Sigma^\infty_W EW.$$

Here the prespectrum level colimit is already a spectrum, so that the colimit is constructed directly, without use of the functor L. Given a G-spectrum E', there results a valuable $\lim^1$ exact sequence

$$(9.3)\qquad 0 \longrightarrow \lim{}^1[\Sigma EV, E'V]_G \longrightarrow [LE, E']_G \longrightarrow \lim[EV, E'V]_G \longrightarrow 0$$

for the calculation of maps in $\bar{h}G\mathscr{S}$ in terms of maps in $\bar{h}G\mathscr{T}$.

To avoid nuisance about inverting weak equivalences here, we introduce an equivariant version of the classical CW prespectra.

DEFINITION 9.4. A G-CW prespectrum is a Σ-cofibrant G-prespectrum E such that each EV has cofibered diagonal and is of the homotopy type of a G-CW complex.

We can insist on actual G-CW complexes, but it would not be reasonable to ask for cellular structure maps. We have the following reassuring result relating this notion to our notion of a G-CW spectrum.

PROPOSITION 9.5. *If E is a G-CW prespectrum, then LE has the homotopy type of a G-CW spectrum. If E is a G-CW spectrum, then each component space EV has the homotopy type of a G-CW complex.*

Now return to our original question of how to get into the stable category. The kind of maps of prespectra that we are interested in here are "weak maps" $D \longrightarrow E$, whose components $DV \longrightarrow EV$ are only required to be compatible up to homotopy with the structural maps. If D is Σ-cofibrant, then any weak map is spacewise homotopic to a genuine map. The inverse limit term of (9.3) is given by weak maps, which represent maps between cohomology theories on spaces, and its $\lim^1$ term measures the difference between weak maps and genuine maps, which represent maps between cohomology theories on spectra.

Applying G-CW approximation spacewise, using I.3.6, we can replace any G-prespectrum E by a spacewise weakly equivalent G-prespectrum ΓE whose component spaces are G-CW complexes and therefore have cofibered diagonal maps. However, the structure maps, which come from the Whitehead theorem and are only defined up to homotopy, need not be cofibrations. The following "cylinder construction" converts a G-prespectrum E whose spaces are of the homotopy types of G-CW complexes and have cofibered diagonals into a spacewise equivalent G-CW prespectrum KE. Both constructions are functorial on weak maps.

The composite $K\Gamma$ carries an arbitrary G-prespectrum E to a spacewise equivalent G-CW prespectrum. By Proposition 9.5, $LK\Gamma E$ has the homotopy type of a G-CW spectrum. In sum, the composite $LK\Gamma$ provides a canonical passage from G-prespectra to G-CW spectra that is functorial up to weak homotopy and preserves all homotopical information in the given G-prespectra.

The version of the cylinder construction presented in [LMS] is rather clumsy. The following version is due independently to Elmendorf and Hesselholt. It enjoys much more precise properties, details of which are given in [EKMM].

CONSTRUCTION 9.6 (CYLINDER CONSTRUCTION). Let E be a G-prespectrum indexed on U. Define KE as follows. For an indexing space V, let $\underline{V}$ be the category of subspaces $V' \subset V$ and inclusions. Define a functor E_V from $\underline{V}$ to G-spaces by letting $E_V(V') = \Sigma^{V-V'}EV'$. For an inclusion $V'' \longrightarrow V'$,

$$V - V'' = (V - V') \oplus (V' - V'')$$

and $\sigma : \Sigma^{V'-V''}EV'' \longrightarrow EV'$ induces $E_V(V'') \longrightarrow E_V(V')$. Define

$$(KE)(V) = \operatorname{hocolim} E_V.$$

An inclusion $i : V \longrightarrow W$ induces a functor $\underline{i} : \underline{V} \longrightarrow \underline{W}$, the functor Σ^{W-V} commutes with homotopy colimits, and we have an evident isomorphism $\Sigma^{W-V}E_V \cong E\underline{i}$ of functors $\underline{V} \longrightarrow \underline{W}$. Therefore $\underline{i}$ induces a map

$$\sigma : \Sigma^{W-V} \operatorname{hocolim} E_V \cong \operatorname{hocolim} \Sigma^{W-V} E_V \cong \operatorname{hocolim} E\underline{i} \longrightarrow \operatorname{hocolim} E_W.$$

One can check that this map is a cofibration. Thus, with these structural maps, KE is a Σ-cofibrant prespectrum. The structural maps $\sigma : E_V V' \longrightarrow EV$ specify a natural transformation to the constant functor at EV and so induce a map $r : (KE)(V) \longrightarrow EV$, and these maps r specify a map of prespectra. Regarding the object V as a trivial subcategory of $\underline{V}$, we obtain $j : EV \longrightarrow (KE)(V)$. Clearly $rj = \mathrm{id}$, and $jr \simeq \mathrm{id}$ via a canonical homotopy since V is a terminal object of $\underline{V}$. The maps j specify a weak map of prespectra, via canonical homotopies. Clearly K is functorial and homotopy-preserving, and r is natural. If each space EV has the homotopy type of a G-CW complex, then so does each $(KE)(V)$, and similarly for the cofibered diagonals condition.

A striking property of this construction is that it commutes with smash products: if E and E' are prespectra indexed on U and U', then $KE \wedge KE'$ is isomorphic over $E \wedge E'$ to $K(E \wedge E')$.

CHAPTER XIII

$RO(G)$-graded homology and cohomology theories

1. Axioms for RO(G)-graded cohomology theories

Switching to a homological point of view, we now consider $RO(G)$-graded homology and cohomology theories. There are several ways to be precise about this, and there are several ways to be imprecise. The latter are better represented in the literature than the former. As we have already said, no matter how things are set up, "$RO(G)$-graded" is technically a misnomer since one cannot think of representations as isomorphism classes and still keep track of signs. We give a formal axiomatic definition here and connect it up with G-spectra in the next section.

From now on, we shall usually restrict attention to reduced homology and cohomology theories and shall write them without a tilde. Of course, a $\mathbb{Z}$-graded homology or cohomology theory on G-spaces is required to satisfy the redundant axioms: homotopy invariance, suspension isomorphism, exactness on cofiber sequences, additivity on wedges, and invariance under weak equivalence. Here exactness only requires that a cofiber sequence $X \longrightarrow Y \longrightarrow Z$ be sent to a three term exact sequence in each degree. The homotopy and weak equivalence axioms say that the theory is defined on $\bar{h}G\mathscr{T}$. Such theories determine and are determined by unreduced theories that satisfy the Eilenberg-Steenrod axioms, minus the dimension axiom. Since

$$k_G^{-n}(X) \cong k_G^0(\Sigma^n X),$$

only the non-negative degree parts of a theory need be specified, and a non-negative integer n corresponds to $\mathbb{R}^n$. Indexing on $\mathbb{Z}$ amounts to either choosing a basis for $\mathbb{R}^\infty$ or, equivalently, choosing a skeleton of a suitable category of trivial representations.

Now assume given a G-universe U, say $U = \oplus(V_i)^\infty$ for some sequence of distinct irreducible representations V_i with $V_1 = \mathbb{R}$. An $RO(G;U)$-graded theory can be thought of as graded on the free Abelian group on basis elements corresponding to the V_i. It is equivalent to grade on the skeleton of a category of representations embeddable in U, or to grade on this entire category. The last approach seems to be preferable when considering change of groups, so we will adopt it.

Thus let $\mathscr{RO}(G;U)$ be the category whose objects are the representations embeddable in U and whose morphisms $V \longrightarrow W$ are the G-linear isometric isomorphisms. Say that two such maps are homotopic if their associated based G-maps $S^V \longrightarrow S^W$ are *stably* homotopic, and let $h\mathscr{RO}(G;U)$ be the resulting homotopy category.

DEFINITION 1.1. An $RO(G;U)$-graded cohomology theory is a functor

$$E_G^* : h\mathscr{RO}(G,U) \times (\bar{h}G\mathscr{T})^{op} \longrightarrow \mathscr{A}b,$$

written $(V,X) \longrightarrow E_G^V(X)$ on objects and similarly on morphisms, together with isomorphisms

$$\sigma^W : E_G^V(X) \longrightarrow E_G^{V\oplus W}(\Sigma^W X).$$

The σ^W must be covariantly natural in V and contravariantly natural in X, and the following axioms must be satisfied.

(1) For each representation V, the functor E_G^V is exact on cofiber sequences and sends wedges to products.

(2) If $\alpha : W \longrightarrow W'$ is a map in $h\mathscr{RO}(G,U)$, then the following diagram commutes:

$$\begin{tikzcd} E_G^V(X) \arrow[r, "\sigma^W"] \arrow[d, "\sigma^{W'}"'] & E_G^{V\oplus W}(\Sigma^W X) \arrow[d, "E_G^{\mathrm{id}\,\oplus\alpha}(\mathrm{id})"] \\ E_G^{V\oplus W'}(\Sigma^{W'} X) \arrow[r, "(\Sigma^\alpha\,\mathrm{id})^*"'] & E_G^{V\oplus W'}(\Sigma^W X). \end{tikzcd}$$

(3) $\sigma^0 = \mathrm{id}$ and the σ are transitive in the sense that the following diagram commutes for each pair of representations (W,Z):

$$\begin{tikzcd} E_G^V(X) \arrow[rr, "\sigma^W"] \arrow[dr, "\sigma^{W\oplus Z}"'] & & E_G^{V\oplus W}(\Sigma^W X) \arrow[dl, "\sigma^Z"] \\ & E_G^{V\oplus W\oplus Z}(\Sigma^{W\oplus Z} X). & \end{tikzcd}$$

We extend a theory so defined to "formal differences $V \ominus W$" for any pair of representations (V,W) by setting

$$E_G^{V\ominus W}(X) = E_G^V(\Sigma^W X).$$

We use the symbol $\ominus$ to avoid confusion with either orthogonal complement or difference in the representation ring. Rigorously, we are thinking of $V \ominus W$ as an object of the category $h\mathscr{RO}(G;U) \times h\mathscr{RO}(G;U)^{op}$, and, for each X, we have defined a functor from this category to the category of Abelian groups.

The representation group $RO(G;U)$ relative to the given universe U is obtained by passage to equivalence classes from the set of formal differences $V \ominus W$, where $V \ominus W$ is equivalent to $V' \ominus W'$ if there is a G-linear isometric isomorphism

$$\alpha : V \oplus W' \longrightarrow V' \oplus W;$$

Let $\mathscr{I}O(G;U)$ and $h\mathscr{I}O(G;U)$ be the full subcategories of $\mathscr{R}O(G;U)$ and $h\mathscr{R}O(G;U)$ whose objects are the indexing spaces in U, let

$$\Psi : \mathscr{I}O(G;U) \longrightarrow \mathscr{R}O(G;U)$$

be the inclusion, and also write Ψ for the inclusion $h\mathscr{I}O(G;U) \longrightarrow h\mathscr{R}O(G;U)$. For each representation V that is embeddable in U, choose an indexing space ΦV in U and a G-linear isomorphism $\phi_V : V \longrightarrow \Phi V$. If V is itself an indexing space in U, choose $\Phi V = V$ and let ϕ_V be the identity map. Extend Φ to a functor

$$\Phi : \mathscr{R}O(G;U) \longrightarrow \mathscr{I}O(G;U)$$

by letting $\Phi\alpha$, $\alpha : V \longrightarrow V'$, be the composite

$$\Phi V \xrightarrow{\phi_V^{-1}} V \xrightarrow{\alpha} V' \xrightarrow{\phi_{V'}} \Phi V'.$$

Then $\Phi \circ \Psi = \mathrm{Id}$ and the ϕ_V define a natural isomorphism $\mathrm{Id} \longrightarrow \Psi \circ \Phi$. This equivalence of categories induces an equivalence of categories between $h\mathscr{I}O(G;U)$ and $h\mathscr{R}O(G;U)$. A functor F from $h\mathscr{I}O(G;U)$ to any category $\mathscr{C}$ extends to the functor $F\Phi$ from $h\mathscr{R}O(G;U)$ to $\mathscr{C}$, and we agree to write F instead of $F\Phi$ for such an extended functor.

LEMMA 2.1. *Let E be an ΩG-prespectrum. Then E gives the object function of a functor $E : h\mathscr{R}O(G;U) \longrightarrow \bar{h}G\mathscr{T}$.*

PROOF. By the observations above, it suffices to define E as a functor on $h\mathscr{I}O(G;U)$. Suppose given indexing spaces V and V' in U and a G-linear isomorphism $\alpha : V \longrightarrow V'$. Choose an indexing space W large enough that it contains both V and V' and that $W - V$ and $W - V'$ both contain copies of representations isomorphic to V and thus to V'. Then there is an isomorphism $\beta : W - V \longrightarrow W - V'$ such that

$$\beta \wedge \alpha : S^W \cong S^{W-V} \wedge S^V \longrightarrow S^{W-V'} \wedge S^{V'} \cong S^W$$

is stably homotopic to the identity. (For the verification, one relates smash product to composition product in the zero stem $\pi_0^G(S^0)$, exactly as in nonequivariant stable homotopy theory.) Then define $E\alpha : EV \longrightarrow EV'$ to be the composite

$$EV \xrightarrow{\tilde{\sigma}} \Omega^{W-V}EW \xrightarrow{\Omega^{\beta^{-1}}} \Omega^{W-V'}EW \xrightarrow{\tilde{\sigma}^{-1}} EV'.$$

It is not hard to check that this construction takes stably homotopic maps α and α' to homotopic maps $E\alpha$ and $E\alpha'$ and that the construction is functorial on $\mathscr{I}O(G;U)$. □

PROPOSITION 2.2. *An Ω-G-prespectrum E indexed on a universe U represents an $RO(G;U)$-graded cohomology theory E_G^* on based G-spaces.*

PROOF. For a representation V that embeds in U, define

$$E_G^V(X) = [X, E\Phi V]_G.$$

For each $\alpha : V \longrightarrow V'$, define

$$E_G^\alpha(X) = [X, E\Phi\alpha]_G.$$

This gives us the required functor

$$E_G^* : h\mathscr{R}O(G,U) \times (\bar{h}G\mathscr{T})^{op} \longrightarrow \mathscr{A}b,$$

and it is obvious that Axiom (1) of Definition 1.1 is satisfied.

Next, suppose given representations V and W that embed in U. We may write

$$\Phi(V \oplus W) = V' + W',$$

where $V' = \phi_{V\oplus W}(V)$ and $W' = \phi_{V\oplus W}(W)$. There result isomorphisms

$$\iota_V : \Phi V \xrightarrow{\phi_V^{-1}} V \xrightarrow{\phi'_V} V' \quad \text{and} \quad \iota_W : \Phi W \xrightarrow{\phi_W^{-1}} W \xrightarrow{\phi'_W} W',$$

where $\phi'_V = \phi_{V\oplus W}|_V$ and $\phi'_W = \phi_{V\oplus W}|_W$. Define

$$\sigma^W : E_G^V(X) \longrightarrow E_G^{V\oplus W}(\Sigma^W X)$$

by the commutativity of the following diagram:

$$\begin{tikzcd}
E_G^V X = [X, E\Phi V]_G \arrow[rr, "{[\mathrm{id}, E\iota_V]}"] \arrow[dd, "\sigma^W"'] & & {[X, EV']_G} \arrow[d, "{[\mathrm{id},\tilde{\sigma}]}"] \\
& & {[X, \Omega^{W'} E(V' + W')]_G} \arrow[d, "\cong"] \\
E_G^{V\oplus W}(\Sigma^W X) = [\Sigma^W X, E\Phi(V \oplus W)]_G & & {[\Sigma^{W'} X, E(V' \oplus W')]_G.} \arrow[ll, "{[\Sigma^{\phi'_W} \mathrm{id}, \mathrm{id}]}"']
\end{tikzcd}$$

Diagram chases from the definitions demonstrate that σ^W is natural, that the diagram of Axiom (2) of Definition 1.1 commutes, and that the transitivity diagram of Axiom 3 commutes because of the transitivity condition that we gave as part of the definition of a G-prespectrum. □

The evident analogue for homology theories on G-spaces also holds.

A slight variant of the proof above could be obtained by first replacing the given Ω-G-prespectrum by a spacewise equivalent G-spectrum indexed on U and then specializing the following result to suspension G-spectra. Recall that, for an indexing space V, we have the shift desuspension functor Σ_V^∞ from based G-spaces to G-spectra. It is left adjoint to the Vth space functor:

(2.3) $$[\Sigma_V^\infty X, E]_G \cong [X, EV]_G.$$

DEFINITION 2.4. For representations V and W of G that embed in U, define the sphere G-spectrum $S^{W\ominus V}$ by

(2.5) $$S^{W\ominus V} = \Sigma_{\Phi V}^\infty S^W,$$

where $\Phi : \mathscr{R}O(G;U) \longrightarrow \mathscr{I}O(G;U)$ is the equivalence of categories constructed above.

PROPOSITION 2.6. *A G-spectrum E indexed on U determines an $RO(G;U)$-graded homology theory E_*^G and an $RO(G;U)$-graded cohomology theory E_G^* on G-spectra.*

PROOF. For G-spectra X and representations V and W that embed in U, we define

$$(2.7) \qquad E^G_{W\ominus V}(X) = [S^{W\ominus V}, E\wedge X]_G$$

and

$$(2.8) \qquad E_G^{V\ominus W}(X) = [S^{W\ominus V}\wedge X, E]_G = [S^{W\ominus V}, F(X,E)]_G.$$

Of course, to verify the axioms, we may as well restrict attention to the case $W = 0$. Obviously, the verification reduces to the study of the properties of the G-spheres $\Sigma^\infty_V S^0$, or of the functors Σ^∞_V. First, we need functoriality on $h\mathscr{R}O(G;U)$, but this is immediate from (2.3) and the functoriality of the EV given by Lemma 2.1. With the notations of the previous proof, we obtain the $\sigma^W : E^V_G(X) \xrightarrow{\cong} E^{V\oplus W}_G(\Sigma^W X)$ from the composite isomorphism of functors

$$\Sigma^\infty_{\Phi V} \cong \Sigma^\infty_{V'} \cong \Sigma^{W'}\Sigma^\infty_{V'+W'} \cong \Sigma^W\Sigma^\infty_{\Phi(V\oplus W)},$$

where the three isomorphisms are given by use of ι_V, passage to adjoints from the homeomorphism $\tilde{\sigma} : EV' \longrightarrow \Omega^{W'}E(V'+W')$, and use of ϕ'_W. From here, the verification of the axioms is straightforward. □

3. Brown's theorem and $RO(G)$-graded cohomology

We next show that, conversely, all $RO(G)$-graded cohomology theories on based G-spaces are represented by Ω-G-prespectra and all theories on G-spectra are represented by G-spectra. We then discuss the situation in homology, which is considerably more subtle equivariantly than nonequivariantly.

We first record Brown's representability theorem. Brown's categorical proof applies just as well equivariantly as nonequivariantly, on both the space and the spectrum level. Recall that homotopy pushouts are double mapping cylinders and that weak pullbacks satisfy the existence but not the uniqueness property of pullbacks. Recall that a G-space X is said to be G-connected if each of its fixed point spaces X^H is non-empty and connected.

THEOREM 3.1 (BROWN). *A contravariant set-valued functor k on the homotopy category of G-connected based G-CW complexes is representable in the form $kX \cong [X,K]_G$ for a based G-CW complex K if and only if k satisfies the wedge and Mayer-Vietoris axioms: k takes wedges to products and takes homotopy pushouts to weak pullbacks. The same statement holds for the homotopy category of G-CW spectra indexed on U for any G-universe U.*

COROLLARY 3.2. *An $RO(G;U)$-graded cohomology theory E^*_G on based G-spaces is represented by an Ω-G-prespectrum indexed on U.*

PROOF. Restricting attention to G-connected based G-spaces, which is harmless in view of the suspension axiom for trivial representations, we see that (1) of Definition 1.1 implies the Mayer-Vietoris and wedge axioms that are needed to apply Brown's representability theorem. This gives that E^V_G is represented by a G-CW complex EV for each indexing space V in U. If $V \subset W$, then the suspension isomorphism

$$\sigma^{W-V} : E^V_G(X) \cong E^W_G(\Sigma^{W-V} X)$$

is represented by a homotopy equivalence $\tilde{\sigma} : EV \longrightarrow \Omega^{W-V} EW$. The transitivity of the given system of suspension isomorphisms only gives that the structural maps are transitive up to homotopy, whereas the definition of a G-prespectrum requires that the structural maps be transitive on the point-set level. If we restrict to a cofinal sequence of indexing spaces, then we can use transitivity to define the structural weak equivalences for non-consecutive terms of the sequence. We can then interpolate using loop spaces to construct a representing Ω-G-prespectrum indexed on all indexing spaces. □

We emphasize a different point of view of the spectrum level analog. In fact, we shall exploit the following result to construct ordinary $RO(G)$-graded cohomology theories in the next section.

COROLLARY 3.3. *A $\mathbb{Z}$-graded cohomology theory on G-spectra indexed on U is represented by a G-spectrum indexed on U and therefore extends to an $RO(G;U)$-graded cohomology theory on G-spectra indexed on U.*

PROOF. Since the loop and suspension functors are inverse equivalences on the stable category $\bar{h}G\mathscr{S}U$, we can reconstruct the given theory from its zeroth term, and Brown's theorem applies to represent the zeroth term. □

We showed in the previous chapter that an Ω-G-prespectrum determines a spacewise equivalent G-spectrum, so that a cohomology theory on based G-spaces extends to a cohomology theory on G-spectra. The extension is unique up to non-unique isomorphism, where the non-uniqueness is measured by the $\lim^1$ term in XII.9.3.

Adams proved a variant of Brown's representability theorem for functors defined only on connected finite CW complexes, removing a countability hypothesis that was present in an earlier version due to Brown. This result also generalizes to the equivariant context, with the same proof as Adams' original one.

THEOREM 3.4 (ADAMS). *A contravariant group-valued functor k defined on the homotopy category of G-connected finite based G-CW complexes is representable in the form $kX \cong [X, K]_G$ for some G-CW spectrum K if and only if k converts finite wedges to direct products and converts homotopy pushouts to weak pullbacks of underlying sets. The same statement holds for the homotopy category of finite G-CW spectra.*

Here the representing G-CW spectrum K is usually infinite and is unique only up to non-canonical equivalence. More precisely, maps $g, g' : Y \longrightarrow Y'$ are said to be weakly homotopic if gf is homotopic to $g'f$ for any map $f : X \to Y$ defined on a finite G-CW spectrum X, and K is unique up to isomorphism in the resulting weak homotopy category of G-CW spectra.

Nonequivariantly, we pass from here to the representation of homology theories by use of Spanier-Whitehead duality. A finite CW spectrum X has a dual

DX that is also a finite CW spectrum. Given a homology theory E_* on based spaces or on spectra, we obtain a dual cohomology theory on finite X by setting

$$E^n(X) = E_{-n}(DX).$$

We then argue as above that this cohomology theory on finite X is representable by a spectrum E, and we deduce by duality that E also represents the originally given homology theory.

Equivariantly, this argument works for a complete G-universe U, but it does not work for a general universe. The problem is that, as we shall see later, only those orbit spectra $\Sigma^\infty G/H_+$ such that G/H embeds equivariantly in U have well-behaved duals. For example, if the universe U is trivial, then inspection of definitions shows that $F(G/H_+, S) = S$ for all $H \subseteq G$, where S is the sphere spectrum with trivial G-action. Thus X is not equivalent to DDX in general and we cannot hope to recover $E_*(X)$ as $E^*(DX)$.

COROLLARY 3.5. *If U is a complete G-universe, then an $RO(G;U)$-graded homology theory on based G-spaces or on G-spectra is representable.*

From now on, unless explicitly stated otherwise, we take our given universe U to be complete, and we write $RO(G) = RO(G;U)$. As shown by long experience in nonequivariant homotopy theory, even if one's primary interest is in spaces, the best way to study homology and cohomology theories is to work on the spectrum level, exploiting the virtues of the stable homotopy category.

J. F. Adams. A variant of E. H. Brown's representability theorem. Topology, 10(1971), 185–198.

E. H. Brown, Jr. Cohomology theories. Annals of Math. 75(1962), 467–484.

E. H. Brown, Jr. Abstract homotopy theory. Trans. Amer. Math. Soc. 119(1965), 79–85.

4. Equivariant Eilenberg-Mac Lane spectra

From the topological point of view, a coefficient system is a contravariant additive functor from the stable category of naive orbit spectra to Abelian groups. In fact, it is easy to see that the group of stable maps $G/H_+ \longrightarrow G/K_+$ in the naive sense is the free Abelian group on the set of G-maps $G/H \longrightarrow G/K$.

Recall from IX§4 that a Mackey functor is defined to be an additive contravariant functor $\mathscr{B}_G \longrightarrow \mathscr{A}b$. Clearly the Burnside category $\mathscr{B} = \mathscr{B}_G$ introduced there is just the full subcategory of the stable category whose objects are the orbit spectra $\Sigma^\infty G/H_+$. The only difference is that, when defining $\mathscr{B}_G$, we abbreviated the names of objects to G/H.

From this point of view, the forgetful functor that takes a Mackey functor to a coefficient system is obtained by pullback along the functor i^* from the stable category of genuine orbit spectra to the stable category of naive orbit spectra. In X§4, Waner described a space level construction of an $RO(G)$-graded cohomology theory with coefficients in a Mackey functor M that extends the ordinary $\mathbb{Z}$-graded cohomology theory determined by its underlying coefficient system i^*M. We shall here give a more sophisticated, and I think more elegant and conceptual, spectrum level construction of such "ordinary" $RO(G)$-graded cohomology theories, and similarly for homology.

Our strategy is to construct a genuine Eilenberg-Mac Lane G-spectrum $HM = K(M,0)$ to represent our theory. Just as nonequivariantly, an Eilenberg-Mac Lane G-spectrum HM is one such that $\underline{\pi}_n(HM) = 0$ for $n \neq 0$. Of course, $\underline{\pi}_0(HM) = M$ must be a Mackey functor since that is true of $\underline{\pi}_n(E)$ for any n and any G-spectrum E. We shall explain the following result.

THEOREM 4.1. *For a Mackey functor M, there is an Eilenberg-Mac Lane G-spectrum HM such that $\underline{\pi}_0(HM) = M$. It is unique up to isomorphism in $\bar{h}G\mathscr{S}$. For Mackey functors M and M', $[HM, HM']_G$ is the group of maps of Mackey functors $M \longrightarrow M'$.*

There are several possible proofs. For example, one can exploit projective resolutions of Mackey functors. The proof that we shall give is the original one of Lewis, McClure, and myself, which I find rather amusing.

What is amusing is that, motivated by the desire to construct an $RO(G)$-graded cohomology theory, we instead construct a $\mathbb{Z}$-graded theory. However, this is a $\mathbb{Z}$-graded theory defined on G-spectra. As observed in Corollary 4.3, it can be represented and therefore extends to an $RO(G)$-graded theory. The representing G-spectrum is the desired Eilenberg-Mac Lane G-spectrum HM. What is also amusing is that the details that we shall use to construct the desired cohomology theories are virtually identical to those that we used to construct ordinary theories in the first place.

We start with G-CW spectra X. They have skeletal filtrations, and we define Mackey-functor valued cellular chains by setting

$$\underline{C}_n(X) = \underline{\pi}_n(X^n/X^{n-1}). \tag{4.2}$$

We used homology groups in I§4, but, aside from nuisance with the cases $n = 0$ and $n = 1$, we could equally well have used homotopy groups. Of course, X^n/X^{n-1} is a wedge of n-sphere G-spectra $S^n_H \simeq G/H_+ \wedge S^n$. We see that the $\underline{C}_n(X)$ are projective objects of the Abelian category of Mackey functors by essentially the same argument that we used in I§4. As there, the connecting homomorphism of the triple (X^n, X^{n-1}, X^{n-2}) specifies a map of Mackey functors

$$d : \underline{C}_n(X) \longrightarrow \underline{C}_{n-1}(X),$$

and $d^2 = 0$. Write $\mathrm{Hom}_{\mathscr{B}}(M, M')$ for the Abelian group of maps of Mackey functors $M \longrightarrow M'$. For a Mackey functor M, define

$$C^n_G(X; M) = \mathrm{Hom}_{\mathscr{B}}(\underline{C}_n(X), M), \quad \text{with} \quad \delta = \mathrm{Hom}_{\mathscr{B}}(d, \mathrm{id}). \tag{4.3}$$

Then $C^*_G(X; M)$ is a cochain complex of Abelian groups. We denote its homology by $H^*_G(X; M)$.

The evident cellular versions of the homotopy, exactness, wedge, and excision axioms admit exactly the same quick derivations as on the space level, and we use G-CW approximation to extend from G-CW spectra to general G-spectra: we have a $\mathbb{Z}$-graded cohomology theory on $\bar{h}G\mathscr{S}$. It satisfies the dimension axiom

$$H^*_G(S^0_H; M) = H^0_G(S^0_H; M) = M(G/H), \tag{4.4}$$

these giving isomorphisms of Mackey functors. The zeroth term is represented by a G-spectrum HM, and we read off its homotopy group Mackey functors directly from (4.4):

$$\underline{\pi}_0(HM) = M \quad \text{and} \quad \underline{\pi}_n(HM) = 0 \text{ if } n \neq 0.$$

The uniqueness of HM is evident, and the calculation of $[HM, HM']_G$ follows easily from the functoriality in M of the theories $H^*_G(X; M)$.

We should observe that spectrum level obstruction theory works exactly as on the space level, modulo connectivity assumptions to ensure that one has a dimension in which to start inductions.

For G-spaces X, we have two meanings in sight for the notation $H^n_G(X; M)$: we can regard our Mackey functor as a coefficient system and take ordinary cohomology as in I§4, or we can take our newly constructed cohomology. We know by the axiomatic characterization of ordinary cohomology that these must in fact be isomorphic, but it is instructive to check this directly. At least after a single suspension, we can approximate any G-space by a weakly equivalent G-CW based complex, with based attaching maps. The functor Σ^∞ takes G-CW based complexes to G-CW spectra, and we find that the two chain complexes in sight are isomorphic. Alternatively, we can check on the represented level:

$$[\Sigma^\infty X, \Sigma^n HM]_G \cong [X, \Omega^\infty \Sigma^n HM]_G \cong [X, K(M, n)]_G.$$

What about homology? Recall that a coMackey functor is a covariant functor $N : \mathscr{B} \longrightarrow \mathscr{A}b$. Using the usual coend construction, we define

$$C^G_n(X; N) = \underline{C}_*(X) \otimes_{\mathscr{B}} N, \quad \text{with} \quad \partial = d \otimes \mathrm{id}. \tag{4.5}$$

Then $C^G_*(X; N)$ is a chain complex of Abelian groups. We denote its homology by $H^G_*(X; N)$. Again, the verification of the axioms for a $\mathbb{Z}$-graded homology theory on $\bar{h}\mathscr{G}\mathscr{S}$ is immediate. The dimension axiom now reads

$$H^G_*(S^0_H; N) = H^G_0(S^0_H; N) = N(G/H). \tag{4.6}$$

We define a cohomology theory on finite G-spectra X by

$$H^*_G(X; N) = H^G_{-*}(DX; N). \tag{4.7}$$

Applying Adams' variant of the Brown representability theorem, we obtain a G-spectrum JN that represents this cohomology theory. For finite X, we obtain

$$H^G_*(X; N) = H^{-*}_G(DX; N) \cong [DX, JN]^{-*}_G \cong [S, JN \wedge X]^G_* = JN^G_*(X).$$

Thus JN represents the $\mathbb{Z}$-graded homology theory that we started with and extends it to an $RO(G)$-graded theory. We again see that, on G-spaces X, $H^G_*(X; N)$ agrees with the homology of X with coefficients in the underlying covariant coefficient system of N, as defined in I§4.

What are the homotopy groups of JN? The answer must be

$$\pi^H_n(JN) = H^G_n(D(G/H_+); N).$$

For finite G, orbits are self-dual and the resulting isomorphism of the stable orbit category with its opposite category induces the evident self-duality of the algebraically defined category of Mackey functors to be discussed in XIX§3. This allows us to conclude that

$$JN = H(N^*),$$

where N^* is the Mackey functor dual to the coMackey functor N.

For general compact Lie groups, however, the dual of G/H_+ is $G \ltimes_H S^{-L(H)}$, and it is not easy to calculate the homotopy groups of JN. This G-spectrum is bounded below, but it is not connective. We must learn to live with the fact that we have two quite different kinds of Eilenberg-Mac Lane G-spectra, one that is suitable for representing "ordinary" cohomology and the other that is suitable for representing "ordinary" homology.

G. Lewis, J. P. May, and J. McClure. Ordinary $RO(G)$-graded cohomology. Bulletin Amer. Math. Soc. 4(1981), 208-212.

5. Ring G-spectra and products

Given our precise definition of $RO(G)$-graded theories and our understanding of their representation by G-spectra, the formal apparatus of products in homology and cohomology theories can be developed in a straightforward manner and is little different from the nonequivariant case in classical lectures of Adams. However, in that early work, Adams did not take full advantage of the stable homotopy category. We here recall briefly the basic definitions from the equivariant treatment in [LMS, III§3].

There are four basic products to consider, two external products and two slant products. The reader should be warned that the treatment of slant products in the literature is inconsistent, at best, and often just plain wrong. These four products come from the following four natural maps in $\bar{h}G\mathscr{S}$; all variables are G-spectra.

(5.1)
$$X \wedge E \wedge X' \wedge E' \xrightarrow{\mathrm{id} \wedge \tau \wedge \mathrm{id}} X \wedge X' \wedge E \wedge E'$$

(5.2)
$$F(X,E) \wedge F(X',E') \xrightarrow{\wedge} F(X \wedge X', E \wedge E')$$

(5.3)
$$\begin{tikzcd} F(X \wedge X', E) \wedge X \wedge E' \arrow[r, "/"] \arrow[d, "\cong"'] & F(X', E \wedge E') \\ F(X, F(X',E)) \wedge X \wedge E' \arrow[r, "\varepsilon \wedge \mathrm{id}"'] & F(X',E) \wedge E' \arrow[u, "\nu"'] \end{tikzcd}$$

(5.4)
$$\begin{tikzcd} X \wedge X' \wedge E \wedge F(X,E') \arrow[rr, "\backslash"] \arrow[dr, "\tau"'] & & X' \wedge E \wedge E' \\ & X' \wedge E \wedge F(X,E') \wedge X \arrow[ur, "\mathrm{id} \wedge \mathrm{id} \wedge \varepsilon"'] & \end{tikzcd}$$

The τ are transposition maps and the ε are evaluation maps. The map ν can be described formally, but it is perhaps best understood by pretending that F means Hom and $\wedge$ means $\otimes$ over a commutative ring and writing down the obvious analog. Categorically, such coherence maps are present in any symmetric monoidal category with an internal hom functor. A categorical coherence theorem asserts that any suitably well formulated diagram involving these transformations will commute.

On passage to homotopy groups, these maps give rise to four products in $RO(G)$-graded homology and cohomology. With our details on $RO(G)$-grading, we leave it as an exercise for the reader to check exactly how the grading behaves.

$$E^G_*(X) \otimes E'^G_*(X') \longrightarrow (E \wedge E')^G_*(X \wedge X') \tag{5.5}$$

$$E^*_G(X) \otimes E'^*_G(X') \longrightarrow (E \wedge E')^*_G(X \wedge X') \tag{5.6}$$

$$/ : E^*_G(X \wedge X') \otimes E'^G_*(X) \longrightarrow (E \wedge E')^*_G(X') \tag{5.7}$$

$$\backslash : E^G_*(X \wedge X') \otimes E'^*_G(X) \longrightarrow (E \wedge E')^G_*(X') \tag{5.8}$$

A ring G-spectrum E is one with a product $\phi : E \wedge E \longrightarrow E$ and a unit map $\eta : S \longrightarrow E$ such that the following diagrams commute in $\bar{h}G\mathscr{S}$:

$$\begin{tikzcd}
S \wedge E \arrow[r, "\eta\wedge 1"] \arrow[dr, "\simeq"'] & E \wedge E \arrow[d, "\phi"] & E \wedge S \arrow[l, "1\wedge\eta"'] \arrow[dl, "\simeq"] \\
& E &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
E \wedge E \wedge E \arrow[r, "1\wedge\phi"] \arrow[d, "\phi\wedge 1"'] & E \wedge E \arrow[d, "\phi"] \\
E \wedge E \arrow[r, "\phi"] & E.
\end{tikzcd}$$

The unlabelled equivalences are canonical isomorphisms in $\bar{h}G\mathscr{S}$ that give the unital property, and we have suppressed such an associativity isomorphism in the second diagram. Of course, there is a weaker notion in which associativity is not required; E is commutative if the following diagram commutes in $\bar{h}G\mathscr{S}$:

$$\begin{tikzcd}
E \wedge E \arrow[rr, "\tau"] \arrow[dr, "\phi"'] & & E \wedge E \arrow[dl, "\phi"] \\
& E. &
\end{tikzcd}$$

An E-module is a spectrum M together with a map $\mu : E \wedge M \longrightarrow M$ such that the following diagrams commute in $\bar{h}G\mathscr{S}$:

$$\begin{tikzcd}
S \wedge M \arrow[r, "\eta\wedge 1"] \arrow[dr, "\simeq"'] & E \wedge M \arrow[d, "\mu"] \\
& M
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
E \wedge E \wedge M \arrow[r, "1\wedge\mu"] \arrow[d, "\phi\wedge 1"'] & E \wedge M \arrow[d, "\mu"] \\
E \wedge M \arrow[r, "\mu"] & M.
\end{tikzcd}$$

We obtain various further products by composing the four external products displayed above with the multiplication of a ring spectrum or with its action on a module spectrum. If $X = X'$ is a based G-space (or rather its suspension spectrum), we obtain internal products by composing with the reduced diagonal

$\Delta : X \longrightarrow X \wedge X$. Of course, it is more usual to think in terms of unbased spaces, but then we adjoin a disjoint basepoint. In particular, for a ring G-spectrum E and a based G-space X, we obtain the cup and cap products

$$\cup : E_G^*(X) \otimes E_G^*(X) \longrightarrow E_G^*(X) \tag{5.9}$$

and

$$\cap : E_*^G(X) \otimes E_G^*(X) \longrightarrow E_*^G(X) \tag{5.10}$$

from the external products $\wedge$ and $\backslash$.

It is natural to ask when HM is a ring G-spectrum. In fact, in common with all such categories of additive functors, the category of Mackey functors has an internal tensor product (see Mitchell). In the present topological context, we can define it simply by setting

$$M \otimes M' = \underline{\pi}_0(HM \wedge HM').$$

There results a notion of a pairing $M \otimes M' \longrightarrow M''$ of Mackey functors. By killing the higher homotopy groups of $HM \wedge HM'$, we obtain a canonical map

$$\iota : HM \wedge HM' \longrightarrow H(M \otimes M'),$$

and ι induces an isomorphism on $H_G^0(-;M'') = [-, HM'']_G$. It follows that pairings of G-spectra $HM \wedge HM' \longrightarrow HM''$ are in bijective correspondence with pairings $M \otimes M' \longrightarrow M''$. From here, it is clear how to define the notion of a ring in the category of Mackey functors — such objects are called Green functors — and to conclude that a ring structure on the G-spectrum HM determines and is determined by a structure of Green functor on the Mackey functor M. These observations come from work of Greenlees and myself on Tate cohomology.

There is a notion of a ring G-prespectrum; modulo $\lim^1$ problems, its associated G-spectrum (here constructed using the cylinder construction since one wishes to retain homotopical information) inherits a structure of ring G-spectrum. A good nonequivariant exposition that carries over to the equivariant context has been given by McClure.

J. F. Adams. Lectures on generalized cohomology. in Springer Lecture Notes in Mathematics, Vol. 99, 1-138.

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CHAPTER XIV

An introduction to equivariant K-theory

by J. P. C. Greenlees

1. The definition and basic properties of K_G-theory

The aim of this chapter is to explain the basic facts about equivariant K-theory through the Atiyah-Segal completion theorem. Throughout, G is a compact Lie group and we focus on complex K-theory. Real K-theory works similarly.

We briefly outline the geometric roots of equivariant K-theory. A G-vector bundle over a G-space X is a G-map $\xi : E \longrightarrow X$ which is a vector bundle such that G acts linearly on the fibers, in the sense that $g : E_x \longrightarrow E_{gx}$ is a linear map. Since G is compact, all short exact sequences of G-vector bundles split. If X is a compact space, then $K_G(X)$ is defined to be the Grothendieck group of finite dimensional G-vector bundles over X. Tensor product of bundles makes $K_G(X)$ into a ring.

Many applications arise; for example, the equivariant K-groups are the homes for indices of G-manifolds and families of elliptic operators.

Any complex representation V of G defines a trivial bundle over X and, by the Peter-Weyl theorem, any G-vector bundle over a compact base space is a summand of such a trivial bundle. The cokernel of $K_G(*) \longrightarrow K_G(X)$ can therefore be described as the group of stable isomorphism classes of bundles over X, where two bundles are stably isomorphic if they become isomorphic upon adding an appropriate trivial bundle to each. When X has a G-fixed basepoint $*$, we write $\tilde{K}_G(X)$ for the isomorphic group $\ker(K_G(X) \longrightarrow K_G(*))$.

The definition of a G-vector bundle makes it clear that G-bundles over a free G-space correspond to vector bundles over the quotient under pullback. We deduce the basic reduction theorem:

$$(1.1) \qquad K_G(X) = K(X/G) \text{ if } X \text{ is } G\text{-free.}$$

This is essentially the statement that K-theory is split in the sense to be discussed in XVI§2. It provides the fundamental link between equivariant and nonequivariant K-theory.

Restriction and induction are the basic pieces of structure that link different ambient groups of equivariance.

If $i : H \longrightarrow G$ is the inclusion of a subgroup it is clear that a G-space or bundle can be viewed as an H-space or bundle; we thereby obtain a restriction map

$$i^* : K_G(X) \longrightarrow K_H(X).$$

There is another way of thinking about this map. For an H-space Y,

$$K_G(G \times_H Y) \cong K_H(Y) \tag{1.2}$$

since a G-bundle over $G \times_H Y$ is determined by its underlying H-bundle over Y. For a G-space X, $G \times_H X \cong G/H \times X$, and the restriction map coincides with the map

$$K_G(X) \longrightarrow K_G(G/H \times X) \cong K_H(X)$$

induced by the projection $G/H \longrightarrow *$.

If H is of finite index in G, an H-bundle over a G-space may be made into a G-bundle by applying the functor $\mathrm{Hom}_H(G, -)$. We thus obtain an induction map $i_* : K_H(X) \longrightarrow K_G(X)$. However if H is of infinite index this construction gives an infinite dimensional bundle. There are three other constructions one may hope to use. First, there is smooth induction, which Segal describes for the representation ring and which should apply to more general base manifolds than a point.

Second, there is the holomorphic transfer, which one only expects to exist when G/H admits the structure of a projective variety. The most important case is when H is the maximal torus in the unitary group $U(n)$, in which case a construction using elliptic operators is described by Atiyah. Its essential property is that it satisfies $i_*i^* = 1$. It is used in the proof of Bott periodicity.

Third, there is a transfer map

$$tr : \tilde{K}_H(\Sigma^W X) \cong \tilde{K}_G(G_+ \wedge_H \Sigma^W X) \longrightarrow \tilde{K}_G(\Sigma^V X)$$

induced by the Pontrjagin-Thom construction $t : S^V \longrightarrow G_+ \wedge_H S^W$ associated to an embedding of G/H in a representation V, where W is the complement of the image in V of the tangent H-representation $L = L(H)$ at the identity coset of G/H. Once we use Bott periodicity to set up $RO(G)$-graded K-theory, this may be interpreted as a dimension-shifting transfer $\tilde{K}_H^{q+L}(X) \longrightarrow \tilde{K}_G^q(X)$. Clearly this transfer is not special to K-theory: it is present in any $RO(G)$-graded theory.

M. F.Atiyah. Bott periodicity and the index of elliptic operators. Quart. J. Math. 19(1968), 113-140.
G. B.Segal. Equivariant K-theory. Pub. IHES 34(1968), 129-151.

2. Bundles over a point: the representation ring

Bundles over a point are representations and hence equivariant K-theory is module-valued over the complex representation ring $R(G)$. More generally, any G-vector bundle over a transitive G-space G/H is of the form $G \times_H V \longrightarrow G \times_H * = G/H$ for some representation V of H. Hence $K_G(G/H) = R(H)$. It follows that $K_G(X)$ takes values in the category of $R(G)$-modules, and thus it is important to understand the algebraic nature of $R(G)$.

Before turning to this, we observe that if G acts trivially on X, then

$$K_G(X) \cong R(G) \otimes K(X).$$

Indeed, the map $K(X) \longrightarrow K_G(X)$ obtained by regarding a vector bundle as a G-trivial G-vector bundle extends to a map $\mu : R(G) \otimes K(X)$ of $R(G)$-modules, and this map is the required isomorphism. An explicit inverse can be constructed as follows. For a representation V, let $\boldsymbol{V}$ denote the trivial G-vector bundle $X \times V \longrightarrow X$. The functor that sends a G-vector bundle ξ to the vector bundle $\mathrm{Hom}_G(\boldsymbol{V}, \xi)$ induces a homomorphism $\varepsilon_V : K_G(X) \longrightarrow K(X)$. Let $\{V_i\}$ run through a set consisting of one representation V_i from each isomorphism class $[V_i]$ of irreducible representations. Then a G-vector bundle ξ over X breaks up as the Whitney sum of its subbundles $\boldsymbol{V_i} \otimes \mathrm{Hom}_G(\boldsymbol{V_i}, \xi)$. Define $\nu : K_G(X) \longrightarrow R(G) \otimes K(X)$ by $\nu(\alpha) = \sum_i [V_i] \otimes \varepsilon_{V_i}(\alpha)$. It is then easy to check that μ and ν are inverse isomorphisms.

To understand the algebra of $R(G)$, one should concentrate on the so called "Cartan subgroups" of G. These are topologically cyclic subgroups H with finite Weyl groups $W_G(H) = N_G(H)/H$. Conjugacy classes of Cartan subgroups are in one-to-one correspondence with conjugacy classes of cyclic subgroups of the component group $\pi_0(G)$. Every element of G lies in some Cartan subgroup, and therefore the restriction maps give an injective ring homomorphism

$$R(G) \longrightarrow \prod_{(C)} R(C) \tag{2.1}$$

where the product is over conjugacy classes of Cartan subgroups.

The ring $R(G)$ is Noetherian. Indeed, by explicit calculation, $R(U(n))$ is Noetherian and the representation ring of a maximal torus T is finite over it. Any group G may be embedded in some $U(n)$, and it is enough to show that $R(G)$ is finitely generated as an $R(U(n))$-module. Now $R(G)$ is detected on finitely many topologically cyclic subgroups C, so it is enough to show each $R(C)$ is finitely generated over $R(U(n))$. But each such C is conjugate to a subgroup of T, and $R(C)$ is finite over $R(T)$.

The map (2.1) makes the codomain a finitely generated module over the domain and consequently the induced map of prime spectra is surjective and has finite fibers. By identifying the fibers it can then be shown that for any prime $\wp$ of $R(G)$ the set of minimal elements of

$$\{H \subseteq G \mid \wp \text{ is the restriction of a prime of } R(H)\}$$

constitutes a single conjugacy class (H) of subgroups, with H topologically cyclic. We say that (H) is the *support* of $\wp$. If $R(G)/\wp$ is of characteristic $p > 0$ then the component group of H has order prime to p.

The first easy consequence is that the Krull dimension of $R(G)$ is one more than the rank of G.

A more technical consequence which will become important to us later is that completion is compatible with restriction. Indeed restriction gives a ring homomorphism $\mathrm{res} : R(G) \longrightarrow R(H)$ by which we may regard an $R(H)$-module as an $R(G)$-module. Let $I(G) = \ker\{\dim : R(G) \longrightarrow \mathbb{Z}\}$ be the augmentation ideal. Using supports, we see that the ideals $I(H)$ and $\mathrm{res}(I(G)) \cdot R(H)$ have the same radical. Consequently the $I(H)$-adic and $I(G)$-adic completions of an $R(H)$-module coincide.

Finally, using supports it is straightforward to understand localizations of equivariant K-theory at primes of $R(G)$. In fact if (H) is the support of $\wp$ the inclusion $X^{(H)} \longrightarrow X$ induces an isomorphism of $K_G(-)_\wp$, where $X^{(H)}$ is the union of the fixed point spaces $X^{H'}$ with H' conjugate to H.

G. B.Segal. The representation ring of a compact Lie group. Pub. IHES 34(1968), 113-128.

3. Equivariant Bott periodicity

Equivariant Bott periodicity is the most important theorem in equivariant K-theory and is even more extraordinary than its nonequivariant counterpart. It underlies all of the amazing properties of equivariant K-theory. For a locally compact G-space X, define $K_G(X)$ to be the reduced K-theory of the one-point compactification $X_\#$ of X. That is, writing $*$ for the point at infinity,

$$K_G(X) = \ker(K_G(X_\#)) \longrightarrow K_G(*).$$

When X is compact, $X_\#$ is the union X_+ of X and a disjoint G-fixed basepoint. We issue a warning: in general, for infinite G-CW complexes, $K_G(X)$ as just defined will not agree with the represented K_G-theory of X that will become available when we construct the K-theory G-spectrum in the next section.

THEOREM 3.1 (THOM ISOMORPHISM). *For vector bundles E over locally compact base spaces X, there is a natural Thom isomorphism*

$$\phi : K_G(X) \xrightarrow{\cong} K_G(E).$$

There is a quick reduction to the case when X is compact, and in this case we can use that any G-bundle is a summand of the trivial bundle of some representation V to reduce to the case when $E = V \times X$. Here, with an appropriate description of the Thom isomorphism, one can reinterpret the statement as a convenient and explicit version of Bott periodicity. To see this, let $\lambda(V) \in R(G)$ denote the alternating sum of exterior powers

$$\lambda(V) = 1 - V + \lambda^2 V - \cdots + (-1)^{dimV} \lambda^{dimV} V,$$

let $e_V : S^0 \longrightarrow S^V$ be the based map that sends the non-basepoint to 0, and, taking X to be a point, let $b_V = \phi(1) \in \tilde{K}(S^V)$. Observe that e_V induces

$$e_V^* : \tilde{K}(S^V) \longrightarrow \tilde{K}(S^0) = R(G).$$

THEOREM 3.2 (BOTT PERIODICITY). *For a compact G-space X and a complex representation V of G, multiplication by b_V specifies an isomorphism*

$$\phi : \tilde{K}_G(X_+) = K_G(X) \xrightarrow{\cong} K_G(V \times X) = \tilde{K}(S^V \wedge X_+).$$

Moreover, $e_V^*(b_V) = \lambda(V)$.

The Thom isomorphism can be proven for line bundles, trivial or not, by arguing with clutching functions, as in the nonequivariant case. The essential point is to show that the K-theory of the projective bundle $P(E \oplus \mathbb{C})$ is the free $K_G(X)$-module generated by the unit element $\{1\}$ and the Hopf bundle H. This implies the case when E is a sum of trivial line bundles. If G is abelian, every V is a sum of one dimensional representations so the theorem is proved. This deals with the case of a torus T. The significantly new feature of the equivariant case is the use of holomorphic transfer to deduce the case of $U(n)$. Finally, by change of groups, the result follows for any subgroup of $U(n)$.

For real equivariant K-theory KO_G, the Bott periodicity theorem is true as stated provided that we restrict V to be a Spin representation of dimension divisible by eight. However, the proof is significantly more difficult, requiring the use of pseudo-differential operators.

Now we may extend $K_G(-)$ to a cohomology theory. Following our usual conventions, we shall write K_G^* for the reduced theory on based G-spaces X. Since we need compactness, we consider based finite G-CW complexes, and we then have the notational conventions that in degree zero

$$K_G^0(X_+) = K_G(X) \quad \text{for finite } G\text{-CW complexes } X$$

and

$$K_G^0(X) = \tilde{K}_G(X) \quad \text{for based finite } G\text{-CW complexes } X.$$

Of course we could already have made the definition $K_G^{-q}(X) = K_G^0(\Sigma^q X)$ for positive q, but we now know that these are periodic with period 2 since $\mathbb{R}^2 = \mathbb{C}$. Thus we may take

$$K_G^{2n}(X) = K_G^0(X) \quad \text{and} \quad K_G^{2n+1}(X) = K_G^0(\Sigma^1 X) \quad \text{for all } n.$$

Note in particular that the coefficient ring is $R(G)$ in even degrees. It is zero in odd degrees because all bundles over S^1 are pullbacks of bundles over a point, $GL_n(\mathbb{C})$ being connected. We can extend this to an $RO(G)$-graded theory that is $R(G)$-periodic, but we let the construction of a representing G-spectrum in the next section take care of this for us.

M. F.Atiyah. Bott periodicity and the index of elliptic operators. Quart. J. Math. 19(1968), 113-140.
M. F.Atiyah and R. Bott. On the periodicity theorem for complex vector bundles. Acta math. 112(1964), 229-247.
G. B.Segal. Equivariant K-theory. Pub. IHES 34(1968), 129-151.

4. Equivariant K-theory spectra

Following the procedures indicated in XII§9, we run through the construction of a G-spectrum that represents equivariant K-theory. Recall from VII.3.1 that the Grassmannian G-space $BU(n,V)$ of complex n-planes in a complex inner product G-space V classifies complex n-dimensional G-vector bundles if V is sufficiently large, for example if V contains a complete complex G-universe.

Diverging slightly from our usual notation, fix a complete G-universe $\mathscr{U}$. For each indexing space $V \subset \mathscr{U}$ and each $q \geq 0$, we have a classifying space

$$BU(q, V \oplus \mathscr{U})$$

for q-plane bundles. For $V \subseteq W$, we have an inclusion

$$BU(q, V \oplus \mathscr{U}) \longrightarrow BU(q + |W - V|, W \oplus \mathscr{U})$$

that sends a plane A to the plane $A + (W - V)$. Define

$$BU_G(V) = \coprod_{q \geq 0} BU(q, V \oplus \mathscr{U}).$$

We take the plane V in $BU(|V|, V \oplus \mathscr{U})$ as the canonical G-fixed basepoint of $BU_G(V)$. For $V \subset W$, we then have an inclusion $BU_G(V)$ in $BU_G(W)$ of based G-spaces. Define BU_G to be the colimit of the $BU_G(V)$.

For finite (unbased) G-CW complexes X, the definition of $K_G(X)$ as a Grothendieck group and the classification theorem for complex G-vector bundles lead to an isomorphism

$$[X_+, BU_G]_G \cong K_G(X) = K_G^0(X_+).$$

The finiteness ensures that our bundles embed in trivial bundles and thus have complements. In turn, this ensures that every element of the Grothendieck group is the difference of a bundle and a trivial bundle. For the proof, we may as well assume that X/G is connected. In this case, a G-map $\phi : X \longrightarrow BU_G$ factors through a map $f : BU_G(q, V \oplus \mathscr{U})$ for some q and V. If f classifies the G-bundle ξ, then the isomorphism sends ϕ to $\xi - V$.

The spaces $BU_G(V)$ and BU_G have the homotopy types of G-CW complexes. If we wish, we can replace them by actual G-CW complexes by use of the functor Γ from G-spaces to G-CW complexes. For a complex representation V and based finite G-CW complexes X, Bott periodicity implies a natural isomorphism

$$[X, BU_G]_G \cong K_G^0(X) \cong K_G^0(\Sigma^V X) \cong [X, \Omega^V BU_G]_G.$$

By Adams' variant XIII.3.4 of Brown's representability theorem, this isomorphism is represented by a G-map $\tilde{\sigma} : BU_G \longrightarrow \Omega^V BU_G$, which must be an equivalence. However, we must check the vanishing of the appropriate $\lim^1$-term to see that the homotopy class of $\tilde{\sigma}$ is well-defined. Restricting to a cofinal sequence of representations so as to arrange transitivity (as in XIII.3.2), we have an Ω-G-prespectrum. It need not be Σ-cofibrant, but we can apply the cylinder construction K to make it so. Applying L, we then obtain a G-spectrum K_G. It is related to the Ω-G-prespectrum that we started with by a spacewise equivalence. Of course, the restriction to complex indexing spaces is no problem since we can extend to all real indexing spaces, as explained in XII§2.

Using real inner product spaces, we obtain an analogous G-space BO_G and an analogous isomorphism

$$[X, BO_G]_G \cong KO_G(X).$$

If we start with *Spin* representations of dimension $8n$, those being the ones for which we have real Bott periodicity, the same argument works to construct a G-spectrum KO_G that represents real K-theory.

5. The Atiyah-Segal completion theorem

It is especially important to understand bundles over the universal space EG, because of their role in the theory of characteristic classes. We have already mentioned one very simple construction of bundles. In fact for any representation V we may form the bundle $EG \times V \longrightarrow EG \times *$ and hence we obtain the homomorphism

$$\alpha : R(G) \longrightarrow K_G(EG).$$

Evidently α is induced by the projection map $\pi : EG \longrightarrow *$. The Atiyah-Segal completion theorem measures how near α is to being an isomorphism.

Of course, EG is a free G-CW complex. Any (based) free G-CW complex is constructed from the G-spaces $G_+ \wedge S^n$ by means of wedges, cofibers, and passage to colimits. From the change of groups isomorphism $K^*_G(G_+ \wedge X) \cong K^*(X)$ we see that the augmentation ideal $I = I(G)$ acts as zero on the K-theory of any space $G_+ \wedge X$.

In particular the K-theory of $G_+ \wedge S^n$ is complete as an $R(G)$-module for the topology defined by powers of I. Completeness is preserved by extensions of finitely generated modules, so we that $K^*_G(X)$ is I-complete for any finite free G-CW complex X. Completeness is also preserved by inverse limits so, *provided* lim^1 *error terms vanish*, the K-theory of EG is I-complete.

Remarkably the K-theory of EG is fully accounted for by the representation ring, in the simplest way allowed for by completeness. The Atiyah-Segal theorem can be seen as a comparison between the algebraic process of I-adic completion and the geometric process of "completion" by making a space free.

The map α has a counterpart in all degrees, and it is useful to allow a parameter space, which will be a based G-space X. Thus we consider the map

$$\pi^* : K^*_G(X) \longrightarrow K^*_G(EG_+ \wedge X).$$

The target is isomorphic to the non-equivariant K-theory $K^*(EG_+ \wedge_G X)$, and the following theorem may be regarded as a calculation of this in terms of the more approachable group $K^*_G(X)$.

THEOREM 5.1 (ATIYAH-SEGAL). *Provided that X is a finite G-CW-complex, the map π^* above is completion at the augmentation ideal, so that*

$$K^*_G(EG_+ \wedge X) \cong K^*_G(X)^\wedge_I.$$

In particular,

$$K^0_G(EG_+) = R(G)^\wedge_I \quad \textit{and} \quad K^1_G(EG_+) = 0.$$

We sketch the simplest proof, which is that of Adams, Haeberly, Jackowski, and May. We skate over two technical points and return to them at the end. For simplicity of notation, we omit the parameter space X. We do not yet know that $K^*_G(EG_+)$ is complete since we do not yet know that the relevant $\lim^1$-term vanishes. If we did know this, we would be reduced to proving that $\pi : EG_+ \longrightarrow S^0$ induces an isomorphism of I-completed K-theory.

If we also knew that "completed K-theory" was a cohomology theory it would then be enough to show that the cofiber of π was acyclic. It is standard to let

$\tilde{E}G$ denote this cofiber, which is easily seen to be the unreduced suspension of EG with one of the cone points as base point. That is, it would be enough to prove that $K_G^*(\tilde{E}G) = 0$ after completion.

The next simplification is adapted from a step in Carlsson's proof of the Segal conjecture. If we argue by induction on the size of the group (which is possible since chains of subgroups of compact Lie groups satisfy the descending chain condition), we may suppose the result proved for all proper subgroups H of G. Accordingly, by change of groups, $K_G^*(G/H_+ \wedge Y) = 0$ after completion for any nonequivariantly contractible space Y and hence by wedges, cofibers, and colimits $K_G^*(E \wedge Y) = 0$ after completion for any G-CW complex E constructed using cells $G/H_+ \wedge S^n$ for various proper subgroups H.

Now if G is finite, let V denote the reduced regular representation and let $S^{\infty V}$ be the union of the representation spheres S^{kV}. For a general compact Lie group G, we let $S^{\infty V}$ denote the union of the representation spheres S^V as V runs over the indexing spaces V such that $V^G = 0$ in a complete G-universe U.

Clearly $(S^{\infty V})^H$ is contractible if H is a proper subgroup and $(S^{\infty V})^G = S^0$. Thus $S^{\infty V}/S^0$ has no G-fixed points and may therefore be constructed using cells $G/H_+ \wedge S^n$ for proper subgroups H. Thus, by the inductive hypothesis, $K_G^*(S^{\infty V}/S^0 \wedge \tilde{E}G) = 0$ after completion, and hence

$$K_G^*(S^{\infty V} \wedge \tilde{E}G) \cong K_G^*(S^0 \wedge \tilde{E}G) = K_G^*(\tilde{E}G)$$

after completion. But evidently the inclusion

$$S^{\infty V} = S^{\infty V} \wedge S^0 \longrightarrow S^{\infty V} \wedge \tilde{E}G$$

is an equivariant homotopy equivalence by consideration of the various fixed point sets. This proves a most convenient reduction: it is enough to prove that $K_G^*(S^{\infty V}) = 0$ after completion.

In fact, this is easy to prove. When G is finite, one just notes that (ignoring $\lim^1$ problems again)

$$K_G^*(S^{\infty V}) = \lim_k K_G^*(S^{kV}) = \lim_k K_G^*(S^0),$$

where the second limit is taken over countably many copies of the multiplication map $\lambda(V) : K_G^*(S^0) \longrightarrow K_G^*(S^0)$. Since $\lambda(V)$ acts invertibly on this inverse limit and $\lambda(V) \in I$, $(\lim K_G^*(S^0))^{\wedge}_I = 0$ by the obvious fact that $M^{\wedge}_I = 0$ if $IM = M$. The argument in the general compact Lie case is only a little more elaborate.

To make this proof honest, we must address the two important properties that we used without justification: (a) that completed K-theory takes cofiberings to exact sequences and (b) that the K-theories of certain infinite complexes are the inverse limits of the K-theories of their finite subcomplexes. In other words the points that we skated over were the linked problems of the inexactness of completion and the nonvanishing of $\lim^1$ terms.

Now, since $R(G)$ is Noetherian, completion *is* exact on finitely generated modules, and the K groups of finite complexes are finitely generated. Accordingly, one route is to arrange the formalities so as to only discuss finite complexes: this is the method of pro-groups, as in the original approach of Atiyah. It is elementary and widely useful. Instead of considering the single group $K_G^*(X)$

we consider the inverse system of groups $K_G^*(X_\alpha)$ as X_α runs over the finite subcomplexes of X.

We do not need to know much about pro-groups. A pro-group is just an inverse system of Abelian groups. There is a natural way to define morphisms, and the resulting category is Abelian. The fundamental technical advantage of working in the category of pro-groups is that, in this category, the inverse limit functor is exact. For an Abelian group valued functor h on G-CW complexes or spectra, we define the associated pro-group valued functor $\mathbf{h}$ by letting $\mathbf{h}(X)$ be the inverse system $\{h(X_\alpha)\}$, where X_α runs over the finite subcomplexes of X.

As long as all K-theory is interpreted as pro-group valued, the argument just given is honest. The conclusion of the argument is that, for a finite G-CW complex X, $\pi : EG_+ \wedge X \longrightarrow X$ induces an isomorphism of I-completed pro-group valued K-theory. Here the I-completion of a pro-$R(G)$-module $\mathbf{M} = \{M_\alpha\}$ is just the inverse system $\{M_\alpha / I^r M_\alpha\}$. When $\mathbf{M}$ is a constant system, such as $\mathbf{K}_G^*(S^0)$, this is just an inverse system of epimorphisms and has zero $\lim^1$. It follows from the isomorphism of pro-groups that $\lim^1$ is also zero for the pro-group $K_G^*(EG_+ \wedge X)$, and hence the group $K_G^*(EG_+ \wedge X)$ is the inverse limit of the K-theories of the skeleta of $EG_+ \wedge X$. We may thus simply pass to inverse limits to obtain the conclusion of Theorem 5.1 as originally stated for ordinary rather than pro-$R(G)$-modules.

There is an alternative way to be honest: we could accept the inexactness and adapt the usual methods for discussing it by derived functors. In fact we shall later see how to realize the construction of left derived functors of completion geometrically. This approach leads compellingly to consideration of completions of K_G-module spectra and to the consideration of homology. We invite the interested reader to turn to Chapter XXV (especially Section 7).

J. F.Adams, J.-P.Haeberly, S.Jackowski and J. P.May A generalization of the Atiyah-Segal completion theorem. Topology 27(1988), 1-6.

M. F. Atiyah. Characters and cohomology of finite groups. Pub. IHES 9(1961), 23-64.

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S. Jackowski. Families of subgroups and completions. J. Pure and Applied Algebra 37(1985), 167-179.

6. The generalization to families

The above statements and proofs for the universal free G-space EG and the augmentation ideal I carry over with the given proofs to theorems about the universal $\mathscr{F}$-free space $E\mathscr{F}$ and the ideal

$$I\mathscr{F} = \bigcap_{H \in \mathscr{F}} ker\{res_H^G : R(G) \longrightarrow R(H)\}.$$

The only difference is that for most families $\mathscr{F}$ there is no reduction of $K_G(E\mathscr{F})$ to the nonequivariant K-theory of some other space. Note that, by the injectivity of (2.1), if $\mathscr{F}$ includes all cyclic subgroups then $I\mathscr{F} = 0$.

THEOREM 6.1. *For any family $\mathscr{F}$ and any finite G-CW-complex X the projection map $E\mathscr{F} \longrightarrow *$ induces completion, so that*

$$K_G^*(E\mathscr{F}_+ \wedge X) \cong K_G^*(X)\hat{}_{I\mathscr{F}}.$$

In particular

$$K_G^0(E\mathscr{F}_+) \cong R(G)\hat{}_{I\mathscr{F}} \quad \text{and} \quad K_G^1(E\mathscr{F}_+) = 0.$$

Two useful consequences of these generalizations are that K-theory is detected on finite subgroups and that isomorphisms are detected by cyclic groups.

THEOREM 6.2 (MCCLURE). *(a) If X is a finite G-CW-complex and $x \in K_G(X)$ restricts to zero in $K_H(X)$ for all finite subgroups H of G then $x = 0$. (b) If $f : X \longrightarrow Y$ is a map of finite G-CW-complexes that induces an isomorphism $K_C(Y) \longrightarrow K_C(X)$ for all finite cyclic subgroups C then $f^* : K_G(Y) \longrightarrow K_G(X)$ is also an isomorphism.*

Thinking about characters, one might be tempted to believe that finite subgroups could be replaced by finite cyclic subgroups in (a), but that is false.

J. F.Adams, J.-P.Haeberly, S.Jackowski and J. P.May. A generalization of the Atiyah-Segal completion theorem. Topology 27(1988), 1-6.
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CHAPTER XV

An introduction to equivariant cobordism

by S. R. Costenoble

1. A review of nonequivariant cobordism

We start with a brief summary of nonequivariant cobordism.

We define a sequence of groups $\mathscr{N}_0$, $\mathscr{N}_1$, $\mathscr{N}_2$, ... as follows: We say that two smooth closed k-dimensional manifolds M_1 and M_2 are *cobordant* if there is a smooth $(k+1)$-dimensional manifold W (the *cobordism*) such that $\partial W \cong M_1 \amalg M_2$; this is an equivalence relation, and $\mathscr{N}_k$ is the set of cobordism classes of k-dimensional manifolds. We make this into an abelian group with addition being disjoint union. The 0 element is the class of the empty manifold $\emptyset$; a manifold is cobordant to $\emptyset$ if it bounds. Every manifold is its own inverse, since $M \amalg M$ bounds $M \times I$. We can make the graded group $\mathscr{N}_*$ into a ring by using cartesian product as multiplication. This ring has been calculated: $\mathscr{N}_* \cong \mathbb{Z}/2[x_k \,|\, k \neq 2^i - 1]$. We'll say more about how we attack this calculation in a moment. This is the *unoriented bordism ring*, due to Thom.

Thom also considered the variant in which the manifolds are oriented. In this case, the cobordism is also required to be oriented, and the boundary ∂W is oriented so that its orientation, together with the inward normal into W, gives the restriction of the orientation of W to ∂W. The effect is that, if M is a closed oriented manifold, then $\partial(M \times I) = M \amalg (-M)$ where $-M$ denotes M with its orientation reversed. This makes $-M$ the negative of M in the resulting *oriented bordism ring* Ω_*. This ring is more complicated than $\mathscr{N}_*$, having both a torsion-free part (calculated by Thom) and a torsion part, consisting entirely of elements of order 2 (calculated by Milnor and Wall).

There are many other variants of these rings, including *unitary bordism*, $\mathscr{U}_*$, which uses "stably almost complex" manifolds; M is such a manifold if there is given an embedding $M \subset \mathbb{R}^n$ and a complex structure on the normal bundle to this embedding. The calculation is $\mathscr{U}_* \cong \mathbb{Z}[z_{2k}]$. This and other variants are discussed in Stong.

These rings are actually coefficient rings of certain homology theories, the *bordism theories* (there is a nice convention, due to Atiyah, that we use the name bordism for the homology theory, and the name cobordism for the related cohomology theory). If X is a space, we define the group $\mathscr{N}_k(X)$ to be the set of bordism classes of maps $M \longrightarrow X$, where M is a k-dimensional smooth closed manifold and the map is continuous. Cobordisms must also map into X, and the restriction of the map to the boundary must agree with the given maps on the k-manifolds. Defining the relative groups $\mathscr{N}_*(X, A)$ is a little trickier. We consider maps $(M, \partial M) \longrightarrow (X, A)$. Such a map is cobordant to $(N, \partial N) \longrightarrow (X, A)$ if there exists a triple $(W, \partial_0 W, \partial_1 W)$, where $\partial W = \partial_0 W \cup \partial_1 W$, the intersection $\partial_0 W \cap \partial_1 W$ is the common boundary $\partial(\partial_0 W) = \partial(\partial_1 W)$, and $\partial_0 W \cong M \amalg N$, together with a map $(W, \partial_1 W) \longrightarrow (X, A)$ that restricts to the given maps on $\partial_0 W$. (This makes the most sense if you draw a picture.) It's useful to think of W as having a "corner" at $\partial_0 W \cap \partial_1 W$; otherwise you have to use resmoothings to get an equivalence relation. It is now a pretty geometric exercise to show that there is a long exact sequence

$$\cdots \longrightarrow \mathscr{N}_k(A) \longrightarrow \mathscr{N}_k(X) \longrightarrow \mathscr{N}_k(X, A) \longrightarrow \mathscr{N}_{k-1}(A) \longrightarrow \cdots$$

where the "boundary map" is precisely taking the boundary. There are oriented, unitary, and other variants of this homology theory.

Calculation of these groups is possible largely because we know the representing spectra for these theories. Let TO (the *Thom prespectrum*) be the prespectrum whose kth space is $TO(k)$, the Thom space of the universal k-plane bundle over $BO(k)$. It is an inclusion prespectrum and, applying the spectrification functor L to it, we obtain the *Thom spectrum* MO. Its homotopy groups are given by

$$\pi_k(MO) = \operatorname{colim}_q \pi_{q+k}(TO(q)).$$

Then $\mathscr{N}_* \cong \pi_*(MO)$, and in fact MO represents unoriented bordism.

The proof goes like this: Given a k-dimensional manifold M, embed M in some $\mathbb{R}^{q+k}$ with normal bundle ν. The unit disk of this bundle is homeomorphic to a tubular neighborhood N of M in $\mathbb{R}^{q+k}$, and so there is a collapse map $c : S^{q+k} \longrightarrow T\nu$ given by collapsing everything outside of N to the basepoint. There is also a classifying map $T\nu \longrightarrow TO(q)$, and the composite

$$S^{q+k} \longrightarrow T\nu \longrightarrow TO(q)$$

represents an element of $\pi_k(MO)$. Applying a similar construction to a cobordism gives a homotopy between the two maps obtained from cobordant manifolds. This construction, known as the *Pontrjagin-Thom construction*, describes the map $\mathscr{N}_k \longrightarrow \pi_k(MO)$.

The inverse map is constructed as follows: Given a map $f : S^{q+k} \longrightarrow TO(q)$, we may assume that f is transverse to the zero-section. The inverse image $M = f^{-1}(BO(q))$ is then a k-dimensional submanifold of S^{q+k} (provided that we use Grassmannian manifold approximations of classifying spaces), and the normal bundle to the embedding of M in S^{q+k} is the pullback of the universal bundle. Making a homotopy between two maps transverse provides a cobordism between the two manifolds obtained from the maps. One can now check that these two constructions are well-defined and inverse isomorphisms. The analysis of $\mathscr{N}_*(X, A)$ is almost identical.

In fact MO is a ring spectrum, and the Thom isomorphism just constructed is an isomorphism of rings. The product on MO is induced from the maps

$$TO(j) \wedge TO(k) \longrightarrow TO(j+k)$$

of Thom complexes arising from the classifying map of the external sum of the jth and kth universal bundle. This becomes clearer when one thinks in a coordinate-free way; in fact, it was inspection of Thom spectra that led to the description of the stable homotopy category that May gave in Chapter XII.

Now MO is a very tractable spectrum. To compute its homotopy we have available such tools as the Thom isomorphism, the Steenrod algebra (mod 2), and the Adams spectral sequence for the most sophisticated calculation. (Stong gives a calculation not using the spectral sequence.) The point is that we now have something concrete to work with, and adequate tools to do the job. For oriented bordism, we replace MO with MSO, which is constructed similarly except that we use the universal oriented bundles over the spaces $BSO(k)$. Here we use the fact that an orientation of a manifold is equivalent to an orientation of its normal bundle. Similarly, for unitary bordism we use the spectrum MU, constructed out of the universal unitary bundles.

The standard general reference is
R. E. Stong. Notes on Cobordism Theory. Princeton University Press. 1968.

2. Equivariant cobordism and Thom spectra

Now we take a compact Lie group G and try to generalize everything to the G-equivariant context. This generalization of nonequivariant bordism was first studied by Conner and Floyd. Using smooth G-manifolds throughout we can certainly copy the definition of cobordism to obtain the equivariant bordism groups $\mathcal{N}_*^G$ and, for pairs of G-spaces (X, A), the groups $\mathcal{N}_*^G(X, A)$. We shall concentrate on unoriented bordism. To define unitary bordism, we consider a unitary manifold to be a smooth G-manifold M together with an embedding of M in either V or $V \oplus \mathbb{R}$, where V is a complex representation of G, and a complex structure on the resulting normal bundle. The notion of an oriented G-manifold is complicated and still controversial, although for odd order groups it suffices to look at oriented manifolds with an action of G; the action of G automatically preserves the orientation.

It is also easy to generalize the Thom spectrum. Let $\mathscr{U}$ be a complete G-universe. In view of the description of the K-theory G-spectra in the previous chapter, it seems most natural to start with the universal n-plane bundles

$$\pi(V) : EO(|V|, V \oplus \mathscr{U}) \longrightarrow BO(|V|, V \oplus \mathscr{U})$$

for indexing spaces V in $\mathscr{U}$. Let $TO_G(V)$ be the Thom space of $\pi(V)$. For $V \subseteq W$, the pullback of $\pi(W)$ over the inclusion

$$BO(|V|, V \oplus \mathscr{U}) \longrightarrow BO(|W|, W \oplus \mathscr{U})$$

is the Whitney sum of $\pi(V)$ and the trivial bundle with fiber $W - V$. Its Thom space is $\Sigma^{W-V} TO_G(V)$, and the evident map of bundles induces an inclusion

$$\sigma : \Sigma^{W-V} TO_G(V) \longrightarrow TO_G(W).$$

This construction gives us an inclusion G-prespectrum TO_G. We define the real Thom G-spectrum to be its spectrification $MO_G = LTO_G$. Using complex representations throughout, we obtain the complex analogs TU_G and MU_G. This definition is essentially due to tom Dieck.

The interesting thing is that MO_G does not represent $\mathscr{N}_*^G$. It is easy to define a map $\mathscr{N}_*^G \longrightarrow \pi_*^G(MO_G) = MO_*^G$ using the Pontrjagin-Thom construction, but we cannot define an inverse. The problem is the failure of transversality in the equivariant context. As a simple example of this failure, consider the group $G = \mathbb{Z}/2$, let $M = *$ be a one-point G-set (a 0-dimensional manifold), let $N = \mathbb{R}$ with the nontrivial linear action of G, and let $Y = \{0\} \subset N$. Let $f : M \longrightarrow N$ be the only G-map that can be defined: it takes M to Y. Clearly f cannot be made transverse to Y, since it is homotopic only to itself. This simple example is paradigmatic. In general, given manifolds M and $Y \subset N$ and a map $f : M \longrightarrow N$, if f fails to be homotopic to a map transverse to Y it is because of the presence in the normal bundle to Y of a nontrivial representation of G that cannot be mapped onto by the representations available in the tangent bundle of M. Wasserman provided conditions under which we can get transversality. If G is a product of a torus and a finite group, he gives a sufficient condition for transversality that amounts to saying that, where needed, we will always have in M a nontrivial representation mapping onto the nontrivial representation we see in the normal bundle to Y. Others have given obstruction theories to transversality, for example Petrie and Waner and myself.

Using Wasserman's condition, it is possible (for one of his G) to construct the G-spectrum that *does* represent $\mathscr{N}_*^G$. Again, let $\mathscr{U}$ be a complete G-universe. We can construct a G-prespectrum to_G with associated G-spectrum mo_G by letting V run through the indexing spaces in our complete universe $\mathscr{U}$ as before, but replacing $\mathscr{U}$ by its G-fixed point space $\mathscr{U}^G \cong \mathbb{R}^\infty$ in the bundles we start with. That is, we start with the G-bundles

$$EO(|V|, V \oplus \mathscr{U}^G) \longrightarrow BO(|V|, V \oplus \mathscr{U}^G)$$

for indexing spaces V in $\mathscr{U}$. Again, restricting attention to complex representations, we obtain the complex analogs tu_G and mu_G. The fact that there are so few nontrivial representations present in the bundle $EO(|V|, V \oplus \mathscr{U}^G)$ allows us to use Wasserman's transversality results to show that mo_G represents $\mathscr{N}_*^G$. The inclusion $\mathscr{U}^G \longrightarrow \mathscr{U}$ induces a map

$$mo_G \longrightarrow MO_G$$

that represents the map $\mathscr{N}_*^G \longrightarrow MO_*^G$ that we originally hoped was an isomorphism.

On the other hand, there is also a geometric interpretation of MO_*^G. Using either transversality arguments or a clever argument due to Bröcker and Hook that works for all compact Lie groups, one can show that

$$MO_k^G(X, A) \cong \operatorname{colim}_V \mathscr{N}_{k+|V|}^G((X, A) \times (D(V), S(V))).$$

Here the maps in the colimit are given by multiplying manifolds by disks of representations, smoothing corners as necessary. We interpret this in the simplest case as follows. A class in $MO_k^G \cong \mathrm{colim}_V \mathscr{N}^G_{k+|V|}(D(V), S(V))$ is represented by a manifold $(M, \partial M)$ together with a map $(M, \partial M) \longrightarrow (D(V), S(V))$. This map is equivalent in the colimit to $(M \times D(W), \partial(M \times D(W)))$ together with the map

$$\begin{aligned}(M \times D(W), \partial(M \times D(W))) &\longrightarrow (D(V) \times D(W), \partial(D(V) \times D(W))) \\ &\cong (D(V \oplus W), S(V \oplus W))\end{aligned}$$

that is obtained by crossing the original map with the identity map on $D(W)$. We call the equivalence class of such a manifold over the disk of a representation a *stable manifold.* Its (virtual) dimension is $\dim M - \dim V$. We can then interpret MO_k^G as the group of cobordism classes of stable manifolds of dimension k. A similar interpretation works for $MO_k^G(X, A)$.

With this interpretation we can see clearly one of the differences between $\mathscr{N}_*^G$ and MO_*^G. If V is a representation of G with no trivial summands, then there is a stable manifold represented by $* \longrightarrow D(V)$, the inclusion of the origin. This represents a nontrivial element $\chi(V) \in MO_{-n}^G$ where $n = |V|$. This element is called the *Euler class* of V. Tom Dieck showed the nontriviality of these elements and we'll give a version of the argument below; note that if V had a trivial summand, then $* \longrightarrow D(V)$ would be homotopic to a map into $S(V)$, so that $\chi(V) = 0$. On the other hand, $\mathscr{N}_*^G$ has no nontrivial elements in negative dimensions, by definition.

Here is another, related difference: Stable bordism is periodic in a sense. If V is any representation of G, then, by the definition of MO_G, $MO_G(V) \cong MO_G(|V|)$; the point is that $MO_G(V)$ really depends only on $|V|$. This gives an equivalence $\Sigma^V MO_G \simeq \Sigma^n MO_G$ if $n = |V|$, or

$$MO_G \simeq \Sigma^{V-n} MO_G.$$

One way of defining an explicit equivalence is to start by classifying the bundle $V \longrightarrow *$ and so obtain an associated map of Thom complexes (a *Thom class*)

$$S^V \longrightarrow TO_G(\mathbb{R}^n) \subset MO_G(\mathbb{R}^n).$$

This is adjoint to a map $\mu(V) : S^{V-n} = \Sigma_n^\infty S^V \longrightarrow MO_G$. The required equivalence is the evident composite

$$S^{V-n} \wedge MO_G \longrightarrow MO_G \wedge MO_G \longrightarrow MO_G.$$

Reversing the roles of V and $\mathbb{R}^n$, we obtain an analogous map $S^{n-V} \longrightarrow MO_G$. It is not hard to check that these are inverse units in the $RO(G)$-graded ring MO_*^G. In homology, this gives isomorphisms of MO_*^G-modules

$$MO_*^G(\Sigma^{|V|} X) \cong MO_*^G(\Sigma^V X)$$

and

$$MO_k^G(X) \cong MO_{k+n}^G(\Sigma^V X)$$

for all k. This is really a special case of a Thom isomorphism that holds for every bundle. The *Thom class* of a bundle ξ is the element in cobordism represented by the map of Thom complexes $T\xi \longrightarrow TO_G(|\xi|) \subset MO_G(|\xi|)$ induced by the classifying map of ξ. Another consequence of the isomorphisms above is that

$MO_V^G(X) \cong MO_n^G(X)$, so that the $RO(G)$-graded groups that we get are no different from the groups in integer grading. We can think of this as a periodicity given by multiplication by the unit $\mu(V)$. It should also be clear that, if $|V| = m$ and $|W| = n$, then the composite isomorphism

$$MO_k^G(X) \cong MO_{k+m}^G(\Sigma^V X) \cong MO_{k+m+n}^G(\Sigma^{V\oplus W} X)$$

agrees with the isomorphism $MO_k^G(X) \cong MO_{k+m+n}^G(\Sigma^{V\oplus W} X)$ associated with the representation $V \oplus W$.

We record one further consequence of all this. Consider the inclusion $e : S^0 \longrightarrow S^V$, where $|V| = n$. This induces a map

$$MO_{k+n}^G(X) \longrightarrow MO_{k+n}^G(\Sigma^V X) \cong MO_k^G(X).$$

It is easy to see geometrically that this is given by multiplication by the stable manifold $* \longrightarrow D(V)$, the inclusion of the origin, which represents $\chi(V) \in MO_{-n}^G$. The similar map in cobordism,

$$MO_G^k(X) \cong MO_G^{k+n}(\Sigma^V X) \longrightarrow MO_G^{k+n}(X)$$

is also given by multiplication by $\chi(V) \in MO_G^n$, as we can see by representing $\chi(V)$ by the stable map

$$S^0 \longrightarrow S^V \longrightarrow \Sigma^V MO_G \simeq \Sigma^n MO_G.$$

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3. Computations: the use of families

For computations, we start with the fact that $\mathcal{N}_*^G(X)$ is a module over $\mathcal{N}_*$ (the nonequivariant bordism ring, which we know) by cartesian product. The question is then its structure as a module. We'll take a look at the main computational techniques and at some of the simpler known results.

The main computational technique was introduced by Conner and Floyd. Recall that a *family of subgroups* of G is a collection of subgroups closed under conjugacy and taking of subgroups (in short, under subconjugacy). If $\mathcal{F}$ is such a family, we define an *$\mathcal{F}$-manifold* to be a smooth G-manifold all of whose isotropy groups are in $\mathcal{F}$. If we restrict our attention to closed $\mathcal{F}$-manifolds and cobordisms that are also $\mathcal{F}$-manifolds, we get the groups $\mathcal{N}_*^G[\mathcal{F}]$ of cobordism classes of manifolds with restricted isotropy. Similarly, we can consider the bordism theory $\mathcal{N}_*^G[\mathcal{F}](X, A)$. Now there is a relative version of this as well. Suppose that $\mathcal{F}' \subset \mathcal{F}$. An *$(\mathcal{F}, \mathcal{F}')$-manifold* is a manifold $(M, \partial M)$ where M is an $\mathcal{F}$-manifold and ∂M is an $\mathcal{F}'$-manifold (possibly empty, of course). To define cobordism of such manifolds, we must resort to manifolds with multipart boundaries, or manifolds with corners. Precisely, $(M, \partial M)$ is cobordant to

$(N, \partial N)$ if there is a manifold $(W, \partial_0 W, \partial_1 W)$ such that W is an $\mathscr{F}$-manifold, $\partial_1 W$ is an $\mathscr{F}'$-manifold, and $\partial_0 W = M \amalg N$, where as usual $\partial W = \partial_0 W \cup \partial_1 W$ and $\partial_0 W \cap \partial_1 W$ is the common boundary of $\partial_0 W$ and $\partial_1 W$. With this definition we can form the relative bordism groups $\mathscr{N}_*^G[\mathscr{F}, \mathscr{F}']$. Of course, there is also an associated bordism theory, although to describe the relative groups of that theory requires manifolds with 2-part boundaries, and cobordisms with 3-part boundaries!

From a homotopy theoretic point of view it's interesting to notice that

$$\mathscr{N}_*^G[\mathscr{F}] \cong \mathscr{N}_*^G(E\mathscr{F}),$$

since a manifold over $E\mathscr{F}$ must be an $\mathscr{F}$-manifold, and any $\mathscr{F}$-manifold has a unique homotopy class of maps into $E\mathscr{F}$. Similarly,

$$\mathscr{N}_*^G[\mathscr{F}](X) \cong \mathscr{N}_*^G(X \times E\mathscr{F}),$$

and so on. For the purposes of computation, it is usually more fruitful to think in terms of manifolds with restricted isotropy, however. Notice that this gives us an easy way to define $MO_*^G[\mathscr{F}]$: it is $MO_*^G(E\mathscr{F})$. We can also interpret this in terms of stable manifolds with restricted isotropy.

As a first illustration of the use of families, we give the promised proof of the nontriviality of Euler classes.

LEMMA 3.1. *Let G be a compact Lie group and V be a representation of G without trivial summands. Then $\chi(V) \neq 0$ in MO_{-n}^G, where $n = |V|$.*

PROOF. Let $\mathscr{A}$ be the family of all subgroups, and let $\mathscr{P}$ be the family of proper subgroups. Consider the map $MO_*^G \longrightarrow MO_*^G[\mathscr{A}, \mathscr{P}]$. We claim that the image of $\chi(V)$ is invertible in $MO_*^G[\mathscr{A}, \mathscr{P}]$ (which is nonzero), so that $\chi(V) \neq 0$. Thinking in terms of stable manifolds, $\chi(V) = [* \longrightarrow D(V)]$. Its inverse is $D(V) \longrightarrow *$, which lives in the group $MO_*^G[\mathscr{A}, \mathscr{P}]$ because $\partial D(V) = S(V)$ has no fixed points. It's slightly tricky to show that the product, which is represented by $D(V) \longrightarrow * \longrightarrow D(V)$, is cobordant to the identity $D(V) \longrightarrow D(V)$, as we have to change the interpretation of the boundary $S(V)$ of the source from being the "$\mathscr{P}$-manifold part" to being the "maps into $S(V)$ part". However, a little cleverness with $D(V) \times I$ does the trick. □

Returning to our general discussion of the use of families, note that, for a pair of families $(\mathscr{F}, \mathscr{F}')$, there is a long exact sequence

$$\cdots \longrightarrow \mathscr{N}_k^G[\mathscr{F}'] \longrightarrow \mathscr{N}_k^G[\mathscr{F}] \longrightarrow \mathscr{N}_k^G[\mathscr{F}, \mathscr{F}'] \longrightarrow \mathscr{N}_{k-1}^G[\mathscr{F}'] \longrightarrow \cdots,$$

where the boundary map is given by taking boundaries. (This is of course the same as the long exact sequence associated with the pair of spaces $(E\mathscr{F}, E\mathscr{F}')$.) We would like to use this exact sequence to calculate $\mathscr{N}_*^G$ inductively. To set this up a little more systematically, suppose that we have a sequence $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots$ of families of subgroups whose union is the family of all subgroups. If we can calculate $\mathscr{N}_k^G[\mathscr{F}_0]$ and each relative term $\mathscr{N}_k^G[\mathscr{F}_p, \mathscr{F}_{p-1}]$, we may be able to calculate every $\mathscr{N}_k^G[\mathscr{F}_p]$ and ultimately $\mathscr{N}_*^G$. We can also introduce the machinery of spectral sequences here: The long exact sequences give us an exact couple

$$\begin{tikzcd}
\mathcal{N}_*^G[\mathscr{F}_{p-1}] \arrow[rr] & & \mathcal{N}_*^G[\mathscr{F}_p] \arrow[dl] \\
& \mathcal{N}_*^G[\mathscr{F}_p, \mathscr{F}_{p-1}] \arrow[ul] &
\end{tikzcd}$$

and hence a spectral sequence with $E^1_{p,q} = \mathcal{N}_q^G[\mathscr{F}_p, \mathscr{F}_{p-1}]$ that converges to $\mathcal{N}_*^G$.

This would all be academic if not for the fact that $\mathcal{N}_*^G[\mathscr{F}_p, \mathscr{F}_{p-1}]$ is often computable. Let us start off with the base of the induction: $\mathcal{N}_*^G[\{e\}, \emptyset] = \mathcal{N}_k^G[\{e\}]$. This is the bordism group of free closed G-manifolds. Now, if M is a free G-manifold, then M/G is also a manifold, of dimension $\dim M - \dim G$. There is a unique homotopy class of G-maps $M \longrightarrow EG$, which passes to quotients to give a map $M/G \longrightarrow BG$. Moreover, given the map $M/G \longrightarrow BG$ we can recover the original manifold M, since it is the pullback in the following diagram:

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & EG \arrow[d] \\
M/G \arrow[r] & BG.
\end{tikzcd}$$

This applies equally well to manifolds with or without boundary, so it applies to cobordisms as well. This establishes the isomorphism

$$\mathcal{N}_k^G[\{e\}] \cong \mathcal{N}_{k-\dim G}(BG).$$

Now the bordism of a classifying space may or may not be easy to compute, but at least this is a nonequivariant problem.

The inductive step can also be reduced to a nonequivariant calculation. Suppose that G is finite or Abelian for convenience. We say that $\mathscr{F}$ and $\mathscr{F}'$ are adjacent if $\mathscr{F} = \mathscr{F}' \cup (H)$ for a single conjugacy class of subgroups (H), and it suffices to restrict attention to such an adjacent pair. Suppose that $(M, \partial M)$ is an $(\mathscr{F}, \mathscr{F}')$-manifold. Let $M^{(H)}$ denote the set of points in M with isotropy groups in (H); $M^{(H)}$ lies in the interior of M, since ∂M is an $\mathscr{F}'$-manifold, and $M^{(H)} = \cup_{K\in(H)} M^K$ is a union of closed submanifolds of M. Moreover, these submanifolds are pairwise disjoint, since (H) is maximal in $\mathscr{F}$. Therefore $M^{(H)}$ is a closed G-invariant submanifold in the interior of M, isomorphic to $G \times_{NH} M^H$. (Here is where it is convenient to have G finite or Abelian.) Thus $M^{(H)}$ has a G-invariant closed tubular neighborhood in M, call it N. Here is the key step: *$(M, \partial M)$ is cobordant to $(N, \partial N)$ as an $(\mathscr{F}, \mathscr{F}')$-manifold.* The cobordism is provided by $M \times I$ with corners smoothed (this is easiest to see in a picture).

As usual, let $WH = NH/H$. Now $(N, \partial N)$ is determined by the free WH-manifold M^H and the NH-vector bundle over it which is its normal bundle. Since WH acts freely on the base, each fiber is a representation of H with no trivial summands and decomposes into a sum of multiples of irreducible representations. This also decomposes the whole bundle: Suppose that the nontrivial irreducible representations of H are $V_1, V_2, \ldots$. Then $\nu = \oplus\nu_i$, where each fiber of each ν_i is a sum of copies of V_i. Clearly ν_i is completely determined by the free WH-bundle $\mathrm{Hom}_G(V_i, \nu_i)$, which has fibers $\mathbb{F}^n$ where $\mathbb{F}$ is one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, depending on V_i. Notice, however, that the NH-action on ν induces certain isomorphisms among the ν_i: If V_i and V_j are conjugate representations under the action of NH, then ν_i and ν_j must be isomorphic.

The upshot of all of this is that $\mathscr{N}_k^G[\mathscr{F}, \mathscr{F}']$ is isomorphic to the group obtained in the following way. Suppose that the dimension of V_i is d_i and that $\mathrm{Hom}_G(V_i, V_i) = \mathbb{F}_i$, where $\mathbb{F}_i = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Consider free WH-manifolds M, together with a sequence of WH-bundles ξ_1, ξ_2, $\cdots$ over M, one for each V_i, the group of ξ_i being $O(\mathbb{F}_i, n_i)$ (i.e., $O(n_i)$, $U(n_i)$, or $Sp(n_i)$). If V_i and V_j are conjugate under the action of NH, then we insist that ξ_i and ξ_j be isomorphic. The dimension of $(M; \xi_1, \xi_2, \cdots)$ is $\dim M + \sum n_i d_i$; that is, this should equal k. Now define $(M; \xi_1, \xi_2, \cdots)$ to be cobordant to $(N; \zeta_1, \zeta_2, \cdots)$ if there exists some $(W; \theta_1, \theta_2, \cdots)$ such that $\partial W = M \amalg N$ and the restriction of θ_i to ∂W is $\xi_i \amalg \zeta_i$. It should be reasonably clear from this description that we have an isomorphism

$$\mathscr{N}_k^G[\mathscr{F}, \mathscr{F}'] \cong \sum_{j + \sum n_i d_i = k} \mathscr{N}_j^{WH}(EWH \times (\times_i BO(\mathbb{F}_i, n_i)))$$

where WH acts on $\times_i BO(\mathbb{F}_i, n_i)$ via its permutation of the representations of H. One more step and this becomes a nonequivariant problem: We take the quotient by WH, which we can do because the argument $EWH \times (\times_i BO(\mathbb{F}_i, n_i))$ is free (this being just like the case $\mathscr{N}_*^G[\{e\}]$ above). This gives

$$(3.2)\quad \mathscr{N}_k^G[\mathscr{F}, \mathscr{F}'] \cong \sum_{\dim WH + j + \sum n_i d_i = k} \mathscr{N}_j(EWH \times_{WH} (\times_i BO(\mathbb{F}_i, n_i))).$$

Notice that, if G is Abelian or if WH acts trivially on the representations of H for some other reason, then the argument is $BWH \times (\times_i BO(\mathbb{F}_i, n_i)))$.

P. E. Conner and E. E. Floyd. Differentiable periodic maps. Academic Press, Inc. 1964.

4. Special cases: odd order groups and $\mathbb{Z}/2$

If G is a finite group of odd order, then the differentials in the spectral sequence for $\mathscr{N}_*^G$ all vanish, and $\mathscr{N}_*^G$ is the direct sum over (H) of the groups displayed in (3.2). This is actually a consequence of a very general splitting result that will be explained in XVII§6. The point is that $\mathscr{N}_*^G$ is a $\mathbb{Z}/2$-vector space and, away from the order of the group, the Burnside ring $A(G)$ splits as a direct sum of copies of $\mathbb{Z}[1/|G|]$, one for each conjugacy class of subgroups of G. This induces splittings in all modules over the Burnside ring, including all $RO(G)$-graded homology theories (that is, those homology theories represented by spectra indexed on complete universes). The moral of the story is that, away from the order of the group, equivariant topology generally reduces to nonequivariant topology.

This observation can also be used to show that the spectra mo_G and MO_G split as products of Eilenberg-Mac Lane spectra, just as in the nonequivariant case. Remember that this depends on G having odd order.

Conner and Floyd computed the additive structure of $\mathscr{N}_*^{\mathbb{Z}/2}$, and Alexander computed its multiplicative structure. There is a split short exact sequence

$$0 \longrightarrow \mathscr{N}_k^{\mathbb{Z}/2} \longrightarrow \oplus_{0\leq n\leq k}\mathscr{N}_{k-n}(BO(n)) \longrightarrow \mathscr{N}_{k-1}(B\mathbb{Z}/2) \longrightarrow 0,$$

which is part of the long exact sequence of the pair $(\{\mathbb{Z}/2, e\}, \{e\})$. The first map is given by restriction to $\mathbb{Z}/2$-fixed points and the normal bundles to these. The second map is given by taking the unit sphere of a bundle, then taking the quotient by the antipodal map (a free $\mathbb{Z}/2$-action) and classifying the resulting $\mathbb{Z}/2$-bundle. This map is the only nontrivial differential in the spectral sequence. Now

$$\oplus_{0\leq n\leq k}\mathscr{N}_{k-n}(BO(n)) \cong \mathscr{N}_*[x_1, x_2, \cdots],$$

where $x_k \in \mathscr{N}_{k-1}(BO(1))$ is the class of the canonical line bundle over $\mathbb{R}P^{k-1}$. On the other hand,

$$\mathscr{N}_*(B\mathbb{Z}/2) \cong \mathscr{N}_*\{r_0, r_1, r_2, \cdots\}$$

is the free $\mathscr{N}_*$-module generated by $\{r_k\}$, where r_k is the class of $\mathbb{R}P^k \longrightarrow B\mathbb{Z}/2$. The splitting is the obvious one: it sends r_k to x_{k+1}. In fact, the x_k all live in the summand $\mathscr{N}_*(B\mathbb{Z}/2) = \mathscr{N}_*(BO(1))$, and the splitting is simply the inclusion of this summand. It follows that $\mathscr{N}_*^{\mathbb{Z}/2}$ is a free module over $\mathscr{N}_*$, and one can write down explicit generators. Alexander writes down explicit multiplicative generators.

A similar calculation can be done for $MO_*^{\mathbb{Z}/2}$. The short exact sequence is then

$$0 \longrightarrow MO_k^{\mathbb{Z}/2} \longrightarrow \oplus_n\mathscr{N}_{k-n}(BO) \longrightarrow \mathscr{N}_{k-1}(B\mathbb{Z}/2) \longrightarrow 0,$$

where now k and n range over the integers, positive and negative, and the sum in the middle is infinite. In fact,

$$\oplus_n\mathscr{N}_{*-n}(BO) \cong \mathscr{N}_*[x_1^{-1}, x_1, x_2, \cdots],$$

where the x_i are the images of the elements of the same name from the geometric case. Here x_1^{-1} is the image of χ_L, where L is the nontrivial irreducible representation of $\mathbb{Z}/2$.

It is natural to ask whether or not $mo_{\mathbb{Z}/2}$ and $MO_{\mathbb{Z}/2}$ are products of Eilenberg-Mac Lane $\mathbb{Z}/2$-spectra, as in the case of odd order groups. I showed that the answer turns out to be no.

J. C. Alexander. The bordism ring of manifolds with involution. Proc. Amer. Math. Soc. 31(1972), 536-542.

P. E. Conner and E. E. Floyd. Differentiable periodic maps. Academic Press, Inc. 1964.

S. Costenoble. The structure of some equivariant Thom spectra. Trans. Amer. Math. Soc. 315(1989), 231-254.

CHAPTER XVI

Spectra and G-spectra; change of groups; duality

In this and the following three chapters, we return to the development of features of the equivariant stable homotopy category. The basic reference is [LMS], and specific citations are given at the ends of sections.

1. Fixed point spectra and orbit spectra

Much of the most interesting work in equivariant algebraic topology involves the connection between equivariant constructions and nonequivariant topics of current interest. We here explain the basic facts concerning the relationships between G-spectra and spectra and between equivariant and nonequivariant cohomology theories.

We restrict attention to a complete G-universe U and we write $RO(G)$ for $RO(G;U)$. Given the details of Chapter XIII, we shall be more informal about the $RO(G)$-grading from now on. In particular, we shall allow ourselves to write $E_G^\alpha(X)$ for $\alpha \in RO(G)$, ignoring the fact that, for rigor, we must first fix a presentation of α as a formal difference $V \ominus W$. We write S^α instead of $S^{V\ominus W}$ and, for G-spectra X and E, we write

$$E_\alpha^G(X) = [S^\alpha, E \wedge X]_G \tag{1.1}$$

and

$$E_G^\alpha(X) = [S^{-\alpha} \wedge X, E]_G = [S^{-\alpha}, F(X,E)]_G. \tag{1.2}$$

To relate this to nonequivariant theories, let $i : U^G \longrightarrow U$ be the inclusion of the fixed point universe. Recall that we have the forgetful functor

$$i^* : G\mathscr{S}U \longrightarrow G\mathscr{S}U^G$$

obtained by forgetting the indexing G-spaces with non-trivial G-action. The "underlying nonequivariant spectrum" of E is i^*E with its action by G ignored. Recall too that i^* has a left adjoint

$$i_* : G\mathscr{S}U^G \longrightarrow G\mathscr{S}U$$

that builds in non-trivial representations. Explicitly, for a naive G-prespectrum D and an indexing G-space V,

$$(i_*D)(V) = D(V^G) \wedge S^{V-V^G}.$$

For a naive G-spectrum D, $i_*D = Li_*\ell D$, as usual. These change of universe functors play a subtle and critical role in relating equivariant and nonequivariant phenomena. Since, with G-actions ignored, the universes are isomorphic, the following result is intuitively obvious.

LEMMA 1.3. *For $D \in G\mathscr{S}U^G$, the unit G-map $\eta : D \longrightarrow i^*i_*D$ of the (i_*, i^*) adjunction is a nonequivariant equivalence. For $E \in G\mathscr{S}U$, the counit G-map $\varepsilon : i_*i^*E \longrightarrow E$ is a nonequivariant equivalence.*

We define the fixed point spectrum D^G of a naive G-spectrum D by passing to fixed points spacewise, $D^G(V) = (DV)^G$. This functor is right adjoint to the trivial G-action functor from spectra to naive G-spectra:

(1.4) $G\mathscr{S}U^G(C, D) \cong \mathscr{S}U^G(C, D^G)$ for $C \in \mathscr{S}U^G$ and $D \in G\mathscr{S}U^G$.

It is essential that G act trivially on the universe to obtain well-defined structural homeomorphisms on D^G. For $E \in G\mathscr{S}U$, we define $E^G = (i^*E)^G$. Composing the (i_*, i^*)-adjunction with (1.4), we obtain

(1.5) $G\mathscr{S}U(i_*C, E) \cong \mathscr{S}U^G(C, E^G)$ for $C \in \mathscr{S}U^G$ and $D \in G\mathscr{S}U^G$.

The sphere G-spectra $G/H_+ \wedge S^n$ in $G\mathscr{S}U$ are obtained by applying i_* to the corresponding sphere G-spectra in $G\mathscr{S}U^G$. When we restrict (1.1) and (1.2) to integer gradings and take $H = G$, we see that (1.5) implies

$$E^G_n(X) \cong \pi_n((E \wedge X)^G) \tag{1.6}$$

and

$$E_G^n(X) \cong \pi_{-n}(F(X, E)^G). \tag{1.7}$$

As in the second isomorphism, naive G-spectra D represent $\mathbb{Z}$-graded cohomology theories on naive G-spectra or on G-spaces. In contrast, as already noted in XIII§3, we cannot expect to represent interesting homology theories on G-spaces X in the form $\pi_*((D \wedge X)^G)$ for a naive G-spectrum D: here smash products commute with fixed points, hence such theories vanish on X/X^G. For genuine G-spectra, there is a well-behaved natural map

$$E^G \wedge (E')^G \longrightarrow (E \wedge E')^G, \tag{1.8}$$

but, even when E' is replaced by a G-space, it is not an equivalence. In Section 3, we shall define a different G-fixed point functor that does commute with smash products.

Orbit spectra D/G of naive G-spectra are constructed by first passing to orbits spacewise on the prespectrum level and then applying the functor L from prespectra to spectra. Here $(\Sigma^\infty X)/G \cong \Sigma^\infty(X/G)$. The orbit functor is left adjoint to the trivial G-action functor from spectra to naive G-spectra:

(1.9) $\mathscr{S}U^G(D/G, C) \cong G\mathscr{S}U^G(D, C)$ for $C \in \mathscr{S}U^G$ and $D \in G\mathscr{S}U^G$.

For a genuine G-spectrum E, it is tempting to define E/G to be $L((i^*E)/G)$, but this appears to be an entirely useless construction. For free actions, we will shortly give a substitute.

[LMS, especially I§3]

2. Split G-spectra and free G-spectra

The calculation of the equivariant cohomology of free G-spectra in terms of the nonequivariant cohomology of orbit spectra is fundamental to the passage back and forth between equivariant and nonequivariant phenomena. This requires the subtle and important notion of a "split G-spectrum".

DEFINITION 2.1. A naive G-spectrum D is said to be split if there is a nonequivariant map of spectra $\zeta : D \longrightarrow D^G$ whose composite with the inclusion of D^G in D is homotopic to the identity map. A genuine G-spectrum E is said to be split if i^*E is split.

The K-theory G-spectra K_G and KO_G are split. Intuitively, the splitting is obtained by giving nonequivariant bundles trivial G-action. The cobordism spectra MO_G and MU_G are also split. The Eilenberg-Mac Lane G-spectrum HM associated to a Mackey functor M is split if and only if the canonical map $M(G/G) \longrightarrow M(G/e)$ is a split epimorphism; this implies that G acts trivially on $M(G/e)$, which is usually not the case. The suspension G-spectrum $\Sigma^\infty X$ of a G-space X is split if and only if X is stably a retract up to homotopy of X^G, which again is usually not the case. In particular, however, the sphere G-spectrum $S = \Sigma^\infty S^0$ is split. The following consequence of Lemma 1.3 gives more examples.

LEMMA 2.2. *If $D \in G\mathscr{S}U^G$ is split, then $i_*D \in G\mathscr{S}U$ is also split.*

The notion of a split G-spectrum is defined in nonequivariant terms, but it admits the following equivariant interpretation.

LEMMA 2.3. *If E is a G-spectrum with underlying nonequivariant spectrum D, then E is split if and only if there is a map of G-spectra $i_*D \longrightarrow E$ that is a nonequivariant homotopy equivalence.*

Recall that a based G-space is said to be free if it is free away from its G-fixed basepoint. A G-spectrum, either naive or genuine, is said to be free if it is equivalent to a G-CW spectrum built up out of free cells $G_+ \wedge CS^n$. The functors $\Sigma^\infty : \mathscr{T} \longrightarrow G\mathscr{S}U^G$ and $i_* : G\mathscr{S}U^G \longrightarrow G\mathscr{S}U$ carry free G-spaces to free naive G-spectra and free naive G-spectra to free G-spectra. In all three categories, X is homotopy equivalent to a free object if and only if the canonical G-map $EG_+ \wedge X \longrightarrow X$ is an equivalence. A free G-spectrum E is equivalent to i_*D for a free naive G-spectrum D, unique up to equivalence; the orbit spectrum D/G is the substitute for E/G that we alluded to above. A useful mnemonic slogan is that "free G-spectra live in the trivial universe". Note, however, that we cannot take $D = i * E$: this is not a free G-spectrum. For example, $\Sigma^\infty G_+ \in G\mathscr{S}U^G$ clearly satisfies $(\Sigma^\infty G_+)^G = *$, but we shall see later that $i_*\Sigma^\infty G_+$, which is the genuine suspension G-spectrum $\Sigma^\infty G_+ \in G\mathscr{S}U$, satisfies $(i^*\Sigma^\infty G_+)^G = S$.

THEOREM 2.4. *If E is a split G-spectrum and X is a free naive G-spectrum, then there are natural isomorphisms*

$$E_n^G(i_*X) \cong E_n((\Sigma^{Ad(G)}X)/G) \quad \textit{and} \quad E_G^n(i_*X) \cong E^n(X/G),$$

where $Ad(G)$ is the adjoint representation of G and E_* and E^* denote the theories represented by the underlying nonequivariant spectrum of E.

The cohomology isomorphism holds by inductive reduction to the case $X = G_+$ and use of Lemma 2.3. The homology isomorphism is quite subtle and depends on a dimension-shifting transfer isomorphism that we shall say more about later. This result is an essential starting point for the approach to generalized Tate cohomology theory that we shall present later.

In analogy with (1.8), there is a well-behaved natural map

$$(2.5) \qquad \Sigma^\infty(X^G) \longrightarrow (\Sigma^\infty X)^G,$$

but it is not an equivalence.

[LMS, especially II.1.8, II.2.8, II.2.12, II.8.4]

3. Geometric fixed point spectra

There is a "geometric fixed-point functor"

$$\Phi^G : G\mathscr{S}U \longrightarrow \mathscr{S}U^G$$

that enjoys the properties

$$(3.1) \qquad \Sigma^\infty(X^G) \simeq \Phi^G(\Sigma^\infty X)$$

and

$$(3.2) \qquad \Phi^G(E) \wedge \Phi^G(E') \simeq \Phi^G(E \wedge E').$$

To construct it, recall the definition of $\tilde{E}\mathscr{F}$ for a family $\mathscr{F}$ from V.4.6 and set

$$(3.3) \qquad \Phi^G E = (E \wedge \tilde{E}\mathscr{P})^G,$$

where $\mathscr{P}$ is the family of all proper subgroups of G. Here $E \wedge \tilde{E}\mathscr{P}$ is H-trivial for all $H \in \mathscr{P}$.

The name "geometric fixed point spectrum" comes from an equivalent description of the functor Φ^G. There is an intuitive "spacewise G-fixed point functor" Φ^G from G-prespectra indexed on U to prespectra indexed on U^G. To be precise about this, we index G-prespectra on an indexing sequence $\{V_i\}$, so that $V_i \subset V_{i+1}$ and $U = \cup V_i$, and index prespectra on the indexing sequence $\{V_i^G\}$. Here we use indexing sequences to avoid ambiguities resulting from the fact that different indexing spaces in U can have the same G-fixed point space. For a G-prespectrum $D = \{DV_i\}$, the prespectrum $\Phi^G D$ is given by

$(\Phi^G D)(V_i^G) = (DV_i)^G$, with structural maps $\Sigma^{V_{i+1}^G - V_i^G}(DV_i)^G \longrightarrow (DV_{i+1})^G$ obtained from those of D by passage to G-fixed points. We are interested in homotopical properties of this construction, and when applying it to spectra regarded as prespectra, we must first apply the cylinder functor K and CW approximation functor Γ discussed in XII§9. The relationship between the resulting construction and the spectrum-level construction (3.3) is as follows. Remember that ℓ denotes the forgetful functor from spectra to prespectra and L denotes its left adjoint.

THEOREM 3.4. *For Σ-cofibrant G-prespectra D, there is a natural weak equivalence of spectra*

$$\Phi^G LD \longrightarrow L\Phi^G D.$$

For G-CW spectra E, there is a natural weak equivalence of spectra

$$\Phi^G E \longrightarrow L\Phi^G K\Gamma\ell E.$$

It is not hard to deduce the isomorphisms (3.1) and (3.2) in the stable homotopy category $\bar{h}\mathscr{S}U^G$ from this prespectrum level description of Φ^G.

[LMS, II§9]

4. Change of groups and the Wirthmüller isomorphism

In the previous sections, we discussed the relationship between G-spectra and e-spectra, where we write e both for the identity element and the trivial subgroup of G. We must consider other subgroups and quotient groups of G. First, consider a subgroup H. Since any representation of NH extends to a representation of G and since a WH-representation is just an H-fixed NH-representation, the H-fixed point space U^H of our given complete G-universe U is a complete WH-universe. We define

$$(4.1) \qquad E^H = (i^*E)^H, \quad i : U^H \subset U.$$

This gives a functor $G\mathscr{S}U \longrightarrow (WH)\mathscr{S}U^H$. Of course, we can also define E^H as a spectrum in $\mathscr{S}U^G$. The forgetful functor associated to the inclusion $U^G \longrightarrow U^H$ carries the first version of E^H to the second, and we use the same notation for both. For $D \in (NH)\mathscr{S}U^H$, the orbit spectrum D/H is also a WH-spectrum.

Exactly as on the space level in I§1, we have induced and coinduced G-spectra generated by an H-spectrum $D \in H\mathscr{S}U$. These are denoted by

$$G \ltimes_H D \quad \text{and} \quad F_H[G, D).$$

The "twisted" notation $\ltimes$ is used because there is a little twist in the definitions to take account of the action of G on indexing spaces. As on the space level, these functors are left and right adjoint to the forgetful functor $G\mathscr{S}U \longrightarrow H\mathscr{S}U$: for $D \in H\mathscr{S}U$ and $E \in G\mathscr{S}U$, we have

$$(4.2) \qquad G\mathscr{S}U(G \ltimes_H D, E) \cong H\mathscr{S}U(D, E)$$

and

$$H\mathscr{S}U(E,D) \cong G\mathscr{S}U(E, F_H[G,D)). \tag{4.3}$$

Again, as on the space level, for $E \in G\mathscr{S}U$ we have

$$G \ltimes_H E \cong (G/H)_+ \wedge E \tag{4.4}$$

and

$$F_H[G,E) \cong F(G/H_+, E). \tag{4.5}$$

As promised in XII§6, we can now deduce as in (1.6) that

$$\pi_n^H(E) \equiv [G/H_+ \wedge S^n, E]_G \cong [S^n, E]_H \cong \pi_n(E^H). \tag{4.6}$$

In cohomology, the isomorphism (4.2) gives

$$E_G^*(G \ltimes_H D) \cong E_H^*(D). \tag{4.7}$$

We shall not go into detail, but we can interpret this in terms of $RO(G)$ and $RO(H)$ graded theories via the evident functor $\mathscr{R}O(G) \longrightarrow \mathscr{R}O(H)$. The isomorphism (4.3) does not have such a convenient interpretation as it stands. However, there is a fundamental change of groups result — called the Wirthmüller isomorphism — which in its most conceptual form is given by a calculation of the functor $F_H[G,D)$. It leads to the following homological complement of (4.7). Let $L(H)$ be the tangent H-representation at the identity coset of G/H. Then

$$E_*^G(G \ltimes_H D) \cong E_*^H(\Sigma^{L(H)} D). \tag{4.8}$$

THEOREM 4.9 (GENERALIZED WIRTHMÜLLER ISOMORPHISM). *For H-spectra D, there is a natural equivalence of G-spectra*

$$F_H[G, \Sigma^{L(H)} D) \longrightarrow G \ltimes_H D.$$

Therefore, for G-spectra E,

$$[E, \Sigma^{L(H)} D]_H \cong [E, G \ltimes_H D]_G.$$

The last isomorphism complements the isomorphism from (4.2):

$$[G \ltimes_H D, E]_G \cong [D, E]_H. \tag{4.10}$$

We deduce (4.8) by replacing E in Theorem 4.9 by a sphere, replacing D by $E \wedge D$, and using the generalization

$$G \ltimes_H (D \wedge E) \cong (G \ltimes_H D) \wedge E$$

of (4.4).

[LMS, II§§2-4]
K. Wirthmüller. Equivariant homology and duality. Manuscripta Math. 11(1974), 373-390.

5. Quotient groups and the Adams isomorphism

Let N be a normal subgroup of G with quotient group J. In practice, one is often thinking of a quotient map $NH \longrightarrow WH$ rather than $G \longrightarrow J$. There is an analog of the Wirthmüller isomorphism — called the Adams isomorphism — that compares orbit and fixed-point spectra. It involves the change of universe functors associated to the inclusion $i : U^N \longrightarrow U$ and requires restriction to N-free G-spectra. We note first that the fixed point and orbit functors $G\mathscr{S}U^N \longrightarrow J\mathscr{S}U^N$ are right and left adjoint to the evident pullback functor from J-spectra to G-spectra: for $D \in J\mathscr{S}U^N$ and $E \in G\mathscr{S}U^N$,

$$G\mathscr{S}U^N(D,E) \cong J\mathscr{S}U^N(D,E^N) \tag{5.1}$$

and

$$J\mathscr{S}U^N(E/N,D) \cong G\mathscr{S}U^N(E,D). \tag{5.2}$$

Here we suppress notation for the pullback functor $J\mathscr{S}U^N \longrightarrow G\mathscr{S}U^N$. An N-free G-spectrum E indexed on U is equivalent to i_*D for an N-free G-spectrum D indexed on U^N, and D is unique up to equivalence. Thus our slogan that "free G-spectra live in the trivial universe" generalizes to the slogan that "N-free G-spectra live in the N-fixed universe". This gives force to the following version of (5.2). It compares maps of J-spectra indexed on U^N with maps of G-spectra indexed on U.

THEOREM 5.3. *Let $J = G/N$. For N-free G-spectra E indexed on U^N and J-spectra D indexed on U^N,*

$$[E/N, D]_J \cong [i_*E, i_*D]_G.$$

The conjugation action of G on N gives rise to an action of G on the tangent space of N at e; we call this representation $Ad(N)$, or $Ad(N;G)$. The following result complements the previous one, but is very much deeper. When $N = G$, it is the heart of the proof of the homology isomorphism of Theorem 2.4. We shall later describe the dimension-shifting transfer that is the basic ingredient in its proof.

THEOREM 5.4 (GENERALIZED ADAMS ISOMORPHISM). *Let $J = G/N$. For N-free G-spectra $E \in G\mathscr{S}U^N$, there is a natural equivalence of J-spectra*

$$E/N \longrightarrow (\Sigma^{-Ad(N)} i_*E)^N.$$

Therefore, for $D \in J\mathscr{S}U^N$,

$$[D, E/N]_J \cong [i_*D, \Sigma^{-Ad(N)} i_*E]_G.$$

This result is another of the essential starting points for the approach to generalized Tate cohomology that we will present later. The last two results cry out for general homological and cohomological interpretations, like those of Theorem 2.4. Looking back at Lemma 2.3, we see that what is needed for this are analogs of the underlying nonequivariant spectrum and of the characterization of split G-spectra that make sense for quotient groups J. What is so special about the trivial group is just that it is naturally both a subgroup and a quotient group of G.

The language of families is helpful here. Let $\mathscr{F}$ be a family. We say that a G-spectrum E is $\mathscr{F}$-free, or is an $\mathscr{F}$-spectrum, if E is equivalent to a G-CW spectrum all of whose cells are of orbit type in $\mathscr{F}$. Thus free G-spectra are $\{e\}$-free. We say that a map $f : D \longrightarrow E$ is an $\mathscr{F}$-equivalence if $f^H : D^H \longrightarrow E^H$ is an equivalence for all $H \in \mathscr{F}$ or, equivalently by the Whitehead theorem, if f is an H-equivalence for all $H \in \mathscr{F}$.

Returning to our normal subgroup N, let $\mathscr{F}(N) = \mathscr{F}(N; G)$ be the family of subgroups of G that intersect N in the trivial group. Thus an $\mathscr{F}(N)$-spectrum is an N-free G-spectrum. We have seen these families before, in our study of equivariant bundles. We can now state precise generalizations of Lemma 2.3 and Theorem 2.4. Fix spectra

$$D \in J\mathscr{S}U^N \quad \text{and} \quad E \in G\mathscr{S}U.$$

LEMMA 5.5. *A G-map $\xi : i_*D \longrightarrow E$ is an $\mathscr{F}(N)$-equivalence if and only if the composite of the adjoint $D \longrightarrow (i^*E)^N$ of ξ and the inclusion $(i^*E)^N \longrightarrow i^*E$ is an $\mathscr{F}(N)$-equivalence.*

THEOREM 5.6. *Assume given an $\mathscr{F}(N)$-equivalence $i_*D \longrightarrow E$. For any N-free G-spectrum $X \in G\mathscr{S}U^N$,*

$$E_*^G(\Sigma^{-Ad(N)}(i_*X)) \cong D_*^J(X/N) \quad \text{and} \quad E_G^*(i_*X) \cong D_J^*(X/N).$$

Given E, when do we have an appropriate D? We often have theories that are defined on the category of all compact Lie groups, or on a suitable subcategory. When such theories satisfy appropriate naturality axioms, the theory E_J associated to J will necessarily bear the appropriate relationship to the theory E_G associated to G. We shall not go into detail here. One assumes that the homomorphisms $\alpha : H \longrightarrow G$ in one's category induce maps of H-spectra $\xi_\alpha : \alpha^*E_G \longrightarrow E_H$ in a functorial way, where some bookkeeping with universes is needed to make sense of α^*, and one assumes that ξ_α is an H-equivalence if α is an inclusion. For each $H \in \mathscr{F}(N)$, the quotient map $q : G \longrightarrow J$ restricts to an isomorphism from H to its image K. If the five visible maps,

$$H \subset G,\ K \subset J,\ q : G \longrightarrow J,\ q : H \longrightarrow K, \ \text{ and } \ q^{-1} : K \longrightarrow H,$$

are in one's category, one can deduce that $\xi_q : q^*E_J = i_*E_J \longrightarrow E_G$ is an $\mathscr{F}(N)$-equivalence. This is not too surprising in view of Lemma 2.3, but it is a bit subtle: there are examples where all axioms are satisfied, except that q^{-1} is not in the category, and the conclusion fails because ξ_q is not an H-equivalence. However, this does work for equivariant K-theory and the stable forms of equivariant cobordism, generalizing the arguments used to prove that these theories split. For K-theory, the Bott isomorphisms are suitably natural, by the specification of the Bott elements in terms of exterior powers. For cobordism, we shall explain in XXVI§5 that MO_G and MU_G arise from functors, called "global $\mathscr{I}_*$-functors with smash product", that are defined on all compact Lie groups and their representations and take values in spaces with group actions.

All theories with such a concrete geometric source are defined with suitable naturality on all compact Lie groups G.

J. F. Adams. Prerequisites (on equivariant theory) for Carlsson's lecture. Springer Lecture Notes in Mathematics Vol. 1051, 1984, 483-532.
[LMS, II§§8-9]

6. The construction of G/N-spectra from G-spectra

A different line of thought leads to a construction of J-spectra from G-spectra, $J = G/N$, that is a direct generalization of the geometric fixed point construction $\Phi^G E$. The appropriate analog of $\mathscr{P}$ is the family $\mathscr{F}[N]$ of those subgroups of G that do not contain N. Note that this is a family since N is normal. For a spectrum E in $G\mathscr{S}U$, we define

$$\Phi^N E \equiv (E \wedge \tilde{E}\mathscr{F}[N])^N. \tag{6.1}$$

We have the expected generalizations of (3.1) and (3.2): for a G-space X,

$$\Sigma^\infty(X^N) \simeq \Phi^N(\Sigma^\infty X) \tag{6.2}$$

and, for G-spectra E and E',

$$\Phi^N(E) \wedge \Phi^N(E') \simeq \Phi^N(E \wedge E'). \tag{6.3}$$

We can define $\Phi^H E$ for a not necessarily normal subgroup H by regarding E as an NH-spectrum. Although the Whitehead theorem appears naturally as a statement about homotopy groups and thus about the genuine fixed point functors characterized by the standard adjunctions, it is worth observing that it implies a version in terms of these Φ-fixed point spectra.

THEOREM 6.4. *A map $f : E \longrightarrow E'$ of G-spectra is an equivalence if and only if each $\Phi^H f : \Phi^H E \longrightarrow \Phi^H E'$ is a nonequivariant equivalence.*

Note that, for any family $\mathscr{F}$ and any G-spectra E and E',

$$[E \wedge E\mathscr{F}_+, E' \wedge \tilde{E}\mathscr{F}]_G = 0$$

since $E\mathscr{F}$ only has cells of orbit type G/H with $H \in \mathscr{F}$ and $\tilde{E}\mathscr{F}$ is H-contractible for such H. Therefore the canonical G-map $E \longrightarrow E \wedge \tilde{E}\mathscr{F}$ induces an isomorphism

$$[E \wedge \tilde{E}\mathscr{F}, E' \wedge \tilde{E}\mathscr{F}]_G \cong [E, E' \wedge \tilde{E}\mathscr{F}]_G. \tag{6.5}$$

In the case of $\mathscr{F}[N]$, $E \longrightarrow E \wedge \tilde{E}\mathscr{F}[N]$ is an equivalence if and only if E is concentrated over N, in the sense that E is H-contractible if H does not contain N. Maps into such G-spectra determine and are determined by the J-maps obtained by passage to Φ^N-fixed point spectra. In fact, the stable category of J-spectra is equivalent to the full subcategory of the stable category of G-spectra consisting of the G-spectra concentrated over N.

THEOREM 6.6. *For J-spectra $D \in J\mathscr{S}U^N$ and G-spectra $E \in G\mathscr{S}U$ concentrated over N, there is a natural isomorphism*

$$[D, E^N]_J \cong [i_*D \wedge \tilde{E}\mathscr{F}[N], E]_G.$$

For J-spectra D and D', the functor $i_*(-) \wedge \tilde{E}\mathscr{F}[N]$ induces a natural isomorphism

$$[D, D']_J \cong [i_*D \wedge \tilde{E}\mathscr{F}[N], i_*D \wedge \tilde{E}\mathscr{F}[N]]_G.$$

For general G-spectra E and E', the functor $\Phi^N(-)$ induces a natural isomorphism

$$[\Phi^N E, \Phi^N E']_J \cong [E, E' \wedge \tilde{E}\mathscr{F}[N]]_G.$$

PROOF. The first isomorphism is a consequence of (5.1) and (6.5). The other two isomorphisms follow once one shows that the unit

$$D \longrightarrow (i_*D \wedge \tilde{E}\mathscr{F}[N])^N = \Phi^N(i_*D)$$

and counit

$$(i_*E^N) \wedge \tilde{E}\mathscr{F}[N] \longrightarrow E$$

of the adjunction are equivalences. One proves this by use of a spacewise N-fixed point functor, also denoted Φ^N, from G-prespectra to J-prespectra. This functor is defined exactly as was the spacewise G-fixed point functor in Section 3. It satisfies $\Phi^N(i_*D) = D$, and it commutes with smash products. The following generalization of Theorem 3.4, which shows that the prespectrum level functor Φ^N induces a functor equivalent to Φ^N on the spectrum level, leads to the conclusion. □

THEOREM 6.7. *For Σ-cofibrant G-prespectra D, there is a natural weak equivalence of J-spectra*

$$\Phi^N LD \longrightarrow L\Phi^N D.$$

For G-CW spectra E, there is a natural weak equivalence of J-spectra

$$\Phi^N E \longrightarrow L\Phi^N K\Gamma\ell E.$$

As an illuminating example of the use of $RO(G)$-grading to allow calculational descriptions invisible to the $\mathbb{Z}$-graded part of a theory, we record how to compute the cohomology theory represented by $\Phi^N(E)$ in terms of the cohomology theory represented by E. This uses the Euler classes of representations, which appear ubiquitously in equivariant theory. For a representation V, we define $e(V) \in E^V_G(S^0)$ to be the image of $1 \in E^0_G(S^0) \cong E^V_G(S^V)$ under e^*, where $e : S^0 \longrightarrow S^V$ sends the basepoint to the point at ∞ and the non-basepoint to 0.

PROPOSITION 6.8. *Let E be a ring G-spectrum. For a finite J-CW spectrum X, $(\Phi^N E)^*_J(X)$ is the localization of $E^*_G(X)$ obtained by inverting the Euler classes of all representations V such that $V^N = \{0\}$.*

PROOF. By (6.3), $\Phi^N(E)$ inherits a ring structure from E. In interpreting the grading, we regard representations of J as representations of G by pullback. A check of fixed points, using the cofibrations $S(V) \longrightarrow B(V) \longrightarrow S^V$, shows that we obtain a model for $\tilde{E}\mathscr{F}[N]$ by taking the colimit of the spaces S^V as V ranges over the representations of G such that $V^N = \{0\}$. This leads to a colimit description of $(\Phi^N E)^*_J(X)$ that coincides algebraically with the cited localization. □

With motivation from the last few results, the unfortunate alternative notation $E_J = \Phi^N(E_G)$ was used in [LMS] and elsewhere. This is a red herring from the point of view of Theorem 5.6, and it is ambiguous on two accounts. First, the J-spectrum $\Phi^N(E_G)$ depends vitally on the extension $J = G/N$ and not just on the group J. Second, in classical examples, the spectrum "E_J" will generally not agree with the preassigned spectrum with the same notation. For example, the subquotient J-spectrum "K_J" associated to the K-theory G-spectrum K_G is not the K-theory J-spectrum K_J. However, if S_G is the sphere G-spectrum, then the subquotient J-spectrum S_J is the sphere J-spectrum. We shall see that this easy fact plays a key conceptual role in Carlsson's proof of the Segal conjecture.

[LMS, II§9]

7. Spanier-Whitehead duality

We can develop abstract duality theory in any symmetric monoidal category, such as $\bar{h}G\mathscr{S}$ for our fixed complete G-universe U. While the elegant approach is to start from the abstract context, we shall specialize to $\bar{h}G\mathscr{S}$ from the start since we wish to emphasize equivariant phenomena. Define the dual of a G-spectrum X to be $DX = F(X, S)$ and note that there is a natural evaluation map $\varepsilon : (DX) \wedge X \longrightarrow S$. There is also a natural map

(7.1) $$\nu : F(X,Y) \wedge Z \longrightarrow F(X, Y \wedge Z).$$

Using the unit isomorphism, it specializes to

(7.2) $$\nu : (DX) \wedge X \longrightarrow F(X,X).$$

The adjoint of the unit isomorphism $S \wedge X \longrightarrow X$ gives a natural map $\eta : S \longrightarrow F(X,X)$. We say that X is "strongly dualizable" if there is a coevaluation map $\eta : S \longrightarrow X \wedge (DX)$ such that the following diagram commutes, where γ is the commutativity isomorphism.

(7.3) $$\begin{tikzcd} S \arrow[r, "\eta"] \arrow[d, "\eta"'] & X \wedge (DX) \arrow[d, "\gamma"] \\ F(X,X) & (DX) \wedge X \arrow[l, "\nu"] \end{tikzcd}$$

It is a categorical implication of the definition that the map ν of (7.1) is an isomorphism if either X or Z is strongly dualizable, and there are various other such formal consequences, such as $X \cong DD(X)$ when X is strongly dualizable. In particular, if X is strongly dualizable, then the map ν of (7.2) is an isomorphism.

Conversely, if the map ν of (7.2) is an isomorphism, then X is strongly dualizable since the coevaluation map η can and must be defined to be the composite $\gamma\nu^{-1}\eta$ in (7.3).

THEOREM 7.4. *A G-CW spectrum is strongly dualizable if and only if it is equivalent to a wedge summand of a finite G-CW spectrum.*

PROOF. The evaluation map of X induces a natural map

$$(*)\qquad \varepsilon_{\#} : [Y, Z \wedge DX]_G \longrightarrow [Y \wedge X, Z]_G$$

via $\varepsilon_{\#}(f) = (\mathrm{id} \wedge \varepsilon)(f \wedge \mathrm{id})$, and X is strongly dualizable if and only if $\varepsilon_{\#}$ is an isomorphism for all Y and Z. The Wirthmüller isomorphism implies that $D(\Sigma^\infty G/H_+)$ is equivalent to $G \ltimes_H S^{-L(H)}$, and diagram chases show that it also implies that $\varepsilon_{\#}$ is an isomorphism. Actually, this duality on orbits is the heart of the Wirthmüller isomorphism, and we shall explain it in direct geometric terms in the next section. If X is strongly dualizable, then so is ΣX. The cofiber of a map between strongly dualizable G-spectra is strongly dualizable since both sides of $(*)$ turn cofibrations in X into long exact sequences. By induction on the number of cells, a finite G-CW spectrum is strongly dualizable, and it is formal that a wedge summand of a strongly dualizable G-spectrum is strongly dualizable. For the converse, which was conjectured in [LMS] and proven by Greenlees (unpublished), let X be a strongly dualizable G-CW spectrum with coevaluation map η. Then η factors through $A \wedge DX$ for some finite subcomplex A of X, the following diagram commutes, and its bottom composite is the identity:

$$\begin{tikzcd}
 & A \wedge (DX) \wedge X \arrow[r, "\mathrm{id}\wedge\varepsilon"] \arrow[d] & A \wedge S \cong A \arrow[d] \\
X \cong S \wedge X \arrow[ur] \arrow[r, "\eta\wedge\mathrm{id}"'] & X \wedge (DX) \wedge X \arrow[r, "\mathrm{id}\wedge\varepsilon"'] & X \wedge S \cong X.
\end{tikzcd}$$

Therefore X is a retract up to homotopy and thus a wedge summand up to homotopy of A. $\square$

In contrast to the nonequivariant case, wedge summands of finite G-CW spectra need not be equivalent to finite G-CW spectra.

COROLLARY 7.5 (SPANIER-WHITEHEAD DUALITY). *If X is a wedge summand of a finite G-CW spectrum and E is any G-spectrum, then*

$$\nu : DX \wedge E \longrightarrow F(X, E)$$

is an isomorphism in $\bar{h}G\mathscr{S}$. Therefore, for any representation α,

$$E_\alpha^G(DX) \cong E_G^{-\alpha}(X).$$

So far, we have concentrated on the naturally given dual DX. However, it is important to identify the homotopy types of duals concretely, as we did in the case of orbits. There are a number of equivalent criteria. The most basic one goes as follows.

THEOREM 7.6. *Suppose given G-spectra X and Y and maps*

$$\varepsilon : Y \wedge X \longrightarrow S \quad \textit{and} \quad \eta : S \longrightarrow X \wedge Y$$

such that the composites

$$X \cong S \wedge X \xrightarrow{\eta \wedge \mathrm{id}} X \wedge Y \wedge X \xrightarrow{\mathrm{id} \wedge \varepsilon} X \wedge S \cong X$$

and

$$Y \cong Y \wedge S \xrightarrow{\mathrm{id} \wedge \eta} Y \wedge X \wedge Y \xrightarrow{\varepsilon \wedge \mathrm{id}} S \wedge Y \cong Y$$

are the respective identity maps. Then the adjoint $\tilde{\varepsilon} : Y \longrightarrow DX$ of ε is an equivalence and X is strongly dualizable with coevaluation map $(\mathrm{id} \wedge \tilde{\varepsilon})\eta$.

It is important to note that the maps η and ε that display the duality are not unique, although each determines the other. Thus a duality between X and Y should be interpreted as a choice of duality maps ε and η — much of the literature on duality is quite sloppy about this point.

This criterion admits a homological interpretation, but we will not go into that here. It entails a reinterpretation in terms of the slant products relating homology and cohomology that we defined in XIII§5, and it works in the same way equivariantly as nonequivariantly.

[LMS, III§§1-3]

8. V-duality of G-spaces and Atiyah duality

There is a concrete space level version of the duality criterion just given. To describe it, let X and Y be G-spaces and let V be a representation of G. Suppose given G-maps

$$\varepsilon : Y \wedge X \longrightarrow S^V \quad \text{and} \quad \eta : S^V \longrightarrow X \wedge Y$$

such that the following diagrams are stably homotopy commutative, where $\sigma : S^V \longrightarrow S^V$ is the sign map, $\sigma(v) = -v$, and the γ are transpositions.

$$\begin{tikzcd} S^V \wedge X \arrow[r, "\eta \wedge \mathrm{id}"] \arrow[dr, "\gamma"'] & X \wedge Y \wedge X \arrow[d, "\mathrm{id} \wedge \varepsilon"] \\ & X \wedge S^V \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \wedge S^V \arrow[r, "\mathrm{id} \wedge \eta"] \arrow[d, "\gamma"'] & Y \wedge X \wedge Y \arrow[d, "\varepsilon \wedge \mathrm{id}"] \\ S^V \wedge Y \arrow[r, "\sigma \wedge \mathrm{id}"'] & S^V \wedge Y. \end{tikzcd}$$

On application of the functor Σ^∞_V, we find that $\Sigma^\infty X$ and $\Sigma^\infty_V Y$ are strongly dualizable and dual to one another by our spectrum level criterion.

For reasonable X and Y, say finite G-CW complexes, or, more generally, compact G-ENR's (ENR = Euclidean neighborhood retract), we can use the space level equivariant suspension and Whitehead theorems to prove that a pair of G-maps (ε, η) displays a V-duality between X and Y, as above, if and only if the fixed point pair (ε^H, η^H) displays an $n(H)$-duality between X^H and Y^H for each $H \subset G$, where $n(H) = dim(V^H)$.

If X is a compact G-ENR, then X embeds as a retract of an open set of a G-representation V. One can use elementary space level methods to construct an explicit V-duality between X_+ and the unreduced mapping cone $V \cup C(V - X)$.

For a G-cofibration $A \longrightarrow X$, there is a relative version that constructs a V-duality between $X \cup CA$ and $(V - A) \cup C(V - X)$. The argument specializes to give an equivariant version of the Atiyah duality theorem, via precise duality maps. Recall that the Thom complex of a vector bundle is obtained by fiberwise one-point compactification followed by identification of the points at infinity. When the base space is compact, this is just the one-point compactification of the total space.

THEOREM 8.1 (ATIYAH DUALITY). *If M is a smooth closed G-manifold embedded in a representation V with normal bundle ν, then M_+ is V-dual to the Thom complex $T\nu$. If M is a smooth compact G-manifold with boundary ∂M, $V = V' \oplus \mathbb{R}$, and $(M, \partial M)$ is properly embedded in $(V' \times [0,\infty), V' \times \{0\})$ with normal bundles ν' of ∂M in V' and ν of M in V, then $M/\partial M$ is V-dual to $T\nu$, M_+ is V-dual to $T\nu/T\nu'$, and the cofibration sequence*

$$T\nu' \longrightarrow T\nu \longrightarrow T\nu/T\nu' \longrightarrow \Sigma T\nu'$$

is V-dual to the cofibration sequence

$$\Sigma(\partial M)_+ \longleftarrow M/\partial M \longleftarrow M_+ \longleftarrow (\partial M)_+.$$

We display the duality maps explicitly in the closed case. By the equivariant tubular neighborhood theorem, we may extend the embedding of M in V to an embedding of the normal bundle ν and apply the Pontrjagin-Thom construction to obtain a map $t : S^V \longrightarrow T\nu$. The diagonal map of the total space of ν induces the Thom diagonal $\Delta : T\nu \longrightarrow M_+ \wedge T\nu$. The map η is just $\Delta \circ t$. The map ε is equally explicit but a bit more complicated to describe. Let $s : M \longrightarrow \nu$ be the zero section. The composite of $\Delta : M \longrightarrow M \times M$ and $s \times \mathrm{id} : M \times M \longrightarrow \nu \times M$ is an embedding with trivial normal bundle. The Pontrjagin-Thom construction gives a map $t : T\nu \wedge M_+ \longrightarrow M_+ \wedge S^V$. Let $\xi : M_+ \longrightarrow S^0$ collapse all of M to the non-basepoint. The map ε is just $(\xi \wedge \mathrm{id}) \circ t$. This explicit construction implies that the maps $\xi : M_+ \longrightarrow S^0$ and $t : S^V \longrightarrow T\nu$ are dual to one another.

Let us specialize this discussion to orbits G/H (compare IX.3.4). Recall that $L = L(H)$ is the tangent H-representation at the identity coset of G/H. We have

$$\tau = G \times_H L(H) \quad \text{and} \quad T\tau = G_+ \wedge_H S^{L(H)}.$$

If G/H is embedded in V with normal bundle ν, then $\nu \oplus \tau$ is the trivial bundle $G/H \times V$. Let W be the orthogonal complement to $L(H)$ in the fiber over the identity coset, so that $V = L \oplus W$ as an H-space. Since G/H_+ is V-dual to $T\nu$, $\Sigma^\infty G/H_+$ is dual to $\Sigma^\infty_V T\nu$. Since $S^W \wedge S^{-V} \simeq S^{-L}$ as H-spectra, we find that $\Sigma^\infty_V T\nu \simeq G \ltimes_H S^{-L}$.

[LMS, III§§3-5]

9. Poincaré duality

Returning to general smooth G-manifolds, we can deduce an equivariant version of the Poincaré duality theorem by combining Spanier-Whitehead duality, Atiyah duality, and the Thom isomorphism.

DEFINITION 9.1. Let E be a ring G-spectrum and let ξ be an n-plane G-bundle over a G-space X. An E-orientation of ξ is an element $\mu \in E_G^\alpha(T\xi)$ for some $\alpha \in RO(G)$ of virtual dimension n such that, for each inclusion $i : G/H \longrightarrow X$, the restriction of μ to the Thom complex of the pullback $i^*\xi$ is a generator of the free $E_H^*(S^0)$-module $E_G^*(Ti^*\xi)$.

Here $i^*\xi$ has the form $G \times_H W$ for some representation W of H and $Ti^*\xi = G_+ \wedge S^W$ has cohomology $E_G^*(Ti^*\xi) \cong E_H^*(S^W) \cong E_H^{*-w}(S^0)$. Thus the definition makes sense, but it is limited in scope. If X is G-connected, then there is an obvious preferred choice for α, namely the fiber representation V at any fixed point of X: each W will then be isomorphic to V regarded as a representation of H. In general, however, there is no preferred choice for α and the existence of an orientation implies restrictions on the coefficients $E_H^*(S^0)$: there must be units in degree $\alpha - w \in RO(H)$. If $\alpha \neq w$, this forces a certain amount of periodicity in the theory. There is a great deal of further work, largely unpublished, by Costenoble, Waner, Kriz, and myself in the area of orientation theory and Poincaré duality, but the full story is not yet in place. Where it applies, the present definition does have the expected consequences.

THEOREM 9.2 (THOM ISOMORPHISM). *Let $\mu \in E_G^\alpha(T\xi)$ be an orientation of the G-vector bundle ξ over X. Then*

$$\cup\mu : E_G^\beta(X_+) \longrightarrow E_G^{\alpha+\beta}(T\xi)$$

is an isomorphism for all β.

There is also a relative version. Specializing to oriented manifolds, we obtain the Poincaré duality theorem as an immediate consequence. Observe first that, for bundles ξ and η over X, the diagonal map of X induces a canonical map

$$T(\xi \oplus \eta) \longrightarrow T(\xi \times \eta) \cong T\xi \wedge T\eta.$$

There results a pairing

$$(*) \qquad E_G^\alpha(T\xi) \otimes E_G^\beta(T\eta) \longrightarrow E_G^{\alpha+\beta}(T(\xi \oplus \eta)).$$

We say that a smooth compact G-manifold M is E-oriented if its tangent bundle τ is oriented, say via $\mu \in E_G^\alpha(T\tau)$. In view of our discussion above, this makes most sense when M is a V-manifold and we take α to be V. If M has boundary, the smooth boundary collar theorem shows that the normal bundle of ∂M in M is trivial, and we deduce that an orientation of M determines an orientation $\partial\mu$ of ∂M in degree $\alpha - 1$ such that, under the pairing $(*)$, the product of $\partial\mu$ and the canonical orientation $\iota \in E_G^1(\Sigma(\partial M)_+)$ of the normal bundle is the restriction of μ to $T(\tau|\partial M)$. Similarly, if M is embedded in V, then μ determines an orientation ω of the normal bundle ν such that the product of μ and ω is the canonical orientation of the trivial bundle in $E_G^v(\Sigma^V M_+)$.

DEFINITION 9.3 (POINCARÉ DUALITY). If M is a closed E-oriented smooth G-manifold with orientation $\mu \in E_G^\alpha(T\tau)$, then the composite

$$D : E_G^\beta(M_+) \longrightarrow E_G^{V-\alpha+\beta}(T\nu) \longrightarrow E^G_{\alpha-\beta}(M)$$

of the Thom and Spanier-Whitehead duality isomorphisms is the Poincaré duality isomorphism; the element $[M] = D(1)$ in $E^G_\alpha(M)$ is called the fundamental class associated to the orientation. If M is a compact E-oriented smooth G-manifold with boundary, then the analogous composites

$$D : E^\beta_G(M_+) \longrightarrow E^{V-\alpha+\beta}_G(T\nu) \longrightarrow E^G_{\alpha-\beta}(M, \partial M)$$

and

$$D : E^\beta_G(M, \partial M) \longrightarrow E^{V-\alpha+\beta}_G(T\nu, T(\nu|\partial M)) \longrightarrow E^G_{\alpha-\beta}(M)$$

are called the relative Poincaré duality isomorphisms. With the Poincaré duality isomorphism for ∂M, they specify an isomorphism from the cohomology long exact sequence to the homology long exact sequence of $(M, \partial M)$. Here the element $[M] = D(1)$ in $E^G_\alpha(M, \partial M)$ is called the fundamental class associated to the orientation.

One can check that these isomorphisms are given by capping with the fundamental class, as one would expect.

S. R. Costenoble, J. P. May, and S. Waner. Equivariant orientation theory. Preprint.

S. R. Costenoble and S. Waner. Equivariant Poincaré duality. Michigan Math. J. 39(1992).

[LMS, III§6]