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Quotient Rings of Noncommutative Rings in the First Half of the 20th Century

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Abstract

A keystone of the theory of noncommutative noetherian rings is the theorem that establishes a necessary and sufficient condition for a given ring to have a quotient ring. We trace the development of this theorem, and its applications, from its first version for noncommutative domains in the 1930s to Goldie's theorems in the late 1950s.

1. Introduction

The now standard construction of the rational numbers from the integers was obtained as part of the effort to arithmetize analysis that occupied so many mathematicians in the 19th century. This construction was first extended to integral domains in the paper that marks the beginning of the theory of fields, Ernst Steinitz's *Algebraische Theorie der Körper* [49] of 1910. Steinitz begins by defining an integral domain $\mathfrak{I}$ (Integritätsbereich), essentially as a ring without zero divisors [49, p. 177]. Then [49, p. 178] he calls attention to the fact that if $a, b \in \mathfrak{I}$ and $\mathfrak{I}$ is contained in a field $\mathfrak{L}$ then, so long as $b \neq 0$, one can form the quotient $a/b \in \mathfrak{L}$; and he gives the usual formulae for the equality, addition and multiplication of such elements. He then points out that these elements form a field $\mathfrak{R}$, which contains $\mathfrak{I}$, and he notes that such an $\mathfrak{R}$ must exist whenever $\mathfrak{I}$ can be embedded in a field. In his terminology, $\mathfrak{R}$ is obtained from $\mathfrak{I}$ by the quotient construction (Quotientenbildung). The next step is obvious: define $\mathfrak{R}$ directly from $\mathfrak{I}$. Steinitz constructs the quotient field of $\mathfrak{I}$ as we would do today. First he defines equality of two symbols of the form a/b, and proves that it is transitive, symmetric and reflexive. Then he claims that the formulae for the the sum and product of fractions in $\mathfrak{L}$ that he had already given determine a well-defined addition and multiplication of the symbols a/b. He finishes by noting that the field determined by this process is unique up to isomorphism.

Steinitz's construction is given in detail by van der Waerden in Chapt. III of *Moderne Algebra* [52, §12, pp. 46–49], without any reference to its source. This may seem somewhat surprising because van der Waerden was familiar with Steinitz's paper, which he had read at Emmy Noether's suggestion, when he first visited Göttingen in 1924

[53, p. 33]. However, he later referred to the material in chapters II and III, which include the quotient construction, as 'generally known' at the time he wrote the book, and said that in these chapters he followed the courses of E. Noether and E. Artin [53, pp. 38–39]. This suggests that by 1930 the quotient construction had become so well known that mathematicians may have lost sight of its source.

In the meantime, E. Artin and E. Noether had been developing their approach to noncommutative algebra, which included an extension of Wedderburn's structure theorems to rings with descending chain condition. Noether's approach, in particular, was developed in lectures she gave in Göttingen in the Winter semester of 1927/28, under the title *Hyperkomplexe Grössen und Darstellungstheorie*. Her aim was to use modules over rings with chain conditions to unify the structure theory of algebras and the representation theory of groups [54, pp. 244–251].

The material of this course was published by Noether in 1929 [39]. This paper is based on the notes that van der Waerden took when he attended her course [54, p. 244], [53, p. 37]. Thus, by the time he wrote *Moderne Algebra*, van der Waerden had a very thorough knowledge of the work on noncommutative algebra that was being done in Germany at that time. This is made most obvious in Vol. 2 of [52], but we also see instances of it in Vol. 1. One of these is a comment van der Waerden added to [52, §12]. After giving the proof that every (commutative) domain can be embedded in a field (its quotient field), he says that it is an open problem, except in some very special cases, whether the same is true for a noncommutative ring without zero divisors [52, p. 49].

This question was settled by A. Malcev, who had heard of the problem through Kolmogorov. Malcev first gave an example of a noncommutative domain whose multiplicative semigroup cannot be embedded in a group [33]. Then, in a paper published two years later, he gave necessary and sufficient conditions for a semigroup to be embeddable in a group [34]. Later editions of van der Waerden's book contain a footnote explaining that noncommutative domains cannot always be embedded in division rings, and giving a reference to [33].

Malcev's example shows, in particular, that it is not true that every noncommutative domain has a quotient field. So one may ask whether there is a simple necessary and sufficient condition that a (not necessarily commutative) domain must satisfy in order to have a quotient ring. The solution of the problem in this form is usually attributed to O. Ore [41], but as we shall see, the question was also solved about the same time (if in a less complete form) by D. E. Littlewood and J. H. M. Wedderburn.

Our aim in this paper is to to chart the early development of the theory of quotient rings of noncommutative rings. We centre our discussion around the developments that began in the 1931 papers of Ore and Littlewood and that found their epitome in Alfred Goldie's work in the late 1950s. In fact we stop short of discussing Goldie's work, mainly because it is very well represented in the literature, and does not pose any great historical problems. Moreover, a detailed account of the history of Goldie's discovery is available in [13].

On the other hand, we do not deal with the more general question of when a ring is embeddable in a field, since its solution followed a very different path. Indeed, Malcev's necessary and sufficient conditions for a semigroup to be embedded in a group did not lead to similar conditions for the ring problem. The question was finally settled by P. M. Cohn in the 1970s; see [9, Corollary 1, p. 281].

The paper is divided in three sections. The first section is a discussion of the three papers that in 1931/32 dealt with the quotient ring problem for domains. In Sect. 2 we follow as mathematicians generalize these results and apply them to various special classes of rings. The main conclusions are summarised in Sect. 3, where we also try to put in perspective the effect of Goldie's theorems in the development of the theory.

2. The foundation papers

Although most often attributed to Ore, the theorem on the existence of quotient rings of domains appeared also in the work of two other mathematicians, the already famous Wedderburn, and the novice D. E. Littlewood. At the end of the section we discuss possible reasons why these three mathematicians arrived at this result almost simultaneously.

2.1. Oystein Ore

2.1.1. Background

Oystein Ore was born in Oslo in 1899 and died there in 1968, while attending the 15th Scandinavian Mathematical Congress. He was a student at the Cathedral School and at the University of Oslo. Immediately after the conclusion of his university studies, Ore left for a trip to Göttingen and Paris, the main mathematical centres of the day. Between his stays abroad, he took a doctorate from the University of Oslo with a thesis on algebraic number fields that was published in 1923 [40].

In 1925 Ore was appointed to a position at the University of Oslo, but two years later he moved to the United States and joined the Yale University faculty as an Assistant Professor in Mathematics. In 1931 he was named Sterling Professor at Yale, and retained that post until his retirement in 1968.

In his lifetime, Ore published 9 books and more than 40 papers, most of which on algebra and graph theory. Ore's earliest papers deal with algebraic number theory, and it was only in 1930 that he began to work on noncommutative algebra. This lasted until 1935, when he again changed direction and began a study of lattices. Finally, from 1950 onwards he worked mostly on graph theory. Ore's books range from research monographs such as *Theory of graphs*, to more popular works like *Number theory and its history* and his biographies of N. H. Abel and G. Cardano. Together with R. Fricke and Emmy Noether, Ore also edited Dedekind's *Gesammelte Mathematische Werke*, which was published between 1930 and 32. More details of Ore's life and work may be found in [7].

The three papers that we discuss in detail here belong to the period 1930–1935, and display very clearly the influence of the abstract algebra movement that was taking shape under the leardership of Emmy Noether and Emil Artin.

2.1.2. The existence of the multiplum

Ore's theorem on the embeddability of a domain in a quotient ring appears in his paper *Linear equations in non-commutative fields* [41] of 1931. In the late 1920s, several mathematicians had made attempts to generalize the theory of determinants to division rings, as a means of solving systems of linear equations. Ore begins with a brief review of the work of A. R. Richardson, A. Heyting and E. Study. He discusses their shortcomings and proposes to find a better solution to the problem. So Ore's starting point was not a desire to construct quotient rings, but rather to improve on current work on the solution of linear equations. However, he was aware of the embedding problem, as he makes clear in his introductory remarks:

> In the commutative case all domains of integrity (rings without zero divisors) have a uniquely defined quotient-field, which is the least field containing the ring. For the non-commutative case v. d. Waerden[7] has recently indicated this problem as unsolved

The superscript 7 refers to a footnote which contains a reference to the 1930 edition of van der Waerden's *Moderne Algebra*.

The paper opens with a subsection in which Ore defines what he means by a regular ring S. First he lists the axioms for a ring (without a unit element). Besides the usual axioms, he includes the four axioms that define equality: determination, reflexivity, symmetry and transitivity. These are required for proving that the equality of fractions (in Subsect. 2) is a well-defined equivalence relation. This is followed by a paragraph in which Ore notes that the coefficients of a linear system must satisfy these axioms if the system is to be solved by the usual method of elimination. Then comes the crucial comment:

> The main operation for the usual elimination is however to multiply one equation by a factor and another equation by another factor to make the coefficients of one of the unknowns equal in two equations. We must therefore also demand:
>
> M_V. *Existence of common multiplum* When $a \neq 0$, $b \neq 0$ are two elements of S, then it is always possible to determine two other elements $m \neq 0$, $n \neq 0$ such that
>
> $$an = bm. \tag{1}$$

This axiom is named $\mathrm{M}_{V.}$ because it is the fifith axiom of multiplication in Ore's list. The property stated in $\mathrm{M}_{V.}$ is now often called the *Ore condition* or the *common multiple property*. We will use this terminology here as a shorthand, since most authors up to the 1960s stated the condition in full every time they referred to it.

The sixth of Ore's axioms excludes zero divisors from regular rings. It should be noted that although the adjective "regular" is still often applied to rings, it is never now used in Ore's sense. Indeed, what he would have called a regular ring (with a unit element) is now known as a *right Ore domain*.

We will now break away from the order that Ore follows in his paper in order to explain how condition M_V. is used to solve linear equations. This is done in Subsect. 3, which begins with the system

$$x_1a_{11} + x_2a_{12} = b_1,$$
$$x_1a_{21} + x_2a_{22} = b_2$$

whose coefficients are supposed to belong to a regular ring S. This hypothesis allows us to find A_{12} and A_{22} in S such that $a_{12}A_{22} = a_{22}A_{12}$. Right multiplying the first equation by A_{22} and the second equation by A_{12}, and subtracting one from the other, we end up with

$$x_1(a_{11}A_{22} - a_{21}A_{12}) = b_1A_{22} - b_2A_{12}.$$

Ore writes

$$\left|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}\right\| = a_{11}A_{22} - a_{21}A_{12} \quad \text{and} \quad \left|\begin{matrix} b_1 & a_{12} \\ b_2 & a_{22} \end{matrix}\right\| = b_1A_{22} - b_2A_{12},$$

and calls these the right-hand determinants of the second order. So the Ore condition is actually necessary for Ore's definition of determinant.

Higher order determinants are defined in Subsect. 4, where it is also proved that these expressions satisfy many of the properties that we expect of a determinant, including the fact that a linear system has one and only one solution if its determinant does not vanish [41, p. 477].

2.1.3. Quotient fields

We now turn to Subsect. 2 of [41]. This subsection, entitled *quotient fields*, begins with what would become the most famous result of the whole paper.

Theorem 1. *All regular rings can be considered as subrings* (*more exactly*: *are isomorphic to a subring*) *of a non-commutative field.*

This is Ore's statement of the theorem exactly as it appears in [41, p. 466]. Note that it is not meant to imply that all "regular rings" can be embedded into the same non-commutative field.

Given elements a and $b \neq 0$ in S, Ore writes

$$\left(\frac{a}{b}\right) = (a \cdot b^{-1}).$$

The proof of Theorem 1 consists in extending the usual definitions of equality, addition and multiplication of fractions to the objects defined above. This has now become standard, and can be found in several textbooks, such as [10, Chap. 12, Subsect. 12.1] and [18, Theorem 22, p. 66], so we need not go into detail here. However, we will briefly discuss the definitions of addition and multiplication of (noncommutative) fractions for future reference.

Addition is defined by reducing the two fractions to the same denominator, thus

$$\left(\frac{a}{b}\right) + \left(\frac{a_1}{b_1}\right) = \left(\frac{a\beta_1 + a_1\beta}{b\beta_1}\right),$$

where $b\beta_1 = b_1\beta$ by the Ore condition. Multiplication is defined by

$$\left(\frac{a}{b}\right)\left(\frac{a_1}{b_1}\right) = \left(\frac{a\alpha_1}{b_1\beta}\right)$$

where $b\alpha_1 = a_1\beta$ and $\beta \neq 0$; but Ore gives no rationale for this definition; cf. 2.3.1.

The converse of this result is proved in Theorem II.

Theorem II. *Let a and $b \neq 0$ run through all elements of a ring S without divisors of zero. If then the formal solutions of all equations*

$$xb = a \tag{19}$$

form a field, the ring S must be a regular ring.

At the end of Subsect. 2, Ore gives an example of a "ring" that is not regular, but that can be embedded in a field. The example is in fact the ring of polynomial functions with complex coefficients, with the usual addition, but with composition of polynomials as its multiplication. However, as Ore himself points out, the distributive law does not hold for left-hand multiplication. So this is not really a ring.

From our present point of view the most obvious example of a ring that does not satisfy the Ore condition is the free algebra $\mathcal{F}_2$ in two generators. Indeed, if R is a domain that does not satisfy Ore's condition then it must contain a copy of $\mathcal{F}_2$. Moreover, $\mathcal{F}_2$ is a subring of a division ring. This was first shown [37] by Ruth Moufang in 1937. She did it in two steps. First she proved that the group ring of a free metabelian group on two generators x and y can be embedded in a field. Then she showed that the subalgebra generated by x and y is isomorphic to $\mathcal{F}_2$. For more developments in this direction see [11].

2.1.4. Ore extensions

The most blatant gap in Ore's paper [41] was the absence of examples of "regular rings". He supplied these in 1933 in his *Theory of non-commutative polynomials* [44]. As in the previous paper, Ore is not primarily concerned with quotient rings, his aim is to develop a theory of noncommutative polynomials.

The paper is divided in two parts. In Chapt. I, he defines "noncommutative polynomial rings", which we now call *Ore extensions*, when the base ring is a division ring. This is followed by a discussion of the division and Euclidean algorithms for these rings. This leads to common multiples and, from them, to quotient rings. The last subsections of chapter one deal with conjugates and divisibility properties. In Chapt. II he is concerned mainly with the factorization properties of elements in Ore extensions. One striking thing about this paper is its quaint terminology. Thus the greatest common divisor is called the "cross-cut" and monic polynomials are said to be "reduced". However, in our discussion we will use the terminology that has become standard.

At the beginning of Chapt. I, Subsect. 1, Ore explains how he arrived at the definition of his noncommutative polynomial rings. Suppose, he says, that K is a division ring and let us call a formal object of the form

$$F(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

a polynomial with coefficients in K. We know how to add and subtract these polynomials, and how to left multiply them by elements of K. Moreover, if $a_0 \neq 0$ then we say that $F(x)$ has *degree* n. Since we want these polynomials to form a ring, we may assume, from the start, that the multiplication is going to be associative and distributive. But, under these hypotheses, multiplication is completely defined if we give a meaning to the product $x \cdot a$, where a is an element of K. In order to further restrict the possibilities, Ore postulates that

The degree of a product shall be equal to the sum of the degrees of the factors.

Therefore, we must have that $x \cdot a = \overline{a} \cdot x + a'$, where $\overline{a}, a' \in K$. Ore calls $\overline{a}$ the *conjugate*, and a' the *derivative* of a. He goes on to show that taking the conjugate is an endomorphism of K and taking the derivative a derivation of K relative to conjugation.

In Subsect. 2 Ore explains how one can perform right-hand division of noncommutative polynomials. Suppose that $F(x)$ is as above and that $G(x)$ is a polynomial of degree $m \leq n$. We may assume without loss of generality that $G(x)$ is monic. Then $F(x) - a_0x^{n-m}G(x)$ has smaller degree than $F(x)$. Iterating this we show that $F(x) = Q(x)G(x) + R(x)$, where $Q(x)$ has degree $n - m$ and the degree of $R(x)$ does not exceed $m - 1$. Once we have a division algorithm, we can perform the sequence of divisions that is characteristic of the Euclidean algorithm. This means that given two noncommutative polynomials $F_1(x)$ and $F_2(x)$, one can compute a unique (monic) polynomial $D(x)$ which is their common divisor. Moreover, if $C(x)$ is a polynomial divisible by $D(x)$ then there exist polynomials $A_1(x)$ and $A_2(x)$ such that

$$A_1(x)F_1(x) + A_2(x)F_2(x) = C(x).$$

[44, Theorem 4]. This Subsect. ends with a discussion of the conditions under which it is possible to perform *left-hand* division of noncommutative polynomials – the answer is that conjugation must be an automorphism of K; [44, Theorem 6].

2.1.5. Union and quotient field

This is actually the title of Subsect. 3 of [41] – "union" is Ore's word for the least common multiple. The subsection begins with a formula of which Ore notes: "this formula for the union does not seem to have been observed even in special cases". In order to describe the formula, let $F_1(x)$ and $F_2(x)$ be two noncommutative polynomials and suppose that, applying the Euclidean algorithm to them we get

$$\begin{aligned} F_1(x) &= Q_1(x)F_2(x) + F_3(x), \\ F_2(x) &= Q_2(x)F_3(x) + F_4(x), \\ &\cdots \\ F_{n-2}(x) &= Q_{n-2}(x)F_{n-1}(x) + F_n(x), \\ F_{n-1}(x) &= Q_{n-1}(x)F_n. \end{aligned}$$

Denoting by $[F_1(x), F_2(x)]$ the least common multiple of $F_1(x)$ and $F_2(x)$, the formula of [44, Theorem 8], is

$$[F_1(x), F_2(x)]$$
$$= aF_{n-1}(x)F_n(x)^{-1}F_{n-2}(x)F_{n-1}^{-1}(x)\cdots F_3(x)F_4(x)^{-1}F_2(x)F_3(x)^{-1}F_1(x),$$

where $a \in K$ must be chosen so that the resulting polynomial is monic. This formula has an odd look to it because Ore has yet to prove that noncommutative polynomials can be inverted in a meaningful way. However, the inverses in the formula are a mere shorthand. Thus if a polynomial $B(x)$ is a (right) factor of another polynomial $A(x)$, then $A(x)B(x)^{-1}$ stands for the polynomial $C(x)$ such that $A(x) = C(x)B(x)$. However, this is never explained in the paper. In order to prove the formula, write

$$\phi_i(x) = F_{n-1}(x)F_n(x)^{-1}F_{n-2}(x)\cdots F_i(x)$$

and show, by reverse induction, that it is divisible by $F_{i+1}(x)$ and $F_i(x)$.

Having proved that any two noncommutative polynomials $F_1(x)$ and $F_2(x)$ have a least common multiple we conclude that there exist $B_1(x)$ and $B_2(x)$ such that

$$B_1(x)F_1(x) = [F_1(x), F_2(x)] = B_2(x)F_2(x).$$

But this is the left hand version of the "common multiplum" condition of [41]. In particular, we have proved that the ring of noncommutative polynomials always has a (left) quotient ring.

2.1.6. "Formal theory of differential equations"

In August 1931 Ore submitted to Crelle's Journal a paper [42] on rings of differential operators over a field that is a prelude to his 1933 paper on noncommutative polynomials. Indeed, many of the results of the later paper are foreshadowed in this paper [42] of 1932, and its sequel [43].

In the introduction of the paper, Ore cites work of A. Loewy, H. Blumberg, and E. Noether and W. Schmeidler on differential operators, and claims that he is going to give a different and simpler approach. More important, however, from the point of view of what would come later, is his comment that similar methods would be applicable to any ring with a euclidean algorithm.

Section 1 contains some basic definitions and facts, beginning with that of a "differentiation". Given a field K, a *derivation* of K is a linear operator which satisfies Leibniz's rule for the differentiation of a product. He denotes the derivative of an element $a \in K$ by a'. The field of constants K^0 is the set of elements of K whose derivative is zero. Then he proves that an algebraic number field must be equal to its field of quotients (Satz 1); or what amounts to the same, an algebraic number field does not admit a nonzero derivation. There follows a discussion of the existence of derivations for fields of characteristic zero and of positive characteristic.

A *differential polynomial* is defined in Sect. 2 as an operator of the form

$$A(y) = a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny,$$

where the coefficients a_i belong to the field K. The sum and difference of two differential polynomials are defined in the usual way. If $A(y)$ and $B(y)$ are differential polynomials, then Ore writes

$$A(y) \times B(y) = A(B(y)).$$

He then claims that the differential polynomials form a noncommutative ring without zero divisors.

After these preliminary comments Ore turns to the basic divisibility properties of differential polynomials. He follows exactly the same approach that he would later use in [44]. Thus we find in the remainder of Sect. 2 the definition of divisibility, a description of the division algorithm and the euclidean algorithm, and a proof that the greatest common divisor of $A(y)$ and $B(y)$ can be written as a combination of these two polynomials.

As in the case of [44], Sect. 3 is devoted to the proof that any two differential polynomials have a least common multiple ("Hülle"). The approach is essentially the same as that of [44], and includes the long product formula for the least common multiple. Having proved that two differential polynomials always have a common multiple, Ore appeals to [42] to prove that this ring must have a quotient field. The remainder of the paper discusses many topics that were to be generalized in [44], including transformations and the factorization of elements into "primes".

This paper gives a glimpse of how Ore may have been led to his "noncommutative polynomials". Indeed, if $A(y) = y'$ and $B(y) = ay$ are differential polynomials then

$$A(B(y)) = (ay)' = a'y + ay' = \frac{a'}{a}B(y) + B(A(y)).$$

which gives the formula

$$A(y) \times B(y) = B(y) \times A(y) + \frac{a'}{a}B(y).$$

Identifying $B(y)$ with a, this becomes the defining formula for the ring of noncommutative polynomials whose conjugation is the identity, and whose derivative is a'.

2.2. D. E. Littlewood

2.2.1. Infinite dimensional algebras

By 1910 the structure theory of finite dimensional semisimple algebras over a division ring had reached a very complete state [54, p. 210]. In a paper of 1924 [57], J. H. M. Wedderburn notes that

> It is noteworthy how little place the finiteness of the basis – or indeed the presence of any basis at all – has in the principal theorems of linear algebras.

Wedderburn concludes that one ought to try and develop a theory of algebras without a finite basis, and that is exactly what he attempts in this paper. However, he admits to having great difficulty in extending the structure theorems of [55] to infinite dimensional algebras. Indeed the results of [57] depend heavily on some rather artificial hypotheses on the existence of idempotents. Remarkably, the immediate way ahead was actually implicit in his comment, where he says "or indeed the presence of any basis at all". For the next major development would come from the school of E. Noether and E. Artin, under which the old structure theorems would be extended to rings with descending chain conditions. Artin first began to use this condition systematically by 1927 [3]; therefore only three years after Wedderburn's comment.

At the end of his 1924 paper Wedderburn gives a number of illustrative examples, not all of which verify the postulates that he required in the paper. These examples include tensor algebras, algebras of functions and a special case of Ore's noncommutative polynomials. Their patchiness points to the lack of natural examples of infinite dimensional noncommutative algebras for which a systematic theory could be developed.

The first example of such an algebra would come from an unexpected source: the newly discovered quantum mechanics. Heisenberg's seminal paper of 1925 [25] led Born and Jordan to matrix mechanics [8], in which the amplitudes of Heisenberg's paper appeared as entries of an infinite matrix. At about the same time, Dirac independently developed an alternative interpretation of the quantum formalism in which the main players were the so-called q-numbers. These "quantum" numbers formed an algebra whose elements verified the relation

$$p \cdot q - q \cdot p = \imath \frac{h}{2\pi} 1$$

where p and q are the q-numbers that play the rôle of momentum and position in quantum mechanics, and h is Planck's constant. Dirac also considered some of the algebraic properties of this algebra [16].

It would not be long before the algebraists caught up and began their own study of the properties of Dirac's quantum algebra. One of the first in the field was D. E. Littlewood.

Dudley Ernest Littlewood was born in London in 1903 and died in 1979. From 1948 until his retirement in 1970 he was Head of the Department of Mathematics (and later the Department of Pure Mathematics) of the University College of North Wales.

Littlewood graduated as Wrangler at Trinity College in 1925, and was a contemporary of P. Hall and W. V. D. Hodge. He stayed at Cambridge for a year, but was forced to look for employement elsewhere in 1926. After a number of temporary posts he began work as a part-time lecturer at University College, Swansea, and remained there, except for a short period in 1930, until 1947, when he moved to Cambridge for a year, before taking up a professorship in Bangor. At Swansea, Littlewood came under the influence of A. R. Richardson, who was the Professor of Pure Mathematics. Richardson, whom we have already met in 2.1.2, was one of the few British algebraists of the time [51]. Five of the twelve papers that Littlewood published between 1930 and 1935 were joint work with Richardson. In his later years Littlewood would say of his early papers that they did not fire his imagination, [36, p. 59]. This comment included one of his independent papers, called *On the classification of algebras*, which appeared in 1931. This paper contains the first detailed analysis of Dirac's algebra of q-numbers, including a proof that it has a quotient ring. We analyse the paper in more detail in 2.2.2. Littlewood went on to become an established research mathematician, but his interest shifted towards invariant theory [36, p. 60 ff]. He is most often remembered nowadays as one of the discoverers of the *Littlewood-Richardson rule* in the representation theory of the symmetric group.

The rule first appeared in his joint paper with Richardson *Group characters and algebra* [32].

2.2.2. *"On the classification of algebras"*

Littlewood sets the scene for this paper in the first two paragraphs of the introduction:

> In Mathematical Physics many quantities of a non-commutative nature are used. For the most part these do not conform to any algebra that has been specially studied from an algebraic point of view. One reason for this is that the non-commutative quantities in question have very special properties that are far from fundamental algebraically. For example, certain quantities are used only as left-hand multipliers, and others only as right-hand multipliers†. Nevertheless, in some cases a comparatively simple non-commutative algebra is involved, though all such algebras seem to involve an infinite basis. So far, with the exception of a paper by Wedderburn‡, very little is known of algebras with infinite bases.
>
> In this paper an attempt is made towards the classification of non-commutative algebras, so as to include algebras with an infinite basis. A few of the simpler algebras are studied in detail, including some of the algebras studied by Mathematical Physicists. Though the treatment is purely algebraic and no special attempt is made to justify various assumptions that have been made in Quantum Theory, it is hoped that some doubtful points may be made clear incidentally.

There are two references in the first paragraph above. The first (†) is to the first edition of Dirac's famous book [17]. Littlewood's comment on right-hand and left-hand multipliers probably refers to Dirac's bras and kets. The second (‡) is a reference to Wedderburn's paper [57], which we discussed in 2.2.1. These comments suggest that, unlike Wedderburn, Littlewood went to the new physics for natural examples of infinite dimensional algebras, which he then studied from a purely algebraic perspective. It is interesting to note that Littlewood always cultivated an interest in physics which resurfaced in the 1950s [36, p. 65].

A detailed analysis of Littlewood's paper would take us too far. However, we give a general idea of its content before we turn to his definition of quotient rings. First of all, It should be noted that Littlewood uses the word *algebra* in the sense of Dickson's *linear algebra with a modulus* [15, Sect. 5 and 7], that is an algebra (with a unit element) over a field. Moreover, he only considers algebras over the real or complex fields.

Littlewood constructs his algebras as quotients of the free algebra. In his terminology, the free algebra on r generators is the *general polynomial algebra of Class* r. Moreover, there is no talk of ideals, not even of *invariant subalgebras*, which is Dickson's terminology. This is surprising since both Dickson [15] and Wedderburn [57] are cited in Littlewood's paper. Instead, he talks of residues of the general polynomial algebra "*modulo p* expressions $\mu_1, \ldots, \mu_p$". In other words, he concentrates on the generators, rather then the ideal they generate. Indeed, in its style and terminology, Littlewood's paper is closer to the 19th century rather than to the abstract algebra movement that would take over algebra after the Second World War. The "classification" of the title is in terms of the "class of the algebra" (its number of generators) and of the degree of the moduli.

Most of Part I of the paper is dedicated to general definitions and to the study of algebras of Class I, that is quotients of the polynomial ring in one variable. This part ends with a proof that all (real or complex) finite dimensional algebras are of class II.

In Part II Littlewood turns his attention to two important algebras of Class II and degree 2. The first is the algebra generated by p and x with modulus $px - xp - 1$, which we now call the *first Weyl algebra*. The second is the algebra generated by x and y with modulus $xy - \mu yx$, where $\mu \neq 0, 1$. Note that both of them are (iterated) Ore extensions. In both cases he constructs vector space bases for the algebras and proves many of their algebraic properties. In the case of the Weyl algebra these include the fact that it is simple (Theorem X), and that it is a domain (Theorem XII). Finally, in Part III, he deals with the classification of algebras of class II and degree II.

2.2.3. "The algebra of rational expressions"

Littlewood's definition of quotient ring comes up in his discussion of the properties of the Weyl algebra in Part II of [31]. However, he assumes in Part III that the same results apply to other algebras of class II and degree II.

The context is rather unexpected. Having described the basic properties of the Weyl algebra, Littlewood considers in which ways one might "extend" this algebra. He begins by arguing that since the algebra has an infinite basis one might take infinite series. After some adjustments he manages to get a well-defined algebra. However, this algebra has zero divisors. Moreover, although x and p have right-hand inverses, they do not have left-hand inverses. Then he comments

> Instead of extending our original algebra thus by means of infinite series, we can extend it by defining new quantities which are both right-hand and left-hand inverses of polynomial expressions in x and p.

He proceeds to investigate this ring, which he calls the "algebra of rational expressions" or, more briefly, the "rational algebra". Since he assumes that all elements of the Weyl algebra have both a right and a left inverse, it follows that every right-hand fraction must be equal to a left-hand fraction. Let P, Q, R, S be elements of the Weyl algebra. Littlewood defines PQ^{-1} to be equal to $R^{-1}S$ if $RP = SQ$. This is followed by a theorem in which he gives his version of the Ore condition.

Theorem XIX. *If P and Q are polynomials in x and p, then non-zero polynomials R and S can be found such that*

$$RP = SQ.$$

Littlewood's proof of this result is an interesting exercise in linear algebra. Assume that the degree of P and Q in both x and p is at most r, and take R and S to be elements of degree $3r$ in both x and p, with undetermined coefficients. There will be, on the whole, $2(3r+1)^2$ undetermined coefficients. Since $RP - SQ$ has total degree $4r$, it will have $(4r + 1)^2$ coefficients. Equating these to zero we obtain $(4r + 1)^2$ linear equations whose variables are the $2(3r + 1)^2$ coefficients of R and S. However,

$$(4r + 1)^2 < 2(3r + 1)^2,$$

so this linear system has more variables than equations. Thus the system must be indeterminate. But this means that we may choose R and S not both zero such that $RP - SR = 0$, which proves the theorem.

Corollary I merely states the right Ore condition. Corollary II seems rather odd at first, for it says that

$$R^{-1}(P+Q) = R^{-1}P + R^{-1}Q,$$

which looks like a definition of sum of fractions. However, in the previous page $(P+Q)R^{-1}$ had been defined to be equal to $PR^{-1} + QR^{-1}$. So the proof of Corollary II consists in turning left-hand fractions into their right-handed versions and checking that the expected equality holds true. This is followed by Theorem XX in which Littlewood proves that "the operations of addition, subtraction, multiplication and division are closed for the expressions PQ^{-1} and $R^{-1}S$". Since Littlewood has defined the sum of fractions with the same denominator, part of his proof consists in showing that the general case can be reduced to this one. However, he does not define the multiplication of fractions in a satisfactory way. Indeed, one might say that he assumes that fractions can be multiplied and proceeds to show that a satisfactory definition would lead to the fact that the product of two fractions is indeed a fraction. Then comes a result that sums up all that he has already proved, and more.

Theorem XXI. *The algebra of rational expressions in p and x is a division algebra.*

However, the next paragraph reads:

> Every quantity PQ^{-1} has an inverse QP^{-1}. To prove that the algebra is a division algebra we have only to show that there are no factors of zero.

He seems to miss the fact that the existence of inverses precludes zero-divisors, for he goes on to explain that this follows from the fact that the Weyl algebra is a domain and from Theorem XIX (the Ore condition).

Even though Littlewood "proves" the existence of a quotient ring only for the Weyl algebra, he assumes in Part III of his paper that the result applies to other algebras of class II and degree II. Indeed, in Theorem XXXI he shows that algebras with different moduli may have the same quotient rings.

2.3. J. H. M. Wedderburn

In 1932 Wedderburn published a paper called *Non-commutative domains of integrity*. Even though he claims in the first line of the introduction that

> The present paper forms an introduction to the study of the general properties of domains of integrity.

most of the results in the paper apply only to what he calls a *Euclidean domain*. This is a domain which admits a right and a left division algorithm. Euclidean domains are defined in Subsect. 6; the previous 5 subsections are very short and deal only on very basic definitions and examples.

Subsection 7 of the paper is concerned with a very general "Euclidean algorism", independent of any division properties. This is applied to Euclidean domains in Subsect. 8, in which Wedderburn proves the existence of greatest common divisors (Theorem

8.1) and least common multiples (Theorem 8.3) in these domains. The subsection is entitled “Factorization in a Euclidean domain”, but it does not contain the statements of any factorization theorems, as we now use the term.

Subsection 9 contains Wedderburn’s proof that Euclidean domains have a quotient ring, and will be analysed in greater detail below. Finally, Subsect. 10 proves the normal form theorem for matrices over a Euclidean domain.

2.3.1. Quotient rings

Wedderburn begins Subsect. 9, on quotient algebras, saying that he will “take first the case of Euclidean domains”. Then he uses the results of Subsect. 8 on least common multiples to claim that, given any pair of elements (“integers” in his terminology) a and $b \neq 0$ of a Euclidean domain E, there exist $a_1, b_1 \in E$ such that $a_1 b = b_1 a$, which is the Ore condition. Next he defines (a, b) as “the class of all pairs p, q such that $a_1 q = b_1 p$ ”, where $q \neq 0$.

There is no mention of the fact that this is an equivalence relation. Instead, he proceeds directly to showing that given any two pairs

> we can always assume that the second members of the representative pairs are identical; similar reasoning shows that we may, if we please, make the first members equal instead of the second.

This is then used to define the addition and multiplication of pairs in the usual manner.

Unlike Ore’s, Wedderburn’s definition of the multiplication of pairs enables one to see how it was put together. For, suppose that we want to multiply two pairs (a, b) and (c, d). Using the fact that any two elements have a right common multiple, he writes $f = bf_1 = cf_2$, so that

$$(a, b) = (af_1, bf_1) = (af_1, f) \quad \text{and} \quad (c, d) = (cf_2, df_2) = (f, df_2).$$

Then he defines

$$(a, b)(c, d) = (af_1, f)(f, df_2) = (af_1, df_2).$$

Since (α, β) is to be thought as $\alpha\beta^{-1}$, the equation that defines the product of pairs is

$$\alpha\beta^{-1} \cdot \beta\gamma^{-1} = \alpha\gamma^{-1},$$

which holds in any ring. Although Wedderburn explains what one has to prove in order to show that the quotient ring so defined is a division ring, he gives the details only of the proof of the associativity of the multiplication.

The subsection ends with two very interesting comments. The first one is that it is possible to construct a quotient ring for what he calls a *Hamiltonian domain*. By that he means a domain H such that, for each $a \in H$ there exists $\overline{a} \in H$ such that $a\overline{a} = \alpha$, an element of the centre (which he calls the *scalar*) J of H. He concludes that

> Since J is scalar, it is readily extended so that α has an inverse and then $\overline{a}\alpha^{-1}$ is an inverse of a.

Similar ideas would be used by other mathematicians, for example Jacobson [29] (cf. 3.2.1), and finally entered mainstream ring theory with Posner’s theorem in [45].

The second comment makes clear that Wedderburn had understood the significance of the common multiple condition for the quotient ring problem. In the last paragraph of the subsection he proposes what he calls “a somewhat more general problem”, namely, given an element b of a domain find under what circunstances it can be extended to another domain in which b has an inverse. He remarks that “in view of the discussion given above” it is sufficient to assume that for every element a of the domain “there correspond elements a_b and b_a such that $ba_b = ab_a$’, which is the Ore condition in this particular case.

2.3.2. Of Ore, Littlewood and Wedderburn

One of the surprising things about the papers in which quotient rings were introduced is the fact that they were published within a few months of each other. Ore’s [41] was received by the publisher on December, 8 1930 and appeared in Annals of Mathematics in 1931. Littlewood read a preliminary version of his paper to the London Mathematical Society in March, 13 1930, and it appeared in a revised version in the Society’s Proceedings in June, 1931. Finally, Wedderburn’s was accepted on August, 20 1931 and published in 1932 in Crelle’s Journal. There is no evidence that any one of them knew what the others were doing; indeed the approaches of the three authors differ significantly.

Ore’s starting point was an attempt to generalize results of the late 1920s on solving linear systems over noncommutative algebras. He begins by stating very general results on quotient rings of domains and it is only in a second paper that he proceeds to give examples. Moreover, his papers are very carefully written and follow the definition-theorem-proof style that was becoming standard.

Littlewood, on the other hand, draws his inspiration from the algebras that had appeared in the new quantum mechanics. He writes in a more leisurely style in which the reader is carried in the flow of his ideas. This makes the paper, on the whole, more interesting to read but, unlike Ore, he leaves many loose ends.

Wedderburn’s is the least known of the three papers in which quotient rings were first defined. Like Littlewood, he discusses the common multiple condition only for a very restricted class of domains, the Euclidean domains. However, his approach is actually closer to Ore’s because he was aware that the condition was sufficient for the existence of a quotient ring of a general domain. Moreover, his exposition is as detailed as Ore’s, even though he leaves most of the routine verifications to the reader.

Another interesting point concerns the examples of Ore domains that appear in each paper. First, although the two algebras studied in greatest detail in Littlewood’s paper are examples of Ore extensions, they are not immediately covered by the construction in [44], because Ore allows only for “polynomials” (in one variable) over a field. Moreover, Littlewood defines these algebras as quotients of the free algebra, and not as deformations of the ring of polynomials, which is what Ore’s “noncommutative polynomials” ultimately are. Wedderburn, on the other hand, deals with a class of rings which contains Ore’s examples in [44], or at least those for which the “conjugation” is an automorphism.

We should also try to explain why three different mathematicians, working on different sides of the Atlantic, and with such different objectives, arrived simultaneously at the same results. In this context, the first thing to note is the tight web of influences that worked as a background to the three foundation papers. Thus Ore was influenced by Richardson, who was Littlewood's mentor, and all three of them worked within the framework provided by the work of Dickson and Wedderburn.

In fact we have direct evidence that both Ore and Wedderburn were aware of the importance of adjoining inverses. For Ore this came in the form of Richardson's theory, where inverses were used in defining the determinant. Moreover, Ore knew that van der Waerden had posed this as a problem in [52, §12]. In the case of Wedderburn, the evidence may be found in [56], a paper that is quoted as a reference to [58]. As he says at the beginning of [56]:

> The object of this note is to investigate the properties of simple continued fractions when the terms are not necessarily commutative with one another.

As examples of such noncommutative 'terms' he gives, as one might expect, matrices and hypercomplex numbers, but also "a function of x and a differential operator D_x". Wedderburn takes a very cavalier attitude to the problem of inverses, he simply assumes that they do exist. He elaborates this point further saying that:

> In the demonstrations it is always assumed that an inverse exists, but, as the relations given are integral identities, the truth of the theorems does not depend on the existence of an inverse: the use of the inverse could therefore no doubt have been avoided, as by studying Euclid's algorithm in place of continued fractions for instance, but it does not seem that there would be any material gain in doing so.

The last sentence illustrates how algebraists' attitudes to rigour changed between 1900 and 1930. Perhaps, Wedderburn himself felt this change, for in [58, §7] he goes back to the problems he had dealt with in [56], only this time he uses the Euclidean algorithm. Moreover, as we have seen, this algorithm is then used to prove the existence of inverses [58, §9].

Littlewood poses a more serious problem. Of course he had been working with Richardson, and he must have known that inverses were necessary for defining the determinant. Indeed, this is one of the conditions explicitly stated at the beginning of [46]. Moreover, since he approaches the problem from the point of view of extending the ring, he may have been aiming at finding a way to embed the Weyl algebra in a division ring, which would give immediate access to Richardson's methods. However, it is interesting to note that Littlewood first attempts to extend the algebra by considering infinite sums. Of course passing from polynomials to power series one is indirectly adjoining inverses, so the two problems may not have been unconnected in Littlewood's mind.

Finally, we should mention the claim made by Ludwig Schwarz in [48, Zusatz, p. 276] that he had solved the quotient ring problem before Ore. He says that he did this after the problem had been posed by Emmy Noether, whose student he was. In the paper [48], based on his dissertation, Schwarz tries to adjoin inverses to the free algebra. It is interesting to note that Wedderburn's [58] gets quoted (in footnote 3 at p. 285).

3. The next thirty years

The next turning point in the theory of quotient rings of noncommutative rings would come only in the late 1950s with Alfred Goldie's structure theorems [22], [23]. However, several mathematicians actively pursued the subject between 1932 and 1958, and their work not only kept the early work visible, but also added a number of important results to the theory. These fit into two main lines that we now explore, namely: the extension of the main results to rings with zero divisors, and the addition of various important examples to the list of rings that satisfy the common multiple condition. Thus we see a trend towards generalization, pursued mostly during the 1940s, and a trend towards specialisation, which is more characteristic of the 1950s.

3.1. Rings with zero divisors

One common characteristic of the three foundation papers on quotient rings was the fact that they dealt only with domains. This is not very surprising when we realise that both Littlewood and Wedderburn were concerned with the structure of special rings, while Ore's work was an attempt to generalize current methods for solving linear systems over division rings. However, as algebra became more and more abstract it was inevitable that the problem would be posed for more general classes of rings. Indeed, when this finally occurred, the problem was dealt with as a special case of an even more general problem.

3.1.1. Paul Dubreil

As we saw in the introduction, Malcev solved van der Waerden's problem by giving an example of a domain whose multiplicative semigroup is not embeddable in any group. Two years later he returned to the problem and gave necessary and sufficient conditions for a semigroup to be embeddable in a group. The first to realise the potential of Ore's work on this more general problem was Paul Dubreil in a paper [20] published in 1943.

Dubreil had been a student at the Ècole Normale Supérieure, where he became a lecturer in 1927. Having earned a Rockefeller scholarship in 1929, he went to Hamburg where he expected to work with Emil Artin. There he met Emmy Noether, who was to have a great impact on his work. At the time Dubreil was interested in algebraic varieties, and it was only in 1936 that he turned his attention to general algebraic structures. In particular, he became interested in the problem of generalising the elementary group properties to more general structures. That is how he was finally led to consider semigroups and their embeddability properties.

In his first attempt at applying Ore's condition, Dubreil assumed that the semigroup was right regular. This meant that the semigroup had to satisfy the right cancellation property and the right common multiple property [20, p. 626]. However, he would soon go beyond that.

Dubreil's approach to the general embeddability question appeared in Chapt. 5 ("Corps") of his book *Algèbre* [21] published in 1946. He first handles the case of a right regular semigroup S. He proves that a sufficient condition for embeddability of S in a group G, established by Malcev in [34] holds for such a semigroup. Then he points out that the regularity hypothesis comes up if one

> particularises, with O. Ore, the general embeddability problem, by requiring that each element ξ of the group G has to admit at least one representation of the form
>
> $$\xi = ab^{-1} \quad \text{with } a \text{ and } b \in S.$$

[20, p. 138][1]. He calls this *condition* $\mathcal{C}$, and goes on to prove that if it is satisfied then, indeed, the semigroup S must be right regular. The converse is the content of his Theorem 1. Although Dubreil's proof is essentially a version of the proof Ore gave in [41], it looks somewhat different because he tends to reap every general result he finds on the way. Thus, he defines a set Δ of couples a/b, where a and b are elements of the semigroup S, and calls them *right fractions*. Then he defines the usual (equality) relation between fractions (which he calls $\mathcal{E}_\Delta$) and proves that it is reflexive, symmetric and transitive. However, instead of defining multiplication of equivalence classes of fractions, as Ore had done, he defines a partial multiplication in Δ. Namely, assuming that a/b and c/d are right fractions, *and that* $b = c$, he defines

$$a/b \cdot c/d = a/d,$$

and claims that one may easily check that Δ is a Brandt groupoid under this partial multiplication [20, p. 626]. Here we note the influence of Wedderburn's 1932 paper; an influence, moreover, which is openly acknowledged in footnote 1 of p. 139, at the very beginning of Chapt. V. From that, Dubreil goes on to prove that the quotient semigroup $\Delta/\mathcal{E}_\Delta$ is indeed a group in which S can be embedded in the expected way.

The general result comes in Theorem 2. It is preceded by the following definition. Let D be a semigroup and let S be a subset of D for which the following conditions are satisfied:

I. S is multiplicatively closed.
II. Every element of S can be right and left cancelled in D.
III. If b and $b' \in S$, there exists at least one pair of elements s and $s' \in S$ such that $bs = b's'$.
IV. If $s \in S$ and $t \in D$, there exists at least one pair of elements $m \in D$ and $n \in S$ such that $sm = tn$.

These conditions imply that S is a regular semigroup, so from Theorem 1, it has a group of quotients G.

Then comes the main embeddability theorem for semigroups ("thèoréme 2"). It claims that there exists a semigroup $\mathcal{D}$, which is an extension of both D and the group of

[1] si l'on particularise, avec O. Ore, le problème général d'immersion, en imposant à tout élémént ξ du group G d'admettre au moins une represéntation de la forme

$$\xi = ab^{-1} \quad \text{avec } a \text{ et } b \in S.$$

quotients G of S, such that every element of $\mathcal{D}$ can be written in the form ab^{-1}, where $a \in D$ and $b \in S$. The proof is a generalization of that of Theorem 1, and we need not go into details.

This leads to the embeddability theorem for rings.

Theorem II. *If a subset S of a ring R verifies hypotheses I, II, III, IV, there exists a ring K, which is an extension of R, in which every element of S has an inverse: K is called the right quotient ring of R with respect to S; every element of K is of the form ab^{-1}, where $a \in R$, $b \in S$. If R is right regular and $S = R - \{0\}$, K is a field, called the right quotient field of R.*

In order to prove the theorem Dubreil applies Theorem 2 to the multiplicative semigroup R and the subset S. He writes K for the semigroup $\mathcal{D}$ constructed in Theorem 2, and proceeds to define the addition of fractions and to prove that the required properties hold.

Examples are conspicuously absent from Dubreil's exposition, although he begins the next subsection (under the title "Cas abélien. Corps des quotients d'un domaine d'intégrité; nombres rationnels") explaining that every abelian semigroup with cancellation is regular.

3.1.2. K. Asano

As we saw in the previous subsection, Dubreil's work can be easily traced back to two of the foundation papers, since he quotes both Ore and Wedderburn, and their influence is evident in his proofs. The same cannot be said of Asano's work, for his motivation is quite different. As he explains in the introduction of [4], Asano wanted to develop an arithmetic ideal theory for noncommutative rings along the lines of the corresponding commutative theory of E. Noether [38]. Noether's theory can be applied to any integral domain R that satisfies the following three conditions:

- R is integrally closed in its quotient field,
- R satisfies the ascending chain condition,
- every prime ideal of R is maximal.

In trying to transpose these conditions to a noncommutative setting one is confronted right away with the quotient ring question.

Asano responded to the challenge in Sect. 1 of [4]. His approach is not restricted to domains; he defines quotient rings for any ring with respect to its set of regular elements – that is, nonzero divisors. He proves that if the ring satisfies the Ore condition with respect to its semigroup of nonzero divisors then it has a quotient ring (Satz 1). The elements of the quotient ring of R are represented by pairs of elements (a, α), where $a, \alpha \in R$ and α is a regular element. He establishes the equivalence relation between pairs of elements, defines addition and multiplication of pairs and proves that these operations are compatible with the equivalence relation. However, there is no detailed proof that these operations satisfy the required properties.

One striking thing about this paper is that, although it was published in 1939, it contains no reference to any of the foundation papers. This occurs, despite the fact that Asano is prodigal in citing his predecessors, including Noether, Hasse, Jacobson, Speiser, Brandt, Artin and Deuring. Is it conceivable that he had not heard of the work on quotient rings that had been carried on in the early 1930s? The statement at the beginning of Sect. 1 seems to support this:

> The problem of how to embed a ring in a quotient ring is well-known to be unsolved in general in the non-commutative case.[2]

One also has to take into account that Asano worked in Japan, thus outside the main centres, and that most of his references are to papers by German mathematicians, while most of the work on quotient rings had been pursued in the United States and Britain. Moreover, we have seen that the main existence theorem was independently discovered by three different mathematicians (four, if we count Schwarz) so why not also by Asano? It is interesting to note that this paper was reviewed by Jacobson in the first volume of *Mathematical Reviews* [28], but he does not even mention the results on quotient rings that appear in Sect. 1 of [4].

The rest of Asano's paper is concerned with his attempt to develop an arithmetic theory of noncommutative orders, and we need not go into it here. This was not his last paper on this subject. Indeed, his ideas were further extended in [5], and through it became an established part of the theory of noncommutative noetherian rings [24, Sect. 6, p. 148], [35, Chapt. 3].

In 1949 Asano returned to the quotient ring problem in another paper [6], in which he settled it in a form equivalent to Dubreil's. Once again, there is no reference to the foundation papers or to Dubreil, indeed the only work he cites is his own paper of 1939 [4] that we just discussed. Moreover, the proof of the main theorem is completely different from anything that had been done before.

After introducing the problem and claiming (just as in [4, Sect. 1]) that it has not yet been solved in complete generality, Asano proceeds to define a quotient ring of a ring $\mathfrak{o}$ with respect to a semigroup $\mathfrak{m}$ of nonzero divisors to be a ring $\mathfrak{S}$ with 1, such that every element of $\mathfrak{m}$ is invertible in $\mathfrak{S}$ and for each element $x \in \mathfrak{S}$ there exists $\lambda \in \mathfrak{m}$ with $\lambda x \in \mathfrak{o}$. He goes on to say that the element x may be written in the form $\lambda^{-1}a$, for some $a \in \mathfrak{o}$.

Next comes a lemma in which he shows that the Ore condition (with respect to $\mathfrak{m}$) implies that any finite set has a common multiple in $\mathfrak{o}$. This is followed by the main theorem (Satz 1) which states that the ring $\mathfrak{o}$ has a quotient ring with respect to the semigroup $\mathfrak{m}$ if and only if, for each $\lambda \in \mathfrak{m}$ and each $a \in \mathfrak{o}$ there exists $a' \in \mathfrak{o}$ and $\lambda' \in \mathfrak{m}$ such that $a'\lambda = \lambda' a$.

After the standard proof that the condition is necessary, comes a very surprising proof that it is sufficient. The proof is indirect, in the sense that fractions are not constructed as equivalence classes of pairs in $\mathfrak{o} \times \mathfrak{m}$. Instead, Asano constructs a new ring $\mathfrak{S}_0$ and proves that it satisfies all the required properties. The construction is in terms of

[2] Im nichtkommutativen ist es bekanntlich ein im allgemeinen nicht gelöstes Problem, einen Ring in einen Quotientenbereich einzubetten.

equivalence classes of operators, and we describe its key ideas. For a detailed modern proof of the existence of quotient rings based on Asano's approach see [35, 2.1.12].

Asano begins by declaring a left ideal of $\mathfrak{o}$ to be *regular* if it contains an element of $\mathfrak{m}$. Now let $\mathfrak{a}$ be a regular left ideal of $\mathfrak{o}$, and define an *operator* θ on $\mathfrak{a}$ to be an $\mathfrak{o}$-module homomorphism from $\mathfrak{a}$ to $\mathfrak{o}$. In keeping with Asano's notation we write the argument of the operators on the left.

It turns out that one has to deal with operators whose domains of definition need not be the same. However, we want to add operators, and the natural definition of addition requires the operators to have the same domain. This can be circunvented by restricting the two operators to a left ideal that belongs to the intersection of their domains. But to do this we must assume that two operators are "equal" if they coincide in this common domain. In other words, if θ_1 is defined on $\mathfrak{a}_1$ and θ_2 on $\mathfrak{a}_2$, and if

$$x\theta_1 = x\theta_2 \quad \text{for all} \quad x \in \mathfrak{a}_1 \cap \mathfrak{a}_2,$$

then $\theta_1 = \theta_2$. This turns out to be an equivalence relation, as Asano shows in his paper. The proof makes use of the common multiple condition.

The sum and product of two operators θ_1 and θ_2, with domains $\mathfrak{a}_1$ and $\mathfrak{a}_2$, is given by

$$x(\theta_1 + \theta_2) = x\theta_1 + x\theta_2$$

for all $x \in \mathfrak{a}_1 \cap \mathfrak{a}_2$, and

$$x(\theta_1\theta_2) = (x\theta_1)\theta_2$$

for all $x \in \mathfrak{a}$, where $\mathfrak{a}$ is mapped into $(\mathfrak{a}_1)\theta_1 \cap \mathfrak{a}_2$ by θ_1. Note that $\mathfrak{a}$ is necessarily a regular left ideal because, given $\lambda \in \mathfrak{a}_1 \cap \mathfrak{m}$, the common multiple condition implies that there exists $\mu \in \mathfrak{m}$ such that $\mu(\lambda\theta_1) \in \mathfrak{a}_2$; thus $\mu\lambda \in \mathfrak{a} \cap \mathfrak{m}$. Asano then proves that these operations are well-defined.

Let $\mathfrak{S}_0$ be the set of equivalence classes of operators with the above defined addition and multiplication. Given an element $c \in \mathfrak{o}$ let ϕ_c be the operator which is defined on any regular left ideal $\mathfrak{a}$ by $x\phi_c = xc$, for every $x \in \mathfrak{a}$. Then mapping c onto ϕ_c we get a ring homomorphism from $\mathfrak{o}$ into $\mathfrak{S}_0$. It is now easy to check that $\mathfrak{S}_0$ satisfies the required conditions. For example, if $\lambda \in \mathfrak{m}$, then consider the map θ defined by $x\lambda \to x$ for every $x \in \mathfrak{o}$. This is an operator defined on the regular left ideal $\mathfrak{o}\lambda$. However,

$$x(\phi_\lambda\theta) = (x\phi_\lambda)\theta = (x\lambda)\theta = x,$$

so that $\phi_\lambda\theta = 1$. Since we are identifying λ in $\mathfrak{o}$ with $\phi_\lambda \in \mathfrak{S}_0$, we have proved that θ is a right inverse of λ in $\mathfrak{S}_0$.

This is followed by several corollaries, in which he proves, among other things, that an Ore domain always has a quotient division ring (Satz 4) and that a left principal ideal domain satisfies the common multiple condition (Satz 5). The paper ends with a theorem (Satz 7) on the embedding of torsion-free modules.

3.2. *Examples of Ore domains*

The forties were a time of generalizations in ring theory, its main achievement being Jacobson's structure theorems. Even though Jacobson's *Structure of rings* was only published in 1956, the bulk of the research had been done in the 1940s. This may be one of the reasons why in the 1950s some algebraists began a more systematic study of some special classes of rings, like enveloping algebras of Lie algebras and rings with polynomial identities. Thus it is not surprising to find quotient rings being put to use as a tool to tease apart the structure of these rings. By this time we also note that Ore reigns supreme, and both Littlewood's and Wedderburn's papers have quietly slid into oblivion.

3.2.1. *Enveloping algebras*

Jacobson had been interested in Lie algebras since the 1930s. He had begun the study of Lie algebras over general fields in [27], and since then he had published several papers on the subject. In 1951 he published a note on finite dimensional Lie algebras over fields of positive characteristic [29]. The note is mainly concerned with finite representations of these algebras (Theorems 1 and 2), and, as one would expect, Jacobson's main tool is the enveloping algebra $\mathfrak{A}$ of the Lie algebra. As an offshoot he proves that $\mathfrak{A}$ must be embeddedable in a division algebra, which turns out to be the quotient field of $\mathfrak{A}$.

Jacobson's proof depends on the fact [29, Proposition 2] that the enveloping algebra $\mathfrak{A}$ is finite over its centre. Thus if Δ is the ring obtained by inverting all non-zero elements of the centre $\mathfrak{C}$ of $\mathfrak{A}$, then it follows that Δ is finite over the quotient field of $\mathfrak{C}$. But this implies that Δ is a division ring. Since Δ contains $\mathfrak{A}$, the proof is complete.

In order to prove Proposition 2, Jacobson uses a central polynomial argument. This depends on proposition 1, according to which, if a is an element of a finite dimensional Lie algebra over a field k of positive characteristic then there exists a polynomial $\phi(\lambda) \in k[\lambda]$ such that $\phi(a)$ belongs to the centre of $\mathfrak{A}$. This is proved using only elementary linear algebra.

The case of an enveloping algebra of a finite dimensional solvable Lie algebra over a field of characteristic zero was settled by Charles W. Curtis, who was then one of Jacobson's PhD students. His proof, which appears in a paper [14] published in 1952, is based on the fact that this algebra is what we now call an *iterated Ore extension*. In order to do this, Curtis had first to generalise Ore's noncommutative polynomials to cover the case where the base ring is not a division ring. He calls these *extensions of type O*.

With hindsight, the key result of the paper is the lemma on p. 966, which states, in modern parlance, that if R is an Ore domain, and S is an Ore extension of R, then S is also an Ore domain; cf. [10, Theorem 3, p. 439]. The proof of the lemma can be broken into two parts. First, one proves that S admits a division algorithm, up to multiplication by an element of R. Then one uses this algorithm, and induction on the degree, to prove that the Ore condition holds in S. In a footnote, Curtis explains that this line of argument was suggested by Jacobson; his original proof had consisted in showing that S could be embedded in a principal ideal domain.

Unfortunately, these arguments do not apply to all finite dimensional Lie algebras over a field of characteristic zero. However, this more general case was soon settled in a paper [50] of 1953 by Dov Tamari, who was then at the Institute Henri Poincaré in Paris.

Tamari actually proves his result for rings that are generalisations of enveloping algebras of finite dimensional Lie algebras, the key point being that every element of such a ring must have a "standard form" as in the Poincaré-Birkhoff-Witt Theorem. The existence of the quotient ring follows from an application of Ore's result, and [41] is one of Tamari's references. The proof that the Ore condition holds (the theorem of Sect. 5) is a "simple enumeration" in the style of Littlewood's proof for the Weyl algebra (see 2.2.3).

3.2.2. Rings with polynomial identities

Quotient rings enter the theory of rings with a polynomial identity (PI rings) with S. A. Amitsur's paper [2] of 1955. Since Amitsur is dealing with general PI rings, he could not use an argument that depended on the ring being finite over its centre. Thus Amitsur, like Tamari, uses Ore's results to prove the existence of a quotient ring.

Let S be an algebra over a (commutative) integral domain Ω. We say that S is a *PI ring* if it satisfies an identity $f(x_1, \ldots, x_n) = 0$, where f is a polynomial in (non-commutative) variables over Ω. In his paper Amitsur investigates PI rings without zero divisors and their embeddabibility in matrix rings, a recurrent theme in PI theory. His very first result (Lemma 1) states that if S is a PI domain then it is "right and left regular", in the sense of Ore.

We give Amitsur's proof in the case that S contains a unit element. First of all, since S is a PI ring, it must satisfy an identity in only two variables; see [10, Proposition 4, p. 454] for a proof. Let a and b be non-zero elements of S, and among all polynomials f, in two variables, for which $f(a, b) = 0$ choose one with the smallest possible degree, say g. Now

$$g(a, b) = \alpha + ag_1(a, b) + bg_2(a, b) = 0,$$

and either $g_1(a, b) \neq 0$ or $g_2(a, b) \neq 0$, by the minimality of g. Assume that $g_1(a, b) \neq 0$, and multiply $g(a, b)$ on the right by b. This gives

$$b(-\alpha - g_2(a, b)b) = ag_1(a, b)b \neq 0$$

because S is a domain. This proves that S is right regular. The proof for rings without unit element is more convoluted, but equally elementary.

Amitsur also proves [2, Theorem 1, p. 466] that the quotient ring D of a PI domain S is a central simple algebra over its centre and that D and S satisfy the same identities. As part of the proof of this result he shows that $D = SZ$.

4. Conclusion

A. W. Goldie's structure theorems [22], [23] opened the way for a systematic study of the theory of noncommutative noetherian rings. Since quotient rings played a very important rôle in the theory, the results of the first half of the century became a subject for textbooks and surveys [24], [26], [18], [10]. However, as is often the case, the view of the history of the subject that transpires from these accounts does not do justice to

Table 1

Ring	Mathematician	Date	Key argument of proof
Ore extension	O. Ore	1931	division algorithm
Weyl algebra	D. E. Littlewood	1931	counting argument
Enveloping algebra (positive characteristic)	N. Jacobson	1951	central polynomial
Enveloping algebra (solvable Lie algebra)	C. W. Curtis	1952	division algorithm
Enveloping algebra (all fields)	D. Tamari	1952	counting argument
PI domain	S. Amitsur	1955	minimal identity

what actually happened. Thus Ore is always mentioned, and sometimes Asano, but the work of Wedderburn and Dubreil has been almost totally forgotten.

The truth, as we have seen, is that the key results were discovered almost simultaneously by Ore, Littlewood and Wedderburn; with a claim by Ludwig Schwarz [48] that he had found them even before Ore. Since the three papers were published in very well-known journals, it is reasonable to speculate why Ore's work is almost always the only one quoted after 1950. There seem to be two reasons for this. The first is that his approach was far more general and systematic than the other two. The second is that he worked within the framework of modern abstract algebra. Thus, his papers were probably more accessible to the young mathematicians of the 1940s and 1950s who had been brought up on van der Waerden's *Moderne Algebra*.

Indeed, Ore's name became so indelibly connected with quotient rings that he is often credited with results that he did not prove. For all three foundation papers dealt only with domains, and the general case was only tackled more than ten years later, independently by P. Dubreil (in 1946) and K. Asano (in 1949). At this point, the other foundation papers still exerted some influence, and we see Wedderburn's paper appear as a reference in Dubreil's book [21].

Thus by 1950 the result that provided a general necessary and sufficient condition (now most often called the *Ore condition*) for the existence of a quotient ring had been established in an adequate degree of generality. However, the situation was far less satisfactory with regard to examples of rings (even domains) that verified the Ore condition. Even though there was no shortage of examples, the proofs that the condition was verified varied widely in each case, as illustrated in Table 1.

It is at this point that Goldie intervenes. Since most of the work discussed here has reached us through Goldie's work, we finish by reviewing it against the achievements of the 1950s.

Goldie began by trying to prove Jacobson's conjecture that the intersection of all powers of the Jacobson radical in a left and right noetherian ring is zero [30, p. 200]. He decided to tackle first the case of a noetherian domain. Since he was acquainted with the work of Ore, Littlewood and Tamari, it seemed natural to try to prove that such a domain always has a representation as a subring of a division ring. He had already managed to do this by 1956.

Although Goldie did not publish his proof that a noetherian domain must verify the Ore condition, it was so simple and elementary that it has become part of the folklore of ring theory [13, p. 304], [10, Proposition 4, p. 436]. Indeed, it has become so familiar that it is difficult to understand why it was greeted with surprise when it first appeared [13]. But the very qualities that made it so well-known also explain why it came as a surprise in the 1950s. For, first of all, it identifies the property common to most of the known examples that made the existence of the quotient possible, namely the ascending chain condition. Second, it replaced an assortment of *ad hoc* proofs (some simple, some rather complicated) with a single argument, and a very simple one to boot.

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