London Mathematical Society Lecture Note Series. 44

$\mathbb{Z}/2$ – Homotopy Theory

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CAMBRIDGE UNIVERSITY PRESS
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MELBOURNE SYDNEY

Detailed acknowledgments are recorded in the text.

My special thanks are due to Dr. B. Steer, Prof. I. M. James, Prof. J. C. Becker, Dr. L. Woodward, Dr. W. A. Sutherland, Prof. S. Gitler and Dr. K. Knapp;

to New College, Oxford, where I held a Guinness research fellowship from 1974 until 1977;

and to the SFB "Theoretische Mathematik" at the University of Bonn, where I was 'Gastforscher' from 1977 to 1979.

Much of the material described here first appeared in the thesis [25], written under the supervision of Dr. Steer, and was the subject of a course of lectures at Oxford in 1976. I have also drawn from a seminar on Hermitian K-theory held at Bonn in the autumn of 1978.

1. Introduction

The cyclic group, $\mathbb{Z}/2$, of order two plays a leading rôle in the theory of real vector bundles and manifolds. More precisely, it plays many parts, as abstract permutation group, as orthogonal group in dimension one, as Galois group of $\mathbb{C}$ over $\mathbb{R}$, now clearly distinguished, now merging one into another. The plot is not, by any means, fully revealed. We recount here but a few scenes in which $\mathbb{Z}/2$ figures.

The results largely occur in some form in the literature, many in the unpublished thesis [25] . Our purpose here is to present a concise account from the particular viewpoint of $\mathbb{Z}/2$-homotopy theory and without detailed proofs.

We begin with an extremely simple, but fundamental, result; as we shall see in §3, it lies very close to the theorem of Kahn and Priddy. Here and throughout the essay L will denote the real representation $\mathbb{R}$ of $\mathbb{Z}/2$ with the non-trivial action as multiplication by ± 1 .

Proposition (1.1). Let ξ and ξ' be real vector bundles over a compact ENR X. Suppose that the sphere-bundles $S(\xi)$ and $S(\xi')$ are stably fibre-homotopy equivalent. Then $S(L\otimes\xi)$ and $S(L\otimes\xi')$ are $\mathbb{Z}/2$-equivariantly stably fibre-homotopy equivalent.

Notation. Any space may be considered as a $\mathbb{Z}/2$-space with the trivial action. We use the same symbol for the original space and the corresponding $\mathbb{Z}/2$-space. Thus, $S(\xi)$ as $\mathbb{Z}/2$-space is the sphere-bundle with the trivial involution; $S(L\otimes\xi)$ is the same space, but endowed with the antipodal action of $\mathbb{Z}/2$.

As usual, let $J(X)$ be the quotient of $KO(X)$, the real K-theory of X, by the subgroup generated by differences $[\zeta]-[\zeta']$ of vector bundles ζ, ζ' over X whose sphere-bundles are fibre-homotopy equivalent; so $J(\text{point}) = \mathbb{Z}$. $J_{\mathbb{Z}/2}(X)$ is defined similarly as the quotient of $KO_{\mathbb{Z}/2}(X)$ by differences of $\mathbb{Z}/2$-vector bundles whose sphere-bundles are equivariantly fibre-homotopy equivalent. Two $\mathbb{Z}/2$-vector bundles ζ and ζ' define the same class in $J_{\mathbb{Z}/2}(X)$ if and only if their sphere-bundles are stably $\mathbb{Z}/2$-fibre-homotopy equivalent, that is, if $S(\zeta\oplus W)$ and $S(\zeta'\oplus W)$ are equivariantly fibre-homotopy equivalent for some real representation W of $\mathbb{Z}/2$. (The same symbol W will often be used for a vector space and the corresponding trivial bundle $X\times W$ over X.)

The proposition (1.1) states that the linear map $KO(X)\longrightarrow KO_{\mathbb{Z}/2}(X)$ taking $[\xi]$ to $[L.\xi]$ - we shall often write $L.\xi$ for the tensor product $L\otimes\xi$ - induces a map of quotients $J(X)\longrightarrow J_{\mathbb{Z}/2}(X)$. The proof is simply the observation that $\xi\oplus L.\xi$ may be identified with the sum $\xi\oplus\xi$ equipped with the involution which interchanges the factors (by mapping (u,v) to $(u+v,u-v)$ in each fibre).

More formally, let us define "doubling operations"

$$S^2 : KO(X)\longrightarrow KO_{\mathbb{Z}/2}(X) \quad\text{and}\quad S^2 : J(X)\longrightarrow J_{\mathbb{Z}/2}(X)$$

to be the linear maps taking the class of a vector bundle ξ (or sphere-bundle $S(\xi)$) to the sum $\xi\oplus\xi$ (or fibre-wise join $S(\xi)*S(\xi)$)

with the switching involution. Let

$$\sigma : KO(X)\longrightarrow KO_{\mathbb{Z}/2}(X) \quad\text{and}\quad \sigma : J(X)\longrightarrow J_{\mathbb{Z}/2}(X)$$

be the inclusions of direct summands given by regarding any bundle as a $\mathbb{Z}/2$-bundle with the trivial involution. Write

$$\bar{S}^2(x) := S^2(x)-\sigma(x).$$

Then the proposition is proved by the commutativity of the diagram:

$$\begin{array}{ccc} KO(X) & \xrightarrow{\bar{S}^2} & KO_{\mathbb{Z}/2}(X) \\ \downarrow & & \downarrow \\ J(X) & \xrightarrow[\bar{S}^2]{} & J_{\mathbb{Z}/2}(X). \end{array}$$

There is an immediate corollary.

Corollary (1.2). In addition to the hypotheses of (1.1), suppose that λ is a real line bundle over X. Then the sphere-bundles $S(\lambda\otimes\xi)$ and $S(\lambda\otimes\xi')$ are stably fibre-homotopy equivalent.

It is the explicit geometric construction of the operation S^2 upon which the proof of (1.1) depends; in succeeding paragraphs, particularly §§ 3-5, it will be a central theme. The mere definition of the operation lies at a different level. S^2 is simply <u>induction</u>

$$i_* : KO(X)\longrightarrow KO_{\mathbb{Z}/2}(X)$$

corresponding to the inclusion $i : 0 \dashrightarrow \mathbb{Z}/2$ of the trivial subgroup. (The definition is reviewed in (5.3).) It is the restriction to the diagonal of an external operation

$$S^2 : KO(X) \longrightarrow KO_{\mathbb{Z}/2}(X \times X)$$

- $X \times X$ with, as always, the switching involution - defined as the composition

$$KO(X) \xrightarrow[p_1^*]{} KO(X \times X) \xrightarrow[i_*]{} KO_{\mathbb{Z}/2}(X \times X),$$

where $p_1 : X \times X \longrightarrow X$ is the projection onto the first factor. In this form the definition extends to other generalized cohomology theories, in particular to stable cohomotopy.

As is implicit in the notation, a sum operation S^k may be defined for any natural number k. Write the permutation representation $\mathbb{R}^k$ of the symmetric group $\mathfrak{S}_k$ as the direct sum $\mathbb{R} \oplus V_k$ of a trivial summand and an irreducible representation of dimension k-1. The statement (1.1) clearly remains true if we substitute V_k for $L = V_2$ and $\mathfrak{S}_k$ for $\mathbb{Z}/2 = \mathfrak{S}_2$.

In later paragraphs, too, there will sometimes be generalizations, perhaps from $\mathbb{Z}/2$ to $\mathbb{Z}/p$, p an odd prime, perhaps from $\mathbb{Z}/2 = S^0$ to S^1 and S^3. We shall do no more than record the fact. $\mathbb{Z}/2$ is our proper subject.

(1.2) was first proved in answer to a question of Prof. I. M. James, and initiated much of the work described here. His own equivalent solution may be found in [41] Lemma (2.1).

2. The Euler class and obstruction theory

$\mathbb{Z}/2$, as the subgroup $\{\pm 1\}$ of the group of units of $\mathbb{R}$, appears naturally in the study of r-fields, that is, r linearly independent cross-sections, of a real vector bundle, as soon as r is greater than 1. First, we must recall in an appropriate form the obstruction theory for a single cross-section. We need to fix a notation for stable cohomotopy, and, in view of the varying usage, do so with some care.

Notation. Let ξ and η be real vector bundles over a compact ENR X. ω^* will denote <u>unreduced</u> stable cohomotopy, considered as a generalized cohomology theory. A tilde indicates the associated reduced theory for pointed spaces. Thus, $\omega^*(X)$ is a graded ring with identity. Define

$$\omega^*(X; \xi - \eta) := \tilde{\omega}^*(T(\xi - \eta))$$

to be the reduced stable cohomotopy of the Thom space of the virtual bundle $\xi - \eta$. We think of the stable cohomotopy of X as a theory indexed by the <u>category</u> of virtual vector bundles and stable fibre-homotopy equivalences over X. If $Y \subseteq X$ is a closed sub-ENR, the relative group $\omega^*(X,Y; \xi - \eta)$ is defined to be $\tilde{\omega}^*(T(\xi - \eta)/T(\xi - \eta | Y))$. Corresponding notation is used for stable homotopy.

(In the literature $\tilde{\omega}_*$ is often written, reasonably enough as the limit of unstable homotopy groups, π_*^S (without a tilde). The coefficients '$\xi - \eta$' are sometimes written with a change of sign and with the dimension absorbed into the index '*'; the complexity of the indices required here precludes writing them, as is often done, as sub- or superscript.)

A superscript '+' will be used for one-point compactification, the adjunction of a base-point + if the space is already compact. ξ^+ is to be understood as the <u>fibre-wise</u> one-point compactification of ξ. It is a fibre-bundle over X; its fibre is a sphere with base-point. (For example, in the case that $\xi = 0$ is the zero vector bundle, 0^+ is the trivial bundle $X \times S^0$.) Then $\omega^0(X; \xi - \eta)$ may be interpreted as the group, written $\{\xi^+; \eta^+\}_X$, of stable fibre-homotopy classes of maps $\xi^+ \longrightarrow \eta^+$ over X preserving the base-point in each fibre (ex-maps in the terminology of [40]). (As is customary, $[-; -]$ and $\{-; -\}$ will denote respectively the set of homotopy and the group of stable homotopy classes of maps between pointed spaces.)

The way is prepared for the basic definition of obstruction theory.

Definition (2.1). The <u>Euler class</u> of the vector bundle ξ is the class

$$\gamma(\xi) \in \omega^0(X; -\xi) = \{0^+; \xi^+\}_X$$

represented by the inclusion $0^+ \longrightarrow \xi^+$ of the zero section (induced by $0 \subseteq \xi$).

Clearly $\gamma(\xi)$ vanishes if ξ admits a nowhere-zero cross-section. The converse is true in the 'metastable range'. Before describing

this result, we list some elementary formal properties of the Euler class.

Proposition (2.2). Let ξ, ξ' be real vector bundles over X.

(i) Naturality. If $f: X' \longrightarrow X$ is a map, then $\gamma(f^*\xi) = f^*\gamma(\xi)$.

(ii) Multiplicativity. $\gamma(\xi \oplus \xi') = \gamma(\xi).\gamma(\xi')$.

(iii) Suppose that $a: \xi'^+ \longrightarrow \xi^+$ is a fibre map which is 'polar' in the sense that 0 is mapped to 0, + to + in each fibre. It defines a stable class in $\{\xi'^+; \xi^+\}_X = \omega^0(X; \xi' - \xi)$. Then $\gamma(\xi) = [a].\gamma(\xi')$.

The stable cohomotopy Euler class is defined by exact analogy with the classical definition in cohomology. Indeed, the group $\omega^0(D\xi, S\xi; -\xi)$ - the stable cohomotopy of the disc modulo the sphere with coefficients in (the pullback of) ξ - is naturally isomorphic to $\omega^0(X)$, the isomorphism given by a tautological 'Thom class' $u \in \omega^0(D\xi, S\xi; -\xi)$. $\gamma(\xi)$ is just the restriction of u to the zero-section $(X, \emptyset) \subseteq (D\xi, S\xi)$. More generally, if s is a cross-section of $S(\xi)$ over a closed sub-ENR Y, the <u>relative Euler class</u> $\gamma(\xi, s) \in \omega^0(X, Y; -\xi)$ is defined to be $\tilde{s}^*(u)$ for any extension $\tilde{s}: (X,Y) \longrightarrow (D\xi, S\xi)$ of the section s.

If t is another section of $S(\xi)$ over Y agreeing with s on a closed sub-ENR $Z \subseteq Y$, define their <u>difference class</u> $\delta(s,t) \in \omega^{-1}(Y, Z; -\xi) = \omega^0((Y,Z) \times (I, \dot{I}); -\xi)$ to be the relative Euler class of the pullback of ξ to $Y \times I$ with respect to the section over $Y \times \dot{I} \cup Z \times I$ which agrees with s on $Y \times 0$, t on $Y \times 1$ and their common value on $Z \times I$. It is an obstruction to deforming t into s (by a homotopy constant on Z) and determines the variation of the relative Euler class with the choice of cross-section.

Proposition (2.3). The difference class $\delta(s,t) \in \omega^{-1}(Y,Z;-\xi)$ is mapped to $\gamma(\xi,s) - \gamma(\xi,t) \in \omega^{0}(X,Y;-\xi)$ by the connecting homomorphism in the stable cohomotopy exact sequence of the triple (X,Y,Z).

The fundamental result in the subject is a straightforward corollary of Freudenthal's suspension theorem (proved for a cell complex X and subcomplex Y step by step over the cells).

Proposition (2.4). Suppose that the dimensions $\dim X \leq m$ and $\dim \xi = n$ lie in the metastable range: $m < 2(n-1)$.

(i) A section s of $S(\xi)$ over Y extends over the whole of X if and only if $\gamma(\xi,s) \in \omega^{0}(X,Y;-\xi)$ vanishes.

(ii) If s is a section of $S(\xi)$ over X, d an element of $\omega^{-1}(X,Y;-\xi)$, then there is a section t over X coinciding with s on Y and such that $\delta(s,t) = d$.

This leads at once to a classification theorem.

Proposition (2.5). Suppose that $m+1 < 2(n-1)$ and that $S(\xi)$ has a cross-section s. Then the set of fibre-homotopy classes of cross-sections t of $S(\xi)$ extending $s \mid Y$ over Y - the homotopies understood to be constant on Y - is in 1-1 correspondence with $\omega^{-1}(X,Y;-\xi)$ under the map $t \mapsto \delta(s,t)$. (The map is surjective if $m+1 \leq 2(n-1)$.)

This simple device of stabilization, formalized in (2.4) and (2.5), has both conceptual and practical advantages.

We turn to the question of r-fields, beginning with the local classification problem. Let U and $V (\neq 0)$ be real (Euclidean) vector spaces and write $O(V,U\oplus V)$ for the Stiefel manifold of isometric linear maps $V \longrightarrow U\oplus V$ with base-point, j say, the inclusion of the second factor.

Now an element of $O(V,U\oplus V)$ defines, by restriction, a map of spheres $S(V) \longrightarrow S(U\oplus V)$ commuting with the antipodal map, that is, a $\mathbb{Z}/2$-equivariant map $S(L.V) \longrightarrow S(L.(U\oplus V))$ or an equivariant cross-section of the trivial sphere-bundle $S(L.(U\oplus V))$ over the sphere $S(L.V)$. And so we are led to consider the relative Euler class and difference class in $\mathbb{Z}/2$-equivariant stable cohomotopy. The equivariant theory is indicated by a subscript; its definition, [77], is recalled in (4.1).

Definition (2.6). The <u>local</u> <u>obstruction</u>

$$\theta : \quad [X/Y; \quad O(V,U\oplus V)] \longrightarrow \omega^{-1}_{\mathbb{Z}/2}((X,Y) \times S(L.V); -L.(U\oplus V))$$

is defined as follows. A map of pairs $v: (X,Y) \longrightarrow (O(V,U\oplus V),j)$ gives, as above, a $\mathbb{Z}/2$-equivariant cross-section, v' say, of the trivial bundle $L.(U\oplus V)$ over $X \times S(L.V)$. Set $\theta(v) := \delta(v',j')$. (It clearly depends only on the homotopy class of v.)

Recall that if G is a compact Lie group and $P \longrightarrow B$ a principal G-bundle, B a compact ENR, then, just as $KO_G(P)$ is identified with $KO(B)$, [76], so the G-equivariant stable cohomotopy $\omega^*_G(P)$ is identified with $\omega^*(B)$. More generally, if E is a (virtual) G-module, then $\omega^*_G(P; E)$ is identified with $\omega^*(B; P \times_G E)$ - coefficients in the associated vector bundle over B.

Since $\mathbb{Z}/2$ acts freely on $S(L.V)$, the target group of θ may be rewritten as $\omega^{-1}((X,Y) \times P(V); -H.(U\oplus V))$, where H, associated to the representation L, is the Hopf line bundle $(S(L.V) \times L)/\mathbb{Z}/2$ over the projective space $P(V)$. (Had we not wished to stress the equivariant theory, we might have proceeded directly to this step by noticing that an element of the Stiefel manifold determines a cross-section of $H.(U\oplus V)$ over $P(V)$.) This group is <u>canonically</u> isomorphic by S-duality to

$$\{ X/Y; \quad P(U\oplus V)/P(U) \}$$

On the other hand, the stunted projective space $P(U\oplus V)/P(U)$ is included in a standard way in the Stiefel manifold $O(V,U\oplus V)$ by the 'reflection map' R (which takes a line in $U\oplus V$ to the reflection in the orthogonal hyperplane).

Proposition (2.7). The composition $\theta.R$

$$[\, X/Y; \quad P(U\oplus V)/P(U) \,] \longrightarrow [\, X/Y; \quad O(V,U\oplus V) \,] \longrightarrow \{ X/Y; \quad P(U\oplus V)/P(U) \}$$

is the stabilization map.

The proof is by inspection.

(It is clearly enough to consider the case $(X,Y) = (P(U\oplus V),P(U))$ and look at the image of the map which collapses $P(U)$ to a point. The argument is best described in geometric language (as in §5) and for clarity we assume $U = 0$.

For any closed manifold X there is a duality isomorphism $\{X^+; X^+\} \cong \omega^0(X \times X; -\tau_2 X)$, where $\tau_2 X$ is the tangent bundle on

the second factor. The identity $1 \in \{X^+; X^+\}$ corresponds to the 'Atiyah duality class' $\Delta_*(1) \in \omega^0(X \times X; -\tau_2 X)$ represented, according to definition, by the diagonal $\Delta : X \longrightarrow X \times X$ (with the natural identification of the normal bundle with $\Delta^*\tau_2 X$). (In the traditional terminology of homology theory, the duality class is the 'cycle' defined by the diagonal.)

Specialize now to the case $X = P(V)$, abbreviated to P. There is a standard isomorphism $\tau P(V) \oplus \mathbb{R} \cong H.V$. We are required to identify the duality class $\Delta_*(1) \in \omega^{-1}(P \times P; -H_2.V)$ (where H_2 is the Hopf bundle over the second factor) with the difference class $\theta(1) = \delta(s_0,s_1)$ of the two cross-sections of $S(H_2.V)$ over $P(V) \times P(V)$ induced on the orbit space by the $\mathbb{Z}/2$-equivariant maps $P(V) \times S(L.V) \longrightarrow L.V$:

$$([x],\, y) \longmapsto y - 2\langle y,x\rangle x \qquad (x,\, y \in S(V))$$

and $\quad y$

respectively. ($\langle y,x\rangle$ is the scalar product.)

Now s_0 and s_1 are homotopic outside the diagonal $\Delta(P)$. Indeed, $s_0 + s_1$ vanishes precisely on the diagonal and outside we may choose a linear homotopy $ts_0 + (1-t)s_1$ $(t \in I)$. The proof is completed by observing that s_0 and s_1 satisfy the transversality condition of (5.13) so that $\delta(s_0,s_1)$ is actually represented by the diagonal (with a certain isomorphism of the normal bundle with $H_2.V - \mathbb{R}$ which must be checked to be the standard one).

Here is a reformulation without the geometry. To interpret the duality class we follow through the Pontrjagin-Thom construction. The Thom class $u \in \omega^0(D(\tau P),S(\tau P); -\tau P)$ of the tangent bundle τP corresponds under the isomorphism $\tau P \oplus \mathbb{R} \cong H.V$ to a relative Euler class $\gamma(H.V,s)$ in $\omega^0((D(\mathbb{R}),S(\mathbb{R})) \times (D(\tau P),S(\tau P)); -H.V)$, where s is the section given on the orbit space by the $\mathbb{Z}/2$-equivariant map $D(\mathbb{R}) \times D(\tau S(L.V)) \longrightarrow L.V$ which takes the value $tz+v$ at the point specified by $t \in D(\mathbb{R})$, $z \in S(L.V)$, $v \in D(L.V)$ (in the usual representation of the tangent bundle to the sphere as the set of points $(z,v) \in S(V) \times V$ with v perpendicular to z). Embed $D(\tau P)$ as a tubular neighbourhood of the diagonal in $P \times P$, taking the point represented by (z,v) to $([z-\varepsilon v],[z+\varepsilon v])$ for some small positive ε. Then $\Delta_*(1) \in \omega^{-1}(P \times P; -H_2.V)$ is the image in $\omega^0((D(\mathbb{R}),S(\mathbb{R})) \times P \times P; -H_2.V)$ of $\gamma(H.V,s)$ under the excision isomorphism $(D(\tau P),S(\tau P)) \longrightarrow (P \times P, P \times P - \overset{\circ}{D}(\tau P))$ and restriction to $P \times P$. ($\overset{\circ}{D}$ is the open disc.)

$\theta(1)$ on the other hand may be described as $\gamma(H_2.V,\tilde{s})$ in $\omega^0((D(\mathbb{R}),S(\mathbb{R})) \times P \times P; -H_2.V)$, with $\tilde{s}$ given by the equivariant map $D(\mathbb{R}) \times P(V) \times S(L.V) \longrightarrow L.V$: $(t,[x],y) \longmapsto y-(1-t).\langle y,x\rangle x$. The zeros of $\tilde{s}$ occur on $0 \times \Delta(P)$. Its restriction to the tubular neighbourhood is given by $(z,v) \longmapsto tz+(2-t)\varepsilon v+O(\varepsilon^2)$, which agrees up to permitted homotopy with s. This establishes the equality of $\Delta_*(1)$ and $\theta(1)$ and hence the proposition.)

The reflection map R induces an isomorphism of homotopy groups in a certain range. $R_* : \pi_i(P(U \oplus V)/P(U)) \longrightarrow \pi_i(O(V,U \oplus V))$ is an isomorphism if $i < 2 \dim U$, an epimorphism if $i \leqslant 2 \dim U$. (This is proved by induction on $\dim V$, comparing the homotopy exact sequences of the cofibration sequence:

$$P(U \oplus V)/P(U) \longrightarrow P(U \oplus V \oplus \mathbb{R})/P(U) \longrightarrow (U \oplus V)^+$$

and the fibration sequence:

$$O(V,U \oplus V) \longrightarrow O(V \oplus \mathbb{R}, U \oplus V \oplus \mathbb{R}) \longrightarrow (U \oplus V)^+ .$$

See [42] 3.4 .) (2.7) implies a similar result for θ. (It can also be proved directly by induction on $\dim V$.)

Lemma (2.8). The local obstruction θ of (2.6) is a bijection if $\dim X + 1 < 2 \dim U$, a surjection if $\dim X < 2 \dim U$.

With (2.5) this lemma establishes a bijection, in the range $\dim X + 1 < 2 \dim U$, between $[X/Y;\ O(V,U \oplus V)]$ and the set of homotopy classes of nowhere-zero cross-sections of $H.(U \oplus V)$ over $X \times P(V)$ extending the standard section on $Y \times P(V)$. From this it is an easy step to the obstruction theory for r-fields.

Proposition (2.9). (X,ξ) as in (2.4), $r = \dim V$. $m < 2(n-r)$. Then ξ admits V as a trivial summand if and only if the Euler class

$\gamma(L.\xi) \in \omega^0_{\mathbb{Z}/2}(X; -L.\xi)$

of ξ with the antipodal involution is divisible by $\gamma(L.V)$ or, equivalently, if the Euler class $\gamma(H.\xi) \in \omega^0(X \times P(V); -H.\xi)$ of the tensor product of ξ with the Hopf line bundle H over $P(V)$ vanishes.

There is a corresponding classification theorem, of which (2.8) is a special case. The equivalence of the two conditions in (2.9) is given by the following lemma.

Lemma (2.10). There is a long exact sequence:

$$\cdots \to \omega^*_{\mathbb{Z}/2}(X; L.(V-\xi)) \longrightarrow \omega^*_{\mathbb{Z}/2}(X; -L.\xi) \longrightarrow \omega^*(X \times P(V); -H.\xi) \longrightarrow \cdots$$

in which the first step is multiplication by the Euler class $\gamma(L.V) \in \omega^0_{\mathbb{Z}/2}(\text{point}; -L.V)$.

Nothing more than the $\omega^*_{\mathbb{Z}/2}$-exact sequence of the pair $X \times (D(L.V), S(L.V))$, the contractible disc modulo the sphere, with coefficients $-L.\xi$, this sequence, in various guises, will be a recurrent theme.

As obstruction theory (2.9) is not profound. But it is convenient. Here is an illustration.

Example (2.11). Consider a finite covering $\pi : X' \longrightarrow X$ of odd degree. Then, in the metastable range of the proposition (2.9) a stable bundle ξ ($\dim \xi > \dim X$) admits an r-field if and only if its pullback $\pi^*\xi$ to X' does.

The stable cohomotopy Euler class of an odd-dimensional bundle is 2-primary torsion. (Apply (2.2)(iii) to the antipodal involution of the bundle. $1 - [a]$ has degree 2 in each fibre. This is the classical proof that the cohomology Euler class of an odd-dimensional bundle has order 2.) It follows that $\gamma(H.\xi)$ in (2.9) will be 2-primary torsion if ξ admits a sub-bundle of odd dimension. This is certainly true in the example. Now $\gamma(\pi^*\xi) = \pi^*\gamma(\xi)$ and $\pi_*\pi^*\gamma(\xi) = \pi_*(1).\gamma(\xi)$, where π_* is the transfer $\omega^0(X' \times P(V); -H.\pi^*\xi) \longrightarrow \omega^0(X \times P(V); -H.\xi)$ or $\omega^0(X') \longrightarrow \omega^0(X)$. Since $\pi_*(1)$ is invertible at the prime (2), the proof is done.

We resume the discussion of θ. Suppose that $U = 0$.

Write $\theta_s \in \{O(V);\ P(V)^+\}$ for the image under θ of $1 \in [O(V);\ O(V)]$. This stable map $\theta_s : O(V) \longrightarrow P(V)^+$ is a splitting of the reflection map $R : P(V)^+ \longrightarrow O(V)$. $\theta_s.R = 1$. The splitting is natural in the following sense. The projective orthogonal group $PO(V)$, the quotient of $O(V)$ by its centre, acts on $P(V)$ and $O(V)$. R is $PO(V)$-equivariant. The symmetry of θ_s is expressed by working in $PO(V)$-stable homotopy: it is naturally defined as a $PO(V)$-equivariant stable map. The proof of (2.7) respects the symmetry.

Remark (2.12). (James [38]). $P(V)^+$ is a $PO(V)$-equivariant stable retract of $O(V)$.

The construction of θ is also compatible with stabilization. For any vector space V', $O(V)$ is included in $O(V \oplus V')$ (as the subgroup fixing V') and $P(V)$ in $P(V \oplus V')$. Let Z be a compact ENR with base-point $*$. Then there is a commutative diagram:

$$\begin{array}{ccc} [Z;\ O(V)] & \longrightarrow & \{Z;\ P(V)^+\} \\ \downarrow & & \downarrow \\ [Z;\ O(V \oplus V')] & \longrightarrow & \{Z;\ P(V \oplus V')^+\} \end{array} .$$

In the limit (that is, for $\dim V > \dim Z + 1$) we obtain

(2.13) $\quad \theta : \widetilde{KO}^{-1}(Z) = [Z;\ O(\infty)] \longrightarrow \{Z;\ P(\infty)^+\}$,

where $O(\infty)$ is the infinite orthogonal group and $P(\infty)^+$ is the infinite real projective space with a base-point adjoined. This map will be described below, (3.14), in terms of the

An important element in our story will be the interplay between the equivariant and the non-equivariant theories. The group $\omega^{-1}_{\mathbb{Z}/2}((Z,*) \times S(L.V); -L.V) \cong \omega^{-1}((Z,*) \times P(V); -H.V)$ has been interpreted by S-duality as $\{Z;\ P(V)^+\}$. Forgetting the action of $\mathbb{Z}/2$, or lifting from the projective space to the sphere, we obtain a homomorphism i^* to $\omega^{-1}((Z,*) \times S(V); -V)$ $\cong \{Z;\ S(V)^+\}$ (again by duality). The notation refers to the inclusion $i : 0 \longrightarrow \mathbb{Z}/2$, and i^* will often be called restriction. In its dual aspect $i^* : \{Z;\ P(V)^+\} \longrightarrow \{Z;\ S(V)^+\}$ is the transfer with respect to the double cover $S(V) \longrightarrow P(V)$ (essentially by definition - of the transfer or of S-duality, according to taste). (See pp 37-9.)

The $\mathbb{Z}/2$-equivariant obstruction theory for cross-sections of real vector bundles translates easily into an S^1-theory for complex cross-sections of complex vector bundles and an S^3-theory for quaternionic bundles. We shall return to the complex theory in §6.

It is difficult to give appropriate acknowledgment for the obstruction theory outlined here. Most of it is already implicit in the work of A. Haefliger and M. W. Hirsch [33] and of I. M. James [38]. This account, taken from [25], was influenced particularly by M. F. Atiyah and J. L. Dupont [7], from whom I have taken the notation θ for the local obstruction. (The theory has been developed in a differential topological framework by J. P. Dax, H. A. Salomensen and U. Koschorke; see [55]. There is a related current in homotopy theory due to J. C. Becker and L. L. Larmore.)

3. Spherical fibrations

A closer inspection of the proof of (1.1) shows how to lift, in a natural way, a fibre-homotopy equivalence $S(\xi) \longrightarrow S(\xi')$ to an equivariant stable fibre-homotopy equivalence $S(L.\xi) \longrightarrow S(L.\xi')$. The final goal of this paragraph will be the extension of this result from sphere-bundles to spherical fibrations. We begin with the definition and splitting of the equivariant spherical fibration theory.

If V is a real vector space, we write $H(V)$ for the space, with base-point the identity, of homotopy equivalences $S(V) \longrightarrow S(V)$ (with the compact-open topology). If W is a real $\mathbb{Z}/2$-module, $H^{\mathbb{Z}/2}(W)$ denotes the subspace of $H(W)$ consisting of the $\mathbb{Z}/2$-homotopy equivalences. (It is open and closed in the subspace fixed by the involution, but not equal to it. It is, perhaps, neater to replace $H(V)$ by the homotopy fibre, $\tilde{H}(V)$, of the composition $H(V) \times H(V) \longrightarrow H(V)$. $\tilde{H}(V)$ is homotopy equivalent to $H(V)$ and the fixed subspace of $\tilde{H}(W)$ to $H^{\mathbb{Z}/2}(W)$.) Elements of $H^{\mathbb{Z}/2}(L.V)$ may be interpreted as cross-sections of the trivial bundle $L.V$ over the sphere $S(L.V)$. The difference construction defines, as in (2.6), a map

$$\zeta^{\mathbb{Z}/2} : \quad [Z;\ H^{\mathbb{Z}/2}(L.V)] \longrightarrow \{Z;\ P(V)^+\}$$

for any compact pointed ENR Z.

More generally, if, for any vector space E, $H^{\mathbb{Z}/2}(L.V;E)$ denotes the subspace of $H^{\mathbb{Z}/2}(L.V \oplus E)$ of maps $S(L.V \oplus E) \longrightarrow$

$S(L.V\oplus E)$ which extend the inclusion $S(E) \longrightarrow S(L.V\oplus E)$ of the fixed subspace, the difference construction applied to sections of $L.V\oplus E$ over $S(L.V\oplus E)$ agreeing with the standard section on the subspace $S(E)$ defines a map

$$(3.1) \qquad \zeta^{\mathbb{Z}/2} \;:\; [Z;\; H^{\mathbb{Z}/2}(L.V;E)] \longrightarrow \{Z;\; P(V)^+\} \;.$$

There is a corresponding map

$$(3.2) \qquad \zeta \;:\; [Z;\; H(V;E)] \longrightarrow \{Z;\; S(V)^+\}$$

in the non-equivariant case, such that $\zeta.i^* = i^*.\zeta^{\mathbb{Z}/2}$.

Proposition (3.3). Suppose that Z is connected. Then the maps $\zeta^{\mathbb{Z}/2}$ and ζ of (3.1) and (3.2) are bijective if $\dim Z < \dim V - 2$, surjective if $\dim Z < \dim V - 1$.

This is immediate from (2.5); connectivity is required because $H(V)$ is the space of homotopy equivalences, not arbitrary maps, $S(V) \longrightarrow S(V)$.

The next lemma introduces an important phenomenon in $\mathbb{Z}/2$-equivariant spherical fibration and stable cohomotopy theory: the splitting into free and fixed components.

Lemma (3.4). For any compact ENR Z with base-point and vector space V, there is a split short exact sequence of groups:

$$0 \to [Z;\; H^{\mathbb{Z}/2}(L.V;V)] \longrightarrow [Z;\; H^{\mathbb{Z}/2}(V\oplus L.V)] \underset{\sigma}{\overset{\rho}{\rightleftarrows}} [Z;\; H(V)] \longrightarrow 0.$$

The first map is given by the inclusion, ρ by the 'fixed point map' $H^{\mathbb{Z}/2}(V\oplus L.V) \longrightarrow H(V)$ taking an equivariant self-map of $S(V\oplus L.V)$ to the induced self-map of the subspace $S(V)$ fixed by the involution, and σ by the join with the identity (taking $f \in H(V)$ to the self-map $f * 1$ of $S(V) * S(L.V) \equiv S(V\oplus L.V)$). The group structure is given by composition.

This is simply the exact sequence of the fibration:

$$H^{\mathbb{Z}/2}(L.V;V) \longrightarrow H^{\mathbb{Z}/2}(V\oplus L.V) \longrightarrow H(V).$$

See [39].

Now write the <u>spherical fibration theory</u> in dimension -1 of a compact ENR X as

$$Sph^{-1}(X) := \varinjlim_{n\in\mathbb{N}} [X^+;\; H(\mathbb{R}^n)]$$

(the direct limit over the standard inclusions or, better, over the category of all Euclidean vector spaces and inclusions. $H(V)$ is included in $H(V\oplus V')$ by the join with $1 \in H(V')$.) It is an abelian group, the group of units in $\omega^0(X)$, and it will be written additively. More generally, if $Y \subseteq X$ is a closed sub-ENR, set

$$Sph^{-1}(X,Y) := \varinjlim [X/Y;\; H(\mathbb{R}^n)] \;.$$

The equivariant theory is defined to be

$$Sph^{-1}_{\mathbb{Z}/2}(X,Y) := \varinjlim [X/Y;\; H^{\mathbb{Z}/2}(\mathbb{R}^n\oplus L.\mathbb{R}^n)]$$

and the free spherical fibration theory

$$\mathrm{Sph}^{-1}_{\mathrm{free}\,\mathbb{Z}/2}(X,Y) := \varinjlim\ [X/Y;\ H^{\mathbb{Z}/2}(L.\mathbb{R}^n)]\ .$$

With the observation (from (3.3)) that the inclusion $H^{\mathbb{Z}/2}(L.V) \longrightarrow H^{\mathbb{Z}/2}(L.V;V)$ is highly connected if the dimension of V is large, we may rewrite (3.4) as a splitting of the $\mathbb{Z}/2$-theory.

Proposition (3.5). For any compact ENR pair (X,Y) there is a split short exact sequence of abelian groups:

$$0 \longrightarrow \mathrm{Sph}^{-1}_{\mathrm{free}\,\mathbb{Z}/2}(X,Y) \longrightarrow \mathrm{Sph}^{-1}_{\mathbb{Z}/2}(X,Y) \underset{\sigma}{\overset{\rho}{\rightleftarrows}} \mathrm{Sph}^{-1}(X,Y) \longrightarrow 0\ .$$

A doubling operation $S^2:\ \mathrm{Sph}^{-1}(X) \longrightarrow \mathrm{Sph}^{-1}_{\mathbb{Z}/2}(X)$ may be constructed as follows. Given $f \in H(V)$,

$$f * f:\quad S(V) * S(V) \longrightarrow S(V) * S(V)$$

is equivariant with respect to the switching map, and so, by the identification of $S(V \oplus V)$ with the involution which switches the factors with $S(V \oplus L.V)$, gives an element of $H^{\mathbb{Z}/2}(V \oplus L.V)$. This map $H(V) \longrightarrow H^{\mathbb{Z}/2}(V \oplus L.V)$ defines, in the limit, S^2.

Now clearly, since the fixed point set of $S(V) * S(V)$ is the diagonal, $\rho.S^2$ is the identity. Thus, $\bar{S}^2$ defined as in (1.1) by $\bar{S}^2(x) := S^2(x) - \sigma(x)$ is a map from $\mathrm{Sph}^{-1}(X)$ to $\mathrm{Sph}^{-1}_{\mathrm{free}\,\mathbb{Z}/2}(X)$. Moreover, the composition $i^*.S^2$ with the map

$$i^*:\quad \mathrm{Sph}^{-1}_{\mathbb{Z}/2}(X) \longrightarrow \mathrm{Sph}^{-1}(X)$$

which forgets the involution is multiplication by 2.

Theorem (3.6). (Kahn-Priddy according to Becker-Schultz [48], [15])

$i^*:\quad \mathrm{Sph}^{-1}_{\mathrm{free}\,\mathbb{Z}/2}(X) \longrightarrow \mathrm{Sph}^{-1}(X)$ is a split surjection.

$\bar{S}^2$ is the splitting. The original formulation of the theorem will appear in the next paragraph.

This same doubling construction may be performed globally. If $f: S(\xi) \longrightarrow S(\xi')$ is a stable fibre-homotopy equivalence, then $f * f:\ S(\xi) * S(\xi) \equiv S(\xi \oplus L.\xi) \longrightarrow S(\xi') * S(\xi') \equiv S(\xi' \oplus L.\xi')$ is an equivariant stable fibre-homotopy equivalence. Multiplying by a homotopy inverse $S(\xi') \longrightarrow S(\xi)$, we obtain an equivariant stable fibre-homotopy equivalence $\bar{S}^2(f):\ S(L.\xi) \longrightarrow S(L.\xi')$. The result may be stated rather formally as follows.

Theorem (3.7). Let $\mathcal{V}(X)$ be the category of real vector bundles over X with morphisms the stable fibre-homotopy equivalences of the associated sphere-bundles. Let $\mathcal{V}_{\mathrm{free}}(X)$ be the category of real vector bundles over X with the antipodal action of $\mathbb{Z}/2$ and morphisms the equivariant stable fibre-homotopy equivalences that is, a morphism $L.\xi \longrightarrow L.\xi'$ is to be an equivariant fibre-homotopy equivalence $S(L.\xi \oplus L.V) \longrightarrow S(L.\xi' \oplus L.V)$ for some real vector space V. Then there is a natural splitting (natural in X)

$\bar{S}^2:\quad \mathcal{V}(X) \longrightarrow \mathcal{V}_{\mathrm{free}}(X)$

of the restriction functor

$i^*:\quad \mathcal{V}_{\mathrm{free}}(X) \longrightarrow \mathcal{V}(X)$.

As observed in §1, the corresponding KO-theory is very simple.

Recall that the J-homomorphism

$$J: \quad KO^{-1}(X,Y) \longrightarrow Sph^{-1}(X,Y)$$

is defined as the limit of the maps $[X/Y;\ O(V)] \longrightarrow [X/Y;\ H(V)]$ given by the inclusion of $O(V)$ in $H(V)$. There is a similarly defined equivariant J-homomorphism

$$J_{\mathbb{Z}/2}: \quad KO^{-1}_{\mathbb{Z}/2}(X,Y) \longrightarrow Sph^{-1}_{\mathbb{Z}/2}(X,Y).$$

The operation $\bar{S}^2: KO^{-1}(X,Y) \longrightarrow KO^{-1}_{\mathbb{Z}/2}(X,Y)$, defined in §1, is just multiplication by the class $[L] \in KO_{\mathbb{Z}/2}(\text{point})$. $O(V)$ is a subspace of $H^{\mathbb{Z}/2}(L.V)$ and the composition $\bar{S}^2.J\ (= J_{\mathbb{Z}/2}.\bar{S}^2)$: $KO^{-1}(X,Y) \longrightarrow Sph^{-1}(X,Y) \longrightarrow Sph^{-1}_{\text{free }\mathbb{Z}/2}(X,Y)$ - the construction $\bar{S}^2$ extends to the relative group - is the limit of the induced maps $[X/Y;\ O(V)] \longrightarrow [X/Y;\ H^{\mathbb{Z}/2}(L.V)]$.

Again, the doubling construction may be carried through unstably to define a homomorphism $[Z;\ H(V)] \longrightarrow [Z;\ H^{\mathbb{Z}/2}(L.V;V)]$. Denote its composite with $\zeta^{\mathbb{Z}/2}$, (3.1), by

$$\eta: \quad [Z;\ H(V)] \longrightarrow \{Z;\ P(V)^+\}.$$

The corresponding unstable version of the 'free J-homomorphism' $KO^{-1}(Z,*) \longrightarrow Sph^{-1}_{\text{free }\mathbb{Z}/2}(Z,*)$ is the local obstruction θ.

Proposition (3.8). The properties of η are summarized in the commutative diagram:

$$\begin{array}{ccc}
[Z;\ O(V)] & \longrightarrow & [Z;\ H^{\mathbb{Z}/2}(L.V)] \\
\downarrow & \overset{\theta}{\searrow} & \downarrow \zeta^{\mathbb{Z}/2} \\
[Z;\ H(V)] & \xrightarrow{\ \eta\ } & \{Z;\ P(V)^+\} \\
 & \underset{\zeta}{\searrow} & \downarrow i^* \\
 & & \{Z;\ S(V)^+\}
\end{array}$$

More generally, writing $H(V,U\oplus V)$ for the homotopy fibre of the map of classifying spaces of spherical fibrations $BH(U) \longrightarrow BH(U\oplus V)$, one may define a map

$$\eta : [X/Y;\ H(V,U\oplus V)] \longrightarrow \{X/Y;\ P(U\oplus V)/P(U)\}$$

extending the local obstruction θ, (2.6). (For the application below it is enough to work with $\Omega H(V,U\oplus V)$, the homotopy fibre of $H(U) \longrightarrow H(U\oplus V)$, without mentioning classifying spaces.) Then, according to I. M. James [37], η is a bijection if $\dim X < 2(\dim U - 1)$. This leads to an obstruction theory for fibre-homotopy equivalences between sphere-bundles.

Proposition (3.9). If f is a fibre-homotopy equivalence $S(\xi) \longrightarrow S(\xi')$ between sphere-bundles over X, then

$$\gamma(L.\xi') = [\bar{S}^2(f)].\gamma(L.\xi) \quad \in \quad \omega^0_{\mathbb{Z}/2}(X;\ -L.\xi').$$

($\bar{S}^2(f)$ defines an element of $\omega^0_{\mathbb{Z}/2}(X; L.\xi - L.\xi')$.) Conversely, if $\dim X < 2(\dim \xi - 1)$, then a stable fibre-homotopy equivalence between ξ and ξ' satisfying this condition is representable by an actual fibre-homotopy equivalence $S(\xi) \longrightarrow S(\xi')$.

The first statement should be regarded as a generalization of the fibre-homotopy invariance of the Stiefel-Whitney classes of a vector bundle, which is the corresponding statement in equivariant cohomology theory with $\mathbb{F}_2$-coefficients.

Having considered in some detail dimension -1, we come to spherical fibration theory in dimension 0, which, of course, generalizes the former. Call a spherical fibration $E \longrightarrow X$ with an involution on E covering the identity on the base X a $\mathbb{Z}/2$-spherical fibration if every point of X has a neighbourhood U over which the restriction of E is $\mathbb{Z}/2$-fibre-homotopy equivalent to a product $U \times S(W) \longrightarrow U$ for some $\mathbb{Z}/2$-module W. The set of $\mathbb{Z}/2$-stable fibre-homotopy equivalence classes of such $\mathbb{Z}/2$-spherical fibrations is a monoid under the fibre-wise join and is embedded in the associated abelian group $Sph^0_{\mathbb{Z}/2}(X)$. $Sph^0_{free\ \mathbb{Z}/2}(X)$ will be the group of $\mathbb{Z}/2$-spherical fibrations with free involution, and $Sph^0(X)$ the usual group of (non-equivariant) spherical fibrations. Note that $Sph^0(\text{point}) = \mathbb{Z}$, $Sph^0_{free\ \mathbb{Z}/2}(\text{point}) = \mathbb{Z}$, $Sph^0_{\mathbb{Z}/2}(\text{point}) = \mathbb{Z} \oplus \mathbb{Z}$.

Theorem (3.10). $Sph^0_{\mathbb{Z}/2}(X)$ splits naturally as a direct sum $Sph^0(X) \oplus Sph^0_{free\ \mathbb{Z}/2}(X)$. $i^* : Sph^0_{free\ \mathbb{Z}/2}(X) \rightarrow Sph^0(X)$ is a split surjection.

A more precise statement can be made, as in (3.7), in the language of categories. Roughly speaking, a stable spherical fibration has, up to fibre-homotopy equivalence, a natural free involution.

For an odd prime p, there is a similar splitting of the

$\mathbb{Z}/p$-spherical fibration theory into free and fixed components. The operation $\bar{S}^p$ gives, not a splitting, but a lifting of (multiplication by) $p-1 : Sph^0(X) \longrightarrow Sph^0(X)$; this lifting is natural with respect to automorphisms of the group $\mathbb{Z}/p$ - it maps into the subgroup of $Sph^0_{free\ \mathbb{Z}/p}(X)$ fixed by the group of automorphisms $(\mathbb{Z}/p)^{\bullet}$. (3.8) and (3.9) are specific to $\mathbb{Z}/2$, as indeed is much of the impact of the theory; 2 is distinguished as the smallest prime!

The splitting (3.5) translates readily into a splitting of $\mathbb{Z}/2$-stable cohomotopy theory. $Sph^{-1}(X)$ is the group of units $\omega^0(X)^{\bullet}$ in stable cohomotopy.

Lemma (3.11). Let δ be the connecting homomorphism of the exact sequence of the pair $(D(V),S(V))$, (2.10):

$$\{X^+;\ S(V)^+\} = \omega^{-1}(X \times S(V); -V) \longrightarrow \omega^0(X).$$

An element $x \in [X^+; H(V)]$ defines an element $\bar{x} \in \omega^0(X)^{\bullet}$.

$$\delta\zeta(x) = \bar{x} - 1 \in \omega^0(X).$$

This is an elementary application of (2.3). $Sph^{-1}_{\mathbb{Z}/2}(X)$ is the group of units $\omega^0_{\mathbb{Z}/2}(X)^{\bullet}$ in equivariant cohomotopy. Affixing a label '$\mathbb{Z}/2$' where appropriate, we obtain:

Lemma (3.12). Let δ be the connecting homomorphism

$$\{X^+;\ P(V)^+\} = \omega^{-1}_{\mathbb{Z}/2}(X \times S(L.V); -L.V) \longrightarrow \omega^0_{\mathbb{Z}/2}(X)$$

of the pair $(D(L.V),S(L.V))$. An element $x \in [X^+; H^{\mathbb{Z}/2}(L.V)]$ determines an element $\bar{x} \in Sph^{-1}_{free\ \mathbb{Z}/2}(X) \subseteq Sph^{-1}_{\mathbb{Z}/2}(X) = \omega^0_{\mathbb{Z}/2}(X)^{\bullet}$.

$$\delta\zeta^{\mathbb{Z}/2}(x) = \bar{x} - 1 \in \omega^0_{\mathbb{Z}/2}(X).$$

Identify $Sph^{-1}(Z,*)$ and $\omega^0(Z,*)$ respectively with the kernels of the restriction maps $Sph^{-1}(Z) \longrightarrow Sph^{-1}(*) = \mathbb{Z}/2$ and $\omega^0(Z) \longrightarrow \omega^0(*) = \mathbb{Z}$. If Z is connected, then every element of $\omega^0(Z,*)$ is nilpotent and $Sph^{-1}(Z,*)$ is precisely the set $1+\omega^0(Z,*)$. (If Z is a suspension, then $Sph^{-1}(Z,*)$ and $\omega^0(Z,*)$ are isomorphic as groups.) The same is true in the equivariant theory and (3.5) may be rewritten as a splitting of $\omega^0_{\mathbb{Z}/2}(Z,*)$.

Proposition (3.13). For any compact ENR pair (X,Y) there is a split short exact sequence of abelian groups:

$$0 \longrightarrow \{X/Y;\ P(\infty)^+\} \xrightarrow{\delta} \omega^0_{\mathbb{Z}/2}(X,Y) \underset{\sigma}{\overset{\rho}{\rightleftarrows}} \omega^0(X,Y) \longrightarrow 0.$$

δ is the limit of the connecting homomorphisms (3.12).

The translation has been made using (3.3) and (3.12), on the assumption that X/Y is connected. To complete the proof it is sufficient to check the assertion when X is a point. (This may be done, for example, by looking at the set of connected components of the subspace of $H(V\oplus L.V)$ fixed by the involution.)

We are ready for the promised description of θ. Let

$$F: \ KO^{-1}(X) \longrightarrow KO^{-1}_{\mathbb{Z}/2}(X)$$

be multiplication by L; it is a splitting of the restriction i^*. The last proposition identifies $\{X^+;\ P(\infty)^+\}$ with an ideal of $\omega^0_{\mathbb{Z}/2}(X)$. (It is an ideal because δ in (3.12) is an $\omega^0_{\mathbb{Z}/2}(X)$-homomorphism.)

Remark (3.14). $J_{\mathbb{Z}/2}.F: \ KO^{-1}(X) \longrightarrow Sph^{-1}_{\mathbb{Z}/2}(X) = \omega^0_{\mathbb{Z}/2}(X)^{\bullet}$ and $\theta: \ KO^{-1}(X) \longrightarrow \{X^+;\ P(\infty)^+\}$ are related by the formula:

$$J_{\mathbb{Z}/2}.F(x) = 1+\theta(x).$$

θ is thus seen to be quadratic. If $x, y \in KO^{-1}(X)$, then $\theta(x+y) = \theta(x)+\theta(y)+\theta(x).\theta(y) \in \omega^0_{\mathbb{Z}/2}(X)$.

$\delta: \{X^+;\ S(\infty)^+\} \longrightarrow \omega^0(X)$ is an isomorphism, although in accordance with the sign conventions of §2 (adopted implicitly in (2.3) and (2.7)) it is the <u>negative</u> of the obvious one. ($\delta: \{X^+;\ S(V)^+\} \longrightarrow \omega^0(X)$ is evaluation on $-1 \in \omega^0(S(V))$.) With this identification $J(x) = 1+i^*\theta(x) \in \omega^0(X)^{\bullet}$.

Apart from A. Haefliger and M. W. Hirsch [33] and I. M. James [39], the presentation here draws from the paper [15] of J. C. Becker and R. E. Schultz. The splitting, (3.13), was first known from equivariant framed bordism theory, [77]. I owe much insight and, in particular, (3.4) to discussion with Prof. Becker. The proof of the Kahn-Priddy theorem given here is essentially that of G. B. Segal [80]; the translation will be clearer in §4. Independent expositions along these lines were given by the author [25] and L. M. Woodward [90]. It is a pleasure to acknowledge the influence of Dr. Woodward's (published and unpublished) work on the final form of this account.

4. Stable cohomotopy

This paragraph is concerned with the formal framework of $\mathbb{Z}/2$-equivariant stable cohomotopy theory. It has been clear for some time that it should be a bi-graded theory. (See, for example,[59].)

Definition (4.1). Let X be a compact $\mathbb{Z}/2$-ENR. Then the $\mathbb{Z}/2$-stable cohomotopy of X in dimension $(-i,-j)$

$$\omega^{-i,-j}(X) := \varinjlim_{m,n\in\mathbb{N}} [(\mathbb{R}^{j+m}\oplus L^{i-j+n})^+ \wedge X^+ ; (\mathbb{R}^m\oplus L^n)^+]^{\mathbb{Z}/2}.$$

(The limit is taken over the inclusions. L^n means the sum of n copies of L; in some contexts it is more natural to write nL.)

The choice of indexing is determined by the existence of two maps:

<u>restriction</u> $i^* : \omega^{-i,-j}(X) \longrightarrow \omega^{-i}(X)$ (forgetting the involution) and the

<u>fixed point map</u> $\rho : \omega^{-i,-j}(X) \longrightarrow \omega^{-j}(X^{\mathbb{Z}/2})$. (A map $f : (\mathbb{R}^{j+m}\oplus L^{i-j+n})^+ \wedge X^+ \longrightarrow (\mathbb{R}^m\oplus L^n)^+$ restricts to a map $\rho(f) : (\mathbb{R}^{j+m})^+ \wedge (X^{\mathbb{Z}/2})^+ \longrightarrow (\mathbb{R}^m)^+$ of the fixed point sets.)

A product can be defined in $\omega^{**}(X)$ such that the maps i^* and ρ are ring homomorphisms. The ring is 'graded commutative' in the sense that:

$$(4.2)\qquad x.x' = (-t)^{ii'}t^{jj'}x'.x \qquad (x\in\omega^{-i,-j}(X),\ x'\in\omega^{-i',-j'}(X))$$

where $t\in\omega^{0,0}(\text{point})$ is defined as the class of $(-1)^+ : (\mathbb{R}\oplus L)^+ \longrightarrow (\mathbb{R}\oplus L)^+$. $i^*(t) = 1$, $\rho(t) = -1$.

The <u>$\mathbb{Z}/2$-coefficient groups</u> will be written: $\omega_{i,j} := \omega^{-i,-j}(\text{point})$. They are related to the non-equivariant groups ω_i in the following lemma.

Let $b\in\omega_{-1,0}$ be the Euler class of L, that is, the element represented by the inclusion $0^+ \longrightarrow L^+$ (of the zero subspace in L). $i^*(b) = 0$, $\rho(b) = 1$.

Lemma (4.3). There is a long exact sequence:

$$\cdots \to \omega_{i+1,j} \xrightarrow{.b} \omega_{i,j} \xrightarrow{i^*} \omega_i \xrightarrow{i_*} \omega_{i,j-1} \longrightarrow \cdots$$

in which the successive maps are multiplication by $b\in\omega_{-1,0}$, restriction and induction, in the group-theoretic sense, along the inclusion $i : 0 \longrightarrow \mathbb{Z}/2$ of the trivial subgroup in $\mathbb{Z}/2$. The composite $i^*.i_* : \omega_i \to \omega_{i,j} \to \omega_i$ is multiplication by $1+(-1)^{i-j}$.

It is the $\mathbb{Z}/2$-stable cohomotopy exact sequence of the pair (D(L),S(L)), with coefficients.

The rather specialized splitting lemma (3.13) may be reformulated as follows.

Proposition (4.4). Consider, for any compact $\mathbb{Z}/2$-ENR X and integer j, the direct system

$$\cdots \to \omega^{-i,-j}(X) \xrightarrow{.b} \omega^{-i+1,-j}(X) \to \cdots$$

of abelian groups. Then the fixed point map ρ induces an isomorphism

$$\varinjlim_i \omega^{-i,-j}(X) \longrightarrow \omega^{-j}(X^{\mathbb{Z}/2}).$$

If the action is free on X, then the direct limit is easily seen to be zero. The essential case is that in which the action is trivial; assume this to be so. We need to consider relative groups; let $Y \subseteq X$ be a closed sub-ENR. The direct limit of the stable cohomotopy exact sequences of the pairs $(X,Y) \times (D(L^n), S(L^n))$ $(n \geqslant 1)$ is an exact sequence:

$$(4.5) \qquad \cdots \longrightarrow \{X/Y;\ P(\infty)^+\} \longrightarrow \omega^0_{\mathbb{Z}/2}(X,Y) \longrightarrow \varinjlim_n \omega^{-n,0}(X,Y) \longrightarrow \cdots .$$

($\omega^0_{\mathbb{Z}/2}(X,Y)$ is the name that we have given in earlier paragraphs to $\omega^{0,0}(X,Y)$.) Comparison of (4.5) and (3.13) proves the proposition. (There is a direct proof in Appendix A, (A.1).)

(4.4) and (4.5), with coefficients, relate the coefficient ring ω_{**} to the stable homotopy of stunted projective spaces. If $n > 0$, the infinite stunted real projective space P^∞_n is usually defined as the quotient $P(\infty)/P(\mathbb{R}^n)$. For present purposes P^∞_n for any n, positive or negative, is the Thom space P^{N+n}_n of the (virtual) bundle nH over $P(\mathbb{R}^{N+1})$ for 'sufficiently large N'. (Formally, it is the functor $\varinjlim_N \{-\,;\ P^{N+n}_n\}$ on compact pointed ENRs.)

Proposition (4.6). There is a long exact sequence:

$$\cdots \longrightarrow \tilde{\omega}_j(P^\infty_{j-i}) \xrightarrow{\ \delta\ } \omega_{i,j} \xrightarrow{\ \rho\ } \omega_j \longrightarrow \tilde{\omega}_{j-1}(P^\infty_{j-i}) \longrightarrow \cdots .$$

At the level (j,j) there is a splitting $\sigma : \omega_j \longrightarrow \omega_{j,j}$: $\rho.\sigma = 1$.

Remarks (4.7).

(i) $\omega_{0,0} = \mathbb{Z} \oplus \mathbb{Z}t$ $(t^2 = 1)$ is the Burnside ring.

(ii) $\omega_{i,j} = 0$ if $i < 0$ and $j < 0$,

$\tilde{\omega}_j(P^\infty_{j-i})$ if $j < -1$,

$\omega_j \oplus \tilde{\omega}_j(P^\infty_{j-i})$ for $i \leqslant j$ (with splitting given by $b^{j-i}.\sigma$).

(iii) $(\tilde{\omega}_j(P^\infty_{j-i}))_{i,j}$ is a graded ω_{**}-module.

(iv) The induction map i_* factors as a composition

$$\delta.\tilde{i}_* : \quad \omega_i \longrightarrow \tilde{\omega}_j(P^\infty_{j-i}) \longrightarrow \omega_{i,j}$$

of ω_{**}-module homomorphisms (that is, $\tilde{i}_*(i^*(x).y) = x.\tilde{i}_*(y)$ and $\delta(x.z) = x.\delta(z)$ if $x \in \omega_{**}$, $y \in \omega_*$, $z \in \tilde{\omega}_*(P^\infty_*)$).

(v) $i_* : \omega_0 = \mathbb{Z} \longrightarrow \omega_{0,0} = \mathbb{Z} \oplus \mathbb{Z}t$ takes 1 to $1+t$.

(vi) There is a further decomposition:

$$\omega_{j,j} = \omega_j \oplus \omega_j t \oplus \tilde{\omega}_j(P^\infty_1) .$$

($\tilde{i}_* : \omega_j \longrightarrow \tilde{\omega}_j(P^\infty_0) = \omega_j(P(\infty))$ is split by the map which collapses $P(\infty)$ to a point.)

The process of translation from spherical fibration theory to stable cohomotopy gives the Kahn-Priddy theorem (3.6) its original form.

Theorem (4.8). (Kahn-Priddy). Let Z be a connected compact ENR with base-point. Then the transfer

$$i^* : \quad \{Z;\ P(\infty)^+\} \longrightarrow \{Z;\ S(\infty)^+\}$$

is surjective.

(For an odd prime p, the transfer maps the subgroup of $\{Z;\ (B\mathbb{Z}/p)^+\}$ fixed by the group $(\mathbb{Z}/p)^*$ of automorphisms of $\mathbb{Z}/p$ onto $\{Z;\ S(\infty)^+\}$. It follows that the transfer $\{Z;\ (B\mathfrak{S}_p)^+\}[1/(p-1)!] \longrightarrow \{Z;\ S(\infty)^+\}[1/(p-1)!]$ is surjective.)

Let V be a vector space. The transfer $\pi_* : \omega^0(S(V)) \longrightarrow \omega^0(P(V))$ for the double cover $\pi : S(V) \longrightarrow P(V)$ coincides with induction i_* : $\omega^0(S(V)) \longrightarrow \omega^0_{\mathbb{Z}/2}(S(L.V))$. Write $\bar{t} \in \omega^0(P(V))$ for the element defined by $t \in \omega^0_{\mathbb{Z}/2}(\text{point})$; $-\bar{t}$ is represented by the antipodal involution $(-1)^+ : H^+ \longrightarrow H^+$ of the Hopf bundle. Then $\pi_*(1) = 1 + \bar{t}$, (4.7)(v), and, by the formal properties of the transfer, the theorem may be restated as the surjectivity, when $\dim Z < \dim V$, of the map $\{Z;\ P(V)^+\} \rightarrow \tilde{\omega}^0(Z)$ given by evaluation on $-(1+\bar{t}) \in \omega^0(P(V))$ $(= \tilde{\omega}^0(P(V)^+))$.

Corollary (4.9). (Kahn-Priddy). Z connected, $\dim Z < \dim V$. Then evaluation on the torsion element $1 - \bar{t} \in \omega^0(P(V))$:

$$\{Z;\ P(V)^+\}_{(2)} \longrightarrow \tilde{\omega}^0(Z)_{(2)}$$

is surjective at the prime (2).

This is immediate. It follows that the order of $1 - \bar{t}$, to be computed in §7, bounds the exponent of the 2-torsion of $\tilde{\omega}^0(Z)$.

Being surjective and a natural transformation on the stable homotopy category, the transfer i^* of (4.8) has a splitting. However, the translation of $\bar{S}^2$, which we must now discuss, will in general be non-linear.

For a compact ENR X, the doubling operation S^2 of §3 was a homomorphism $\omega^0(X)^\cdot \longrightarrow \omega^0_{\mathbb{Z}/2}(X)^\cdot$ from the group of units in $\omega^0(X)$ to that in $\omega^0_{\mathbb{Z}/2}(X)$. It has an evident extension to a <u>squaring operation</u>

$$P^2 : \omega^0(X) \longrightarrow \omega^0_{\mathbb{Z}/2}(X),$$

or, without restriction to the diagonal in $X \times X$, an external operation

$$P^2 : \omega^0(X) \longrightarrow \omega^0_{\mathbb{Z}/2}(X \times X),$$

taking the class of a map $f : (\mathbb{R}^n)^+ \wedge X^+ \longrightarrow (\mathbb{R}^n)^+$ to that of $f \wedge f : (\mathbb{R}^n \oplus \mathbb{R}^n)^+ \wedge (X \times X)^+ \longrightarrow (\mathbb{R}^n \oplus \mathbb{R}^n)^+$ (with the switching involution). The same construction defines an operation

$$P^2 : \omega^{-j}(X) \longrightarrow \omega^{-2j,-j}(X).$$

It is quite different in character from S^2, a much more sophisticated concept.

Lemma (4.10). The squaring operation $P^2 : \omega_j \longrightarrow \omega_{2j,j}$ on the coefficient ring has the following properties.

(i) $P^2(x.x') = P^2(x).P^2(x')$ $(x, x' \in \omega_*)$.

(ii) $P^2(x+y) = P^2(x) + i_*(x.y) + P^2(y)$ $(x, y \in \omega_j)$.

(iii) $i^*.P^2(x) = x^2$.

(iv) $\rho.P^2(x) = x$.

Corollary (4.11). (Bredon [19]). $\rho : \omega_{i,j} \longrightarrow \omega_j$ is an epimorphism if $i \leqslant 2j$. It is split if $i < 2j$.

Although P^2 is not in general linear, $b.P^2$ is (by (4.7)(iv), that is, Frobenius reciprocity, since $i^*(b) = 0$).

Analogous to $\bar{S}^2$, define

$$\bar{P}^2 : \omega_j \longrightarrow \omega_j(P(\infty)) \subseteq \omega_{j,j}$$

by $\bar{P}^2(x) := b^j.P^2(x) - \sigma(x)$. Then, if $j > 0$, $-\bar{P}^2$ (notice the sign) is a splitting of the restriction map $i^* : \omega_j(P(\infty)) \longrightarrow$

ω_j, and we have reproduced the proof of the Kahn-Priddy theorem, this time exactly following G. B. Segal [80].

For the coefficient ring it is convenient to state the result in the following form; compare (7.11). a(i+2) is the Hurwitz-Radon number.

Corollary (4.12). Suppose that $i > 0$ and $i-j \equiv 0 \pmod{a(i+2)}$. Then $i^* : \omega_{i,j} \longrightarrow \omega_i$ is surjective.

With the machinery at hand it would be a pity to omit an account of Nishida's theorem (at the prime (2)).

Theorem (4.13). (Nishida [71]). Every torsion element in the graded ring ω_* is nilpotent.

We establish the basic lemma from which the theorem, at (2), easily follows.

Lemma (4.14). Suppose that $x \in \omega_p$ with $2^s x = 0$ $(s > 0)$ and $y \in \omega_q$ with <u>$q > 0$</u> and $p \equiv 0 \pmod{a(q+2)}$. Then

$$2^{s-1}x^2y \in 2^s\omega_{2p+q} .$$

Consider the square $P^2 : \omega_p \longrightarrow \omega_{2p,p}$. By (4.10)(ii),

$$0 = P^2(2^s x) = 2^s P^2(x) + 2^{s-1}(2^s-1)i_*(x^2) .$$

According to (4.12) there is a class $\tilde{y} \in \omega_{q,p+q}$ which restricts to y. Then

$$\begin{aligned} 2^{s-1}i_*(x^2y) &= 2^{s-1}i_*(x^2.i^*(\tilde{y})) \\ &= 2^{s-1}i_*(x^2).\tilde{y} \qquad \text{(Frobenius reciprocity)} \\ &= 2^s(P^2(x) + 2^{s-1}i_*(x^2)).\tilde{y} \in \omega_{2p+q,2p+q} . \end{aligned}$$

But $i_* : \omega_j \longrightarrow \omega_{j,j}$ is the inclusion of a direct summand, (4.7)(vi), and the proof is done.

Corollary (4.15). Every torsion element in the $\mathbb{Z}/2$-coefficient ring ω_{**} is nilpotent.

Indeed, consider a torsion element in ω_{**}. By Nishida's theorem, some power x, say, in $\omega_{i,j}$, satisfies $i^*(x) = 0$ and $\rho(x) = 0$. If $i < 0$, there is nothing further to prove: $x = 0$. If $i \geq 0$, then $b^{i+1}x = 0$, (4.6), and x is divisible by b, (4.3); hence x^{i+2} is zero.

The operation $\bar{P}^2$ is well known, either as the (generalized) <u>Hopf invariant</u> or as Segal's operation θ^2, [80]. Its relation to θ^2 will be discussed in the next paragraph. As Hopf invariant $\bar{P}^2$ has the following properties.

(4.16) $j \equiv 1 \pmod 2$. Then the composition with the Hurewicz homomorphism to $\mathbb{F}_2$-homology is the classical Hopf invariant: $\omega_j \longrightarrow \tilde{\omega}_j(P_0^\infty) \longrightarrow \tilde{H}_j(P_0) = \mathbb{F}_2$.

(4.17) $j \equiv 3 \pmod 4$. Then the composition with the Hurewicz homomorphism to real KO_*-theory is the 2-primary e-invariant: $\omega_j \longrightarrow \tilde{\omega}_j(P_0^\infty) \longrightarrow \widetilde{KO}_j(P_0^\infty) = \mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$.

The connection of $\bar{P}^2$ with the Hopf invariant H of the EHP-sequence fits less happily into the present framework; it is discussed in the Appendix A.

The construction of squaring operations goes back to Steenrod's definition in $\mathbb{F}_2$-cohomology. In stable homotopy it was employed by G. E. Bredon in [19]. The notation P^2 is taken from M. F. Atiyah [4]. The proof of (4.13) is Nishida's original proof (clarified for the author by an exposition given by G. B. Segal). (An account in the language of framed manifolds was given by J. Jones in [45]. (4.12) was isolated by J. Mukai in [70].)

5. Framed manifolds

It is not our business here to discuss the bordism theory of $\mathbb{Z}/2$-framed manifolds, but simply to put the homotopy-theoretic concepts of the last paragraph into a geometric setting. All manifolds will be smooth.

For any closed manifold X (understood to be compact), with tangent bundle τX, the Pontrjagin-Thom construction defines what is variously called the <u>index</u>, <u>direct image</u> or <u>Umkehr homomorphism</u>:

$$\omega^*(X; -\tau X) \longrightarrow \omega^*(\text{point}).$$

(Recollect that $\omega^*(X; -\tau X)$ is, by definition, the stable cohomotopy $\tilde{\omega}^{*+n}(T(\nu))$ of the Thom space of the normal bundle ν of some embedding of X in $\mathbb{R}^n$.) And, in general, for any map $f : X \longrightarrow Y$ of closed manifolds one has $f_* : \omega^*(X; -\tau X) \longrightarrow \omega^*(Y; -\tau Y)$, or with coefficients in a virtual bundle α over Y:

$$(5.1) \qquad f_* : \omega^*(X; f^*\alpha - \tau X) \longrightarrow \omega^*(Y; \alpha - \tau Y).$$

f_* is induced by a stable map of Thom spaces $T(\alpha - \tau Y) \longrightarrow T(f^*\alpha - \tau X)$ - the Atiyah S-dual of $f : T(-f^*\alpha) \longrightarrow T(-\alpha)$. (The Atiyah duality is actually defined by an index construction.)

Example (5.2). Consider the unit sphere S(V) in a vector space V. $\tau S(V) \oplus \mathbb{R} = V$ (where, to be definite, $1 \in \mathbb{R}$ is identified with the outward unit normal vector of the embedding $S(V) \subseteq V$). The index map

$$\omega^0(S(V); -\tau S(V)) \longrightarrow \omega^0(\text{point})$$

coincides, up to a minus sign, with the connecting homomorphism

$$\omega^{-1}(S(V);-V) \longrightarrow \omega^{0}(\text{point})$$

of the exact sequence of the pair $(D(V),S(V))$.

There are two generalizations of the index construction: to an equivariant theory and to a fibre-bundle theory. (The account by M. F. Atiyah and I. M. Singer,[10] I and IV, is recommended.)

Example (5.3). Induction. Let $i : H \longrightarrow G$ be the inclusion of a closed subgroup H in a compact Lie group G. Write the Lie algebras, with the adjoint action of the group, as $\mathfrak{h}$ and $\mathfrak{g}$ respectively. The composition

$$\omega^{0}_{G}(G/H;\, G\times_{H}\mathfrak{h}) \longrightarrow \omega^{0}_{H}(G/H;\, G\times_{H}\mathfrak{h}) \longrightarrow \omega^{0}_{H}(\text{point};\mathfrak{h})$$

of the forgetful map i^* and restriction to the base-point $H \in G/H$ is an isomorphism; compare (7.7). Now the tangent bundle $\tau(G/H)$ is $G\times_{H}(\mathfrak{g}/\mathfrak{h})$ and we have an index map $\omega^{0}_{G}(G/H;\, G\times_{H}\mathfrak{h}) \longrightarrow \omega^{0}_{G}(\text{point};\mathfrak{g})$ defining <u>group-theoretic</u> <u>induction</u>:

$$i_* : \omega^{0}_{H}(\text{point};\mathfrak{h}) \longrightarrow \omega^{0}_{G}(\text{point};\mathfrak{g})$$

and more generally

$$i_* : \omega^{0}_{H}(X;\mathfrak{h}) \longrightarrow \omega^{0}_{G}(X;\mathfrak{g})$$

for any compact G-ENR X. (Introducing coefficients $-\mathfrak{h}$ and multiplying by the Euler class $\gamma(\mathfrak{g}/\mathfrak{h}) \in \omega^{0}_{H}(\text{point};-\mathfrak{g}/\mathfrak{h})$ one obtains a map $i_*.\gamma(\mathfrak{g}/\mathfrak{h}) : \omega^{0}_{H}(X) \longrightarrow \omega^{0}_{G}(X)$. The corresponding K-theory is discussed in [75].)

In the fibre-bundle theory one considers <u>manifolds</u> <u>over</u> a fixed compact ENR <u>B</u>, that is, locally trivial fibre-bundles over B with fibre a closed manifold and structure group the group of diffeomorphisms of the manifold. Such a manifold $\pi : E \longrightarrow B$ over B has a bundle $\tau(\pi)$ of tangents along the fibres and there

is an index:

$$\pi_* : \omega^*(E;-\tau(\pi)) \longrightarrow \omega^*(B).$$

Everything works as before, only fibre-wise over B. The index with coefficients in a virtual bundle α over B is represented by a stable map over B : $T_B(\alpha) \longrightarrow T_B(\pi^*\alpha - \tau(\pi))$ between bundles of Thom spaces and this is the S-dual over B of the map $T_B(-\pi^*\alpha) \longrightarrow T_B(-\alpha)$ induced by π. Collapsing the base-points of all the fibres to a single point, we obtain a stable map $T(\alpha) \longrightarrow T(\pi^*\alpha - \tau(\pi))$. If B happens to be a closed manifold and π a smooth fibre-bundle, then this map is S-dual to the map $T(-\pi^*\alpha - \pi^*\tau B) \longrightarrow T(-\alpha - \tau B)$ defined by π; τE splits as $\pi^*\tau B \oplus \tau(\pi)$. In other words, the definition of $\pi_* : \omega^*(E;\pi^*\alpha - \tau(\pi)) \longrightarrow \omega^*(B;\alpha)$ is consistent with (5.1). It is also consistent with the equivariant theory; if $P \longrightarrow B$ is a principal G-bundle and X a closed G-manifold, there is a commutative square:

$$\begin{array}{ccccc}
\omega^*_G(X;-\tau X) & \longrightarrow & \omega^*_G(P\times X;-\tau X) & = & \omega^*(P\times_G X;-P\times_G \tau X) \\
\downarrow & & & & \downarrow \\
\omega^*_G(\text{point}) & \longrightarrow & \omega^*_G(P) & = & \omega^*(B).
\end{array}$$

When π is a finite cover it is customary to call $\pi_* : \omega^*(E) \longrightarrow \omega^*(B)$ the <u>transfer</u>. (The Becker-Gottlieb transfer in general is the composition $\pi_*.\gamma(\tau(\pi)) : \omega^*(E) \longrightarrow \omega^*(B)$ with multiplication by the Euler class.)

Our first subject is the representation of stable homotopy classes by framed manifolds. A framing of a closed manifold X is a stable isomorphism $\tau X \cong \mathbb{R}^i$. The index homomorphism may be written: $\omega^{*+i}(X) \longrightarrow \omega^*(\text{point})$. The class in the stable i-stem ω_i represented by X is the image of $1 \in \omega^0(X)$. In the same way a manifold X with a (smooth) involution and a $\mathbb{Z}/2$-equivariant framing, that is, an isomorphism

$$\tau X \oplus \mathbb{R}^m \oplus L^n \cong \mathbb{R}^{j+m} \oplus L^{i-j+n}$$

for some m, n, represents an element of $\omega_{i,j}$.

In equivariant bordism theory $\omega_{i,i}$ is realized as the bordism group of a restricted class of framed manifolds ([77], [34], [72]). The components of the splitting $\omega_{i,i} = \omega_i \oplus \omega_i(P(\infty))$ may be represented by manifolds with trivial and free involution respectively. The first is clear. For the second, interpret $\omega_i(P(\infty))$ as the framed bordism group of $P(\infty)$. An element is given by a framed manifold X with a map $g: X \longrightarrow P(\infty)$. Let $\pi: \tilde{X} \longrightarrow X$ be the pullback of the universal double-cover $S(\infty) \longrightarrow P(\infty)$. Then $\tilde{X}$ with the covering involution is a free $\mathbb{Z}/2$-manifold; it is equipped with a stable framing $\tau\tilde{X} \cong \mathbb{R}^i$ lifted from the framing of X.

Proposition (5.4). The $\mathbb{Z}/2$-framed manifold $\tilde{X}$ so constructed represents the class $-[X,g] \in \omega_i(P(\infty)) \subseteq \omega_{i,i}$.

In order to use duality we must replace $P(\infty)$ by a finite projective space $P(V)$. $g: X \longrightarrow P(V)$ lifts to a $\mathbb{Z}/2$-map $g: \tilde{X} \longrightarrow S(L.V)$. The proof, using (equivariant) (5.2), is just the transitivity of the index:

$$\begin{array}{ccccc}
\omega^0(X) & \xrightarrow{g_*} & \omega^{-i-1}(P(V); -H.V) & & \\
\cong\big| & & \big|\cong & & \\
\omega^0_{\mathbb{Z}/2}(\tilde{X}) & \xrightarrow{g_*} & \omega^{-i-1}_{\mathbb{Z}/2}(S(L.V); -L.V) & \xrightarrow{\sigma} & \omega^{-i}_{\mathbb{Z}/2}(\text{point}) .
\end{array}$$

$\tilde{X}$ represents $-\sigma.g_*(1)$.

The restriction $i^*: \omega_{i,i} \longrightarrow \omega_i$ (or, in other language, the transfer $\omega_i(P(\infty)) \longrightarrow \omega_i(S(\infty))$) takes the element defined by $\tilde{X}$ to the class $[\tilde{X}]$ of the framed manifold with the involution forgotten. According to the Kahn-Priddy theorem, if $i > 0$ then every element of ω_i may be so represented.

Now the group $KO^{-1}(X) = [X^+; O(\infty)]$ acts, freely and transitively, on the set of framings of X. Let X' denote the manifold X with framing twisted by the element

$$a = R.g : X \longrightarrow P(\infty) \longrightarrow O(\infty)$$

of order 2 in $KO^{-1}(X)$.

Lemma (5.5). (Ray, Brown, Jones [74], [22], [45]).

$$[\tilde{X}] = [X] - [X'] \quad \text{in } \omega_i.$$

It will be sufficient to show that $\pi_*: \omega^0(\tilde{X}) \longrightarrow \omega^0(X)$ takes 1 to $1 - J(a)$, where $J: KO^{-1}(X) \longrightarrow \omega^0(X)^{\bullet}$ is the J-homomorphism to the group of units in the stable cohomotopy ring. For the index $\omega^0(X) \longrightarrow \omega_i$ defined by the framing of X takes $J(a)$ to $[X']$ and $\pi_*(1)$, by the transitivity of the index, to $[\tilde{X}]$. This follows, by taking the balanced product $\tilde{X} \times_{\mathbb{Z}/2}$, from the universal statement (from (4.7)(v)):

Lemma (5.6). The equivariant transfer (or index for the 'standard' framing - there are only two - of S(L))

$$\omega^0_{\mathbb{Z}/2}(S(L)) = \mathbb{Z} \longrightarrow \omega^0_{\mathbb{Z}/2}(\text{point}) = \mathbb{Z} \oplus \mathbb{Z}t$$

maps 1 to 1+t.

It may be instructive to state the generalization of (5.6) from the real/S^0 ($= \mathbb{Z}/2$) to the complex/S^1 and quaternionic/S^3 cases. Write $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$; $G = S(\mathbb{K})$ - the group of units of norm 1, with adjoint representation $\mathfrak{g}$. We can think of $\mathbb{K}$ as a (left) G-module in two ways: with the action of G by left multiplication, denoted by E, or with the action by conjugation, when it splits as $\mathbb{R}\oplus\mathfrak{g}$. The group itself may similarly be considered as a G-manifold either as S(E), written G_1, or as $S(\mathbb{R}\oplus\mathfrak{g})$, written G_{ad}. G_1 will be given the framing arising from the natural trivialization

$$G_1 \times \mathfrak{g} \longrightarrow \tau G_1 \quad : \quad (x,v) \longmapsto v.x \,.$$

(That is to say, the tangent space at x is identified with the tangent space $\mathfrak{g}$ at 1 by right translation.) It represents an element $[G_1] \in \omega^0_G(\text{point};\mathfrak{g})$. As the boundary of the disc $D(\mathbb{R}\oplus\mathfrak{g})$, $S(\mathbb{R}\oplus\mathfrak{g})$ has an obvious framing $\mathbb{R}\oplus\tau \cong \mathbb{R}\oplus\mathfrak{g}$; the framing we choose differs from it by the equivariant twisting r :

$$S(\mathbb{R}\oplus\mathfrak{g}) = G_{ad} \longrightarrow O(E)$$

defining the representation E.

Proposition (5.7). (Knapp, Stolz [54], [83]).
$[G_1] = [S(\mathbb{R}\oplus\mathfrak{g})]$ in $\omega^0_G(\text{point};\mathfrak{g})$.

However, the two manifolds are certainly not equivariantly framed-cobordant.

The key to the proof is the observation that $[G_1]$ and $[G_{ad}]$

lift in the exact sequence of the pair (D(E),S(E)) to $\omega^{-1}_G(S(E);\mathfrak{g}-E)$. For $G_1 = S(E)$ this is clear, by (5.2). To describe $[G_{ad}]$, identify $\mathfrak{g}^+$ with $S(\mathbb{R}\oplus\mathfrak{g}) = G_{ad}$ (with base-point 1) by mapping $x \in \mathfrak{g}$ to $(-1+x)/|-1+x|$. Then

$$[G_{ad}] = J(\bar{r}) - 1 \quad \text{in} \quad \omega^0_G(S(\mathbb{R}\oplus\mathfrak{g}),*) \subseteq \omega^0_G(S(\mathbb{R}\oplus\mathfrak{g})),$$

where J is the equivariant J-homomorphism $KO^{-1}_G(S(\mathbb{R}\oplus\mathfrak{g})) \longrightarrow \omega^0_G(S(\mathbb{R}\oplus\mathfrak{g}))^{\cdot}$ and $\bar{r} \in KO^{-1}_G(S(\mathbb{R}\oplus\mathfrak{g}),*) \subseteq KO^{-1}_G(S(\mathbb{R}\oplus\mathfrak{g}))$ is the class defined by r. ($\bar{r} \in KO_G(\mathbb{R}\oplus\mathfrak{g})$ - K-theory with compact supports - is represented by the endomorphism of the trivial bundle $E = \mathbb{K}$ over $\mathbb{R}\oplus\mathfrak{g} = \mathbb{K}$ given by left multiplication by the element of the base. It has additional structure as an element of real, complex or quaternionic K-theory.) Now $J(\bar{r}) - 1$ is lifted to $\omega^{-1}_G(S(E) \times (S(\mathbb{R}\oplus\mathfrak{g}),*);-E)$ as the difference class of the two cross-sections of S(E) defined by r and 1, (3.11).

Little remains to be proved: $\omega^{-1}_G(S(E);\mathfrak{g}-E) \cong \mathbb{Z}$. (But it is neater not to use this fact. The identification is natural with respect to automorphisms of $\mathbb{K}$ and the equality (5.7) is valid in $\omega^0_{G\rtimes \mathrm{Aut}(\mathbb{K})}(\text{point};\mathfrak{g})$ - the equivariant theory of the semidirect product of G and the automorphism group of $\mathbb{K}$.)

To translate the result into cobordism theory, let $P \longrightarrow X$ be a smooth principal G-bundle over a framed closed manifold. The balanced product $P \times_G$ then takes equivariantly framed G-manifolds to framed manifolds. According to (5.7), $P = P \times_G G_1$ is cobordant to $P \times_G S(\mathbb{R}\oplus\mathfrak{g})$ - so if $G = S^1$ to the product $X \times S^1$ - with a certain framing (as could have been deduced from (2.7) and (3.14)).

The remainder of the paragraph treats of the squaring operation and the Hopf invariant. Cancelling the earlier notation, we let X be a framed manifold representing a class $x \in \omega_m$. Then $X \times X$ with the involution which interchanges the factors has an evident $\mathbb{Z}/2$-framing: $\tau(X \times X)$ is stably isomorphic to $\mathbb{R}^m \oplus \mathbb{R}^m$ with the switching involution, that is, to $\mathbb{R}^m \oplus L^m$. (And, of course, if $E \longrightarrow B$ is a smooth principal $\mathbb{Z}/2$-bundle, the tangent bundle of $E \times_{\mathbb{Z}/2} (X \times X)$ is stably the pullback of the bundle $\tau B \oplus \mathbb{R}^m \oplus \lambda^m$ over B, where λ is the line bundle $E \times_{\mathbb{Z}/2} L$ corresponding to the double cover.)

Lemma (5.8). $X \times X$ represents $P^2(x)$ in $\omega_{2m,m}$.

The description of $\bar{P}^2(x) = b^m P^2(x) - \sigma(x) \in \omega_{m,m}$, pictorially just the square $X \times X$ minus the diagonal, has a new subtlety. Let $\Delta : X \longrightarrow X \times X$ be the inclusion of the diagonal. Since the normal bundle is stably L^m, there is an index homomorphism:

$$\Delta_* : \quad \omega^0_{\mathbb{Z}/2}(X) \longrightarrow \omega^0_{\mathbb{Z}/2}(X \times X; -mL) .$$

By the transitivity of the index,

(5.9) $$\bar{P}^2(x) = \operatorname{index}(b^m - \Delta_*(1)) \in \omega_{m,m} .$$

Consider the now familiar direct limit of the stable cohomotopy exact sequences of $(X \times X) \times (D(nL), S(nL))$. It may be rewritten in the following form.

Lemma (5.10). For any framed manifold X of dimension m,

there is a split short exact sequence:

$$0 \longrightarrow \omega_m(S(\infty) \times_{\mathbb{Z}/2} (X \times X)) \longrightarrow \omega^0_{\mathbb{Z}/2}(X \times X; -mL) \underset{\longleftarrow}{\overset{\rho}{\longrightarrow}} \omega^0(X) \longrightarrow 0.$$

The first term is the S-dual of $\omega^{-1}_{\mathbb{Z}/2}(S(nL) \times (X \times X); -(m+n)L)$ $= \omega^{-1}(S(nL) \times_{\mathbb{Z}/2} (X \times X); -(m+n)H)$ for large n; the final term is given by (4.4). The splitting is the composition

$$\Delta_* \cdot \sigma : \quad \omega^0(X) \to \omega^0_{\mathbb{Z}/2}(X) \longrightarrow \omega^0_{\mathbb{Z}/2}(X \times X; -mL).$$

Hence $b^m - \Delta_*(1)$ actually defines an element of $\omega_m(S(\infty) \times_{\mathbb{Z}/2} (X \times X))$ whose image in $\omega_m(P(\infty))$ under the map $S(\infty) \times_{\mathbb{Z}/2} (X \times X) \longrightarrow S(\infty) \times_{\mathbb{Z}/2} * = P(\infty)$ collapsing $X \times X$ to a point is $\bar{P}^2(x)$. It is illuminating to describe this element directly. Choose an embedding $i : X \longrightarrow \mathbb{R}^{m+n}$ with trivial normal bundle $\mathbb{R}^n$ corresponding to the given stable trivialization of τX. It extends to a tubular neighbourhood $i : X \times \mathbb{R}^n \longrightarrow \mathbb{R}^{m+n}$. Now note that, if V is a real vector space, the complement $V \times V - V$ of the diagonal in $V \times V$ with the switching involution is $(L.V - 0) \times V$, which is diffeomorphic to $S(L.V) \times \mathbb{R} \times V$ and as far as homotopy is concerned just $S(L.V)$. The restriction of $i \times i$ defines a map $(X \times X) \times (\mathbb{R}^n \times \mathbb{R}^n - \mathbb{R}^n) \longrightarrow (\mathbb{R}^{m+n} \times \mathbb{R}^{m+n} - \mathbb{R}^{m+n})$, so up to homotopy a $\mathbb{Z}/2$-map

$$s_1 : \quad (X \times X) \times S(nL) \longrightarrow S((m+n)L) .$$

Let s_0 be the composition $(X \times X) \times S(nL) \longrightarrow S(nL) \longrightarrow S((m+n)L)$ of the projection and the standard inclusion.

Lemma (5.11). The difference class $\delta(s_0, s_1)$ in $\omega^{-1}_{\mathbb{Z}/2}((X \times X) \times S(nL); -(m+n)L)$ of s_0 and s_1 regarded as sections of the trivial bundle $(m+n)L$ maps to $b^m - \Delta_*(1)$ in $\omega^0_{\mathbb{Z}/2}(X \times X; -mL)$.

Here is the reasoning. By (2.3) $\delta(s_0, s_1)$ is mapped to the difference of relative Euler classes

$$\gamma((m+n)L, s_0) - \gamma((m+n)L, s_1) \in \omega^0_{\mathbb{Z}/2}((X \times X) \times (D(nL), S(nL)); -(m+n)L).$$

The first is, by definition, b^m times the 'Thom class' in $\omega^0_{\mathbb{Z}/2}(D(nL), S(nL); -nL)$. As for the second, notice that away from the diagonal $X \subseteq X \times X$ s_1 factors up to homotopy through the projection onto $X \times X - X$ and so extends to a nowhere-zero section on $(X \times X - X) \times D(nL)$. Thus the obstruction is concentrated on the diagonal. Take an equivariant tubular neighbourhood $D(L.\tau X) \subseteq X \times X$ of the diagonal. Then on this neighbourhood $s_1 : D(L.\tau) \times S(nL) \longrightarrow S((m+n)L)$ is simply, up to homotopy, the restriction of the map

$$D(L.\tau) \times D(nL) \;\equiv\; D(L.(\tau \oplus \mathbb{R}^n)) \;\cong\; D(L.\mathbb{R}^{m+n})$$

given by the stable trivialization $\tau \oplus \mathbb{R}^n \cong \mathbb{R}^{m+n}$ of τX. By inspection the relative Euler class is $\Delta_*(1)$.

It is amusing to translate into the language of differential topology the proof of the Kahn-Priddy theorem in the form:

(5.12) Any framed manifold X of dimension $m > 0$ is cobordant to a framed manifold Y admitting a free involution compatible with the framing (that is, $\tau Y \cong \mathbb{R}^m$ stably equivariantly).

Recall the classical representation of the Euler class of a smooth vector bundle ξ over a closed manifold B as the cycle given by the submanifold Z of zeros of a generic smooth cross-section s (generic meaning transverse to the zero-section). The normal bundle of Z in B is identified with the restriction of ξ and in a tubular neighbourhood $D(\xi \mid Z)$ of Z the section s is, up to homotopy, just the 'diagonal' cross-section of the pullback of ξ. We have an index map $\omega^0(Z) \longrightarrow \omega^0(B; -\xi)$ and by definition (of the index map and of the Euler class) the image of 1 is $\gamma(\xi)$. The corresponding description of the difference class of two sections s_0, s_1 of $S(\xi)$ is as follows.

Lemma (5.13). The section $(s_0, -s_1)$ of the fibre-product $S(\xi) \times_B S(\xi)$ is homotopic to a section $(t_0, -t_1)$ transverse to the diagonal $S(\xi)$. $\delta(s_0, s_1) \in \omega^{-1}(B; -\xi)$ is represented by the inverse image C of this diagonal.

Precisely, the normal bundle ν of C in B is equipped with an isomorphism $\nu \oplus \mathbb{R} \cong \xi \mid C$ and the index homomorphism which it defines takes $1 \in \omega^0(C)$ to $\delta(t_0, t_1) \in \omega^{-1}(B; -\xi)$. Outside C t_0 and t_1 are linearly homotopic.

This, then, is the recipe for finding Y in (5.12). Choose the embedding $i : X \longrightarrow \mathbb{R}^{m+n}$ as above and construct the smooth sections s_0, s_1 of the trivial bundle $(m+n)L$ ($= \xi$) over $S(nL) \times (X \times X)$ ($= B$). Deform s_0, s_1 to t_0, t_1 and take Y to be the obstruction submanifold (representing, although this is irrelevant to the proof, $\delta(s_0, s_1)$) ; this may be done equivariantly by working on $S(nL) \times_{\mathbb{Z}/2} (X \times X)$. On the other hand, forgetting the involution, we may deform s_0 to the constant map

s_0' with value $(0,..,0,1)$. If i is a product of an embedding $X \times \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{m+n-1}$ with the identity $\mathbb{R} \longrightarrow \mathbb{R}$, as we may assume it is by adding 1 to n, then $(s_0', -s_1)$ is already transverse and the obstruction submanifold is just $X \times (0,..,0,-1)$. Now $(s_0', -s_1)$ and $(t_0, -t_1)$ are homotopic by a smooth homotopy constant near the end-points and transverse to the diagonal; the inverse image of the diagonal is a cobordism between X and Y.

The construction (5.11) is really very old. It extends the classical definition of the Hopf invariant as a linking number. (Forgetting some of the information in $\delta(s_0, s_1)$ by restricting to the subspace $(X \times X) \times S(L)$ of $(X \times X) \times S(nL)$, we obtain a class in $\omega^{-1}_{\mathbb{Z}/2}((X \times X) \times S(L); -(m+n)L) = \omega^{m+n-1}(X \times X)$. It is represented by the map $(x,y) \longmapsto i(x) - i_+(y) : X \times X \longrightarrow \mathbb{R}^{m+n} - 0$, where i_+ is given by pushing i out along the first positive normal field.)

There is a special case in which $\bar{P}^2(x)$ admits an even simpler description: namely when the framing of X is given by a genuine trivialization of τX. (As is well known, any framed manifold is cobordant to such an X; but it would be inappropriate to quote this corollary of Kervaire's theorem (8.4) here.)

Remark (5.14). (Löffler [63]). Let s be the 'diagonal' cross-section of the trivial bundle mL over a tubular neighbourhood $X \times D(mL)$ of the diagonal X in $X \times X$ defined by the isomorphism $\tau X \cong \mathbb{R}^m$. s extends to an equivariant cross-section transverse to the zero-section. The zero-set of s is the union of X and a manifold with free involution representing $\bar{P}^2(x)$, (5.9).

We next relate the squaring construction to the definition of the Kervaire-Arf invariant of a framed manifold. Recall the definition of a quadratic form. Let P be a module (projective, finitely generated) over a ring R, commutative for ease of notation. Then $\mathbb{Z}/2 = \{1,T\}$ acts on the group $B = \mathrm{Hom}_R(P \otimes P, R)$ of bilinear forms $P \times P \longrightarrow R$ by interchanging the factors. Classically, a symmetric bilinear form on P is an element of the group $B^{\mathbb{Z}/2}$ of invariants, or, in terms of group cohomology - which is the novelty, of $H^0(\mathbb{Z}/2; B)$. A quadratic form on P is an element of the group $B_{\mathbb{Z}/2} = B/(1-T)B$, or $H_0(\mathbb{Z}/2; B)$. (A coset $[b]$ defines a quadratic function $P \longrightarrow R : x \longmapsto b(x,x)$.) Associated to any quadratic form is a symmetric form, according to the symmetrization map

$$B_{\mathbb{Z}/2} \longrightarrow B^{\mathbb{Z}/2} \quad : \quad [b] \longrightarrow b + Tb .$$

Once the definition is formulated in the language of $\mathbb{Z}/2$-homology, it is fairly clear how to define symmetric (A. S. Miščenko [69]) and quadratic (A. A. Ranicki [73]) forms on a complex of modules. We give the translation into algebraic topology for the singular $(\mathbb{F}_2)$ cochain complex of a manifold. Let H denote homology with $\mathbb{F}_2$-coefficients. Borel has defined the $\mathbb{Z}/2$-<u>co</u>homology of a compact $\mathbb{Z}/2$-ENR Y : $H^*_{\mathbb{Z}/2}(Y) := H^*(S(\infty) \times_{\mathbb{Z}/2} Y)$. Then a symmetric form should be an element of $H^m_{\mathbb{Z}/2}(X \times X)$, a quadratic form an element of $H_m(S(\infty) \times_{\mathbb{Z}/2} (X \times X))$. The symmetrization map is the segment

$$H_m(S(\infty) \times_{\mathbb{Z}/2} (X \times X)) \longrightarrow H^m_{\mathbb{Z}/2}(X \times X)$$

of the $\mathbb{F}_2$-cohomology version of the sequence (5.10).

Definition (5.15). $u_X = \Delta_*(1) - b^m \in H^m_{\mathbb{Z}/2}(X \times X)$ will be called the canonical $\mathbb{F}_2$-symmetric form on the closed manifold X. (Δ_* is the direct image in cohomology; b denotes the Hurewicz image of the element of the same name in stable cohomotopy.)

The terminology is justified by the well known fact that the restriction $i^*(u_X) \in H^m(X \times X)$ satisfies:

$$\{i^*(u_X) \cup (x \times y)\}[X \times X] = \{x \cup y\}[X] \quad (x \in H^i(X),\ y \in H^{m-i}(X)).$$

Definition (5.16). An $\mathbb{F}_2$-quadratic form on X is an element $q \in H_m(S(\infty) \times_{\mathbb{Z}/2} (X \times X))$ whose symmetrization is the canonical symmetric form u_X.

If $m = 2k$ is even, it defines a quadratic form $Q : H^k(X) \longrightarrow \mathbb{F}_2$ in the classical sense: $Q(x+y) = Q(x) + Q(y) + \{x \cup y\}[X]$. The squaring operation

$$P^2 : H^k(X) \longrightarrow H^{2k}_{\mathbb{Z}/2}(X \times X)$$

was introduced by Steenrod in his definition of the Steenrod squares (as the square on the cochain level). There is a cap-product pairing

$$\langle\ ,\ \rangle : H^r_{\mathbb{Z}/2}(X \times X) \otimes H_s(S(\infty) \times_{\mathbb{Z}/2} (X \times X)) \longrightarrow H_{s-r}(P(\infty)).$$

$Q(x)$ is equal to $\langle P^2(x), q\rangle \in \mathbb{F}_2$.

Set $H = H^*(X)$ and write K and I respectively for the

kernel and image of $1 + T : H \otimes H \longrightarrow H \otimes H$. The equivariant groups decompose canonically as:

$$H^{2k}_{\mathbb{Z}/2}(X \times X) = \sum_{k \geq r > 0} \{K/I\}^{2(k-r)} \oplus K^{2k},$$

$$H_{2k}(S(\infty) \times_{\mathbb{Z}/2} (X \times X)) = \{H \otimes H/I\}^{2k} \oplus \sum_{0 < r \leq k} \{K/I\}^{2(k+r)}.$$

$P^2(x)$ is then simply $x \otimes x$ in K^{2k}. We shall write Q also for the component of q in $\{(H \otimes H)/I\}^{2k}$; it determines the quadratic function on H^k.

The Hopf invariant $\Delta_*(1) - b^m \in \omega_m(S(\infty) \times_{\mathbb{Z}/2} (X \times X))$ thus provides, by the Hurewicz map, a natural quadratic form for the framed manifold X. If $m \equiv 2 \pmod 4$, the Arf invariant of Q is called the Kervaire-Arf invariant of the framed manifold. It vanishes if X is a framed boundary. (Here is a sketch of the proof. If $X = \partial W$, one must show that Q is zero on elements lifting to $H^k(W)$. In order to avoid discussing manifolds with boundary we consider the closed framed manifold $Y = W \amalg_X W$ obtained by gluing two copies of W along the boundary X; the 'folding map' gives a retraction of Y onto W. The inclusion $j : X \longrightarrow Y$ of X as codimension 1 submanifold represents 0 in $\omega^1(Y)$. In other words, $j_* : \omega^0(X) \longrightarrow \omega^0(Y; -\mathbb{R})$ takes 1 to zero. It follows formally that $(j \times j)_* : \omega^0_{\mathbb{Z}/2}(X \times X; -mL) \longrightarrow \omega^0_{\mathbb{Z}/2}(Y \times Y; -(m+1)L - \mathbb{R})$ kills $\Delta_*(1) - b^m$. Then in homology $(j \times j)_*(q) = 0 \in H_m(S(\infty) \times_{\mathbb{Z}/2} (Y \times Y))$, which implies that Q is zero on $j^*H^k(Y)$.)

With this definition, the dependence of the quadratic form on the choice of framing is particularly easy to describe.

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Consider a change of framing $a: X \longrightarrow O(\infty)$. $\theta(a)$, (2.13), is an element of $\{X^+; P(\infty)^+\}$, or, by S-duality, $\omega_m(P(\infty) \times X)$. Distinguish the new quadratic form by a prime.

Lemma (5.17). $q - q' = h\Delta_*(\theta(a))$, where

$\Delta_*: \omega_m(P(\infty) \times X) \longrightarrow \omega_m(S(\infty) \times_{\mathbb{Z}/2} (X \times X))$

is the diagonal map and h is the Hurewicz homomorphism.

The variation of the quadratic form thus depends on the 'spherical class' $h\theta(a) \in H_m(P(\infty) \times X)$. A classical computation in cohomology, recalled below, using the fact that the total Steenrod square Sq fixes a spherical class, that is, something in the image of stable cohomotopy, establishes:

Proposition (5.18). (Browder [20], Jones-Rees [47])

$Q'(x) - Q(x) = \langle x, w \rangle$ if m+2 is a power of 2,

0 otherwise,

where $w \in H^k(X)$ is the pullback via $\theta(a) \in \{X^+; P(\infty)^+\}$ of the generator of $H^k(P(\infty))$.

It follows from (5.5) and the Kahn-Priddy theorem that the Arf invariant of X is zero unless m+2 is a power of 2. (This is the proof of J. Jones and E. Rees.)

Consider the homomorphism $\{X^+; P(\infty)^+\} \longrightarrow \sum_{i \geq 0} H^i(X)$ taking a stable map to the image under the induced map in cohomology of the several generators of $H^i(P(\infty))$. The dual or inverse total Steenrod square χSq is defined by $Sq.\chi Sq = 1$.

Lemma (5.19). (Steenrod-Whitehead). If (a_i) lies in the

image of $\{X^+; P(\infty)^+\} \longrightarrow \sum_{i \geq 0} H^i(X)$, then

$$(\chi Sq)^{j-i} a_i = \binom{2j-i}{j} a_j .$$

In particular,

$$\sum_{i \leq j} (\chi Sq)^{j-i} a_i = \begin{cases} 0 & \text{if } j+1 \text{ is not a power of 2,} \\ a_j & \text{if it is.} \end{cases}$$

In the application (5.18) Δ_* translates into χSq, by the definition of the Steenrod squares and the triviality of the Wu classes of the framed manifold, and in fact $Q' - Q = w \otimes w$ or 0 in $\{(H \otimes H)/I\}^{2k}$. The success of the method leads us to ask what restrictions are imposed on Q by its origin in stable cohomotopy. The action of $Sq = Sq \otimes Sq$ on $H \otimes H$ passes to the quotient $(H \otimes H)/I$.

Proposition (5.20). (Jones [46]). $k+1 \equiv 2^t \pmod{2^{t+1}}$.

Then $Q + Sq(Q) = 0$ if $k+1 \equiv -2^t \pmod{2^{t+2}}$,

$z \otimes z$ for some $z \in H^{k+2^t}(X)$ if $k+1 \equiv 2^t$ $\pmod{2^{t+2}}$.

The proof is a matter of interpreting the statement: $Sq(q) = q$. It requires knowledge of the action of Sq in $H^*_{\mathbb{Z}/2}(X \times X)$; [67] 3.7. As an immediate consequence of (5.20) one has:

$$Q(\chi Sq^i x) = \sum_{j > 0} \langle \chi Sq^{i+j} x, \chi Sq^{i-j} x \rangle \quad \text{if } x \in H^{k-i}(X),$$

$i > 0$, with a correction term $\langle x, z \rangle$ if $i = 2^t$ and $k+1 \equiv 2^t$ $\pmod{2^{t+2}}$.

The $\mathbb{Z}/2$-homotopy theoretic description of the quadratic form on a framed manifold is also well adapted to another problem. Let $\pi : \tilde{X} \longrightarrow X$ be a double cover. The framing of X lifts to a framing of $\tilde{X}$. How are the quadratic forms related? As usual $\tilde{X}$ is considered as a $\mathbb{Z}/2$-space with the trivial involution. We also require $\tilde{X}$ with the covering involution T and refer to it then as Y; it is equivariantly framed. Now the inverse image under $\pi \times \pi : \tilde{X} \times \tilde{X} \longrightarrow X \times X$ of the diagonal X splits into a fixed, diagonal component $\Delta(\tilde{X})$ and a free, off-diagonal one $\nabla(Y)$, where $\nabla : Y \longrightarrow \tilde{X} \times \tilde{X}$ takes y to (y, Ty). Hence the lift of the 'quadratic form' $\Delta_*(1) - b^m$ in $\omega^0_{\mathbb{Z}/2}(X \times X; -mL)$ to $\omega^0_{\mathbb{Z}/2}(\tilde{X} \times \tilde{X}; -mL)$ is the sum $(\Delta_*(1) - b^m)$ + $\nabla_*(1)$ of the quadratic form on $\tilde{X}$ and an off-diagonal correction term. The second term is substantially simpler than the first, for Y is a free $\mathbb{Z}/2$-manifold and there are isomorphisms:

$$\begin{array}{ccc} \omega_m(S(\infty) \times_{\mathbb{Z}/2} Y) & \longrightarrow & \omega^0_{\mathbb{Z}/2}(Y) \\ \downarrow & & \downarrow \\ \omega_m(X) & \longrightarrow & \omega^0(X) . \end{array}$$

The vertical maps are induced by the projection π, the top row is the Gysin sequence (4.5) with S-duality. So $\nabla_*(1)$ is described directly in the free component as the image under the map $\omega_m(S(\infty) \times_{\mathbb{Z}/2} Y) \longrightarrow \omega_m(S(\infty) \times_{\mathbb{Z}/2} (\tilde{X} \times \tilde{X}))$ induced by ∇ of the element corresponding to the fundamental class $[X]$ in $\omega_m(X)$. The associated quadratic form $r \in H_m(S(\infty) \times_{\mathbb{Z}/2} (\tilde{X} \times \tilde{X}))$ is clearly independent of the framing; its definition uses only Poincaré duality. Let $\tilde{q}$ be the quadratic form determined by the framing. The splitting in cohomotopy translates at once into:

Proposition (5.21). (Brumfiel-Milgram [23]).

$$(\pi \times \pi)^* q \quad = \quad \tilde{q} + r .$$

If $m = 2k$ is even, then the quadratic functions on the middle cohomology satisfy: $Q(\pi_* x) = \tilde{Q}(x) + R(x) \quad (x \in H^k(\tilde{X}))$.

$(\pi \times \pi)^*$ and π_* are the transfer maps in homology and cohomology respectively. Our interest here lies not so much in the result (5.21) as in the method of proof, to which we shall return shortly.

Consider next an immersion $i : X \longrightarrow \mathbb{R}^{m+n}$, with normal bundle ν, of the framed manifold X in Euclidean space. The derivative of i embeds τX as a sub-bundle of the trivial bundle $\mathbb{R}^{m+n}$ and so, as in §2, defines a $\mathbb{Z}/2$-cross-section s of the trivial bundle L^{m+n} over $S(L.\tau X)$. Let $\tilde{s} : X \times X \longrightarrow L^{m+n}$ be the section $(x,y) \longmapsto i(x) - i(y)$ of L^{m+n} over $X \times X$. Choose an equivariant tubular neighbourhood $D(L.\tau X) \subseteq X \times X$ of the diagonal. Then, up to homotopy, s coincides with the restriction of $\tilde{s}$ to $S(L.\tau X)$. We see that the relative Euler class $\gamma((m+n)L, s) \in \omega^0_{\mathbb{Z}/2}(X \times X - \mathring{D}(L.\tau X), S(L.\tau X); -(m+n)L)$ is an obstruction to the existence of a regular homotopy of i to an embedding. (In a certain metastable range this is the precise obstruction, by the theory of Haefliger-Hirsch [33].) On the other hand, $\gamma((m+n)L, s) \in \omega^0_{\mathbb{Z}/2}(D(L.\tau), S(L.\tau); -(m+n)L)$ is just the Euler class $\gamma(L.\nu)$ of the normal bundle multiplied by the 'Thom class' of $L.\tau$. In this way we obtain a decomposition of $\gamma((m+n)L) = b^{m+n} \in \omega^0_{\mathbb{Z}/2}(X \times X; -(m+n)L)$ into a diagonal and an off-diagonal component. This may be given geometric content in the following manner.

The normal bundle ν is equipped with a stable trivialization, say $\nu \oplus \mathbb{R}^N \cong \mathbb{R}^{n+N}$ for some $N \gg m$, classified by a map $f : X \longrightarrow V_{n+N,N}$ to the Stiefel manifold of N-frames in $\mathbb{R}^{n+N}$. $\theta(f)$, (2.6), is an element of $\omega^{-1}_{\mathbb{Z}/2}(X \times S(NL); -(n+N)L)$ $= \{X^+;\ P^\infty_n\}$. It is equal to the difference of Euler classes $\gamma(L.\nu) - \gamma(nL) \in \{X^+;\ P^\infty_n\} \subseteq \omega^0_{\mathbb{Z}/2}(X; -nL)$. (The coefficients $L.\nu$ and nL are identified via the stable isomorphism.) The index of $\theta(f)$ in $\tilde{\omega}_m(P^\infty_n) \subseteq \omega_{m-n,m}$ is called the (stable) Smale invariant of the immersion.

We turn to the obstruction to embedding. The restriction of $\tilde{s}$ to $X \times X - \mathring{D}(L.\tau X)$ is $\mathbb{Z}/2$-equivariantly homotopic through a homotopy constant on $S(L.\tau X)$ to a section transverse to the zero-section. The zero-set of this cross-section is a manifold Y with free involution and a stable framing of τY as $\mathbb{R}^m - L^n$. It represents the off-diagonal component of b^{m+n} in $\omega^0_{\mathbb{Z}/2}(X \times X; -(m+n)L)$:

(5.22) $$b^{m+n} = \Delta_* \gamma(L.\nu) + [Y] .$$

The stable class in $\omega_{m-n,m}$ defined by Y is called the double-point invariant of the immersion. If the only singularities of i are transverse double intersections, then Y may be taken as $\{(x,y) \in X \times X \mid i(x) = i(y),\ x \neq y\}$; whence the name.

Proposition (5.23). (Kervaire [53]). Consider an immersion $X \longrightarrow \mathbb{R}^{m+n}$ of a framed manifold in Euclidean space. Then the sum of the Smale invariant and the double-point invariant is equal to (-) the Hopf invariant $b^n.\bar{P}^2(x) \in \tilde{\omega}_m(P^\infty_n)$ of the class $x \in \omega_m$ defined by X.

The off-diagonal term $[Y]$ lies in the free summand of $\omega^0_{\mathbb{Z}/2}(X \times X; -(m+n)L)$, and just as in the example (5.21), we may define it directly as an obstruction class, $o(i)$ say, in $\omega_m(S(\infty) \times_{\mathbb{Z}/2} (X \times X); nH)$ to deforming i into an embedding. ($\gamma((m+n)L,s)$ is an element of the group $\omega^0_{\mathbb{Z}/2}(X \times X - \mathring{D}(L.\tau), S(L.\tau); -(m+n)L)$, which is isomorphic to $\omega_m(S(\infty) \times_{\mathbb{Z}/2} (X \times X - \mathring{D}(L.\tau)); nH)$. $o(i)$ is defined by mapping $X \times X - \mathring{D}(L.\tau)$ into $X \times X$.) We shall regard $o(i)$ as an element of the S-dual group $\omega^{-1}_{\mathbb{Z}/2}(X \times X \times S(NL); -(m+n)L - NL)$ for $N \gg m$. It is defined here whether X is framed or not.

Before compounding the elements of (5.21) and (5.23) in a final example, we must indicate the connection between $\bar{P}^2$ and Segal's operation θ^2. Let $\pi : E \longrightarrow B$ be a finite cover of a compact ENR B. There is an index map, or transfer, π_* : $\omega^0(E) \longrightarrow \omega^0(B)$. $\pi_*(1)$ is the element of $\omega^0(B)$ represented by the finite cover in the sense of [80]. A finite cover with an involution similarly represents an element of $\omega^0_{\mathbb{Z}/2}(B)$.

Proposition (5.24). $\bar{P}^2(\pi_*(1)) \in \omega^0_{\mathbb{Z}/2}(B)$ is represented by the finite cover

$$\{(x,y) \in E \times_B E \mid x \neq y\} \longrightarrow B$$

with the free involution which interchanges the factors of the fibre-product.

If B is a point, this is (5.9) in dimension zero. The fibre-bundle version is no more difficult.

Now let $\pi : \tilde{X} \longrightarrow X$ be a finite d-fold covering of a closed connected m-manifold X, $m > 0$, defining a class u =

$\pi_*(1) \in \omega^0(X)$. An immersion $i : X \longrightarrow \mathbb{R}^{m+n}$ induces, by composition with π, an immersion π^*i of $\tilde{X}$. The obstructions $o(i)$ and $o(\pi^*i)$ are related by the equation:

$$(5.25) \qquad P^2(u).o(i) = (\pi \times \pi)_* o(\pi^* i) - \Delta_*(\bar{P}^2(u).\gamma(L.\nu))$$

in $\omega^{-1}_{\mathbb{Z}/2}(X \times X \times S(NL); -(m+n)L - NL)$, for $N \gg m$. $P^2(u)$ lies in $\omega^0_{\mathbb{Z}/2}(X \times X)$, $\bar{P}^2(u)$ in $\omega^{-1}_{\mathbb{Z}/2}(X \times S(NL); -NL)$, that is $\{X^+;\ P(\infty)^+\}$. Δ is the diagonal of X, ν the normal bundle.

The equality is proved by lifting $o(i)$ to $(\pi \times \pi)^* o(i)$ on $\tilde{X} \times \tilde{X}$, where it splits into two terms corresponding to the diagonal $\tilde{X}$ and off-diagonal $\tilde{X} \times_X \tilde{X} - \tilde{X}$ parts of $(\pi \times \pi)^{-1}(X) = \tilde{X} \times_X \tilde{X}$. The first is $o(\pi^*i)$; the second should be thought of as an obstruction to embedding $\tilde{X}$ fibre-wise in the normal bundle ν over X.

We specialize to the case $m = n$. The Poincaré dual of the Hurewicz image of $o(i)$ in $\mathbb{F}_2$-cohomology lies in $H_0(S(\infty) \times_{\mathbb{Z}/2} (X \times X))$. Let $I(i)$ in $H_0(P(\infty)) = \mathbb{F}_2$ be its image on collapsing $X \times X$ to a point. For a generic immersion I is the number modulo 2 of self-intersections.

Proposition (5.26). (Brown [21]). Let $\pi : \tilde{X} \longrightarrow X$ be a finite d-fold cover of closed manifolds of dimension $m > 0$ and i an immersion of X in $\mathbb{R}^{2m}$ lifting to an immersion π^*i of $\tilde{X}$. Then

$$I(\pi^*i) = d.I(i) + \begin{cases} 0 & \text{if } m+1 \text{ is not a power of } 2, \\ \alpha_m(\pi)[X] & \text{if it is,} \end{cases}$$

where α_m is a certain characteristic class of finite coverings. For a double covering π classified by $a \in H^1(X; \mathbb{F}_2)$, $\alpha_m(\pi) = a^m$.

We have only to describe the Hurewicz image of the final term in (5.25). Let $(\alpha_i(\pi))_{i \geqslant 0}$ be the image of $\bar{P}^2(u)$ under the map of (5.19). Then the term $\Delta_*(\bar{P}^2(u).\gamma(L.\nu))$ reduces to $\sum_{0 \leqslant i \leqslant m} \alpha_i(\pi)\bar{w}_{m-i}[X]$, where $\bar{w}$ is the normal Stiefel-Whitney class, that is, to $\sum_{0 \leqslant i \leqslant m} (\chi Sq)^{m-i}\alpha_i(\pi)[X]$ (by the standard property of the Wu class). And (5.19) completes the proof.

My account of quadratic forms and the Kervaire-Arf invariant owes much to J. Jones [45], [46] and is dependent on the definition given by A. A. Ranicki. The definition of the double-point invariant of an immersion and the whole geometric approach to the Hopf invariant comes from the work of U. Koschorke and B. Sanderson [57] and R. M. W. Wood [89]. The original version of (5.26) in [21] dealt with the double cover $S^m \longrightarrow P^m$; this is now seen to be typical. (The more general problem was posed by L. Smith.) There is a systematic study of constructions such as (5.13) in [55].

A. Appendix: On the Hopf invariant

We shall need a generalization of (4.4).

Lemma (A.1). Let A be a pointed $\mathbb{Z}/2$-space and (X,Y) a compact $\mathbb{Z}/2$-ENR pair. Then the fixed point map gives an isomorphism ρ :

$$\underset{\substack{\longrightarrow \\ n}}{\lim} \{X/Y;\ (L^n)^+ \wedge A\}^{\mathbb{Z}/2} \longrightarrow \{(X/Y)^{\mathbb{Z}/2};\ A^{\mathbb{Z}/2}\} .$$

(The direct limit is taken over successive inclusions.)

It is simplest to work from first principles. As in (4.4) we may assume that $\mathbb{Z}/2$ acts trivially on X, and then there is a splitting σ of ρ. Consider the special case of a $\mathbb{Z}/2$-map $f:\ (X/Y)\wedge(L^n)^+ \longrightarrow A$ such that $\rho(f):\ X/Y \longrightarrow A^{\mathbb{Z}/2}$ is null-homotopic. Then f is annihilated by composition with the inclusion $A \longrightarrow (L^n)^+\wedge A$, for it lifts to the first term of the homotopy exact sequence: $[(X/Y)\wedge \mathbb{R}^+ \wedge S(L^n)^+;\ A]^{\mathbb{Z}/2} \longrightarrow [(X/Y)\wedge(L^n)^+;\ A]^{\mathbb{Z}/2} \longrightarrow [X/Y;\ A]^{\mathbb{Z}/2}$ of the pair $(D(L^n), S(L^n))$. This is converted into a general proof by re-labelling.

If V is a real vector space, write $\underline{\underline{P}}_V(A)$ for the quotient: $(S(L.V)\times_{\mathbb{Z}/2} A)/(S(L.V)\times_{\mathbb{Z}/2} *)$.

Lemma (A.2). Suppose that the involution on X is trivial. Then there is a canonical isomorphism:

$$\{X/Y;\ S(L.V)^+\wedge A\}^{\mathbb{Z}/2} \cong \{X/Y;\ \underline{\underline{P}}_V(A)\} .$$

Its definition is an exercise in S-duality (in its $\mathbb{Z}/2$-equivariant and fibre-bundle over P(V) manifestations). (A.1) and (A.2) together yield a splitting theorem extending (3.13).

Corollary (A.3). X with trivial involution. Then there is a split short exact sequence:

$$0 \longrightarrow \{X/Y;\ \underline{\underline{P}}_\infty(A)\} \xrightarrow{\ \varsigma\ } \{X/Y;\ A\}^{\mathbb{Z}/2} \underset{\sigma}{\overset{\rho}{\rightleftarrows}} \{X/Y;\ A^{\mathbb{Z}/2}\} \longrightarrow 0 .$$

The relation of the operation $\bar{P}^2$ introduced in §4 to the Hopf invariant H of the EHP-sequence is best understood by generalization. Let $\mathfrak{M}(V^+,V^+)$ denote the space of all base-point preserving maps $V^+ \longrightarrow V^+$, with the standard topology and the zero map as base-point. Corresponding to the unstable version η of $\bar{S}^2$ in §3, there is an operation

$$\bar{P}^2_V:\ [Z;\ \mathfrak{M}(V^+,V^+)] \longrightarrow \{Z;\ P(V)^+\} .$$

Now let B be a compact ENR with base-point. Then the construction, via the difference class and S-duality, extends to define

$$\bar{P}^2_V:\ [Z;\ \mathfrak{M}(V^+,V^+\wedge B)] \longrightarrow \{Z;\ \underline{\underline{P}}_V(B\wedge B)\} .$$

$(-)\bar{P}^2_V$ is the Hopf invariant of the EHP-sequence, [67] 1.11. The verification is facilitated by the observation:

Remark (A.4). The projection $B\times B \longrightarrow B\wedge B$ ($= (B\times B)/(B\vee B)$) has a natural $\mathbb{Z}/2$-equivariant stable splitting.

There is an operation $S^2:\ \{B;\ B\vee B\} \longrightarrow \{B\times B;\ B\vee B\}^{\mathbb{Z}/2}$.

The image of the inclusion $B \longrightarrow B \vee B$ of the first factor defines the splitting.

The splitting (A.3) interprets the stable Hopf invariant $\bar{P}^2 : \{Z; B\} \longrightarrow \{Z; P_\infty(B \wedge B)\}$ as:

$$(A.5) \qquad \delta.\bar{P}^2(x) = \Delta_Z^*(P^2 x) - (\Delta_B)_*\sigma(x) \qquad (x \in \{Z; B\}),$$

where $P^2 : \{Z; B\} \longrightarrow \{Z \wedge Z; B \wedge B\}^{\mathbb{Z}/2}$ is the squaring operation and Δ_Z and Δ_B the diagonal maps $Z \longrightarrow Z \wedge Z$, $B \longrightarrow B \wedge B$.

The definition of the Hopf invariant $H = -\bar{P}^2$ in full generality allows us to refine the statement (5.9). $\Delta_*(1) - b^m \in \omega_m(S(\infty) \times_{\mathbb{Z}/2} (X \times X))$ is equal to $H[X]$, the Hopf invariant of the fundamental class $[X] \in \omega_m(X)$. (The geometric description (5.11) may also be made more precise. $-\delta(s_0, s_1) \in \omega_m(S(nL) \times_{\mathbb{Z}/2} (X \times X))$ is the Hopf invariant of the unstable class in $\pi_{m+n}(S^n)$ defined by the embedded manifold with trivial normal bundle.)

The naïve account given here of the Hopf invariant (or cyclic pth power operation for an odd prime p) is founded on the free action of $\mathbb{Z}/2$ (or $\mathbb{Z}/p$) on a sphere and the consequent interpretation of the classifying space $B\mathbb{Z}/2$ (or $B\mathbb{Z}/p$) as a projective (or lens) space. At the heart of the sophisticated theory is the theorem of Barratt-Priddy-Quillen relating finite sets (or symmetric groups) to stable homotopy theory. Then comes the sequence of operations or stable splittings, one for each symmetric group, beginning with the classical Hopf invariant: V. P. Snaith [82], M. G. Barratt-P. J. Eccles [11], G. B. Segal [80] (in terms of covering spaces), U. Koschorke-B. Sanderson [58] (self-intersections of immersed manifolds), H. Hauschild [35] (equivariant homotopy, generalizing (A.5)). Of course, localizing at the prime (2) we may reduce (by passing from a symmetric group to its Sylow 2-subgroup) to an iterated Hopf invariant, and in practice it may be convenient to do so. The Arf invariant is an example.

Remark (A.6). k odd. There is a commutative diagram:

$$\begin{array}{ccc}
\tilde{\omega}_{2k}(P_0^\infty) & \xrightarrow{\quad b^k \quad} & \tilde{\omega}_{2k}(P_k^\infty) \\
H \uparrow \downarrow i^* & & \downarrow H \\
\omega_{2k} & \xrightarrow{\quad \mathrm{Arf} \quad} & \mathbb{Z}/2 = \omega_{2k}(B(\mathbb{Z}/2 \wr \mathbb{Z}/2); k\gamma)
\end{array}$$

$S(\infty) \times_{\mathbb{Z}/2} (P(\infty) \times P(\infty))$ is the classifying space of the wreath product $\mathbb{Z}/2 \wr \mathbb{Z}/2$. More generally, $\underline{P}_\infty(P_k^\infty \wedge P_k^\infty)$ is the Thom space of $k\gamma$, where γ is the 2-dimensional real vector bundle over $B(\mathbb{Z}/2 \wr \mathbb{Z}/2)$ associated to the representation $S^2(L)$ of $\mathbb{Z}/2 \wr \mathbb{Z}/2$. (The construction in §1 gives an operator $S^2 : RO(\mathbb{Z}/2) \longrightarrow RO(\mathbb{Z}/2 \wr \mathbb{Z}/2)$ on the real representation rings.)

In the notation of (5.18), $\mathrm{Arf}(Q) - \mathrm{Arf}(Q') = Q(w)$ if $k+1$ is a power of 2. With (5.5) this provides a manageable description of the Arf invariant of a framed manifold with free involution and leads to a geometric proof of the remark.

(A.3) is the simplest example of a general splitting in equivariant stable homotopy theory; see, for example, [35].

6. K-theory

Notations and concepts will be carried over from equivariant stable cohomotopy to real KO-theory without comment. Thus, the $KO_{\mathbb{Z}/2}$-theory of a space is a $\mathbb{Z} \times \mathbb{Z}$-graded ring, commutative in the sense of (4.2). The Hurewicz homomorphism

$$\omega_{**} \longrightarrow KO_{**}$$

of the coefficient rings takes elements of stable cohomotopy to elements of KO-theory denoted by the same symbol. Parallel local coefficient notation will be used.

Let ξ be a real vector bundle over a compact ENR X. $C(\xi)$ will be the associated bundle of $\mathbb{Z}/2$-graded Clifford algebras. (The standard reference is [6].) A Clifford module is a graded real vector bundle μ over X with a structure homomorphism $C(\xi) \longrightarrow \mathrm{End}\,\mu$ (of graded algebras). The Grothendieck group of such Clifford modules will be written $KO_{C(\xi)}(X)$. Now we can state the basic 'periodicity' theorems of $\mathbb{Z}/2$-equivariant K-theory.

Theorem (6.1). (Karoubi-Segal [49] p.193, [7] Theorem 3.3) There is a natural isomorphism

$$KO_{C(\xi)}(X) \longrightarrow KO_{\mathbb{Z}/2}(X; L.\xi).$$

If $C(\xi)^o$ denotes the opposite algebra (opposite in the graded sense), then similarly

$$KO_{C(\xi)^o}(X) \cong KO_{\mathbb{Z}/2}(X; -L.\xi).$$

The duality between modules over $C(\xi)$ and those over $C(\xi)^o$ gives a natural identification:

(6.2) $$\tau : KO_{\mathbb{Z}/2}(X; L.\xi) \equiv KO_{\mathbb{Z}/2}(X; -L.\xi).$$

Proposition (6.3). Let ξ be an oriented real vector bundle of dimension a multiple of 4. Then there is a Bott isomorphism

$$KO_{\mathbb{Z}/2}(X) \longrightarrow KO_{\mathbb{Z}/2}(X; L.\xi - \xi)$$

defined by multiplication by a canonical periodicity class $\alpha(\xi)$ such that $i^*\alpha(\xi) \in KO(X; \xi - \xi) \equiv KO(X)$ is the identity.

The proposition, reduced to algebra by (6.1), may be generalized by introducing coefficients in an arbitrary $\mathbb{Z}/2$-vector bundle. In particular,

(6.4) $$KO_{\mathbb{Z}/2}(X; -L.\xi) \equiv KO_{\mathbb{Z}/2}(X; -\xi)$$

for an oriented bundle ξ of dimension a multiple of 4.

This periodicity reduces $KO_{\mathbb{Z}/2}$-theory to a $\mathbb{Z}/8 \times \mathbb{Z}/4$-graded theory. $\alpha(\mathbb{R}^4)$ defines an element u in $KO_{0,-4}$. The identification (6.2) leads to an involution

$$\tau : KO_{i,j} \longrightarrow KO_{2j-i,j}$$

of the coefficient ring. The periodicity is defined by multiplication by the central units $u^{-1} \in KO_{0,4}$ and $u.\tau(u)^{-1} \in KO_{8,0}$. For reference we recall from [26] the tabulation of the coefficient ring KO_{**} and the restriction map $i^* : KO_{**} \longrightarrow KO_*$.

Table (6.5). KO-theory coefficient rings.

j mod 4									
	3	0	0	0	0	0	0	0	0
	2	0	0	0	$\mathbb{Z}d$	$Ab\eta^2$	$A\eta^2 \oplus At\eta^2$	$Aa\eta^2$	$\mathbb{Z}e$
	1	0	0	0	$Ab\eta$	$A\eta \oplus At\eta$	$Aa\eta$	0	0
	0	$\mathbb{Z}b^3$	$\mathbb{Z}b^2$	$\mathbb{Z}b$	$\mathbb{Z}1 \oplus \mathbb{Z}t$	$\mathbb{Z}a$	$\mathbb{Z}a^2$	$\mathbb{Z}a^3$	$\mathbb{Z}c \oplus \mathbb{Z}tc$
$\underline{KO}_{i,j}$		$\underline{-3}$	$\underline{-2}$	$\underline{-1}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$
					i mod 8				
$\underline{KO}_i$		0	0	0	$\mathbb{Z}1$	$A\eta$	$A\eta^2$	0	$\mathbb{Z}c$

(For typographical reasons $\mathbb{Z}/2$ has been abbreviated to A.)

$i^*(a) = \eta$, $i^*(d) = 2$, $i^*(e) = c$. $d = \tau(e)$, $a = \tau(b)$. $ab = 1-t$. η has been written for $r(\eta)$ and c for $r(c)$.

The tensor product of vector bundles defines a squaring operation

$$P^2 : KO^{-j}(X) \longrightarrow KO^{-2j,-j}(X).$$

It is compatible with the square in stable cohomotopy and has the same formal properties (4.10)(i)-(iii). If $j = 0$, then $P^2[\xi] = [\xi \otimes \xi]$ with, of course, the involution which switches the factors. The equivariant K-theory $KO_{\mathbb{Z}/2}(X)$ of a space with trivial involution splits as a direct sum $KO(X) \oplus KO(X)t$ of two copies of $KO(X)$ corresponding to the splitting of $\mathbb{Z}/2$-vector bundles into positive and negative eigenspaces. $\xi \otimes \xi$ is decomposed as the sum of the symmetric and the exterior square: $P^2[\xi] = \sigma^2(\xi) + \lambda^2(\xi)t$.

[illegible]

Indeed, $x^2 = i^*P^2(x)$. But $i^* : KO^{-2j,-j}(X) \longrightarrow KO^{-2j}(X)$ is zero if $j \equiv 3 \pmod 4$; (6.1) interprets it as a map $KO^1(X) \longrightarrow KO^2(X)$ extending to a $KO^*(X)$-homomorphism $KO^*(X) \longrightarrow KO^{*+1}(X)$, which vanishes since $KO_{-1} = 0$.

A similar argument shows that the square of an element $x \in KO^{-1}(X)$ is described by the formula $x^2 = \eta . \lambda^2(x) \in KO^{-2}(X)$. ($KO^{-2,-1}(X)$ is isomorphic to $KO^{-1}(X)$. $P^2(x)$ translates into $\sigma^2(x) = -\lambda^2(x)$. i^* becomes multiplication by η.)

So we have the basic machinery of $\mathbb{Z}/2$-equivariant KO-theory. One of the merits of the theory is its accessibility. The Euler class has played an important part in the investigation of cross-sections of vector bundles. We wish to describe here its relation to the perhaps more familiar rational cohomology characteristic classes.

Define a $\mathbb{Z}/4$-graded cohomology theory R by

$$R^i(X) := \sum_{n \equiv i \bmod 4} H^n(X; \mathbb{Q}).$$

The Chern (Pontrjagin) character defines a natural transformation $ch : KO^* \longrightarrow R^*$ of cohomology theories.

Proposition (6.7). Let ξ be an oriented real vector bundle of even dimension n over X. Identify $R^0(X; -\xi)$ with $R^n(X)$ by the Thom isomorphism. Then

$$ch : KO^0(X; -\xi) \longrightarrow R^0(X; -\xi)$$

maps the KO-theory Euler class $\gamma(\xi)$ to the classical rational cohomology Euler class $e(\xi) \in H^n(X; \mathbb{Q}) \subseteq R^n(X)$.

If the dimension of ξ is a multiple of 4, then $\gamma(L.\xi) \in KO_{\mathbb{Z}/2}(X; -L.\xi)$ corresponds, by (6.4), to a class in $KO_{\mathbb{Z}/2}(X; -\xi)$. This latter group splits as $KO(X; -\xi) \oplus KO(X; -\xi)t$, because $\mathbb{Z}/2$ acts trivially on X and ξ. Write the image of $\gamma(L.\xi)$ as $\gamma_+ + t\gamma_-$; $\gamma_+, \gamma_- \in KO(X; -\xi)$. $\gamma_+ + \gamma_- = \gamma(\xi)$.

Proposition (6.8). ξ oriented of dimension 4k. Then
$ch: KO^0(X; -\xi) \longrightarrow R^0(X; -\xi)$
maps $\gamma_+ - \gamma_-$ to $(2^{2(k-i)} L_{4i}(\xi))_{i \geq 0} \in R^0(X)$, where $(L_{4i}(\xi))$ is the Hirzebruch L-class ([36] 1.5).

The rational L-class of an oriented vector bundle is thus described in terms of the $KO_{\mathbb{Z}/2}$-Euler class of the vector bundle with the antipodal involution.

In the discussion above KO-theory has been presented as an abstract cohomology theory and the analytic character intrinsic in its very definition ignored. Atiyah's work on 'K-theory and Reality' showed that, in order to prove theorems in real KO-theory, one should, by analogy with algebraic geometry, extend it to the Real KR-theory defined on a category of spaces with involution. In this context $\mathbb{R}$ with the involution -1 is traditionally written $i\mathbb{R}$. The basic periodicity theorem $KR(X) \cong KR(X; \mathbb{R} \oplus i\mathbb{R})$, and hence $KR(X) \cong KR(X; \xi \oplus i\xi)$ for any real vector bundle ξ, then permits the definition of the cohomology theory KO* and gives sense to $KO(X; -\xi)$ as $KR(X; i\xi)$. This is the way to understand the statements (6.2) and (6.3): in terms of $KR_{\mathbb{Z}/2}$-theory. Here the abstract $\mathbb{Z}/2$-action in $KO_{\mathbb{Z}/2}$ and the Galois action of $\mathbb{Z}/2$ in KR are conceptually quite

different. However, KR-theory does define in itself a $\mathbb{Z}/2$-equivariant cohomology theory. This will be interpreted later as complex Hermitian K-theory. In the example which follows the 'Real' character is the more apparent, although the distinction is by no means clear-cut. Notice that, in the indexing of (4.1), the KR-theory in dimension (-i,-j) of a compact $\mathbb{Z}/2$-ENR X is what is normally written as $KR^{i-2j}(X)$.

We are aiming now at a Real version of the local obstruction, or 'free J-homomorphism', Θ, but must begin with a résumé of the complex theory. As in §§2 and 3 (X,Y) will be a compact ENR-pair, Z a compact ENR with base-point. The notation used in §3 for the free spherical fibration theory is cumbersome. Let G be a finite cyclic group or S^1. We set

$$L(G)^{-1}(X,Y) := \varinjlim \; [X/Y; \; H^G(W)] ,$$

where the limit is taken over all real G-modules W for which the action of G on the sphere S(W) is free and $H^G(W)$ is the space of G-homotopy equivalences $S(W) \longrightarrow S(W)$. The elementary proof of the splitting lemma (3.5) may be generalized, by an inductive argument, to show that the obvious map

$$L(G)^{-1}(X) \longrightarrow \omega_G^0(X)^{\cdot}$$

to the group of units in equivariant stable cohomotopy is a split injection.

Let V be a real vector space. Write $V_{\mathbb{C}}$ for $\mathbb{C} \otimes_{\mathbb{R}} V$, $U(V_{\mathbb{C}})$ for the unitary group, $\mathbb{C}P(V_{\mathbb{C}})$ for the complex projective space and $Q(V_{\mathbb{C}})$ for the 'quasi-projective space' $(i\mathbb{R})^+ \wedge \mathbb{C}P(V_{\mathbb{C}})^+$. $i\mathbb{R}$ is the Lie algebra of S^1 - the complex numbers of unit modulus. E, as in (5.7), will be the standard complex representation of S^1.

The complex θ is a map

$$[Z;\ U(V_{\mathbb{C}})] \longrightarrow \{Z;\ Q(V_{\mathbb{C}})\} \equiv \omega^{-1}_{S^1}((Z,*) \times S(E \otimes_{\mathbb{R}} V);\ -E \otimes_{\mathbb{R}} V),$$

defining in the limit

$$(6.9) \qquad \theta\ :\ \tilde{K}^{-1}(Z) = [Z;\ U(\infty)] \longrightarrow \{Z;\ Q(\infty)\}\ .$$

As in the real case, there is a map:

$$\zeta^{S^1}:\quad L(S^1)^{-1}(Z,*) \longrightarrow \{Z;\ Q(\infty)\}\ ,$$

which is always a bijection (even when Z is not connected). It identifies $\{X^+;\ Q(\infty)\}$ with an ideal (and direct summand) in $\omega^0_{S^1}(X)$. Now consider the commutative diagram:

$$\begin{array}{ccc} K^{-1}(X) & \longrightarrow & L(S^1)^{-1}(X) \\ {\scriptstyle F}\downarrow & & \downarrow \\ K^{-1}_{S^1}(X) & \xrightarrow[J_{S^1}]{} & \omega^0_{S^1}(X)^{\bullet} \end{array},$$

in which the top row is defined by the inclusion of $U(V_{\mathbb{C}})$ in $H^{S^1}(E \otimes_{\mathbb{R}} V)$ and F is the tensor product with E. We have $J_{S^1}.F(x) = 1 + \theta(x)\ (x \in K^{-1}(X))$.

This description shows that θ, (6.9), is quadratic. There is a linear map in the opposite direction

$$\varepsilon:\quad \{Z;\ Q(\infty)\} \longrightarrow [Z;\ U(\infty)] = \tilde{K}^{-1}(Z)$$

defined using the infinite loop space-structure of the infinite unitary group. If $f : Z \longrightarrow Q(V_{\mathbb{C}})$ is a stable map, set $\varepsilon(f) := f^*(r) \in \tilde{K}^{-1}(Z)$, where $r \in \tilde{K}^{-1}(Q(V_{\mathbb{C}}))$ is represented by the reflection map $R :\ Q(V_{\mathbb{C}}) \subseteq U(V_{\mathbb{C}})$. (Recall that R is defined by writing $Q(V_{\mathbb{C}})$ as $(S^1 \times \mathbb{C}P(V_{\mathbb{C}}))/(1 \times \mathbb{C}P(V_{\mathbb{C}}))$. $R(z,[x])$ $(z \in S^1$, $x \in S(V_{\mathbb{C}}))$ acts on x as multiplication by z and fixes vectors orthogonal to x.) The composition $\varepsilon.\theta$ is linear, because $\tilde{K}^{-1}(Q(V_{\mathbb{C}}) \wedge Q(V_{\mathbb{C}})) = K^1(\mathbb{C}P(V_{\mathbb{C}}) \times \mathbb{C}P(V_{\mathbb{C}}))$ is zero. By construction and the complex version of (2.7), $(\varepsilon.\theta).R = R$; more is true.

Proposition (6.10). (Segal [79]). The composition $\varepsilon.\theta$

$$[Z;\ U(\infty)] \longrightarrow \{Z;\ Q(\infty)\} \longrightarrow [Z;\ U(\infty)]$$

is the identity.

It suffices to consider $Z = U(V_{\mathbb{C}})$ and the inclusion $U(V_{\mathbb{C}}) \longrightarrow U(\infty)$. Write $n = \dim V$. Since the multiplication map $Q(V_{\mathbb{C}})^n \subseteq U(V_{\mathbb{C}})^n \longrightarrow U(V_{\mathbb{C}})$ induces a monomorphism in K-theory, we can reduce to the case $Z = Q(V_{\mathbb{C}})^n$, which is easily checked.

(The identity $\varepsilon.\theta = 1$ is natural. The projective unitary group $PU(V_{\mathbb{C}})$ acts on $U(V_{\mathbb{C}})$ and on $Q(V_{\mathbb{C}})$. θ is defined by a $PU(V_{\mathbb{C}})$-equivariant stable splitting $\theta_s : U(V_{\mathbb{C}}) \longrightarrow Q(V_{\mathbb{C}})$ of R. Now note that the $U(V_{\mathbb{C}})$-equivariant K-theory of a compact $PU(V_{\mathbb{C}})$-ENR splits as a direct sum of components corresponding to the irreducible representations of the centre S^1 of $U(V_{\mathbb{C}})$.

r is given by an element in $K^{-1}_{U(V_{\mathbb{C}})}(Q(V_{\mathbb{C}}))$ of weight 1. $\theta_s{}^*(r)$ is the canonical class (of weight 1) in $K^{-1}_{U(V_{\mathbb{C}})}(U(V_{\mathbb{C}}))$.)

We turn to the Real theory. $U(V_{\mathbb{C}})$ and $Q(V_{\mathbb{C}})$ are given the involution induced by conjugation; the fixed point sets are $O(V)$ and the real projective space $P(V)^+$ (with disjoint base-point). There is little to change, save the notation. X, Y and Z will be $\mathbb{Z}/2$-ENRs. $L(S^1)^{-1}_{\mathbb{Z}/2}(X)$ will be a subgroup of the group of units in $S^1 \rtimes \mathbb{Z}/2$-stable cohomotopy. The fixed sub-space of Z must be connected if $\zeta^{S^1} : L(S^1)^{-1}_{\mathbb{Z}/2}(Z,*) \longrightarrow \{Z;\ Q(\infty)\}^{\mathbb{Z}/2}$ is to be a bijection.

Proposition (6.11). Let Z be a compact $\mathbb{Z}/2$-ENR with base-point. Then there is a splitting $\varepsilon.\theta = 1$:

$$\widetilde{KR}^{-1}(Z) = [Z;\ U(\infty)]^{\mathbb{Z}/2} \longrightarrow \{Z;\ Q(\infty)\}^{\mathbb{Z}/2} \longrightarrow [Z;\ U(\infty)]^{\mathbb{Z}/2}.$$

The construction of ε uses KR as $\mathbb{Z}/2$-cohomology theory, or, equivalently, the $\mathbb{Z}/2$-equivariant infinite loop space-structure of $U(\infty)$.

If Z is a suspension $(i\mathbb{R})^+ \wedge B$ with the involution on B trivial, then $[Z;\ U(\infty)]^{\mathbb{Z}/2}$ is identified with $\widetilde{KO}(B)$ by KR-periodicity and $\{Z;\ Q(\infty)\}^{\mathbb{Z}/2} = \{B;\ \mathbb{C}P(\infty)^+\}^{\mathbb{Z}/2}$ splits as a direct sum $\{B;\ P(\infty)^+\} \oplus \{B;\ BO(2)^+\}$ of the fixed point and free components, (A.3). (Recall that the classifying space of $O(2) = S^1 \rtimes \mathbb{Z}/2$ may be written as $S(\infty) \times_{\mathbb{Z}/2} \mathbb{C}P(\infty)$.) (6.11) is essentially a desuspension (in effect if not in spirit) of a result of J. C. Becker and D. H. Gottlieb [13] and (6.10) a desuspension of Segal's original result.

The Adams conjecture is another topic that is profitably studied by Real methods; again we start with the complex theory. Fix a prime l and let $j : \mathbb{Z}/l^N \longrightarrow S^1$ be the inclusion of the subgroup of l^N-th roots of unity. We consider the J-homomorphism, f say:

$$j^*.J_{S^1}.F :\ K^{-1}(X) \longrightarrow K^{-1}_{S^1}(X) \longrightarrow \omega^0_{S^1}(X)^{\cdot} \longrightarrow \omega^0_{\mathbb{Z}/l^N}(X)^{\cdot}.$$

The automorphism group $(\mathbb{Z}/l^N)^{\cdot}$ of $\mathbb{Z}/l^N$ acts on $\omega^0_{\mathbb{Z}/l^N}(X)$. (Precisely, any endomorphism is given by multiplication by an integer q and induces an endomorphism q^* of $\omega^0_{\mathbb{Z}/l^N}(X)$; an automorphism defined by a q prime to l shall act by $q_* := (q^*)^{-1}$.) This 'Galois symmetry' lifts (at (l)) to the action of the Adams operations in K-theory.

Proposition (6.12). Let q be an integer prime to l. Then, at the prime (l),

$$f(\psi^q(x)) = q_*.f(x) \in \omega^0_{\mathbb{Z}/l^N}(X)_{(l)} \quad (x \in K^{-1}(X)).$$

The reduction to be used in the proof is also applicable to (6.10). Let B be the space of maximal tori of a compact connected Lie group G. The torus-bundle $E = \{(T,g) \in B \times G \mid g \in T\}$ over B projects onto G by a map, say $\pi : E \longrightarrow G$, of degree 1. Since both E and G are framed manifolds, $\pi^* : \omega^0(G) \longrightarrow \omega^0(E)$ is a split monomorphism. (The index $\pi_* : \omega^0(E) \longrightarrow \omega^0(G)$ defined by the framings supplies a splitting; $\pi_*.\pi^*$ is multiplication by the unit $\pi_*(1) \in \omega^0(G)$.)

It is sufficient to verify the assertion (6.12) when X is $U(V_{\mathbb{C}})$ and x is the universal class. To do this we may lift from

$G = U(V_{\mathbb{C}})$ to $E = U(V_{\mathbb{C}}) \times_N T$, where T is a maximal torus and N its normalizer in G. Here the identity may be checked quite explicitly, by writing T as $(S^1)^n$ and N as the wreath product $S^1 \int \mathfrak{S}_n$. (Compare the Becker-Gottlieb solution of the Adams conjecture [14].)

It is more satisfactory to state the proposition in terms of the free summand $L(\mathbb{Z}/_l N)^{-1}(X)$. This is mapped by the $\mathbb{Z}/_l N$-ζ to the ideal $\{X^+;\ B(\mathbb{Z}/_l N)^+\}$ in $\omega^0_{\mathbb{Z}/_l N}(X)$ and identified by the map: $x \longmapsto 1 + \zeta(x)$ with the group of all units in $\omega^0_{\mathbb{Z}/_l N}(X)$ which lie in the coset $1 + \{X^+;\ B(\mathbb{Z}/_l N)^+\}$. ($B(\mathbb{Z}/_l N)$ is the classifying space, the direct limit of the lens spaces $S(E \otimes V)/(\mathbb{Z}/_l N)$.) Now let $L^{\wedge}_{(l)}(\mathbb{Z}/_l N)^{-1}(X)$ be the intersection of $1 + \{X^+;\ B(\mathbb{Z}/_l N)^+\}^{\wedge}_{(l)}$ with the group of units in the l-adic completion $\omega^0_{\mathbb{Z}/_l N}(X)^{\wedge}_{(l)}$. It is a finitely generated $\mathbb{Z}^{\wedge}_{(l)}$-module (the sum if X is connected, and not empty, of a finite group and $L^{\wedge}_{(l)}(\mathbb{Z}/_l N)^{-1}(\text{point}) = \operatorname{Ker}\{\mathbb{Z}^{\wedge\bullet}_{(l)} \twoheadrightarrow (\mathbb{Z}/_l N)^{\bullet}\}$). f is continuous for the l-adic topology - its image is finite - and defines a $\mathbb{Z}^{\wedge}_{(l)}$- linear map

$$K^{-1}(X)^{\wedge}_{(l)} \longrightarrow L^{\wedge}_{(l)}(\mathbb{Z}/_l N)^{-1}(X),$$

which, according to (6.12), is equivariant with respect to the action of the group $\mathbb{Z}^{\wedge\bullet}_{(l)}$ of l-adic units on the left by the Adams operations and on the right through the projection onto $(\mathbb{Z}/_l N)^{\bullet}$.

We now recover the S^1-theory as an inverse limit. An homology argument shows that the restriction maps give an isomorphism:

$$\{X^+;\ (i\mathbb{R})^+ \wedge (BS^1)^+\}^{\wedge}_{(l)} \longrightarrow \varprojlim_N \{X^+;\ B(\mathbb{Z}/_l N)^+\}^{\wedge}_{(l)}.$$

The inverse limit is taken over the transfer maps defined by the inclusions $\mathbb{Z}/_l N \to \mathbb{Z}/_l N{+}1$. For the sake of consistency, $\mathbb{C}P(\infty)$ is interpreted as the classifying space of S^1. The action of $\mathbb{Z}^{\wedge\bullet}_{(l)}$ lifts to the inverse limit and θ extends to a $\mathbb{Z}^{\wedge\bullet}_{(l)}$-equivariant map

$$K^{-1}(X)^{\wedge}_{(l)} \longrightarrow \{X^+;\ (i\mathbb{R})^+ \wedge (BS^1)^+\}^{\wedge}_{(l)}.$$

(Again, this is better understood as a J-homomorphism by introducing a theory $L^{\wedge}_{(l)}(S^1)^{-1}(X)$.)

This completes the discussion of the complex theory; the parallel development of the Real theory is straightforward. Only the prime l = 2 is interesting, (9.6)(ii).

Remark (6.13). Let X be a compact $\mathbb{Z}/2$-ENR. An element $x \in \omega^0_{\mathbb{Z}/2}(X)_{(2)}$ is invertible if and only if $i^*(x) \in \omega^0(X)_{(2)}$ is invertible.

Hence the reduction employed in the proof of (6.12) generalizes at once; an involution on the group G induces an involution on E and $\pi^* : \omega^0_{\mathbb{Z}/2}(G)_{(2)} \to \omega^0_{\mathbb{Z}/2}(E)_{(2)}$ is a split monomorphism. However, it is not necessary to localize; the restriction of $\pi : E^{\mathbb{Z}/2} \to G^{\mathbb{Z}/2}$ has degree 1 on each component of the fixed point set. (Notice, too, that both the space B of maximal tori and the subspace $B^{\mathbb{Z}/2}$ fixed by the involution have Euler characteristic 1.)

One further point deserves mention. $\mathbb{C}P(\infty)$ with the action of $\mathbb{Z}/2$ by conjugation is the classifying space of the group S^1 with the involution: $z \mapsto z^{-1}$. See pp 105,6. The classifying space of the subgroup with involution $\mathbb{Z}/_1 N$ is, similarly, realized as an infinite lens space.

The Real (6.12) establishes, by passage to fixed subspaces, the real Adams conjecture for a suspension, although that was not our primary concern. Real methods are appropriate in the general case, too. If ξ is a Real vector bundle over a compact $\mathbb{Z}/2$-ENR X, then, for odd q, there is a stable $\mathbb{Z}/2$-map $\xi^+ \longrightarrow (\psi^q\xi)^+$ over X with the (non-equivariant) degree in each fibre odd. (See (B.2).)

Both the formulation of the periodicity theorem (6.3) and the construction of the KO-Euler class $\gamma(L.\xi)$ stem from the work of M. F. Atiyah and J. L. Dupont [7]. (6.6) is well known; the argument has the merit of immediate extension to equivariant KO_G-theory, G a compact Lie group. (See J. Berrick [17] for an application.) The method of proof, if not the statement, of (6.7) and (6.8) is now standard; see [10] III. The Real J-homomorphism has also been studied by H. Minami [68].

7. The image of J

As a first step in understanding $\mathbb{Z}/2$-equivariant stable cohomotopy, and hence as a means of defining universal elements in any $\mathbb{Z}/2$-cohomology theory, we shall describe the computation of the $KO_{\mathbb{Z}/2}$-theory d and e invariants. The methods and many of the results are to be found in the fundamental work of J. F. Adams.

$\omega_{**} \otimes \mathbb{Z}[\frac{1}{2}]$ is uninteresting: it follows from (4.3) and (4.4) that

(7.1) $\quad \omega_{i,j} \otimes \mathbb{Z}[\frac{1}{2}]$ is equal to $\omega_i \otimes \mathbb{Z}[\frac{1}{2}]$ if i-j is odd,
$(\omega_i \oplus \omega_j) \otimes \mathbb{Z}[\frac{1}{2}]$ if i-j is even.

We shall work throughout this paragraph <u>modulo odd torsion</u>.

According to the split exact sequence:

$$0 \longrightarrow \omega_{1,0} \xrightarrow{\ .b\ } \omega_{0,0} \underset{\sigma}{\overset{i^*}{\rightleftarrows}} \omega_0 \longrightarrow 0,$$

$\omega_{1,0}$ is a free abelian group on one generator, a say, such that ab = 1-t. From KO-theory, (6.5), $i^*(a)$ is the Hopf element $\eta \in \omega_1$. (In fact, the classical Hopf map $S^3 \longrightarrow S^2$, written as $S(\mathbb{C}^2) \longrightarrow \mathbb{C}P(\mathbb{C}^2)$, is equivariant with respect to complex conjugation and so may be regarded as a map $(\mathbb{R}\oplus L^2)^+ \longrightarrow (\mathbb{R}\oplus L)^+$.) This element a is the first of an infinite family generating the torsion-free part of $\omega_{i,0}$, $i > 0$.

We write a(i) for the Hurwitz-Radon number, the order of $\widetilde{KO}(P(\mathbb{R}^i))$, and set $a'(i) := a(i)$ if $i \not\equiv 0 \pmod 4$, $2a(i)$ if $i \equiv 0 \pmod 4$.

Theorem (7.2). There is a 'natural' graded subring M_* of ω_{*0} with the following properties.

$M_{-i} = \mathbb{Z}b^i$ if $i > 0$,

$M_0 = \mathbb{Z} \oplus \mathbb{Z}t$,

$M_i = \mathbb{Z}\mu_i$, if $i > 0$, is free on a generator μ_i such that $b^i\mu_i = a'(i)(1-t)$.

The kernel of the Hurewicz homomorphism $d : \omega_{*0} \longrightarrow KO_{*0}$ is precisely the torsion subgroup and $\omega_{*0} = M_* \oplus \mathrm{Ker}(d)$.

The restriction $i^*\mu_i \in \omega_i$ is:
of order 2 and detected by the d-invariant if $i \equiv 1$ or $2 \pmod 8$;
equal to the element of order 2 in the image of the J-homomorphism, $J : \mathbb{Z} = KO_{i+1} \longrightarrow \omega_i$ if $i \equiv 3$ or $7 \pmod 8$;
zero otherwise.

The epithet 'natural' in the statement of the theorem is to be taken both in the technical sense that the μ_i are defined, rather than their existence postulated, and in the colloquial sense. The method of proof may be seen as an attempt to realize the KO-theory periodicity operators in stable cohomotopy.

Proposition (7.3). In the coefficient ring ω_{**} there are operators, defined if i-j is odd,

$A : \omega_{i,j} \longrightarrow \omega_{i+8,j}$ and

$T : \omega_{i,j} \longrightarrow \omega_{i+2k,j}$ if $j-i \equiv k \pmod 4$, $k = 1$ or 3,

with the properties listed.

(i) $A = T^2$.

(ii) Regard the d-invariant $\omega_{**} \longrightarrow KO_{**}$ as a map from a $\mathbb{Z} \times \mathbb{Z}$- to a $\mathbb{Z}/8 \times \mathbb{Z}/4$-graded ring. Then $d.A = d$ and $d.T = \tau.d$.

(iii) $T(b) = a$; $A(b)$ generates the free component of $\omega_{7,0}$.

(iv) $x \in \omega_{i,j}$. If i-j is odd, then $i^*A(x)$ lies in the Toda bracket $\langle i^*x, 2, 8\sigma \rangle$. ($8\sigma$ is the element of order 2 in ω_7.) If $i-j \equiv 3 \pmod 4$, then $i^*T(x) \in \langle i^*x, 2, \eta \rangle$.

Remark (7.4). If $x \in \omega_{i,j}$ and i-j is odd, then $(1+t)x$ is zero (by inspection, because x is fixed by the involution). Thus, $2i^*(x) = 0$.

M_* will be the smallest subring of ω_{*0} containing b and closed under the action of the operator T. We illustrate the construction of T in the simpler case, $k = 1$. Let V be a real vector space. $a \in \omega_{1,0} = \omega^0_{\mathbb{Z}/2}(\text{point}; L)$ lifts to an element of $\omega^0_{\mathbb{Z}/2}(S(L.V); L)$, that is, of $\omega^0(P(V); H)$. As observed in §2, elements of this latter group may be thought of as stable fibre-homotopy classes of maps $H^+ \longrightarrow 0^+$ over P(V). By collapsing the base-points in each fibre to a single base-point, we obtain a stable map

$$P(V)^H \longrightarrow P(V)^+$$

of Thom spaces. The whole procedure is functorial and so produces an O(V)-equivariant stable map. In particular, if V is equal to E, the basic representation $\mathbb{H}$ of S^3 with the action by left multiplication, (5.7), a leads to an S^3-equivariant stable map

$$a_* : \; P(E)^H \longrightarrow P(E)^+ .$$

From b comes, similarly, the 'inclusion' $b_*: P(E)^{-H} \longrightarrow P(E)^+$.

Lemma (7.5). There is a unique S^3-equivariant stable map

$$T: P(E)^H \longrightarrow P(E)^{-H}$$

such that $b_*.T = a_*$.

In fact, T generates the group of stable S^3-maps $P(E)^H \longrightarrow P(E)^{-H}$.

Remark (7.6). Notice that, for any quaternionic vector space V, the balanced product construction '$S(V) \times_{S} 3$' applied to T will produce a stable Sp(V)-equivariant map $P(V)^H \longrightarrow P(V)^{-H}$. (Sp(V) is the symplectic group and P still the real projective space.)

The statement (7.5) on the lifting of a_*, on the face of it an S^3-equivariant result, is converted to $\mathbb{Z}/2$-homotopy theory by a simple observation.

Remark (7.7). Let H (not the Hopf bundle) be a closed subgroup of a compact Lie group G, X and Y respectively an H- and a G-ENR. Then G-maps $G \times_H X \longrightarrow Y$ (with G acting on the first space by left multiplication) correspond, by restriction, to H-maps $X \longrightarrow Y$. The same is true stably and, in particular, if X is compact and W a (virtual) coefficient H-module, then $\omega_G^*(G \times_H X; G \times_H W) \equiv \omega_H^*(X; W)$.

The reduction is effected by identifying P(E) with the homogeneous space $S^3/(\mathbb{Z}/2)$; the Hopf bundle is $S^3 \times_{\mathbb{Z}/2} L$. The same method is used for the construction of the operator T

in the stable cohomotopy ring. The map T induces

$$T^*: \omega_{S^3}^{-j}(P(E); -H) \longrightarrow \omega_{S^3}^{-j}(P(E); H),$$

that is, an operator $\omega_{j-1,j} = \omega_{\mathbb{Z}/2}^{-j}(\text{point}; -L) \longrightarrow \omega_{j+1,j} = \omega_{\mathbb{Z}/2}^{-j}(\text{point}; L)$. The extension to an operator $\omega_{i,j} \longrightarrow \omega_{i+2,j}$ for $j-i \equiv 1 \pmod 4$ depends on the fact that 4H over P(E) is equivariantly trivial (isomorphic to E), so that T defines a stable map $P(E)^{(4N+1)H} \longrightarrow P(E)^{(4N-1)H}$ for any integer N.

T in the case $k = 3$ is defined by constructing a stable S^3-map $P(E)^{3H} \longrightarrow P(E)^{-3H}$. The torsion subgroup of $\omega_{6,0}$ is of order 2, generated by x say, and there are two natural candidates for the map T differing by x_*. x is killed by multiplication by a or b, and the choice of T does not affect M_*.

The operator A is constructed from an S^1-equivariant stable map $P(E)^{7H} \longrightarrow P(E)^{-H}$, where E is now $\mathbb{C}$ with the action of S^1 by multiplication; it is an equivariant version of the original operator 'A' of Adams [1]. (There is also a stable S^1-map $P(E)^{3H} \longrightarrow P(E)^{-H}$ defining the operator $a^2.T$ or $T.a^2$.)

(7.2) describes the image of the d-invariant $\omega_{i,j} \longrightarrow KO_{i,j}$ for $j = 0$. The same method will give a description of the remaining, torsion cases. We turn our attention to the e-invariant. The Adams operation ψ^3 extends to a stable operation on the bigraded $KO_{\mathbb{Z}/2}$-theory $KO^{**}_{(2)}$ localized at the prime (2). The associated e-invariant will be a homomorphism:

$\mathrm{Ker}(d)\ (\subseteq \omega_{i,j}) \longrightarrow (KO_{i+1,j+1})_{(2)}/(\psi^3 - 1)(KO_{i+1,j+1})_{(2)}$.

When $j \equiv 3 \pmod 4$ $KO_{i,j} = 0$ and it takes the form ($j \neq -1$):

$$(7.8) \quad e : \omega_{i,j} \longrightarrow \mathbb{Z}/2^{v(j+1)+1} \quad \text{if } i+1 \not\equiv 0 \pmod 4,$$
$$(\mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(2)}t)/(2^{v(i+1)}(1+t) - 2^{v(j+1)}(1-t)) \quad \text{if } i+1 \equiv 0 \pmod 4.$$

Here v is the 2-adic valuation: $v(2^s.\mathrm{odd}) = s$. The second group is the quotient of the ring $\mathbb{Z}_{(2)}[t]/(t^2-1)$ by a principal ideal.

Theorem (7.9). (Mahowald). $j+1 \equiv 0 \pmod 4$, $j \neq -1$ or 3. Then the image of e, (7.8), is as follows. Set i_0: $\max\{ i \mid j+1 \equiv 0 \pmod{a(i+1)} \}$. e is surjective if $i \geq i_0$. If $i < i_0$, then the image is generated by:

$2^{v(j+1)}/a(i+1)$ if $i+1 \not\equiv 0 \pmod 4$,

$1+t$ and $(2^{v(j+1)}/2a(i+1)).(1-t)$ if $i+1 \equiv 0 \pmod 4$,

with the exception of the case $i = 7$, $j+1 \equiv 4 \pmod 8$, when it is generated by $1+t$ and $4(4-(1-t))$.

There is an M_*-submodule B_{*j} of ω_{*j} such that $\omega_{*j} = \mathrm{Ker}(e) \oplus B_{*j}$; $\rho B_{*j} = 0$; and the restriction $i^*B_{*j} \subseteq \omega_*$ lies in the image of J, with the exception of the cases i = 6, 8 and 9, $j+1 \equiv 4 \pmod 8$, when the image in ω_i is, respectively,

$\mathbb{Z}/2.\nu^2$, $\mathbb{Z}/2.(\bar{\nu} + \eta\sigma)$ and $\mathbb{Z}/2.(\eta\bar{\nu} + \eta^2\sigma)$.

($\bar{\nu}$ is the generator of the kernel of e in ω_8; σ is the generator of ω_7.)

This theorem is a generalization of the vector field theorem of Adams in the form:

(7.10) $j+1 \equiv 0 \pmod 4$. Then $b^i : \tilde{\omega}_j(P^\infty_{j-i}) \longrightarrow \tilde{\omega}_j(P^\infty_j) = \mathbb{Z}/2$ is non-zero if and only if $j+1 \equiv 0 \pmod{a(i+1)}$.

The point is that the e-invariant detects $\tilde{\omega}_j(P^\infty_j)$.

We indicate the idea of the proof by defining a 'Clifford element' $c_j \in \omega_{i_0,j}$ which under the action of T and b generates a substantial part of the summand B_{*j}. If $i_0+1 \equiv 0$ or $1 \pmod 8$ and $j+1 \equiv 0 \pmod 8$, this will include the whole of $B_{i,j}$ in dimensions $i \not\equiv 3 \pmod 4$. (In the other two cases $i_0+1 \equiv 2$ or $4 \pmod 8$, it is necessary to introduce another generator in dimension i_0+5 or i_0+3 ($\equiv j+7 \pmod 8$) to achieve the same effect. This depends on a constructive reading of Lemma (4.8) of [27].)

Write $V := \mathbb{R}^{i_0+1}$. $(\mathbb{R}^{j+1}, \mathbb{R}^{j+1})$ admits the structure of a $\mathbb{Z}/2$-graded module over the Clifford algebra C(V). The multiplication defines a trivialization of the bundle (j+1)H over the real projective space P(V) and so an isomorphism

$$(7.11) \qquad \omega_j(P(V); (j+1)H - H.V) \cong \omega_{-1}(P(V); -H.V)$$

(in the local coefficient notation of §2). The second group contains a canonical element corresponding under S-duality to $1 \in \omega^0(P(V))$. c_j will be its image under the map (7.11), the (surjective) stabilization map $(P(V) \longrightarrow P(\infty))$ and δ of (4.6):

$$\omega_j(P(V); (j+1)H - H.V) \longrightarrow \tilde{\omega}_j(P^\infty_{j-i_0}) \longrightarrow \omega_{i_0,j} .$$

i^*c_j is non-zero; it generates the image of J in ω_{i_0}. (This is proved, for example, in [27] (4.8).) c_j is well defined, up to sign, if $i_0+1 \not\equiv 0 \pmod 4$. If $i_0+1 \equiv 0 \pmod 4$, then c_j depends on the choice of the Clifford module-structure, but only within the limits of B_{*j}. In fact, the variation lies within the subgroup of $\omega^0(P(V))$ generated by the image of the J-homomorphism $KO^{-1}(P(V))$ (= $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}) \longrightarrow \omega^0(P(V))$ and this subgroup is mapped onto $B_{i_0,j}$.

For any positive $k \equiv 0 \pmod 4$, let I be the subgroup of $\omega^0(P(\mathbb{R}^k))$ generated by the image of the J-homomorphism. There is a restriction map $\omega_{k-1} \cong \omega^0(P(\mathbb{R}^k),P(\mathbb{R}^{k-1})) \longrightarrow \omega^0(P(\mathbb{R}^k))$; let x be the image of a generator of the (2-primary component of the) image of J. The description of I is surprisingly delicate; it is, however, an easy consequence of (7.2).

Remark (7.12). $I \subseteq \omega^0(P(\mathbb{R}^k))$ is a direct summand. It is generated by 1, $\bar{t}$ (as in (4.9)) and x, with relations:

$$2a(k).(1-\bar{t}) = 0, \quad a(k).(1-\bar{t}) = 2^{v(k)}.x .$$

B_{**} is the 'image of J' of the title. There is an obvious candidate for inclusion in any such image of J. If $j \equiv 3 \pmod 4$, we have $\theta : \pi_j(O(\infty)) = \mathbb{Z} \longrightarrow \omega_j(P(\infty)) \subseteq \omega_{j,j}$, (2.13), and the image, $\tilde{\rho}_j$ say, of a generator restricts to a generator ρ_j of the image of J in ω_j. (From (3.8) $\bar{P}^2(\rho_j)$ is equal to $\tilde{\rho}_j - i_*(\rho_j)$ and so adds nothing new to the discussion.) It is tempting to guess that $\tilde{\rho}_j$ lies in $B_{j,j}$ (properly defined). This would imply at once the Barratt-Mahowald theorem [12], which asserts the vanishing of $b^{1+\frac{1}{2}(j+1)}.\tilde{\rho}_j$ if $j > 15$. More generally, one could try to relate B_{**} to the image of the equivariant J-map : $KO_{i+1,j+1} \longrightarrow \omega_{i,j}$ defined if

$i \geqslant j \geqslant 0$. (The image in $\omega_{j,j}$ lies in the subgroup generated by $\sigma(\rho_j)$, $i_*(\rho_j)$ and $\tilde{\rho}_j$.)

And then one might seek a representation of elements of $B_{i,j}$ when $0 \leqslant i \leqslant j$, perhaps by a non-singular bilinear map

$$\mathbb{R}^{j+1} \times \mathbb{R}^{i+N} \longrightarrow \mathbb{R}^{j+N}$$

for some integer N. (Such a bilinear map defines an element of $\pi_j(V_{j+N,i+N})$ and so, via θ, of $\tilde{\omega}_j(P^{\infty}_{j-i})$.) Both c_j and $\tilde{\rho}_j$ are so represented, by Clifford multiplication. In the metastable range Lam's theorem [60] may be applied to give an abstract existence result.

Remark (7.13). $0 \leqslant 2i < j$. $x \in \tilde{\omega}_j(P^{\infty}_{j-i})$. Then $(1-(-1)^j)x$ and $(1+(-1)^j t)x$ are both representable by bilinear maps. (Compare the statement $i^*.i_* = 1+(-1)^{i-j}$ of (4.3). The argument can be found in K. Y. Lam [61] or L. Smith [81].) If further $i > 0$, $j+1 \equiv 0 \pmod{a(i+2)}$ and $b^i x = 0$, then the Kahn-Priddy theorem allows the representation of x itself.

In the non-equivariant theory the d and e invariants are most neatly described in terms of the fibre of $\psi^3 - 1$. There is a generalized cohomology theory J^* fitting into an exact sequence:

$$\cdots \longrightarrow J^* \longrightarrow KO^*_{(2)} \xrightarrow{\psi^3 - 1} KO^*_{(2)} \longrightarrow \cdots .$$

On the coefficient groups the Hurewicz map $\omega_i \longrightarrow J_i$ is a split surjection if $i > 1$. The splitting is defined by a solution of the Adams conjecture, which, for any compact ENR X, extends the J-homomorphism on $KO^{-1}(X)$ to a map (not, in general, a homomorphism):

$$KO^{-1}(X)_{(2)} \longrightarrow J^0(X) \longrightarrow \omega^0(X)^{\cdot}_{(2)} .$$

(This form of the Adams conjecture is needed for the verification in (7.2) that $i^*\mu_i$ is in the image of J for $i \equiv 3 \pmod 4$.)

The $\mathbb{Z}/2$-equivariant J-theory is defined in the same way, as the fibre of $\psi^3 - 1$. We have been concerned in this paragraph with the Hurewicz map $\omega_{**} \longrightarrow J_{**}$ on the coefficient groups and have seen, at least in certain dimensions, that its kernel has a natural complementary summand (containing M_{*0} and B_{**}), the analogue of the 'image of J'. $\omega_{i,j} \longrightarrow J_{i,j}$ is not always surjective and only in the range $i \geqslant j \geqslant 0$ is a J-map available to construct a splitting. It is conceivable that both these defects could be remedied by replacing J^{**} by a connective theory, connective in a sense to be made precise.

The element a which began our story may be described geometrically as follows. Consider S^1 as a Lie group with the trivialization of its tangent bundle by right translation (as in (5.7)). This framing is natural and so compatible with the action of $\mathbb{Z}/2$ on S^1 as the group of automorphisms and τS^1 is trivialized as L. S^1 so framed represents $a \in \omega_{1,0}$. The same procedure may be applied to any compact connected Lie group. For example, S^3 is framed as an $SO(3)$ ($= \mathrm{Aut}(S^3)$) -manifold. Restricting from $SO(3)$ to a subgroup of order two,

we obtain a $\mathbb{Z}/2$-framing of τS^3 as $\mathbb{R} \oplus L^2$. This defines an element of $\omega_{3,1}$ restricting to the generator ν of ω_3 and mapping under the fixed point map to $\eta \in \omega_1$ represented by the fixed subgroup $S^1 \subseteq S^3$ (or $SO(2) \subseteq SU(2)$). Thus enters the rich theory of symmetric spaces into $\mathbb{Z}/2$-homotopy theory.

In the discussion which follows there is no need to neglect odd torsion.

1-t is divisible by b in ω_{**}; indeed a was introduced by the factorization $ab = 1-t$. It is a recent and beautiful theorem of W. H. Lin that any element of $\omega_{i,j}$ divisible by a sufficiently high power of b (depending on (i,j)) is divisible by 1-t.

Theorem (7.14). (Lin). For each (i,j) the two filtrations $((1-t)^N.\omega_{i,j})_{N \geqslant 0}$ and $(b^{2N}.\omega_{i+2N,j})_{N \geqslant 0}$ of $\omega_{i,j}$ define the same topology.

(J. F. Adams has given an account in [2]. There are references there to the work of W. H. Lin and of D. M. Davis and M. E. Mahowald. The theorem answers a long-standing question posed, in different forms, by G. B. Segal and M. E. Mahowald.)

The principal (graded) ideal (b) $\subseteq \omega_{**}$ is the kernel of the restriction map $i^* : \omega_{**} \longrightarrow \omega_*$. Using (7.1) and (7.2) (to deal with the odd torsion and free summand respectively), it is easy to reduce (7.14) to the statement:

(7.15) $$\bigcap_{N \geqslant 0} (b)^N = 0 \text{ at the prime (2).}$$

The theorem has an important corollary, deduced by a routine manipulation of the $\mathbb{Z}/2$-stable cohomotopy exact sequences of the pairs $(D(NL), S(NL))$, $N > 0$.

Corollary (7.16). (Lin). Let X be a compact $\mathbb{Z}/2$-ENR. Fix (i,j). Then there are isomorphisms of pro-abelian groups:

$$(\omega^{-i,-j}(X)/(1-t)^N.\omega^{-i,-j}(X))_{N \geqslant 0} \longrightarrow$$

$$(\omega^{-i,-j}(X)/b^N.\omega^{-i-N,-j}(X))_{N \geqslant 0} \longrightarrow (\omega^{-i,-j}(X \times S(NL)))_{N \geqslant 0}.$$

The structure maps of the inverse systems are respectively the algebraic projections and the topological restriction maps.

In particular, $\varprojlim \omega^{-i,-j}(X \times S(NL))$ is isomorphic to the (1-t)-adic completion of $\omega^{-i,-j}(X)$ and $R^1\varprojlim = 0$. The result is also true with KO-theory substituted for stable cohomotopy; the formulation of (7.16) has been carried over directly from the K-theory [9] of M. F. Atiyah and G. B. Segal. It is worth noting how the K-theory version of (7.14) is proved: in KO_{**}, b^8 is divisible by $(1-t)^4$. (They differ by multiplication by a Bott periodicity class in $KO_{-8,0}$.)

A $\mathbb{Z}/p$-equivariant version, for an odd prime p, of the theory described in this paragraph is lacking. A class in $\omega^0_{\mathbb{Z}/p}(\text{point}; V_p)$ corresponding to a is defined by a torus of rank p-1, the kernel of multiplication $(S^1)^p \longrightarrow S^1$ with $\mathbb{Z}/p$ permuting the factors of the p-fold product. However, the construction of a $\mathbb{Z}/p$-Adams-Toda operator is an open problem.

The primary references are the papers of H. Toda [85] (and also [44]) and J. F. Adams [1]. The application of the Adams-Toda operators A and T to the proof of Mahowald's theorem [66] (Theorem D) is due to S. Feder, S. Gitler and K. Y. Lam [30].

I am especially grateful to Prof. Gitler for conversations on this subject. (7.2) is a lifting to $\mathbb{Z}/2$-stable cohomotopy from stable cohomotopy with coefficients in $\mathbb{Z}/2$ of the first definition by Adams of generators in ω_i (i = 1,2,3 and 7 (mod 8)). The existence of the lifting follows from [85] and the work of G. E. Bredon [19]; see also (5.13) of [7]. The representation of the generator a by the framed Lie group S^1 appears in the 'figure-of-eight construction' of U. Koschorke [56]. Such a significant result as Lin's theorem could not be omitted from an account of $\mathbb{Z}/2$-homotopy theory; but the exposition here is regrettably brief. I have placed the emphasis on the apparently more elementary statement (7.14).

8. The Euler characteristic

Let X be a closed orientable manifold of dimension m. The stable cohomotopy Euler class $\gamma(L.\tau X) \in \omega^0_{\mathbb{Z}/2}(X; -L.\tau X)$ of the tangent bundle with the antipodal involution was introduced in §2, for an arbitrary vector bundle, as an obstruction to the existence of cross-sections. For the tangent bundle it has a wider significance.

Suppose that $f : X \longrightarrow X'$ is a homeomorphism between closed manifolds. By looking at the restriction of $f \times f$ to the complement of the diagonal $X \times X - X \longrightarrow X' \times X' - X'$, A. Haefliger and M. W. Hirsch [33] defined a natural $\mathbb{Z}/2$-fibre-homotopy equivalence $S(L.\tau X) \longrightarrow S(L.f^*\tau X')$, up to homotopy. If f is merely a homotopy equivalence, M. F. Atiyah [3] defined a natural stable fibre-homotopy equivalence, the 'homotopy derivative', $df : S(\tau X) \rightarrow S(f^*\tau X')$. ($f_* : \omega^0(X; f^*\tau X' - \tau X) \longrightarrow \omega^0(X'; \tau X' - \tau X') = \omega^0(X')$ is an isomorphism; the inverse image of 1 is $(df)^{-1}$.) More properly, by (3.7), we should now regard the homotopy derivative as an equivariant stable fibre-homotopy equivalence

(8.1) $\qquad df : S(L.\tau X) \longrightarrow S(L.\tau X')$,

agreeing with the Haefliger-Hirsch definition if f is a homeomorphism. The index homomorphism, (5.1), defined by the pair (f,df)

$$(f,df)_* : \omega^0(X) \longrightarrow \omega^0(X')$$

maps 1 to 1; it is, almost by definition, the inverse of f*.

Theorem (8.2). (Sutherland [84], Benlian-Wagoner [16], Dupont [29]). Let $f : X \longrightarrow X'$ be a homotopy equivalence between orientable closed manifolds. Then

$$(f,df)_* : \omega^0_{\mathbb{Z}/2}(X; -L.\tau X) \longrightarrow \omega^0_{\mathbb{Z}/2}(X'; -L.\tau X'),$$

inverse to $(f,df)^*$, maps $\gamma(L.\tau X)$ to $\gamma(L.\tau X')$.

This statement of the homotopy invariance of the Euler class is equivalent, by (3.9), to the original formulation of the theorem, namely, that $S(\tau X)$ and $S(f^*\tau X')$ are fibre-homotopy equivalent. We make one or two remarks on the proof below.

The Euler characteristic E(X) of the manifold X is the image of $\gamma(\tau X)$ under the index homomorphism

$$\omega^0(X; -\tau X) \longrightarrow \omega^0(\text{point}) = \omega_0 = \mathbb{Z}.$$

A framing of X defines an index

$$\omega^0_{\mathbb{Z}/2}(X; -L.\tau X) \longrightarrow \omega^0_{\mathbb{Z}/2}(\text{point}; \mathbb{R}^m - L^m) = \omega_{0,m},$$

for $-L.\tau X$ may be written as $-\tau X + \mathbb{R}^m - L^m$.

Definition (8.3). The <u>$\mathbb{Z}/2$-Euler characteristic</u> of the framed manifold X is the index in $\omega_{0,m}$ of $\gamma(L.\tau X)$.

It restricts to the classical Euler characteristic in ω_0 and is described by the following 'curvatura integra' theorem of M. Kervaire [52].

Theorem (8.4). (Kervaire). Let $x \in \omega_m$ be the class represented by the framed manifold X. Then the Euler characteristic of X in

$$\omega_{0,m} = \omega_m \oplus \begin{cases} \mathbb{Z} & \text{if } m \text{ is even} \\ \mathbb{Z}/2 & \text{if } m \text{ is odd (by (4.7)(ii))} \end{cases}$$

may be written as $x + \begin{cases} \frac{1}{2}E(X) & m \text{ even} \\ R(X) + \text{Hopf invariant of } x & m \text{ odd.} \end{cases}$

$R(X)$ is the mod 2 semicharacteristic $\sum_{i \text{ even}} \dim H^i(X; \mathbb{F}_2)$ (mod 2). The Hopf invariant $b^m\tilde{P}^2(x) \in \tilde{\omega}_m(P_m^\infty) = \mathbb{Z}/2$ is zero unless $m = 1, 3$ or 7 (by the theorem of Adams, (7.10)).

(Perhaps the simplest proof uses the quadratic form on a framed manifold. We use the notation of §5.

Lemma (8.5). Let X be a framed manifold of dimension $m = 2k \equiv 2 \pmod 4$ and $Y \subseteq X$ a framed submanifold of dimension k with normal bundle ν representing a class $y \in H^k(X)$. The stable trivializations define an index map $\omega^0_{\mathbb{Z}/2}(Y; -L.\nu) \longrightarrow \omega_{0,k}$. The image of $\gamma(L.\nu)$ is equal to:

$$[Y] + (Q(y) + \text{Hopf invariant of } [Y]) \in \omega_k \oplus \mathbb{Z}/2 = \omega_{0,k}.$$

The essence of the proof is this: $\Delta^X_*(1) \in \omega^0_{\mathbb{Z}/2}(X \times X; -mL)$ restricts to $\Delta^Y_*(\gamma(L.\nu)) \in \omega^0_{\mathbb{Z}/2}(Y \times Y; -mL)$, where Δ^X and Δ^Y are the respective inclusions of the diagonal.

(8.4) is proved by applying the lemma to the diagonal submanifold X of $X \times X$ with normal bundle τX.)

Without the requirement that the manifold be framed, but merely that it be oriented, we may define the $KO_{\mathbb{Z}/2}$-Euler characteristic in $KO_{0,m}$. For (6.3), with coefficients, gives an isomorphism: $KO_{\mathbb{Z}/2}(X; -L.\tau X) \cong KO_{\mathbb{Z}/2}(X; \mathbb{R}^m - L^m - \tau X) \equiv KO^{0,-m}(X; -\tau X)$. The computation of the Euler characteristic

and its interpretation as an obstruction to the existence of vector fields formed the subject of the important paper [7] of M. F. Atiyah and J. L. Dupont.

Theorem (8.6). (Atiyah-Dupont). The $KO_{\mathbb{Z}/2}$-Euler characteristic of an oriented m-manifold X in $KO_{0,m}$ is:

$\frac{1}{2}(E(X)+S(X)) + \frac{1}{2}(E(X)-S(X))t \in \mathbb{Z} \oplus \mathbb{Z}t$ if $m \equiv 0 \pmod 4$,

$R(X) \in \mathbb{Z}/2$ if $m \equiv 1 \pmod 4$,

$\frac{1}{2}E(X) \in \mathbb{Z}$ if $m \equiv 2 \pmod 4$,

$0 \in 0$ if $m \equiv 3 \pmod 4$,

where $S(X)$ is the signature and $R(X)$ the Kervaire semicharacteristic $\sum_{i \text{ even}} \dim H^i(X; \mathbb{R})$ (mod 2).

(According to G. Lusztig, J. Milnor, F. P. Peterson [65], the two definitions of $R(X)$ in (8.4) and (8.6) agree for a framed manifold of dimension 1 (mod 4).)

The proof of (8.2) given by W. A. Sutherland for even m relies on the fact that the image of

$$i^*((f,df)_*\gamma(L.\tau X) - \gamma(L.\tau X')) = f_*\gamma(\tau X) - \gamma(\tau X')$$

under the index map $\omega^0(X'; -\tau X') \longrightarrow \omega_0 = \mathbb{Z}$ is $E(X) - E(X')$, which vanishes when X and X' are homotopy equivalent, because of the expression of $E(X)$ as the alternating sum $\sum_i (-1)^i \dim H^i(X; \mathbb{R})$. If $m \equiv 1 \pmod 4$, then a similar proof, in the style of [29], can be given by observing that the $KO_{\mathbb{Z}/2}$-index of

$$(f,df)_*\gamma(L.\tau X) - \gamma(L.\tau X') \in KO_{\mathbb{Z}/2}(X'; -L.\tau X')$$

is $R(X) - R(X') \in \mathbb{Z}/2$, which vanishes for the same reason. This is not quite immediate. In general, the diagram

$$\begin{array}{ccc} KO_{\mathbb{Z}/2}(X; -L.\tau X) & \xrightarrow{(f,df)_*} & KO_{\mathbb{Z}/2}(X'; -L.\tau X') \\ \cong \big| & & \big| \cong \\ KO^{0,-m}(X; -\tau X) & \xrightarrow[f_*]{} & KO^{0,-m}(X'; -\tau X') \end{array}$$

will not be commutative. The deviation from commutativity will be multiplication by a unit in $KO_{\mathbb{Z}/2}(X)$ restricting to 1 in $KO(X)$, that is, of the form $1 + (1-t)w$. But $(1-t)KO_{0,1} = 0$.

The equivariant $KO_{\mathbb{Z}/2}$-theory may be complexified in two ways, either directly to $K_{\mathbb{Z}/2}$-theory, or, by extending the involution antilinearly, to KR-theory. One may again consider $\gamma(L.\tau X) \in KR(X; -L.\tau X)$ (writing L rather than $i\mathbb{R}$). This, too, was introduced into obstruction theory by Atiyah and Dupont. The orientability condition necessary to define the $\mathbb{Z}/2$-Euler characteristic is now weakened to the existence of a (flat) square root, ε say, of the complexified line bundle of m-forms $\Lambda^m \tau^* X \otimes \mathbb{C}$; topologically $w_1(X)^2 = 0$. The KR-Euler characteristic, depending upon the choice of ε, is:

$$E(X;\varepsilon) \in \mathbb{Z},\ R(X;\varepsilon) \in \mathbb{Z}/2,\ \tfrac{1}{2}E(X;\varepsilon) \in \mathbb{Z},\ 0 \in 0$$

according as $m \equiv 0, 1, 2, 3 \pmod 4$. E and R are the Euler characteristic and Kervaire semicharacteristic with coefficients in the flat bundle and are invariants of non-singular symmetric bilinear forms defined over $\mathbb{C}$. E, S, R in (8.6) are likewise invariants of real symmetric forms. This will become clearer in the next paragraph, in which $KO_{\mathbb{Z}/2}$- and KR-theory are

interpreted respectively as the Hermitian K-theory of $\mathbb{R}$ and $\mathbb{C}$ (with the trivial involution).

9. Topological Hermitian K-theory

The topological Hermitian K-theory is now well understood. It will serve as model for the algebraic theory of the next paragraph. Let X be a compact Hausdorff space with involution and μ a (finite dimensional) complex vector bundle over X. The involution on X is written $^{-}$. In the context of symmetric forms $\bar{\mu}$ will be not the complex conjugate, but the complex bundle with fibre $\bar{\mu}_x = \mu_{\bar{x}}$ $(x \in X)$ induced from μ by the involution.

A non-singular symmetric form on μ is a bundle map $g : \mu \otimes \bar{\mu} \longrightarrow \mathbb{C}$ satisfying the conditions of

non-singularity: the adjoint $g' : \mu \longrightarrow \bar{\mu}^*$ $(= \mathrm{Hom}(\bar{\mu},\mathbb{C}))$ is a bundle isomorphism,

and symmetry: $g_x(u,v) = g_{\bar{x}}(v,u)$ if $u \in \mu_x = \bar{\mu}_{\bar{x}}$, $v \in \mu_{\bar{x}} = \bar{\mu}_x$.

$K^{0,0}(X)$ will be the Grothendieck group of isomorphism classes of such bundles with non-singular symmetric forms. There is a forgetful map $K^{0,0}(X) \longrightarrow K^0(X)$ to complex K-theory and a hyperbolic map $K^0(X) \longrightarrow K^{0,0}(X)$ taking a complex bundle ν to $\nu \oplus \bar{\nu}^*$ with the form $(\nu \oplus \bar{\nu}^*) \otimes (\bar{\nu} \oplus \nu^*) \longrightarrow \mathbb{C}$: $(u,\alpha) \otimes (v,\beta) \longmapsto \alpha(v) + \beta(u)$. Both the insight into Hermitian K-theory as a $\mathbb{Z}/2$-equivariant cohomology theory and the basic computational tool of the theory are supplied by the identification:

Lemma (9.1). $K^{0,0}(X) = KR(X)$.

Indeed, choosing a positive-definite Hermitian metric $\langle -,- \rangle$ on μ (unique up to equivalence or homotopy), we may write

$$g_x(u,v) = \langle \varphi_x(u),v \rangle \qquad (u \in \mu_x,\ v \in \mu_{\bar{x}}) ,$$

where φ_x is a conjugate linear map $\mu_x \longrightarrow \mu_{\bar{x}}$. $\varphi_{\bar{x}} \cdot \varphi_x$ is a positive-definite $\mathbb{C}$-linear automorphism of μ_x and has a canonical positive-definite square-root. $j_x := \varphi_x(\varphi_{\bar{x}} \cdot \varphi_x)^{-\frac{1}{2}}$ is a Real structure for μ – $j^2 = 1$, and is determined up to equivalence by the symmetric form.

To develop the theory and, in particular, to state a periodicity theorem, one is forced to introduce forms defined on complexes of vector bundles and with coefficients in a complex line bundle. So let $\mu : \rightarrow \mu^r \xrightarrow{d} \mu^{r+1} \rightarrow$ be a complex of $\mathbb{C}$-vector bundles (with $\mu^r = 0$ for all but finitely many $r \in \mathbb{Z}$) and ε a $\mathbb{C}$-line bundle with involution (that is, a $\mathbb{Z}/2$-line bundle) over X. Write $\varepsilon(m)$ for the complex with component ε in degree m and otherwise zero. The involution on ε will be written once again as $^{-}$. A non-singular $\varepsilon(m)$-valued symmetric bilinear form on the complex μ is a chain map $g : \mu \otimes \bar{\mu} \longrightarrow \varepsilon(m)$ of degree zero, non-singular in the sense that its adjoint $g' : \mu \longrightarrow \mathrm{Hom}(\bar{\mu},\varepsilon(m))$ is a chain homotopy equivalence and symmetric in the graded sense: $g_x(u,v) = (-1)^{rs}\overline{g_{\bar{x}}(v,u)}$ $(u \in \mu^r_x,\ v \in \mu^s_{\bar{x}})$. (The tensor product is the product of complexes and homomorphisms are of degree zero.)

One introduces a notion of equivalence for such forms: generated by

(o) isomorphism (in the strict sense),

(i) homotopy - if (μ, d_t, g_t) $(t \in [0,1])$ is a family of non-singular forms, then (μ, d_0, g_0) is equivalent to (μ, d_1, g_1),

(ii) addition of an acyclic (that is, exact) complex.

The set of equivalence classes is a group under direct sum and will be denoted by $K_\varepsilon^{0,-m}(X)$. A surgery argument shows that this is consistent with the primitive definition if $m = 0$ and $\varepsilon = \mathbb{C}$; a trivial coefficient bundle will sometimes be dropped from the notation.

The tensor product of complexes and coefficient bundles defines a product:

$$K_\varepsilon^{0,-m}(X) \otimes K_{\varepsilon'}^{0,-m'}(X') \longrightarrow K_{\varepsilon\otimes\varepsilon'}^{0,-(m+m')}(X \times X').$$

There is a basic periodicity class in $K_{L\otimes\mathbb{C}}^{0,-2}(\text{point})$ represented by the complex $\mathbb{C}$ concentrated in degree 1 with the symmetric form $\mathbb{C}\otimes\mathbb{C} \longrightarrow L\otimes\mathbb{C}$ given by multiplication. Its square gives periodicity in m modulo 4, re-indexing a complex by a shift of degree 2.

As in the usual K-theory it is convenient to extend the definition to locally compact spaces by considering complexes with compact support. To accommodate the coefficient bundle one must consider locally compact spaces over X. The most important example is the Bott class. Let ζ be a complex $\mathbb{Z}/2$-vector bundle of dimension n over X. Exterior multiplication defines a $\Lambda^n\zeta = (\det\zeta)(n)$-valued symmetric form on the exterior algebra $\Lambda\zeta$. Lifted to the total space $E(L\otimes\zeta)$ of ζ with the negative of the given involution and equipped with the standard differential, this defines a class

$$\lambda_\zeta \in K_{\det(\zeta)}^{0,-n}(E(L.\zeta))$$

which restricts to the periodicity class in $K^0(E(\zeta))$.

Proposition (9.2). Multiplication by λ_ζ gives a periodicity isomorphism in Hermitian K-theory

$$K_\varepsilon^{0,-m}(X) \longrightarrow K_{\varepsilon\otimes\det(\zeta)}^{0,-(m+n)}(E(L.\zeta)).$$

This result, or its generalization to locally compact spaces over X, permits the definition, in the usual way, of a periodic $\mathbb{Z}/2$-cohomology theory K^{**} (or with coefficients K_ε^{**}). $K^{-i,-j}(X)$ is just $KR^{i-2j}(X)$. It is related to complex K-theory by:

Lemma (9.3). $K_\varepsilon^{-i,-j}(X \times S(L)) = K^{-i}(X)$.

And then the exact sequence of the pair $(D(L),S(L))$, as in stable cohomotopy (4.3), has the following interpretation. Let T_ε be the involution in complex K-theory which takes a vector bundle ν over X to $\mathrm{Hom}(\bar{\nu},\varepsilon)$.

Lemma (9.4). The linear and Hermitian K-theory are related by the exact sequence: $\ldots \longrightarrow K_\varepsilon^{-(i+1),-j}(X) \xrightarrow{.b} K_\varepsilon^{-i,-j}(X) \xrightarrow{r} K^{-i}(X) \xrightarrow{h_\varepsilon} K_\varepsilon^{-i,-j+1}(X) \longrightarrow \ldots$, in which r is the forgetful (restriction) map and h_ε the hyperbolic (induction) map. The composition $r.h_\varepsilon$: $K^{-i}(X) \longrightarrow K_\varepsilon^{-i,-j}(X) \longrightarrow K^{-i}(X)$ is $1 + (-1)^{i-j}T_\varepsilon$.

Now define formally the <u>Witt cohomology theory</u> of a compact Hausdorff space Y (no involution) as:

$$(9.5) \qquad W^{-j}(Y) := \varinjlim_{i} K^{-i,-j}(Y).$$

(It is a cohomology theory because the direct limit is an exact functor.) The limit is taken over successive multiplication by b, as in (4.4).

Lemma (9.6). (Formal properties of the Witt theory).

(i) Restriction to the fixed subspace gives an isomorphism $\varinjlim_{i} K^{-i,-j}(X) \longrightarrow W^{-j}(X^{\mathbb{Z}/2})$.

(ii) $K^{-i,-j}(X)[\frac{1}{2}]$ splits as the direct sum of $W^{-j}(X^{\mathbb{Z}/2})[\frac{1}{2}]$ and the subspace of $K^{-i}(X)[\frac{1}{2}]$ fixed by $(-1)^{i-j}T$.

As it happens, $b^3 = 0$ in this example - b is just $\eta \in KR^{-1}(\text{point})$ - and the Witt theory must vanish identically. This will not always be so; see (9.11).

All this is rather superficial. At a deeper level is a 'periodicity theorem' corresponding to (6.3).

Proposition (9.7). Let ξ be a real vector bundle of dimension n over X. Its orientation bundle is a principal $\mathbb{Z}/2$-bundle; let ω be the complex line bundle associated to it by the action of $\mathbb{Z}/2$ on $\mathbb{C}$ as ± 1. Then there is a canonical isomorphism

$$K_{\varepsilon}^{-i,-j}(E(L.\xi)) \cong K_{\varepsilon\otimes\omega}^{-i,-j+n}(E(\xi))$$

lifting the identity on $K^{-i}(E(\xi))$.

The proposition follows from (9.2) and the periodicity isomorphism $KR^*(X) \cong KR^*(E(\xi \oplus i\xi))$ in KR-theory.

Continuing our survey, we note that there is a squaring operation

$$(9.8) \qquad P^2 : \quad K^0(X) \longrightarrow K^{0,0}(X),$$

mapping a complex bundle ν to $\nu\otimes\bar{\nu}^*$ with bilinear form: $\langle u\otimes\alpha, v\otimes\beta\rangle := \alpha(v).\beta(u)$. $P^2(x+y) = P^2(x) + i_*(x.Ty) + P^2(y)$; $i^*P^2(x) = x.Tx$ $(x, y \in K^0(X))$. ($i_* = h$ and $i^* = r$ in (9.4).)

The coefficient groups $K^{-i}(\text{point})$ of complex K-theory may be regarded as the topological K-groups $K_i(\mathbb{C})$ of $\mathbb{C}$ and the cohomology functor $X \mapsto K^*(X)$ as the 'topological K-theory of $\mathbb{C}$'. On the other hand, the groups $K^{-i}(X)$ may themselves be thought of as the topological K-groups $K_i(R)$ of the C*-algebra R of continuous complex-valued functions on X; the corresponding K-theory is $K^*(X \times -)$. In the Hermitian case, the coefficient groups $K^{-i,-j}(\text{point})$ will, similarly, be the topological Hermitian K-groups $K_{i,j}(\mathbb{C})$ of $\mathbb{C}$ with the trivial involution; and the $\mathbb{Z}/2$-equivariant theory K^{**} is the 'topological Hermitian K-theory of $\mathbb{C}$'. The involution on X translates into a $\mathbb{C}$-linear involution of R (commuting with the *-operator), the coefficient bundle ε to the invertible R-module E of cross-sections, and the involution on ε to an involution of E. $K_{\varepsilon}^{-i,-j}(X)$ becomes the topological Hermitian K-group $K_{i,j}(R;E)$ of the C*-algebra with involution R, with coefficients in the invertible R-module with involution E. From this point of view the definition of the Witt theory must be completed by setting:

$$(9.9) \qquad W_j(R;E) := \varinjlim_{i} K_{i,j}(R;E).$$

A topological Hermitian K-theory for $\mathbb{R}$ may be developed in the same way. KR-theory is replaced by $KO_{\mathbb{Z}/2}$-theory.

Lemma (9.10). $K^{0,0}_{\mathbb{R}}(X) = KO_{\mathbb{Z}/2}(X)$.

This is the classical splitting of a non-singular symmetric form over $\mathbb{R}$ into a positive- and a negative-definite component.

The fundamental properties (9.3)-(9.8) are valid too. The Witt theory is more interesting.

Lemma (9.11). $W^{0}_{\mathbb{R}}(Y) = KO(Y)[\frac{1}{2}]$.

In the analogous theory for $\mathbb{C}$ with the non-trivial involution, conjugation, $KO_{\mathbb{Z}/2}$ is replaced by $K_{\mathbb{Z}/2}$.

The paper [5] of M. F. Atiyah is doubtless the source of much of the theory described here. The formal structure of Hermitian K-theory was laid down by C. T. C. Wall, [87] and [88]. G. Lusztig studied the real theory in [64] and M. F. Atiyah and E. Rees the complex theory, [8] (not to mention Poincaré and Serre). The splitting (9.6)(ii) away from the prime (2) is in [50].

10. Algebraic Hermitian K-theory

Our subject is the Karoubi-Quillen algebraic Hermitian K-theory of a ring with involution in which 2 is invertible. Once again, the squaring or quadratic construction will play a vital part.

Let R be a ring (not necessarily commutative, but with identity) with involution; this involution, written $\bar{\ }$, is additive and satisfies $\overline{rs} = \bar{s}.\bar{r}$ $(r,s \in R)$. Let E be an invertible R-bimodule with involution: R acts on the left and on the right and the involution $\bar{\ }$ of E is additive with $\overline{res} = \bar{s}.\bar{e}.\bar{r}$ $(e \in E)$. (The prime example is $E = R$ with the given involution.) E defines a (contravariant) involution $\mathfrak{D}_E$ of the category of finitely generated projective (left) R-modules: $\mathfrak{D}_E(P) = \{f \in \mathrm{Hom}_{\mathbb{Z}}(P,E) \mid f(sy) = f(y)\bar{s} \text{ for } s \in R,\ y \in P\}$ with the left-action of R. The involution on E identifies $\mathfrak{D}_E(\mathfrak{D}_E P)$ with P.

The primitive object of interest in Hermitian K-theory is a non-singular Hermitian form on a finitely generated projective module, that is, a $\mathbb{Z}$-bilinear map $g : P \times P \longrightarrow E$ satisfying:

(i) $g(rx,sy) = rg(x,y)\bar{s}$ $(r,s \in R;\ x,y \in P)$;

(ii) $g(x,y) = \overline{g(y,x)}$;

(iii) the adjoint $g' : P \longrightarrow \mathfrak{D}_E(P)$ $(g'(x) = g(x,-))$ is an isomorphism.

Equivalently and better for our purposes, in terms of the adjoint alone, it is an isomorphism $g' : P \longrightarrow \mathfrak{D}_E(P)$ of R-modules such that $\mathfrak{D}_E(g') = g'$.

It is sensible if 2 is invertible in R to define $K_{0,0}(R;E)$ to be the Grothendieck group of isomorphism classes of such non-singular Hermitian forms under direct sum. It maps, by forgetting the structure, to the group $K_0(R)$ of finitely generated projective R-modules. (There are two general comments: first, functoriality, which is easier to describe in the commutative case. Let $\varphi : R \longrightarrow R'$ be a homomorphism of commutative rings with trivial involution. Then the tensor product $R' \otimes_R -$ defines $\varphi_* : K_{0,0}(R;E) \longrightarrow K_{0,0}(R';R'\otimes_R E)$. Under the restrictive hypotheses that R' be finitely generated and projective as R-module and that $Hom_R(R',R)$ be an invertible R'-module, restriction defines a map $\varphi^* : K_{0,0}(R';Hom_R(R',E)) \longrightarrow K_{0,0}(R;E)$ in the opposite direction. Second, notice that the coefficient module E is important only up to 'squares'. If F is an invertible R-bimodule with conjugate bimodule $\bar{F}$ (defined so that $\overline{rfs} = \bar{s}.\bar{f}.\bar{r}$ for $r,s \in R$ and $f \in F$), then $F \otimes_R E \otimes_R \bar{F}$ has an involution $f \otimes e \otimes \bar{g} \longmapsto g \otimes \bar{e} \otimes \bar{f}$ and $F \otimes_R -$ gives a 'periodicity isomorphism' $K_{0,0}(R;E) \longrightarrow K_{0,0}(R;F \otimes E \otimes \bar{F})$.)

The hyperbolic functor takes a module Q to $H_E(Q) = Q \oplus \mathfrak{D}_E(Q)$ with the form defined by the identity map : $Q \oplus \mathfrak{D}_E(Q) \longrightarrow \mathfrak{D}_E(Q \oplus \mathfrak{D}_E(Q)) = Q \oplus \mathfrak{D}_E(Q)$. Now the group Aut(P) of automorphisms of the R-module P has an involution 'inverse transpose' $T_E : x \longmapsto g'^{-1}.\mathfrak{D}_E(x)^{-1}.g'$. This is true in particular of $Aut(H_E(Q))$, which contains $Aut(Q) \times Aut(Q)$ as the subgroup of diagonal matrices. The inclusion

(10.1) $$h : Aut(Q) \times Aut(Q) \longrightarrow Aut(H_E(Q))$$
$$(x,y) \longmapsto \begin{bmatrix} x & 0 \\ 0 & \mathfrak{D}_E(y)^{-1} \end{bmatrix}$$

is equivariant with respect to the map which switches the two factors Aut(Q). This is the starting point for the construction of the equivariant Hermitian K-theory.

We must first recall Quillen's original definition of algebraic K-theory. (The reader is referred to the exposition by J.-L. Loday [62] .) Set $GL(R) := \varinjlim Aut(R^n)$ $(n \in \mathbb{N})$. Quillen gave a construction (the 'plus' construction, but the notation has been pre-empted by compactification)

$$BGL(R) \longrightarrow B^{ab}GL(R)$$

which abelianizes the fundamental group GL(R) of the classifying space and induces an isomorphism in homology (with arbitrary coefficients). The algebraic K-group $K_i(R)$, for $i > 0$, is defined to be the homotopy group $\pi_i(B^{ab}GL(R))$.

It is clear how one should adapt the construction to Hermitian K-theory. GL(R) is intuitively the limit of Aut(Q) for all finitely generated projective modules Q and we redefine it as the direct limit $\varinjlim Aut(H_E(R^n))$ (with respect to the standard inclusions). It is a discrete topological group with involution T_E.

So consider a topological group G with (continuous) involution T. We make the hypothesis, which is trivially fulfilled by a discrete group, that the connected component of the identity be a Lie group (with smooth involution). (But we cannot afford any countability condition.)

Non-equivariantly, the functor which assigns to a compact ENR-pair (X,Y) the set $P(G)(X,Y)$ of isomorphism classes of principal G-bundles $P \longrightarrow X$ equipped with a trivialization $P|Y \longrightarrow Y \times G$ over Y is represented by the <u>pointed</u> classifying space of G: $P(G)(X,Y) = [X/Y;\ BG]$. (To define BG it is, of course, necessary to work in a larger category of spaces.) There is a corresponding functor $P(G)_{\mathbb{Z}/2}$ on compact $\mathbb{Z}/2$-ENR pairs. (A $\mathbb{Z}/2$-principal G-bundle $P \longrightarrow X$ has an involution on P covering that on the base and such that the operation of G: $P \times G \longrightarrow P$ is a $\mathbb{Z}/2$-map.) The basic properties follow readily from the definition.

Lemma (10.2). Let (X,Y) be a compact ENR-pair (with the trivial action of $\mathbb{Z}/2$ when appropriate).

(i) $P(G)_{\mathbb{Z}/2}((X,Y) \times S(L)) = P(G)(X,Y)$.

(ii) If X is connected and Y non-empty, then

$$P(G)_{\mathbb{Z}/2}(X,Y) = P(G^{\mathbb{Z}/2})(X,Y).$$

(iii) $P(G)_{\mathbb{Z}/2}(\text{point}) = H^1(\mathbb{Z}/2;\ G)$ (that is, the set of elements $g \in G$ such that $gT(g) = 1$ modulo the equivalence relation $g \sim hgT(h)^{-1}$ $(h \in G)$).

The definition and construction of a $\mathbb{Z}/2$-classifying space parallel the non-equivariant theory. (Let $EG \longrightarrow BG$ be the universal bundle constructed by Milnor as an infinite join. $\mathbb{Z}/2$ acts on $EG * EG$ by switching the factors. The projection $EG * EG \longrightarrow (EG * EG)/G$ to the orbit space of the G-action: $[x,t,y].g = [xg,t,yT(g)]$ $(x,y \in EG,\ t \in [0,1])$ is a universal $\mathbb{Z}/2$-bundle.) (10.2) may be refined to a statement about the classifying space, and (i) allows us, without ambiguity, to call this BG, the name of its underlying (pointed) space.

A convenient category, and the one we shall use, for representing-spaces in $\mathbb{Z}/2$-topology is that of the (metrisable) $\mathbb{Z}/2$-ANRs. There is an indispensable criterion for identifying homotopy equivalences. Let A and B be pointed $\mathbb{Z}/2$-ANRs. Then a $\mathbb{Z}/2$-map $f : A \longrightarrow B$ is a $\mathbb{Z}/2$-homotopy equivalence if (and only if) f and its restriction to the fixed subspaces $A^{\mathbb{Z}/2} \longrightarrow B^{\mathbb{Z}/2}$ are (non-equivariant) homotopy equivalences. (Bredon [18], James-Segal [43].) Often one is concerned only with the functors $Z \longmapsto [Z;\ A]^{\mathbb{Z}/2}$, $[Z;\ B]^{\mathbb{Z}/2}$ on the category of compact pointed $\mathbb{Z}/2$-ENRs. To show that f defines an equivalence, it is enough to check when Z has the form X^+ or $(X \times S(L))^+$, X a compact ENR (with trivial involution). (Here A and B can be arbitrary topological spaces.)

We end this topological digression with a transparent, but significant, remark.

Remark (10.3). The space of pointed $\mathbb{Z}/2$-maps from L^+ to A is homeomorphic to the homotopy-fibre of the inclusion $A^{\mathbb{Z}/2} \longrightarrow A$ of the fixed subspace.

Consider now the classifying space of the group with involution $GL(R)$. (10.2) supplies the following information about the fixed subspace. The connected component of the base-point is the classifying space $BO(R)$ of the infinite orthogonal group $O(R)$, the fixed subgroup. (iii) identifies the set of components with the kernel of the restriction map $K_{0,0}(R;E) \longrightarrow K_0(R)$. Here and in the remainder of the paragraph, 2 is supposed to be invertible in R. (Then every element of the kernel may be written as a difference $[H_E(R^n)] - [P]$ with P and $H_E(R^n)$

isomorphic as modules. The symmetric form on P defines an element of $H^1(\mathbb{Z}/2; Aut(H_E(R^n))).$)

An equivariant Quillen construction gives a map of $\mathbb{Z}/2$-ANRs (with base-point)

$$BGL(R) \longrightarrow B^{ab}GL(R)$$

abelianizing the fundamental group and inducing an isomorphism of homology (with any twisted coefficients) of BGL(R) and of every component of the fixed subspace. Direct sum defines a structure of homotopy-commutative and associative $\mathbb{Z}/2$-H-space on $B^{ab}GL(R)$. The monoid of path-components of the fixed subspace is a group: Ker $\{K_{0,0}(R;E) \longrightarrow K_0(R)\}$. The component of the base-point in $(B^{ab}GL(R))^{\mathbb{Z}/2}$ is $B^{ab}O(R)$, the result of applying Quillen's construction to the classifying space of the orthogonal group.

Definition (10.4). $i \geqslant j \geqslant 0$ and $i > 0$.

$$K_{i,j}(R;E) := [(\mathbb{R}^j \oplus L^{i-j})^+;\ B^{ab}GL(R)]^{\mathbb{Z}/2} .$$

In certain dimensions this is exactly Karoubi's original non-equivariant definition, [50]. For $j > 0$,

$$(10.5) \qquad K_{j,j}(R;E) = \pi_j(B^{ab}O(R)),$$
$$K_{j+1,j}(R;E) = \pi_{j+1}(B^{ab}GL(R), B^{ab}O(R)) .$$

The second equality follows from (10.3).

The method of Gersten-Wagoner will be used to complete

the definition. Let C ('cone') be the ring of infinite matrices $[a_{pq}]$ $(p,q \in \mathbb{N})$ over $\mathbb{Z}$ which have only finitely many non-zero elements in each row and each column, K ('kompakt') the two-sided ideal of matrices with only finitely many non-zero entries altogether, and S ('suspension') the quotient C/K. The matrix: $a_{pq} = 1$ if $q = p+1$, 0 if not, represents an invertible element z in S. Think of the group of units $S^{\cdot}$ as the group $Aut(S) \subseteq GL(S)$ of automorphisms of S as left S-module. Then the homomorphism $n \mapsto z^n : \mathbb{Z} \longrightarrow GL(S)$ induces a map $\mathbb{R}^+ = B\mathbb{Z} \longrightarrow BGL(S)$ of classifying spaces and defines an element $z_* \in K_1(S)$. The basic theorem asserts that the tensor product $z_* . :$ $[Z;\ B^{ab}GL(R)] \longrightarrow [\mathbb{R}^+ \wedge Z;\ B^{ab}GL(S.R)]$ - S.R is the tensor product of rings - is a bijection for any <u>connected</u> compact ENR Z with base-point. $K_i(R)$, for any integer i, positive or negative, is defined so that $z_*.: K_i(R) \longrightarrow K_{i+1}(S.R)$ is an isomorphism. This is consistent with the definition of K_0 as a Grothendieck group.

The equivariant theory will be defined by the squaring construction. If S^o is the opposite ring of S, then $S \otimes S^o$ has a canonical involution: $s \otimes t^o \mapsto t \otimes s^o$ $(s,t \in S)$.

Proposition (10.6). For any <u>connected</u> compact $\mathbb{Z}/2$-ENR Z with base-point, there is a natural isomorphism of abelian groups:

$$[Z;\ B^{ab}GL(R)]^{\mathbb{Z}/2} \longrightarrow [(\mathbb{R} \oplus L)^+ \wedge Z;\ B^{ab}GL(S \otimes S^o.R)]^{\mathbb{Z}/2} .$$

It is multiplication by $'P^2(z_*)' \in K_{2,1}(S \otimes S^o)$ (that is, $[(\mathbb{R} \oplus L)^+;\ B^{ab}GL(S \otimes S^o)]^{\mathbb{Z}/2}$). (In just a little more detail, the homomorphism $\mathbb{Z} \times \mathbb{Z} \longrightarrow Aut(H(S \otimes S^o))$:

$$(m,n) \longmapsto \begin{bmatrix} z^m & 0 \\ 0 & (z^o)^{-n} \end{bmatrix}$$

is equivariant with respect to the switching map on $\mathbb{Z} \times \mathbb{Z}$. At the classifying space level it gives rise to a map $\mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow B^{ab}GL(S \otimes S^o) = B$, say. Now $[\mathbb{R}^+ \times \mathbb{R}^+;\ B]^{\mathbb{Z}/2}$ splits as $[\mathbb{R}^+ \wedge \mathbb{R}^+;\ B]^{\mathbb{Z}/2} \oplus [\mathbb{R}^+;\ B]$. We take the first component. The splitting comes from (10.1), which provides a $\mathbb{Z}/2$-map $B \times B \longrightarrow B$ restricting to the identity in $[B;\ B] = [B \vee B;\ B]^{\mathbb{Z}/2}$. Compare (A.4).)

Now $K_{i,j}(R;E)$ for all integers i,j is defined to satisfy:

$$(10.7) \qquad K_{i,j}(R;E) = K_{i+2,j+1}(S \otimes S^o.R;\ S \otimes S^o.E)\ .$$

In order to describe the groups $K_{0,j}(R;E)$, we shall use the invertibility of 2 in a crucial way. Let S' be the ring $S[\frac{1}{2}]$ with the involution given by matrix transposition. The hyperbolic S'-module H(S') splits as an orthogonal direct sum $S' \oplus S'$ with symmetric forms $(s,t) \longmapsto s\bar{t}$, respectively $-s\bar{t}$, on the two factors. The automorphism of H(S') which acts as the image z' of z in $(S')^{\cdot} = \mathrm{Aut}(S')$ on the first summand and trivially on the second defines an element $z'_* \in K_{1,1}(S')$.

Proposition (10.8). Z as in (10.6). Then the product: $[Z;\ B^{ab}GL(R)]^{\mathbb{Z}/2} \longrightarrow [\mathbb{R}^+ \wedge Z;\ B^{ab}GL(S'.R)]^{\mathbb{Z}/2}$ with z'_* is an isomorphism.

The proof has three parts, checking non-equivariantly, when the involution on Z is trivial, and when $Z = L^+$. (Of these,

the first follows from the Gersten-Wagoner theorem, since the restriction $i^*(z'_*) \in K_1(S')$ comes from $z_* \in K_1(S)$, and the second is a result about the orthogonal group, [62] 3.1.6.)

$K_{0,0}(R;E)$ has been defined twice, initially as a Grothendieck group, and now again by (10.7). M. Karoubi and O. Villamayor [51] have identified $K_{1,1}(S'.R;\ S'.E)$ with the Grothendieck group; in conjunction with (10.8) this confirms the equivalence of the two definitions. The groups $K_{i,j}(R;E)$ are the coefficient groups of a $\mathbb{Z}/2$-cohomology theory enjoying all the essentially formal properties of §9. $K_{0,1}(R;E)$ is described by the exact sequence (9.4): $K_1 \longrightarrow K_{1,1} \longrightarrow K_{0,1} \longrightarrow K_0 \longrightarrow K_{0,0}$.

Proposition (10.9).

(i) $K_{0,0}(R;E)$ is the Grothendieck group of non-singular symmetric forms on finitely generated projective R-modules.

(ii) $K_{0,1}(R;E)$ is the quotient of the monoid of isomorphism classes of triples $(Q_0,Q_1;f)$, where Q_0 and Q_1 are finitely generated projective R-modules and f is an isomorphism $H_E(Q_0) \longrightarrow H_E(Q_1)$ (preserving the symmetric form), by the relations:

$$(Q_0,Q_1;f) + (Q_1,Q_2;g) = (Q_0,Q_2;g.f)$$
$$(Q,Q;1) = 0\ .$$

(An isomorphism $(Q_0,Q_1;f) \longrightarrow (Q_0',Q_1';f')$ is given by a pair of isomorphisms $k_i : Q_i \longrightarrow Q_i'$ such that $f'.H_E(k_0) = H_E(k_1).f$.)

There is a 4-fold periodicity in Hermitian K-theory.

Proposition (10.10). (Karoubi periodicity. [50], [62] 3.1.7.) Let $\mathbb{L}$ denote $\mathbb{Z}$ with the involution -1. Then there exist classes $u \in K_{0,2}(\mathbb{Z}[\frac{1}{2}]; \mathbb{L}[\frac{1}{2}])$ and $v \in K_{0,-2}(\mathbb{Z}[\frac{1}{2}]; \mathbb{L}[\frac{1}{2}])$ such that $u.v = 1$ in $K_{0,0}(\mathbb{Z}[\frac{1}{2}]) = \mathbb{Z}1 \oplus \mathbb{Z}t \oplus \mathbb{Z}/2.x \oplus \mathbb{Z}/2.tx$. ($x^2 = 0$.) This implies that the groups $K_{i,j}(R; E)$ are periodic in j (mod 4).

One might hope to do better, namely to define a periodicity isomorphism $K^{-i,-j}(E(L.\xi)) \cong K^{-i,-j+n}(E(\xi))$ for any oriented vector bundle ξ of dimension n, as in (9.7). There is one piece of evidence for the existence of such an isomorphism. Algebraic K-theory has been 'computed' in essentially only one case: that of a finite field. This was the starting point of Quillen's theory. E. M. Friedlander [31] has given an analogous computation of the Hermitian K-theory of a finite field of odd characteristic, with the trivial involution, by relating it to the fibre of the Adams operation $\psi^q - 1$ in the topological Hermitian K-theory of $\mathbb{C}$, with the trivial involution.

Remark (10.11). K denotes the complex theory of §9. Let q be odd. The Adams operation ψ^q defined on $K^0(-)$ and $K^{0,0}(-)$ extends to a multiplicative operation on the localized cohomology theories $K^*(-)_{(2)}$ and $K^{**}(-)_{(2)}$. It respects i* and the 4-fold periodicity. What is more, for any oriented real vector bundle ξ of dimension a multiple of 4, the periodicity map $K^{**}(E(L.\xi))_{(2)} \longrightarrow K^{**}(E(\xi))_{(2)}$, of (9.7), commutes with the action of ψ^q.

This is clear, because the real (or Real) representation ring of SO(n) is embedded in the complex ring by the restriction i*. But the periodicity class restricts to 1. (Of course,

(10.11) is false for the real $KO_{\mathbb{Z}/2}$-theory, and the lack of commutativity has geometric significance.)

(10.9) and (10.10) give a complete description of the groups $K_{0,j}(R; E)$. If the ring R is regular, then $K_i(R)$ is known to vanish for $i < 0$ and the Witt group $W_j(R; E)$, (9.9), is the cokernel of the hyperbolic (induction) map $K_0(R) \longrightarrow K_{0,j}(R; E)$. This has been interpreted by A. A. Ranicki, [73], as a cobordism group of 'algebraic Poincaré complexes'. In the same way, there is a realization of $K_{0,j}(R; E)$ as a group of equivalence classes of complexes with non-singular symmetric form, precisely parallel to the topological theory. It is a notable defect of the algebraic theory presented here that this relationship has appeared so late and, apparently, so fortuitously.

The topological real theory of the last paragraph relates in a rather satisfactory manner the Euler class $\gamma(L.\xi)$ of an oriented real vector bundle with the antipodal involution (and the associated obstruction theory of §2) to the Hirzebruch L-class, (6.8), via the Witt theory (9.11). Further, when ξ is the tangent bundle of a manifold, the $\mathbb{Z}/2$-Euler characteristic, (8.6), appears naturally as an invariant of the non-singular cup-product form on the de Rham cohomology.

One might hope that an algebraic Hermitian K-theory of $\mathbb{Z}$ would have analogous properties. The work of G. Brumfiel and J. Morgan [24] and A. A. Ranicki [73] suggests that the corresponding Witt theory, localized at (2), should be:

$$W^{-j}(Y)_{(2)} = \sum_{r\geq 0} H^{4r-j}(Y;\mathbb{Z}_{(2)}) \oplus \sum_{r\geq 0} H^{4r+1-j}(Y;\mathbb{F}_2),$$

and that $\gamma(L.\xi) \in K^{0,0}(X;-L.\xi) \cong K^{0,-n}(X;-\xi)$ should map to $(L_{4r}(\xi), (v(\xi).\delta v(\xi))_{4r+1})_{r\geq 0}$ in $W^0(X)_{(2)} \cong W^{-n}(X;-\xi)_{(2)}$, where n is the dimension of ξ, v the (total) Wu class and δ the Bockstein (Sq^1). This would, in particular, relate the $\mathbb{Z}_{(2)}$ L-class $L(\xi)$ to obstruction theory, a relation whose existence has been conjectured by M. E. Mahowald. Moreover, the de Rham invariant $v(X).\delta v(X)[X] \in \mathbb{F}_2$ of a (4k+1)-manifold would appear as a component of the $\mathbb{Z}/2$-Euler characteristic. From one point of view it is the discrepancy between the $\mathbb{F}_2$- and $\mathbb{R}$-semi-characteristics, [65]. From another, it is an obstruction (and up to cobordism the only one) to the existence of a 3-field on X; it may be written as $w_{4k-1}(X).w_2(X)[X]$.

Finally, granted such a theory for vector bundles, one might turn to topological bundles. Is the periodicity theorem (6.3) valid in this more general case?

It is the thesis of this paragraph that Karoubi's definition, [50], of the Hermitian K-theory of a ring in which 2 is a unit is the correct one and that it is best formulated in the language of $\mathbb{Z}/2$-homotopy theory. The significance of the bi-grading was made clear by C. T. C. Wall in [88]. C. H. Giffen has announced results on the equivariant theory in [32]. In addition, he investigates the filtration (b^n) of $K_{**}(R)$ in dimension (0,0) for certain rings R.

B. Appendix: On the Hermitian J-homomorphism

In homotopy theory the distinction between real vector bundles and complex bundles with a non-singular symmetric form - the real orthogonal group O(n) is a maximal compact subgroup in both $GL(n,\mathbb{R})$ and $O(n,\mathbb{C})$ - is a subjective rather than a material one. (This is not true in differential geometry. Consider a smooth complex vector bundle of even dimension with a smooth non-singular symmetric form and a compatible connection. Then the Pfaffian of the curvature represents the Euler class, in de Rham cohomology, of the associated, homotopy-theoretic, real vector bundle. If the connection is flat, then the rational Euler class is zero. On the other hand, there are flat real vector bundles with non-vanishing rational Euler class.) In the context of the J-homomorphism, nevertheless, the Hermitian standpoint seems to be preferred to the Real. We redress the balance of §6.

As far as homotopy theory is concerned a non-singular ε-symmetric (ε = +1 or -1) form on a complex vector bundle μ over a compact $\mathbb{Z}/2$-ENR X is a conjugate linear bundle map $j: \mu \to \mu$ covering the involution on X and such that $j^2 = \varepsilon$. The symmetric form is recovered, by choosing an invariant Hermitian metric $\langle -,- \rangle$, as $(u,v) \mapsto \langle j_x u, v\rangle$ $(u \in \mu_x,\ v \in \mu_{\bar{x}})$. (See (9.1).)

We begin with the local obstruction. Let V be a complex vector space with conjugate linear structure map j, $j^2 = \varepsilon$, and invariant metric. The involution T_ε, p 104, of Aut(V) restricts to the involution of the unitary group $U(V)$ given by conjugation with j. We have, for any based compact $\mathbb{Z}/2$-ENR Z:

$$\begin{aligned} \Theta : \quad [Z; U(V)]^{\mathbb{Z}/2} \longrightarrow & \ \omega^{-1}_{D(\varepsilon)}((Z,*) \times S(V); -V) \\ \cong & \ \omega^{-1}_{\mathbb{Z}/2}((Z,*) \times \mathbb{C}P(V); -H.V) \\ \cong & \ \{Z; L^+ \wedge \mathbb{C}P(V)^+\}^{\mathbb{Z}/2}, \end{aligned}$$

where $D(\varepsilon)$ is the extension of S^1 by $\mathbb{Z}/2$: $S^1 \cup S^1 j$, with $jzj^{-1} = z^{-1}$ ($z \in S^1$), $j^2 = \varepsilon \in S^1$. ($D(+)$ is $O(2)$, $D(-)$ the normalizer of S^1 in S^3.) $D(\varepsilon)$ acts on V by the given action of j and complex multiplication by elements of S^1. The quotient $D(\varepsilon)/S^1$ acts on $\mathbb{C}P(V)$, freely if $\varepsilon = -1$, and on the tensor product $H.V$ of V and the complex Hopf bundle (although not on the individual factors in the skew case). $D(\varepsilon)$ acts on Z through the projection onto $\mathbb{Z}/2$. Stably, we have

$$\Theta : \quad \tilde{K}^{-1,-1}_{\varepsilon}(Z) \longrightarrow \{Z; L^+ \wedge \mathbb{C}P_{\varepsilon}(\infty)^+\}^{\mathbb{Z}/2}.$$

$\mathbb{C}P_+(\infty)$ is the classifying space of the group with involution S^1; $\mathbb{C}P_-(\infty)$ can be regarded as the orbit space $(ED(-))/S^1$, where $ED(-) \longrightarrow BD(-)$ is a universal principal $D(-)$-bundle. Again, as in §6, $1 + \Theta$ is a homomorphism to the group of units in $D(\varepsilon)$-stable cohomotopy. By restricting to the subgroup of $D(\varepsilon)$ generated by j, we obtain the basic Hermitian J-homomorphism

$$J : \quad K^{-1,-1}_{\varepsilon}(X) \longrightarrow L(\varepsilon)^{-1}_{\mathbb{Z}/2}(X).$$

To define the target groups, look at the action of $\mathbb{Z}/4$ on X through the projection $p : \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2$. Restriction to subspaces fixed by the kernel of p gives a map $\omega^0_{\mathbb{Z}/4}(X)^{\bullet} \longrightarrow \omega^0_{\mathbb{Z}/2}(X)^{\bullet}$,

which is split by p^* (lifting a $\mathbb{Z}/2$-action to an action of $\mathbb{Z}/4$). $L(+)^{-1}_{\mathbb{Z}/2}(X)$ will be $\omega^0_{\mathbb{Z}/2}(X)^{\bullet}$ and $L(-)^{-1}_{\mathbb{Z}/2}(X)$ the complementary summand in $\omega^0_{\mathbb{Z}/4}(X)^{\bullet}$. There are obvious restriction maps $i^* : L(\varepsilon)^{-1}_{\mathbb{Z}/2}(X) \longrightarrow L(0)^{-1}(X) = \omega^0(X)^{\bullet}$ and slightly less obvious 'induction maps' in the opposite direction. They are compatible, via J, with the restriction and hyperbolic maps in K-theory. i_* is defined by doubling; we illustrate the construction for $\varepsilon = -1$ on the global scale.

Remark (B.1). (Woodward [91]). Let ξ, ξ' be real vector bundles over a compact ENR X, $\dim \xi = \dim \xi' \gg \dim X$, and $f : S(\xi) \longrightarrow S(\xi')$ a fibre-homotopy equivalence. Write E for the irreducible 2-dimensional real representation of $\mathbb{Z}/4$. Then $S(E \otimes \xi)$ and $S(E \otimes \xi')$ are $\mathbb{Z}/4$-equivariantly fibre-homotopy equivalent. (It follows that the mod 4 Pontrjagin classes of the two bundles coincide; they can be recovered from the $\mathbb{Z}/4$-cohomology Euler class of $E \otimes \xi$. Indeed, this is a lifting to stable cohomotopy of the classical argument involving the Pontrjagin square.)

We replace f by $\bar{S}^2(f)$, (3.7), which commutes with the antipodal involution, and then double it. The generator of $\mathbb{Z}/4$ acts on $\xi \oplus \xi$ by $(u,v) \longmapsto (-v,u)$ in each fibre.

If q is an odd integer, there is an Adams operation ψ^q on $K^{0,0}_{\varepsilon}(X)$.

Proposition (B.2). (Hermitian Adams conjecture). q odd. Any class in $(\psi^q - 1)K^{0,0}_{\varepsilon}(X)$ is representable as a difference of complex vector bundles μ, ν with conjugate linear structure maps for which there exists a $\mathbb{Z}/2$ (or $\mathbb{Z}/4$ if $\epsilon = -1$) -equivariant map $\mu^+ \longrightarrow \nu^+$ over X with odd (non-equivariant) degree in each fibre.

This is proved by reduction to the case of a line-bundle (even in the skew case). It may be well to recall the principal step in the Becker-Gottlieb argument (with a modification due to A. Dold [28]). We state it non-equivariantly.

Lemma (B.3). Let $\pi : E \longrightarrow B$ be a manifold over a compact ENR B, p 38, and α an oriented virtual real vector bundle of dimension 0 over B. Assume E to be connected. Let $I(\alpha) \subseteq H^0(B) = \mathbb{Z}$ and $I(\pi^*\alpha) \subseteq H^0(E) = \mathbb{Z}$ be the Hurewicz images of $\omega^0(B;\alpha)$ and $\omega^0(E;\pi^*\alpha)$ respectively. Then

$$\chi . I(\pi^*\alpha) \subseteq \pi^* I(\alpha) \subseteq I(\pi^*\alpha)$$

where χ is the Euler characteristic of the fibre of π.

(For the transfer $\pi_* . \gamma(\tau(\pi)) : \omega^0(E;\pi^*\alpha) \longrightarrow \omega^0(B;\alpha)$ lifts multiplication by χ in integral cohomology.)

A description of the discrete analogue of θ will conclude this survey. Let k be a finite field, with q elements. Consider a locally trivial bundle μ of finite-dimensional k-vector spaces over X. The complement of the zero-section is a finite cover of X with a free action of the group $k^{\cdot}$ of units in k by multiplication. It determines, p 57, an element, $\theta(\mu)$ say, of $\omega^0_{k^{\cdot}}(X)$ - in fact, of the free summand $\{X^+;\ (Bk^{\cdot})^+\}$. Now $\alpha(\mu) = 1 + \theta(\mu)$ is invertible in $\omega^0_{k^{\cdot}}(X)[1/q]$ and $\alpha(\mu \oplus \nu) = \alpha(\mu).\alpha(\nu)$ if ν is a second bundle. From this it follows that θ extends to a linear map:

$$\{X^+;\ \mathbb{Z} \times BGL(k)\}\ (= \{X^+;\ \mathbb{Z} \times B^{ab}GL(k)\}) \longrightarrow \{X^+;(Bk^{\cdot})^+\}[1/q],$$

splitting the inclusion 'R' : $Bk^{\cdot} \longrightarrow 1 \times BGL(k)$ of the line bundles, and α to a homomorphism defined on the algebraic K-theory of k:

$$K^0(X) \longrightarrow \omega^0_{k^{\cdot}}(X)[1/q]^{\cdot}.$$

If the characteristic of k is odd, then there is an Hermitian analogue due to G. B. Segal [78]. Suppose that μ has a non-singular ϵ-symmetric form: $g_x : \mu_x \otimes \mu_{\bar{x}} \longrightarrow k$. The combinatorial information will be contained in the finite covers $S_a(\mu) \longrightarrow X$, one for each $a \in k$, with fibre over $x \in X$

$$(S_a(\mu))_x := \{(u,v) \in \mu_x \times \mu_{\bar{x}} - \{(0,0)\} \mid g_x(u,v) = a\}.$$

Now the extension $D(\epsilon)$: $k^{\cdot} \cup k^{\cdot}j$ (with $jzj^{-1} = z^{-1}$ for $z \in k^{\cdot}$, $j^2 = \epsilon \in k^{\cdot}$) of $k^{\cdot}$ by $\mathbb{Z}/2$ acts on $S_a(\mu)$: $z.(u,v) = (zu, z^{-1}v)$, $j.(u,v) = (\epsilon v, u)$. Define $\theta(\mu) \in \omega^0_{D(\epsilon)}(X)_{(2)}$ to be the difference $[S_0(\mu)] - [S_1(\mu)]$. It lies in the 'free summand'

$\{X^+;\ (Bk^{\cdot})^+\}^{\mathbb{Z}/2}$ ($Bk^{\cdot}$ the classifying space of the group with involution $z \longmapsto z^{-1}$) if $\epsilon = +1$,

$\{X^+;\ (ED(-)/k^{\cdot})^+\}^{\mathbb{Z}/2}$ if $\epsilon = -1$.

Choose a non-square b in $k^{\cdot}$. The involution $\tau : z \longmapsto z^{-1}$, $j \longmapsto bj$ of $D(\epsilon)$ is dependent on the choice of b only up to inner automorphisms. Write ' for the involution of $\omega^0_{D(\epsilon)}(X)_{(2)}$ induced by τ and the involution $^-$ of X. Then $[S_a(\mu)] = [S_1(\mu)]$ or $[S_1(\mu)]'$ according as $a \in k^{\cdot}$ is a square or a non-square.

$\alpha(\mu) = 1 + \theta(\mu)$ is invertible and satisfies the formula: $\alpha(\mu \oplus \nu) = \alpha(\mu).\alpha(\nu) + c(\alpha(\mu)' - \alpha(\mu)).(\alpha(\nu)' - \alpha(\nu))$, where c is equal to $\frac{1}{4}(q-1)$ if $q \equiv 1 \pmod 4$, $\frac{1}{4}(-q-1)$ if $q \equiv 3 \pmod 4$.

$i^*\theta(\mu) \in \{X^+; (Bk^{\cdot})^+\}_{(2)}$, surprisingly, agrees with the non-equivariant $\Theta(\mu)$, at (2). (It suffices to check for line bundles if $q \equiv 1 \pmod 4$, and 2-dimensional bundles if $q \equiv 3 \pmod 4$. Then we can use the doubling construction and the transfer.)

α extends to a map $K_{\varepsilon}^{0,0}(X) \longrightarrow \omega^0_{D(\varepsilon)}(X)^{\cdot}_{(2)}$ such that:

(B.4) $\alpha(x+y) = \alpha(x).\alpha(y).(1 + c(\xi(x)-1)(\xi(y)-1))$,

where $\xi(x) = \alpha(x)/\alpha(x)'$. $K^{0,0}(*) = K_{0,0}(k)$ is $\mathbb{Z}\oplus\mathbb{Z}/2$, generated by 1 and $\langle b\rangle :=$ k with the form $(u,v) \longmapsto buv$ (b a non-square). Let ' denote the involution of $K_{\varepsilon}^{0,0}(X)$ given by the tensor product with $\langle b\rangle$. Then $\alpha(x') = \alpha(x)'$ and, since $2x = 2x'$, $\xi(x)^2 = 1$. (B.4) implies at once that $\xi(x+y) = \xi(x).\xi(y)$.

(B.2) is, of course, a corollary of the $\mathbb{Z}/2$ (or $\mathbb{Z}/4$)-equivariant real Adams conjecture and such results are practically as old as the non-equivariant theorem, [77]. There has been important recent work on the equivariant Adams conjecture for more complicated groups by T. tom Dieck and others. The formula (B.4) is implicit in Segal's work; it is closely related to a result of J. Tornehave [86] on the deviation from additivity of a chosen solution of the Adams conjecture.

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