A couple of notes to: Tim D. Cochran, Kent E. Orr, and Peter Teichner, "Knot concordance, Whitney towers and  $L^2$ -signatures", henceforth [COT99]. Absolutely nothing in these pages is original—except possibly the errors everything is already contained (more or less explicitly) in [COT99].

## 1 Poly-torsion-free-abelian groups

Let M denote the zero-framed surgery on a given knot  $K \subset S^3$ . The following properties are fulfilled by  $\Gamma = \mathbb{Z}$ :

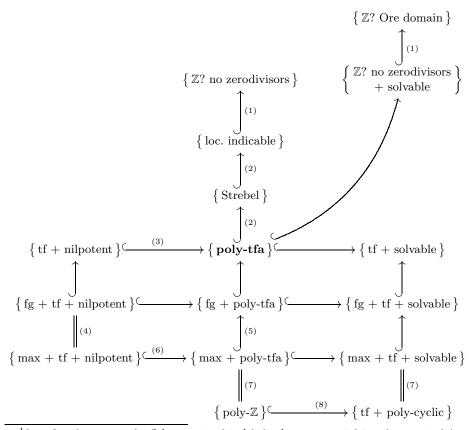
a. there are interesting maps  $\phi: M \to B\Gamma$ ;

b. the integral group ring  $\mathbb{Z}[\Gamma]$  has a skew field of quotients  $\mathcal{K}\Gamma$ ;

c.  $C_*(\tilde{M}) \otimes_{\phi} \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma]} \mathcal{K}\Gamma$  is contractible (as a  $\mathcal{K}\Gamma$ -chain complex).

These properties are also satisfied by any poly-torsion-free-abelian group (polytfa here, PTFA in [COT99, see definition 2.1] **Comment:** a subnormal series should be throughout enough). This is shown in [COT99]: a. in section 3; b. in proposition 2.5—see also below; c. follows from propositions 2.9, 2.11.

Where do poly-tfa groups live? We have the following picture:<sup>1</sup>



 $<sup>{}^{1}</sup>$ fg = finitely generated; tf[a] = torsion-free [abelian]; max = satisfying the maximal (or equivalently the ascending chain) condition on subgroups.

We say that a group  $\Gamma$  has Strebel's property [Str74, definition 1.1 of the class D] if any map  $f: A \to B$  between projective  $R[\Gamma]$ -modules, for which  $f \otimes_{R[\Gamma]} \operatorname{id}_R: A \otimes_{R[\Gamma]} R \to B \otimes_{R[\Gamma]} R$  is injective (R any commutative unital ring), is itself injective. (Note that in this case also  $f \otimes_{\mathbb{Z}[\Gamma]} \operatorname{id}_{\mathcal{K}\Gamma}: A \otimes_{\mathbb{Z}[\Gamma]} \mathcal{K}\Gamma \to B \otimes_{\mathbb{Z}[\Gamma]} \mathcal{K}\Gamma$  is injective.)

All the inclusions in the diagram above that are indicated by hooked arrows are proper. Some remarks are perhaps in order.

- (1) These are proved e.g. in [Pas77, chapter 13, see in particular lemma 3.6.iii, theorem 1.11, and exercise 9.ii].
- (2) These in [Str74, proposition 1.5, corollary 1.8, and proposition 1.9].
- (3) Follows from: if G is any group with tf center  $\mathcal{Z}G$ , then every upper central factor  $\mathcal{Z}_{n+1}G/\mathcal{Z}_nG$  is tf.
- (4) Follows from: if G is nilpotent and G/[G, G] is fg, then G satisfies max.
- (5) To show that this inclusion is proper consider  $H := \{mp^n \mid m, n \in \mathbb{Z}\} < \mathbb{Q}$  for a fixed prime p. H is tfa but not fg. Define  $\eta : H \to H$ ,  $q \mapsto pq$ . Now construct the semidirect product  $G := H \rtimes_{\eta} \mathbb{Z}$ . G is then poly-tfa and fg, but since H does not satisfy max, G cannot satisfy max.
- (6) Examples showing that the inclusion cannot be reversed might be constructed observing the following easy fact: in any nilpotent group, any nontrivial normal subgroup contains a nontrivial central element. Therefore, any nontrivial semidirect product will do the job, for example  $H := \langle x, t | x^t = x^{-1} \rangle \cong \mathbb{Z} \rtimes \mathbb{Z}$ .
- (7)  $\{\max + \text{poly-}\mathcal{P}\} = \{\max + \text{poly-}(\max + \mathcal{P})\} = \{\text{poly-}(\max + \mathcal{P})\}\$  for any group-theoretic property  $\mathcal{P}$ ,  $\{\max + \text{tfa}\} = \{\text{fg} + \text{tfa}\} = \{\text{poly-}\mathbb{Z}\}\$ , and finally  $\{\text{poly-}\mathbb{Z}\} = \{\text{poly-poly-}\mathbb{Z}\}\$ , of course; similarly one can show that the following equality holds:  $\{\max + \text{solvable}\} = \{\text{poly-cyclic}\}\$ .
- (8) The group  $G := \langle x, y, t | x^t = x^{-1}, y^t = y^{-1}, [x, y] = t^4 \rangle$  is tf and polycyclic, fitting in the extension  $1 \to H \to G \to \mathbb{Z}/2 \to 1$ , where H is the group defined in (6), but not poly- $\mathbb{Z}$  since G/[G, G] is finite. G is not locally indicable, and this shows also that not all solvable groups have Strebel's property.

## 2 L-theoretic invariants & signatures

Let M denote the zero-framed surgery on a given knot  $K \subset S^3$ . Assume there is a map  $\phi: M \to B\Gamma$  where  $\Gamma$  is a poly-tfa group. In this case there is a skew field of quotients  $\mathcal{K}\Gamma$  for the integral group ring  $\mathbb{Z}[\Gamma]$  and  $C_*(\tilde{M}) \otimes_{\phi} \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma]} \mathcal{K}\Gamma$  is contractible (see section 1).

This implies that  $\sigma(M,\phi) = (C_*(\tilde{M}) \otimes_{\phi} \mathbb{Z}[\Gamma],\varphi) \in L^3(\mathbb{Z}[\Gamma])$  lifts to an element  $\overline{\sigma}(M,\phi) \in L^4(\mathbb{Z}[\Gamma] \subset \mathcal{K}\Gamma)$ :

$$\cdots \longrightarrow L^4(\mathbb{Z}[\Gamma]) \xrightarrow{i_*} L^4(\mathcal{K}\Gamma) \xrightarrow{\delta} L^4(\mathbb{Z}[\Gamma] \subset \mathcal{K}\Gamma) \xrightarrow{\partial} L^3(\mathbb{Z}[\Gamma]) \longrightarrow \cdots$$

If 
$$[M, \phi] = 0 \in \Omega_3(B\Gamma)$$
 then

$$\overline{\sigma}(M,\phi) \in \ker \partial = \operatorname{im} \delta \cong L^4(\mathcal{K}\Gamma)/i_*(L^4(\mathbb{Z}[\Gamma])).$$

Under this identification  $\overline{\sigma}(M, \phi)$  corresponds to the element  $B(M, \phi)$  constructed in [COT99, section 4]. The independence of this element of the choice of the bounding manifold is a very special case of the localization long exact sequence in *L*-theory: see [Ran81, chapter 3], and also [Ran92, example 3.13].

Consider an intermediate ring  $\mathbb{Z}[\Gamma] \subset \mathcal{R}\Gamma \subset \mathcal{K}\Gamma$ :

If  $\mathcal{R}\Gamma$  is a PID then the image of  $\overline{\sigma}(M,\phi)$  in  $L^4(\mathcal{R}\Gamma \subset \mathcal{K}\Gamma)$  corresponds to the class of the higher order Blanchfield linking form  $B\ell(M,\phi)$  constructed in [COT99, theorem 2.13], under the isomorphism described in [Ran81, section 3.4].

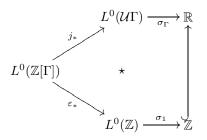
We recall now two deep results of [COT99].

**Theorem ([COT99, theorems 4.2 and 4.4]).** If  $\Gamma$  is a poly-tfa  $\begin{cases} n \\ n-1 \end{cases}$ -step solvable group and M as above is rationally  $\begin{cases} n.5 \\ n \end{cases}$ -solvable via a 4-manifold W over which  $\phi$  extends, then  $\begin{cases} \overline{\sigma}(M,\phi) = 0 \in L^4(\mathbb{Z}[\Gamma] \subset \mathcal{K}\Gamma) \\ B\ell(M,\phi) = 0 \in L^4(\mathcal{R}\Gamma \subset \mathcal{K}\Gamma) \end{cases}$ .

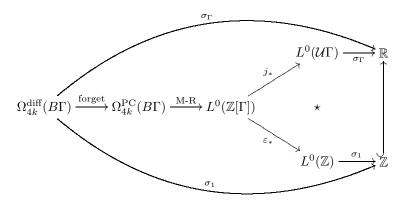
Next we consider the numerical invariants—signatures—used to detect this obstructions.

**Theorem ([COT99, proposition 5.12]).** Suppose that  $\Gamma$  is a poly-tfa group, or more generally a torsionfree amenable group. Then the  $L^2$ -signature and the

ordinary signature coincide on  $L^0(\mathbb{Z}[\Gamma])$ , i.e., the following diagram commutes.



Before the proof, let us point out that in general  $\star$  does not commute. Indeed it fits into the bigger diagram:



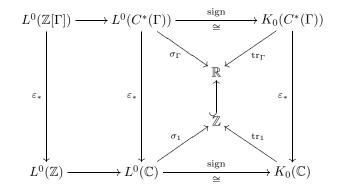
where the outer triangle commutes—this follows from Atiyah's  $L^2$ -index theorem for the signature operator. But Wall produces an example of a Poincaré complex (which does not carry any smooth structure) whose ordinary signature is not multiplicative under finite coverings, hence cannot coincide with the  $L^2$ signature—and this shows in particular the noncommutativity of  $\star$ . (Since the  $L^2$ -signature for smooth manifolds is multiplicative under finite coverings, the  $L^2$ -index theorem implies that the same also holds for the ordinary signature of smooth manifolds.)

*Proof.* Since  $\Gamma$  is amenable,  $C^*_{red}(\Gamma) \cong C^*_{max}(\Gamma) =: C^*(\Gamma)$ , and the augmentation  $\varepsilon : \mathbb{Z}[\Gamma] \to \mathbb{Z}$  induces a map  $\varepsilon : C^*(\Gamma) \to \mathbb{C}$ . We have the following diagram:

$$\begin{array}{ccc} K_0(B\Gamma) & \xrightarrow{A_{\Gamma}} & K_0(C^*(\Gamma)) & \xrightarrow{\operatorname{tr}_{\Gamma}} & \mathbb{R} \\ & & & & \\ &$$

where  $A_{\Gamma}$  is the analytic assembly map (Baum-Connes), which is known to be an isomorphism under our assumptions, namely for torsionfree amenable groups. The outer rectangle commutes, again by Atiyah's  $L^2$ -index theorem. Since  $A_{\Gamma}$  is onto the right-hand square also commutes.

The claim now follows from the diagram:



where we find  $\star$  and everything commutes.

**Comment:** Note that for the argument above we need to use the maximal group  $C^*$ -algebra, and therefore we assume (K-)amenability of  $\Gamma$ . Is it possible to avoid this? Is surjectivity of the analytic assembly map  $A_{\Gamma} \colon K_0(B\Gamma) \to K_0(C^*_{\mathsf{red}}(\Gamma))$  to the K-theory of the reduced group  $C^*$ -algebra enough?

Corollary. In the situation of the theorem above the diagram

induces a homomorphism  $\mu \colon L^0(\mathcal{K}\Gamma)/i_*(L^0(\mathbb{Z}[\Gamma])) \to \mathbb{R}/\mathbb{Z}.$ 

**Comment:** Can one define  $\tilde{\mu}$ :  $L^0(\mathcal{K}\Gamma)/i_*(L^0(\mathbb{Z}[\Gamma])) \to \mathbb{R}$ ?

## References

- [COT99] Tim D. Cochran, Kent E. Orr, and Peter Teichner. Knot concordance, Whitney towers and  $L^2$ -signatures. math.GT/9908117, preprint, 1999.
- [Pas77] Donald S. Passman. The algebraic structure of group rings. Wiley-Interscience [John Wiley & Sons], New York, 1977. Pure and Applied Mathematics.
- [Ran81] Andrew Ranicki. Exact sequences in the algebraic theory of surgery. Princeton University Press, Princeton, N.J., 1981.
- [Ran92] Andrew Ranicki. Algebraic L-theory and topological manifolds. Cambridge University Press, Cambridge, 1992.
- [Str74] Ralph Strebel. Homological methods applied to the derived series of groups. *Comment. Math. Helv.*, 49:302–332, 1974.

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