A couple of notes to: Tim D. Cochran, Kent E. Orr, and Peter Teichner, “Knot concordance, Whitney towers and L^2-signatures”, henceforth [COT99]. Absolutely nothing in these pages is original—except possibly the errors—everything is already contained (more or less explicitly) in [COT99].

1 Poly-torsion-free-abelian groups

Let M denote the zero-framed surgery on a given knot $K \subset S^3$. The following properties are fulfilled by $\Gamma = \mathbb{Z}$:

a. there are interesting maps $\phi\colon M \to B\Gamma$;

b. the integral group ring $\mathbb{Z}[\Gamma]$ has a skew field of quotients $\mathcal{K}\Gamma$;

c. $C_*(\tilde{M}) \otimes_\phi \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma]} \mathcal{K}\Gamma$ is contractible (as a $\mathcal{K}\Gamma$-chain complex).

These properties are also satisfied by any poly-torsion-free-abelian group (poly-tfa here, PTFA in [COT99, see definition 2.1] **Comment:** a subnormal series should be throughout enough). This is shown in [COT99]: a. in section 3; b. in proposition 2.5—see also below; c. follows from propositions 2.9, 2.11.

Where do poly-tfa groups live? We have the following picture:[1]

$$\begin{tikzcd}
 & & \{\mathbb{Z}?\ \text{Ore domain}\} \\
 & \{\mathbb{Z}?\ \text{no zerodivisors}\} & \left\{\begin{array}{c}\mathbb{Z}?\ \text{no zerodivisors}\\ +\ \text{solvable}\end{array}\right\} \arrow[u, hook, "(1)"'] \\
 & \{\text{loc. indicable}\} \arrow[u, hook, "(1)"'] & \\
 & \{\text{Strebel}\} \arrow[u, hook, "(2)"'] & \\
\{\text{tf} + \text{nilpotent}\} \arrow[r, hook, "(3)"] & \{\textbf{poly-tfa}\} \arrow[u, hook, "(2)"'] \arrow[r, hook] \arrow[uuur, hook, bend right] & \{\text{tf} + \text{solvable}\} \\
\{\text{fg} + \text{tf} + \text{nilpotent}\} \arrow[u, hook] \arrow[r, hook] & \{\text{fg} + \text{poly-tfa}\} \arrow[u, hook] \arrow[r, hook] & \{\text{fg} + \text{tf} + \text{solvable}\} \arrow[u, hook] \\
\{\text{max} + \text{tf} + \text{nilpotent}\} \arrow[u, equal, "(4)"'] \arrow[r, hook, "(6)"] & \{\text{max} + \text{poly-tfa}\} \arrow[u, hook, "(5)"'] \arrow[r, hook] & \{\text{max} + \text{tf} + \text{solvable}\} \arrow[u, hook] \\
 & \{\text{poly-}\mathbb{Z}\} \arrow[u, equal, "(7)"'] \arrow[r, hook, "(8)"] & \{\text{tf} + \text{poly-cyclic}\} \arrow[u, equal, "(7)"']
\end{tikzcd}$$

[1] fg = finitely generated; tf[a] = torsion-free [abelian]; max = satisfying the maximal (or equivalently the ascending chain) condition on subgroups.

We say that a group Γ has Strebel's property [Str74, definition 1.1 of the class D] if any map $f\colon A \to B$ between projective $R[\Gamma]$-modules, for which $f \otimes_{R[\Gamma]} \mathrm{id}_R\colon A \otimes_{R[\Gamma]} R \to B \otimes_{R[\Gamma]} R$ is injective (R any commutative unital ring), is itself injective. (Note that in this case also $f \otimes_{\mathbb{Z}[\Gamma]} \mathrm{id}_{\mathcal{K}\Gamma}\colon A \otimes_{\mathbb{Z}[\Gamma]} \mathcal{K}\Gamma \to B \otimes_{\mathbb{Z}[\Gamma]} \mathcal{K}\Gamma$ is injective.)

All the inclusions in the diagram above that are indicated by hooked arrows are proper. Some remarks are perhaps in order.

(1) These are proved e.g. in [Pas77, chapter 13, see in particular lemma 3.6.iii, theorem 1.11, and exercise 9.ii].

(2) These in [Str74, proposition 1.5, corollary 1.8, and proposition 1.9].

(3) Follows from: if G is any group with tf center $\mathcal{Z}G$, then every upper central factor $\mathcal{Z}_{n+1}G/\mathcal{Z}_nG$ is tf.

(4) Follows from: if G is nilpotent and $G/[G,G]$ is fg, then G satisfies max.

(5) To show that this inclusion is proper consider $H := \{\, mp^n \mid m,n \in \mathbb{Z} \,\} < \mathbb{Q}$ for a fixed prime p. H is tfa but not fg. Define $\eta\colon H \to H\ ,\ q \mapsto pq$. Now construct the semidirect product $G := H \rtimes_\eta \mathbb{Z}$. G is then poly-tfa and fg, but since H does not satisfy max, G cannot satisfy max.

(6) Examples showing that the inclusion cannot be reversed might be constructed observing the following easy fact: in any nilpotent group, any nontrivial normal subgroup contains a nontrivial central element. Therefore, any nontrivial semidirect product will do the job, for example $H := \langle\, x,t \mid x^t = x^{-1} \,\rangle \cong \mathbb{Z} \rtimes \mathbb{Z}$.

(7) $\{\, \text{max} + \text{poly-}\mathcal{P} \,\} = \{\, \text{max} + \text{poly-}(\text{max} + \mathcal{P}) \,\} = \{\, \text{poly-}(\text{max} + \mathcal{P}) \,\}$ for any group-theoretic property $\mathcal{P}$, $\{\, \text{max} + \text{tfa} \,\} = \{\, \text{fg} + \text{tfa} \,\} = \{\, \text{poly-}\mathbb{Z} \,\}$, and finally $\{\, \text{poly-}\mathbb{Z} \,\} = \{\, \text{poly-poly-}\mathbb{Z} \,\}$, of course; similarly one can show that the following equality holds: $\{\, \text{max} + \text{solvable} \,\} = \{\, \text{poly-cyclic} \,\}$.

(8) The group $G := \langle\, x,y,t \mid x^t = x^{-1}, y^t = y^{-1}, [x,y] = t^4 \,\rangle$ is tf and polycyclic, fitting in the extension $1 \to H \to G \to \mathbb{Z}/2 \to 1$, where H is the group defined in (6), but not poly-$\mathbb{Z}$ since $G/[G,G]$ is finite. G is not locally indicable, and this shows also that not all solvable groups have Strebel's property.

2 L-theoretic invariants & signatures

Let M denote the zero-framed surgery on a given knot $K \subset S^3$. Assume there is a map $\phi\colon M \to B\Gamma$ where Γ is a poly-tfa group. In this case there is a skew field of quotients $\mathcal{K}\Gamma$ for the integral group ring $\mathbb{Z}[\Gamma]$ and $C_*(\tilde{M}) \otimes_\phi \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma]} \mathcal{K}\Gamma$ is contractible (see section 1).

This implies that $\sigma(M,\phi) = (C_*(\tilde{M}) \otimes_\phi \mathbb{Z}[\Gamma], \varphi) \in L^3(\mathbb{Z}[\Gamma])$ lifts to an element $\overline{\sigma}(M,\phi) \in L^4(\mathbb{Z}[\Gamma] \subset \mathcal{K}\Gamma)$:

$$\cdots \longrightarrow L^4(\mathbb{Z}[\Gamma]) \xrightarrow{i_*} L^4(\mathcal{K}\Gamma) \xrightarrow{\delta} L^4(\mathbb{Z}[\Gamma] \subset \mathcal{K}\Gamma) \xrightarrow{\partial} L^3(\mathbb{Z}[\Gamma]) \longrightarrow \cdots$$

If $[M,\phi] = 0 \in \Omega_3(B\Gamma)$ then

$$\overline{\sigma}(M,\phi) \in \ker\partial = \operatorname{im}\delta \cong L^4(\mathcal{K}\Gamma)/i_*(L^4(\mathbb{Z}[\Gamma])).$$

Under this identification $\overline{\sigma}(M,\phi)$ corresponds to the element $B(M,\phi)$ constructed in [COT99, section 4]. The independence of this element of the choice of the bounding manifold is a very special case of the localization long exact sequence in L-theory: see [Ran81, chapter 3], and also [Ran92, example 3.13].

Consider an intermediate ring $\mathbb{Z}[\Gamma] \subset \mathcal{R}\Gamma \subset \mathcal{K}\Gamma$:

$$\begin{tikzcd}
\cdots \arrow[r] & L^4(\mathbb{Z}[\Gamma]) \arrow[r] \arrow[d] & L^4(\mathcal{K}\Gamma) \arrow[r] \arrow[d, equal] & L^4(\mathbb{Z}[\Gamma] \subset \mathcal{K}\Gamma) \arrow[r] \arrow[d] & L^3(\mathbb{Z}[\Gamma]) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & L^4(\mathcal{R}\Gamma) \arrow[r] & L^4(\mathcal{K}\Gamma) \arrow[r] & L^4(\mathcal{R}\Gamma \subset \mathcal{K}\Gamma) \arrow[r] & L^3(\mathcal{R}\Gamma) \arrow[r] & \cdots
\end{tikzcd}$$

If $\mathcal{R}\Gamma$ is a PID then the image of $\overline{\sigma}(M,\phi)$ in $L^4(\mathcal{R}\Gamma \subset \mathcal{K}\Gamma)$ corresponds to the class of the higher order Blanchfield linking form $B\ell(M,\phi)$ constructed in [COT99, theorem 2.13], under the isomorphism described in [Ran81, section 3.4].

We recall now two deep results of [COT99].

Theorem ([COT99, theorems 4.2 and 4.4]). *If Γ is a poly-tfa $\begin{cases} n \\ n-1 \end{cases}$-step solvable group and M as above is rationally $\begin{cases} n.5 \\ n \end{cases}$-solvable via a 4-manifold W over which ϕ extends, then $\begin{cases} \overline{\sigma}(M,\phi) = 0 \in L^4(\mathbb{Z}[\Gamma] \subset \mathcal{K}\Gamma) \\ B\ell(M,\phi) = 0 \in L^4(\mathcal{R}\Gamma \subset \mathcal{K}\Gamma) \end{cases}$.*

Next we consider the numerical invariants—signatures—used to detect this obstructions.

Theorem ([COT99, proposition 5.12]). *Suppose that Γ is a poly-tfa group, or more generally a torsionfree amenable group. Then the L^2-signature and the* *ordinary signature coincide on $L^0(\mathbb{Z}[\Gamma])$, i.e., the following diagram commutes.*

$$\begin{tikzcd}
 & L^0(\mathcal{U}\Gamma) \arrow[r, "\sigma_\Gamma"'] & \mathbb{R} \\
L^0(\mathbb{Z}[\Gamma]) \arrow[ur, "j_*"] \arrow[dr, "\varepsilon_*"'] \arrow[rr, phantom, "\star"] & & \\
 & L^0(\mathbb{Z}) \arrow[r, "\sigma_1"] & \mathbb{Z} \arrow[uu, hook]
\end{tikzcd}$$

Before the proof, let us point out that in general $\star$ does not commute. Indeed it fits into the bigger diagram:

$$\begin{tikzcd}
 & & & L^0(\mathcal{U}\Gamma) \arrow[r, "\sigma_\Gamma"'] & \mathbb{R} \\
\Omega_{4k}^{\mathrm{diff}}(B\Gamma) \arrow[r, "\text{forget}"] \arrow[urrrr, bend left, "\sigma_\Gamma"] \arrow[drrrr, bend right, "\sigma_1"'] & \Omega_{4k}^{\mathrm{PC}}(B\Gamma) \arrow[r, "\text{M-R}"] & L^0(\mathbb{Z}[\Gamma]) \arrow[ur, "j_*"] \arrow[dr, "\varepsilon_*"'] \arrow[rr, phantom, "\star"] & & \\
 & & & L^0(\mathbb{Z}) \arrow[r, "\sigma_1"] & \mathbb{Z} \arrow[uu, hook]
\end{tikzcd}$$

where the outer triangle commutes—this follows from Atiyah's L^2-index theorem for the signature operator. But Wall produces an example of a Poincaré complex (which does not carry any smooth structure) whose ordinary signature is not multiplicative under finite coverings, hence cannot coincide with the L^2-signature—and this shows in particular the noncommutativity of $\star$. (Since the L^2-signature for smooth manifolds is multiplicative under finite coverings, the L^2-index theorem implies that the same also holds for the ordinary signature of smooth manifolds.)

Proof. Since Γ is amenable, $C^*_{\mathrm{red}}(\Gamma) \cong C^*_{\max}(\Gamma) =: C^*(\Gamma)$, and the augmentation $\varepsilon\colon \mathbb{Z}[\Gamma] \to \mathbb{Z}$ induces a map $\varepsilon\colon C^*(\Gamma) \to \mathbb{C}$. We have the following diagram:

$$\begin{tikzcd}
K_0(B\Gamma) \arrow[r, "A_\Gamma"] \arrow[d, "\mathrm{pr}"'] & K_0(C^*(\Gamma)) \arrow[r, "\mathrm{tr}_\Gamma"] \arrow[d, "\varepsilon_*"'] & \mathbb{R} \\
K_0(\mathrm{pt}) \arrow[r, "A_1", "\cong"'] & K_0(\mathbb{C}) \arrow[r, "\mathrm{tr}_1", "\cong"'] & \mathbb{Z} \arrow[u, hook]
\end{tikzcd}$$

where A_Γ is the analytic assembly map (Baum-Connes), which is known to be an isomorphism under our assumptions, namely for torsionfree amenable groups. The outer rectangle commutes, again by Atiyah's L^2-index theorem. Since A_Γ is onto the right-hand square also commutes.

The claim now follows from the diagram:

$$\begin{tikzcd}
L^0(\mathbb{Z}[\Gamma]) \arrow[r] \arrow[ddd, "\varepsilon_*"'] & L^0(C^*(\Gamma)) \arrow[rr, "\mathrm{sign}", "\cong"'] \arrow[ddd, "\varepsilon_*"'] \arrow[dr, "\sigma_\Gamma"'] & & K_0(C^*(\Gamma)) \arrow[ddd, "\varepsilon_*"'] \arrow[dl, "\mathrm{tr}_\Gamma"] \\
& & \mathbb{R} & \\
& & \mathbb{Z} \arrow[u, hook] & \\
L^0(\mathbb{Z}) \arrow[r] & L^0(\mathbb{C}) \arrow[rr, "\mathrm{sign}", "\cong"'] \arrow[ur, "\sigma_1"] & & K_0(\mathbb{C}) \arrow[ul, "\mathrm{tr}_1"']
\end{tikzcd}$$

where we find $\star$ and everything commutes.

Comment: Note that for the argument above we need to use the maximal group C^*-algebra, and therefore we assume (K-)amenability of Γ. Is it possible to avoid this? Is surjectivity of the analytic assembly map $A_\Gamma \colon K_0(B\Gamma) \to K_0(C^*_{\mathsf{red}}(\Gamma))$ to the K-theory of the reduced group C^*-algebra enough? $\square$

Corollary. *In the situation of the theorem above the diagram*

$$\begin{tikzcd}
0 \arrow[r] & i_*(L^0(\mathbb{Z}[\Gamma])) \arrow[r] \arrow[d, "\sigma_\Gamma|"'] & L^0(\mathcal{K}\Gamma) \arrow[r] \arrow[d, "\sigma_\Gamma"'] & L^0(\mathcal{K}\Gamma)/i_*(L^0(\mathbb{Z}[\Gamma])) \arrow[r] \arrow[d, dotted, "\mu"'] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

induces a homomorphism $\mu \colon L^0(\mathcal{K}\Gamma)/i_*(L^0(\mathbb{Z}[\Gamma])) \to \mathbb{R}/\mathbb{Z}$.

Comment: Can one define $\tilde{\mu} \colon L^0(\mathcal{K}\Gamma)/i_*(L^0(\mathbb{Z}[\Gamma])) \to \mathbb{R}$?

References

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