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Equivariant fixed-point indices of iterated maps

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Abstract. We describe an equivariant version (for actions of a finite group G) of Dold's index theory, [10], for iterated maps. Equivariant Dold indices are defined, in general, for a G-map $U \to X$ defined on an open G-subset of a G-ANR X (and satisfying a suitable compactness condition). A local index for isolated fixed-points is introduced, and the theorem of Shub and Sullivan on the vanishing of all but finitely many Dold indices for a continuously differentiable map is extended to the equivariant case. Homotopy Dold indices, arising from the equivariant Reidemeister trace, are also considered.

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1. Introduction

Suppose, first, that X is a compact ENR and $\phi : X \to X$ is a (continuous) map. Consider the formal power series

$$Z(\phi, X; q) = \exp\Big(-\sum_{k\geq 1} L(\phi^k, X)q^k/k\Big) \in \mathbb{Q}[[q]],$$

where $L(\phi^k, X) \in \mathbb{Z}$ is the Lefschetz number of the kth iterate $\phi^k : X \to X$. Such a formal power series with constant term equal to 1 factorizes, algebraically, as an infinite product

$$Z(\phi, X; q) = \prod_{k\geq 1}(1 - q^k)^{d_k},$$

where $d_k \in \mathbb{Q}$. In [10] Dold showed that the exponents d_k are integers. We shall call them the *Dold indices* and write them as $\mathcal{D}^k(\phi, X)$. (In [16] the term 'multiplicity' is used, rather than 'index'.) Dold's proof involved consideration of the map

$$\pi_k(\phi) : X^k \to X^k, \quad (x_1, x_2, \dots, x_k) \mapsto (\phi(x_k), \phi(x_1), \dots, \phi(x_{k-1})),$$

which had appeared earlier in the work of Fuller [13]. The map $\pi_k(\phi)$ is equivariant with respect to the action of the group $\mathbb{Z}/k\mathbb{Z}$ on X^k by cyclic permutation, and projection to the first (or any) factor $X^k \to X$ gives a homeomorphism from the fixed-subspace of $\pi_k(\phi)$ to that of ϕ^k under which the action of $\mathbb{Z}/k\mathbb{Z}$ corresponds to the operation of the powers of ϕ. A more conceptual proof of Dold's theorem was given, a few years later, by Komiya [17] using equivariant methods.

The equivariant formulation of the proof was recast in [5] as the definition of the Dold indices and generalized to the fibrewise theory (as a prelude to the construction of the topological Fuller index). This paper is written as an exposition of Dold's theory of iterated maps from this point of view. Specifically, we shall consider here the G-equivariant generalization of the Dold indices, where G, in notation used throughout this paper, is a finite group. We shall follow closely the account of the index theory of iterated maps in [5] (in particular, Section 3), concentrating on those places where the equivariant theory exhibits new features, for much of the theory generalizes easily to the G-equivariant situation.

Suppose now that $\phi : X \to X$ is a G-equivariant self-map of a G-ENR X. Let $K \leq G$ be a subgroup and let $g \in N_G(K)$ be an element of the normalizer of K. The action of g restricts to a map $X^K \to X^K$ of the sub-ENR X^K fixed by each element of K, and $\phi(X^K) \subseteq X^K$. For each integer $k \geq 1$, we can consider the classical Lefschetz number $L(g^{-1}\phi^k, X^K) \in \mathbb{Z}$ of the map $g^{-1}\phi^k|X^K : X^K \to X^K$. The sequence of equivariant Dold indices $\mathcal{D}^m(\phi, X)$, $m \geq 1$, in certain groups $B^{(m)}(G)$ will determine, and be determined by, the family of all such Lefschetz numbers.

We shall explain various forms of the index. Section 2 deals, combinatorially, with the case in which X is a finite set and defines the groups $B^{(m)}(G)$. Some results on equivariant stable homotopy theory are collected in Section 3. The topological Dold indices are defined in Section 4 for the general case in which X is a G-ANR. Localization of the index to a neighbourhood of the fixed-subspace is expressed by constructing, for a map $\phi : U \to X$ defined on an open G-subspace U of X (and satisfying a suitable compactness hypothesis), an index $\tilde{\mathcal{D}}^k(\phi, U)$ in the G-equivariant stable homotopy of a certain G-space constructed from U^k. Isolated periodic G-orbits are considered in Section 5, where an equivariant generalization of the Shub–Sullivan theorem for differentiable maps is given. Section 6 is concerned with the homotopy Dold indices, defined via equivariant Reidemeister–Nielsen indices.

A good reference for equivariant fixed-point theory is [23]; see also [6] for a discussion of equivariant stable homotopy theory.

It is a pleasure to dedicate this paper, which relies heavily on the framework for fixed-point theory constructed in [7], [8], [9], and not just on the specific paper [10], to Professor Dold on the occasion of his 80th birthday.

2. Combinatorial fixed-point indices

Consider a finite G-set X and a G-map $\phi : X \to X$. A point $x \in X$ is a *periodic point* of ϕ if $\phi^k(x) = x$ for some $k \geq 1$, and in that case the least such integer k is called the *minimal period* of x. Let us write D_k for the set of all periodic points of ϕ with minimal period equal to k. Clearly D_k is a G-set. Moreover, it has a compatible free action of the cyclic group $\mathbb{Z}/k\mathbb{Z}$ given by the action of the generator $1 \in \mathbb{Z}/k\mathbb{Z}$ as $\phi : D_k \to D_k$, that is, D_k is a $G \times \mathbb{Z}/k\mathbb{Z}$-set.

Now the set $\mathrm{Fix}(\phi^k) = \{x \in X \mid \phi^k(x) = x\}$ of fixed-points of the kth iterate ϕ^k decomposes as a disjoint union of G-sets:

$$\mathrm{Fix}(\phi^k) = \bigsqcup_{l \mid k} D_l,$$

indexed by the divisors $l \geq 1$ of k. These various G-sets will be counted by elements of certain Grothendieck groups which we now describe.

Recall first that the Burnside ring, $A(G)$, of G is defined as the Grothendieck group of isomorphism classes of finite G-sets under disjoint union, with the multiplication arising from the product of sets. Each G-set is a disjoint union of orbits, and each orbit is isomorphic as a G-set to a homogeneous space G/H, where $H \leq G$ is a subgroup of G. Since two coset spaces G/H and G/H' are isomorphic if and only if the subgroups H and H' are conjugate, we can identify $A(G)$ with the free abelian group generated by the conjugacy classes (H) of subgroups of G. For each conjugacy class (H), we have a ring homomorphism

$$\rho_{(H)} : A(G) \to \mathbb{Z}$$

which maps the class of a G-set S to the number, $\#S^H$, of elements in the subset S^H fixed by the subgroup H. Moreover, the product

$$\prod_{(H)} \rho_{(H)} : A(G) \to \prod_{(H)} \mathbb{Z}$$

is injective (and so has finite cokernel).

Definition 2.1. We can now define the *Lefschetz fixed-point index* of the kth iterate $\phi^k : X \to X$ combinatorially as the class

$$L(\phi^k, X) = [\mathrm{Fix}(\phi^k)] \in A(G).$$

To count the G-sets D_k we make the following definition.

Definition 2.2. For each integer $k \geq 1$, let $B^{(k)}(G)$ be the Grothendieck group of isomorphism classes, under disjoint union, of finite $G \times \mathbb{Z}/k\mathbb{Z}$-sets on which the restricted action of $\mathbb{Z}/k\mathbb{Z}$ $(= \{1\} \times \mathbb{Z}/k\mathbb{Z})$ acts freely.

Such a $G \times \mathbb{Z}/k\mathbb{Z}$-set P on which the subgroup $\mathbb{Z}/k\mathbb{Z}$ acts freely may be regarded as a G-equivariant principal $\mathbb{Z}/k\mathbb{Z}$-bundle over a finite base. Indeed, P fibres as the principal $\mathbb{Z}/k\mathbb{Z}$-bundle $P \to P/(\mathbb{Z}/k\mathbb{Z}) = S$. These bundles may be described, up to isomorphism, as follows. The base S decomposes as a union of transitive G-sets of the form G/H. Now a principal $\mathbb{Z}/k\mathbb{Z}$-bundle over G/H has the form

$$P = G \times_H \mathbb{Z}/k\mathbb{Z} \to G/H,$$

where the action of H is given by a homomorphism $\alpha : H \to \mathbb{Z}/k\mathbb{Z}$. This leads us to introduce the notation $\Xi^{(k)}(G)$ for the set of homomorphisms $\alpha : H \to \mathbb{Z}/k\mathbb{Z}$, where $H \leq G$ is a subgroup of G. The group G acts on $\Xi^{(k)}(G)$ by conjugation:

for $g \in G$, $g \cdot \alpha : gHg^{-1} \to \mathbb{Z}/k\mathbb{Z}$ is the homomorphism $ghg^{-1} \mapsto \alpha(h)$.

Let (α) denote the conjugacy class of α and let $[\alpha] \in B^{(k)}(G)$ denote the element defined by the corresponding principal bundle. We thus have the following description of $B^{(k)}(G)$.

Lemma 2.3. *The group $B^{(k)}(G)$ is free abelian on the set $\{[\alpha] \mid (\alpha) \in \Xi^{(k)}(G)/G\}$ of equivalence classes of homomorphisms $\alpha : H \to \mathbb{Z}/k\mathbb{Z}$.* □

An element $\alpha : H \to \mathbb{Z}/k\mathbb{Z}$ of $\Xi^{(k)}(G)$ determines a subgroup

$$T_k(\alpha) = \{(h, -\alpha(h)) \in G \times \mathbb{Z}/k\mathbb{Z} \mid h \in H\} \leq G \times \mathbb{Z}/k\mathbb{Z}$$

and a fixed-point homomorphism, depending only on the conjugacy class (α),

$$\rho_{(\alpha)} : B^{(k)}(G) \to \mathbb{Z},$$

which counts the $T_k(\alpha)$-fixed-points of a $G \times \mathbb{Z}/k\mathbb{Z}$-set with free $\mathbb{Z}/k\mathbb{Z}$-action:

$$[P] \mapsto \frac{1}{k}\#P^{T_k(\alpha)} = \#(P^{T_k(\alpha)}/(\mathbb{Z}/k\mathbb{Z})).$$

Consider an element $\alpha' : H' \to \mathbb{Z}/k\mathbb{Z}$ of $\Xi^{(k)}(G)$, determining a principal bundle $P' = G \times_{H'} \mathbb{Z}/k\mathbb{Z}$. Then

$$(P')^{T_k(\alpha)} = \{[g, i] \mid H \leq gH'g^{-1},\ \alpha = g \cdot \alpha'|H,\ i \in \mathbb{Z}/k\mathbb{Z}\}$$

(where $(g \cdot \alpha')(gkg^{-1}) = \alpha'(k)$ for $k \in H'$). The set $\Xi^{(k)}(G)/G$ is partially ordered by the relation:

$(\alpha) \leq (\alpha')$ if and only if, for some $g \in G$, $H \leq gH'g^{-1}$ and $\alpha = g \cdot \alpha'|H$.

Thus, $\rho_{(\alpha)}([\alpha'])$ is 0 unless $(\alpha) \leq (\alpha')$, and is a unit in $\mathbb{Z}[1/\#G]$ if $(\alpha) = (\alpha')$. It follows that the product of the fixed-point homomorphisms

$$(\rho_{(\alpha)}) : B^{(k)}(G) \to \prod_{(\alpha) \in \Xi^{(k)}(G)/G} \mathbb{Z}$$

is injective and becomes an isomorphism on inverting the order $\#G$ of the group.

Definition 2.4. The *Dold indices* of the G-map $\phi : X \to X$ can be defined combinatorially as

$$\mathcal{D}^k(\phi, X) = [D_k] \in B^{(k)}(G), \qquad k \geq 1.$$

The relationship with the Lefschetz indices $L(\phi^k, X)$ involves the forgetful maps

$$\sigma^{(k)} : B^{(k)}(G) \to A(G)$$

defined by mapping a $G \times \mathbb{Z}/k\mathbb{Z}$-set with free $\mathbb{Z}/k\mathbb{Z}$-action to the underlying G-set. When G is trivial, both $B^{(k)}(G)$ and $A(G)$ are naturally identified with $\mathbb{Z}$, and $\sigma^{(k)}$ is multiplication by k. In general, we have the following integrality condition.

Lemma 2.5. *Let $e(G)$ be the exponent of G, that is, the least integer $e \geq 1$ such that $g^e = 1$ for all $g \in G$. Then*

$$e(G)\sigma^{(k)}(B^{(k)}(G)) \subseteq kA(G).$$

Proof. It is enough to look at an element $[\alpha] \in B^{(k)}(G)$. The homomorphism $\alpha : H \to \mathbb{Z}/k\mathbb{Z}$ must map into the subgroup M with order equal to the highest common factor m of k and $e(G)$. Hence, the corresponding bundle $P = G \times_H \mathbb{Z}/k\mathbb{Z}$ decomposes as a G-set into a disjoint union of k/m subspaces each isomorphic to $G \times_H M$. □

From the decomposition of $\mathrm{Fix}(\phi^k)$ we obtain at once the fundamental identity expressing the Lefschetz numbers of the iterates in terms of the Dold indices.

Proposition 2.6. *Let $\phi : X \to X$ be an equivariant self-map of a finite G-set X. Then*

$$L(\phi^k, X) = \sum_{l \mid k} \sigma^{(l)} \mathcal{D}^l(\phi, X) \in A(G),$$

for each $k \geq 1$. Moreover, the indices $\mathcal{D}^l(\phi, X)$ satisfy the equivariant Dold congruences*:*

$$e(G)\sigma^{(l)}\mathcal{D}^l(\phi, X) \in lA(G).$$ □

If l divides k, we may include $\mathbb{Z}/l\mathbb{Z}$ as a subgroup of $\mathbb{Z}/k\mathbb{Z}$ by mapping the generator 1 to k/l. The associated group-theoretic induction homomorphism $A(G \times \mathbb{Z}/l\mathbb{Z}) \to A(G \times \mathbb{Z}/k\mathbb{Z})$ restricts to an injective map

$$i : B^{(l)}(G) \hookrightarrow B^{(k)}(G).$$

In concrete terms, a G-equivariant principal $\mathbb{Z}/l\mathbb{Z}$-bundle Q is mapped to the principal $\mathbb{Z}/k\mathbb{Z}$-bundle $P = Q \times_{\mathbb{Z}/l\mathbb{Z}} \mathbb{Z}/k\mathbb{Z}$, or a homomorphism $\beta : H \to \mathbb{Z}/l\mathbb{Z}$ representing an element of $\Xi^{(l)}(G)$ is mapped to the composition $\alpha = i \circ \beta : H \to \mathbb{Z}/l\mathbb{Z} \hookrightarrow \mathbb{Z}/k\mathbb{Z}$ giving an element of $\Xi^{(k)}(G)$. When G is trivial, i is the identity map $\mathbb{Z} \to \mathbb{Z}$.

Lemma 2.7. *Suppose that l divides k. Then the composition*

$$\sigma^{(k)} \circ i : B^{(l)}(G) \hookrightarrow B^{(k)}(G) \to A(G)$$

is equal to $(k/l)\sigma^{(l)}$. □

Remark 2.8. Let $\mathbb{T}$ be the circle group of complex numbers of modulus 1, and define $B(G)$, as in [3], to be the Grothendieck group of isomorphism classes of G-equivariant principal $\mathbb{T}$-bundles over finite G-sets. The group $B(G)$ may be identified additively with the free abelian group on the set of conjugacy classes of homomorphisms $\alpha : H \to \mathbb{T}$, where $H \leq G$. It has an associative product given by the group multiplication on $\mathbb{T}$ (or, equivalently, by the tensor product of complex line bundles) and thus becomes a commutative ring.

Including $\mathbb{Z}/k\mathbb{Z}$ in $\mathbb{T}$ by mapping 1 to $e^{2\pi i/k} \in \mathbb{T}$ we can embed $B^{(k)}(G)$ as a subring of $B(G)$ and $\Xi^{(k)}(G)$ as a subset of the set $\Xi(G)$ of homomorphisms

$\alpha : H \to \mathbb{T}$. Notice that $B^{(k)}(G) = B(G)$ if k is divisible by $e(G)$. We may identify $B^{(1)}(G)$ with $A(G)$. Each group $B^{(k)}(G)$ thus has the structure of an $A(G)$-module; indeed, it is the obvious structure given by the Cartesian product. The map $\sigma^{(k)}$ is an $A(G)$-module homomorphism, but not, in general, a ring homomorphism. However, there is a ring homomorphism

$$\tau : B(G) \to A(G)$$

defined by mapping the class of a bundle $P \to S$ to its base $[S] \in A(G)$.

The translation from the discrete to the topological setting requires a change of viewpoint. For an integer $k \geq 1$, we make the k-fold product X^k, thought of as the space of maps $\mathbb{Z}/k\mathbb{Z} \to X$, into a $G \times \mathbb{Z}/k\mathbb{Z}$-set by letting the generator $1 \in \mathbb{Z}/k\mathbb{Z}$ act by the cyclic shift

$$(x_1, x_2, \ldots, x_k) \mapsto (x_k, x_1, \ldots, x_{k-1}).$$

Then, following [5], we write

$$\pi_k(\phi) : X^k \to X^k \quad \text{(where } k \geq 1)$$

for the $G \times \mathbb{Z}/k\mathbb{Z}$-equivariant map $(x_1, \ldots, x_k) \mapsto (\phi(x_k), \phi(x_1), \ldots, \phi(x_{k-1}))$. The fixed-point set of $\pi_k(\phi)$ is naturally identified with the fixed-point set of ϕ^k by the correspondence

$$x \mapsto (x, \phi(x), \ldots, \phi^{k-1}(x)), \quad \mathrm{Fix}(\phi^k) \to \mathrm{Fix}(\pi_k(\phi)),$$

under which the action of $1 \in \mathbb{Z}/k\mathbb{Z}$ on $\mathrm{Fix}(\pi_k(\phi))$ corresponds to the action of ϕ on $\mathrm{Fix}(\phi^k)$. The equivariant Lefschetz index of $\pi_k(\phi)$ is thus defined as an element of $A(G \times \mathbb{Z}/k\mathbb{Z})$. This Burnside ring is easily described in terms of the groups $B^{(k)}(G)$.

Lemma 2.9. *There is a natural isomorphism*

$$A(G \times \mathbb{Z}/k\mathbb{Z}) = \bigoplus_{l \mid k} B^{(l)}(G).$$

Proof. Indeed, a $G \times \mathbb{Z}/k\mathbb{Z}$-set may be thought of as a G-set equipped with a self-map whose kth power is the identity. The result follows from the partition considered earlier into the subsets of points with minimal period equal to l. □

This leads us to re-define the Dold index.

Definition 2.10. The Dold index $\mathcal{D}^k(\phi, X) \in B^{(k)}(G)$ is defined to be the $B^{(k)}(G)$-component of $L(\pi_k(\phi), X^k) \in A(G \times \mathbb{Z}/k\mathbb{Z})$ in the above decomposition, in Lemma 2.9, of $A(G \times \mathbb{Z}/k\mathbb{Z})$.

In order to establish Proposition 2.6 with this new definition, or, more precisely, to show that the components of $L(\pi_k(\phi, X^k)$ are the indices $\mathcal{D}^{(l)}(\phi, X)$, $l \mid k$, we must compare $\pi_k(\phi)$ and $\pi_l(\phi)$.

First, we record an easy observation.

Lemma 2.11. *In terms of the decomposition in Lemma* 2.9 *the (forgetful) restriction map* $A(G \times \mathbb{Z}/k\mathbb{Z}) \to A(G)$ *is given by* $\sigma^{(l)}$ *on the summand* $B^{(l)}(G)$. □

For a divisor l of k, there is a fixed-point homomorphism

$$\rho_{l\mathbb{Z}/k\mathbb{Z}} : A(G \times \mathbb{Z}/k\mathbb{Z}) = \bigoplus_{m \mid k} B^{(m)}(G) \to A(G \times \mathbb{Z}/l\mathbb{Z}) = \bigoplus_{m \mid l} B^{(m)}(G)$$

which maps a finite $G \times \mathbb{Z}/k\mathbb{Z}$-set S to the subset $S^{l\mathbb{Z}/k\mathbb{Z}}$ fixed by the subgroup $l\mathbb{Z}/k\mathbb{Z} \leq \mathbb{Z}/k\mathbb{Z} \leq G \times \mathbb{Z}/k\mathbb{Z}$. The quotient $(G \times \mathbb{Z}/k\mathbb{Z})/(l\mathbb{Z}/k\mathbb{Z})$ acting on the fixed subset is identified with $G \times \mathbb{Z}/l\mathbb{Z}$ in the obvious way.

Lemma 2.12. *The homomorphism* $\rho_{l\mathbb{Z}/k\mathbb{Z}}$ *maps a summand* $B^{(m)}(G)$ *to* 0 *if* m *does not divide* l, *and isomorphically to the summand* $B^{(m)}(G)$ *as the identity on* $B^{(m)}(G)$ *if* m *divides* l. □

Taking fixed-points under $l\mathbb{Z}/k\mathbb{Z}$ maps $\pi_k(\phi) : X^k \to X^k$ to $\pi_l(\phi) : X^l \to X^l$ and so

$$\rho_{l\mathbb{Z}/k\mathbb{Z}} L(\pi_k(\phi), X^k) = L(\pi_l(\phi), X^l).$$

This equality, in conjunction with Lemma 2.11, establishes Proposition 2.6 for the re-defined Dold indices.

Example 2.13. Let $p > 1$ be a prime, and suppose that G is cyclic of order p. The Burnside ring $A(G)$ is generated by $1 = [G/G]$ and $x = [G/1]$:

$$A(G) = \mathbb{Z}[x]/(x^2 - px).$$

The ring $B(G)$, described in Remark 2.8, is generated as an $A(G)$-algebra by an element $y \in B^{(p)}(G)$ corresponding to a chosen isomorphism $G \to \mathbb{Z}/p\mathbb{Z}$:

$$B(G) = A(G)[y]/(xy - x,\ y^p - 1),$$

$\sigma^{(k)}(1) = k$ and, if $p \mid k$, $\sigma^{(k)}(y^i) = kx/p$, $1 \leq i \leq p - 1$, and $\tau(y) = 1$.

The equivariant Dold indices also count the fixed-points of $g^{-1}\phi^k$ for each $g \in G$. Indeed,

$$\mathrm{Fix}(g^{-1}\phi^k) = \{x \in X \mid \phi^k(x) = gx\} \subseteq \mathrm{Fix}(\phi^{ke}),$$

where e is the order of g, and we shall see that $L(g^{-1}\phi^k, X)$ is determined by the indices $\mathcal{D}^l(\phi, X) \in B^{(l)}(G)$ where l is a divisor of ke. On the subgroup $\langle g \rangle \leq G$ generated by g we have an injective homomorphism $\alpha : \langle g \rangle \to \mathbb{Z}/ke\mathbb{Z}$ mapping g to k. Now consider the fixed-point homomorphism

$$\rho_{T_{ke}(\alpha)} : A(G \times \mathbb{Z}/ke\mathbb{Z}) = \bigoplus_{l \mid ke} B^{(l)}(G) \to \mathbb{Z}$$

for the associated subgroup $T_{ke}(\alpha)$ of $G \times \mathbb{Z}/ke\mathbb{Z}$.

On the $B^{(l)}(G)$-summand this corresponds to taking fixed-points of the image of $T_{ke}(\alpha)$ under the projection map $G \times \mathbb{Z}/ke\mathbb{Z} \to G \times \mathbb{Z}/l\mathbb{Z}$. This image is the subgroup $T_l(\alpha_l)$, where α_l is the homomorphism

$$g \mapsto k, \quad \langle g \rangle \to \mathbb{Z}/l\mathbb{Z}.$$

The subspace of X^{ke} fixed by $T_{ke}(\alpha)$ is the set

$$\{(x_1,\ldots,x_{ke}) \in X^{ke} \mid x_{i+k} = gx_i\}$$

(where the subscripts i are interpreted mod ke). We may identify this set with X^k, by projecting to the first k factors, and the restriction of $\pi_{ke}(\phi)$ to this set corresponds to the map:

$$(x_1,\ldots,x_k) \mapsto (g^{-1}\phi(x_k), \phi(x_1), \ldots, \phi(x_{k-1})),$$

since $x_{ke} = g^{e-1}x_k = g^{-1}x_k$. Its fixed-point set is then identified with $\mathrm{Fix}(g^{-1}\phi^k)$ by the correspondence

$$x \mapsto (x, \phi(x), \phi^2(x), \ldots, \phi^{k-1}(x)).$$

This computes the non-equivariant Lefschetz index

$$L(g^{-1}\phi^k, X) = \rho_{T_{ke}(\alpha)} L(\pi_k(\phi), X^k) \in \mathbb{Z}$$

as the sum of the terms $\rho_{(\alpha_l)}\mathcal{D}^l(\phi, X)$. More generally, by only minor changes, we obtain:

Proposition 2.14. *Consider a subgroup $K \leq G$ and an element $g \in N_G(K)$. Let $H = \langle K, g\rangle$ be the subgroup generated by K and g. Suppose that $gK \in W_G(K) = N_G(K)/K$ has order e. Then*

$$L(g^{-1}\phi^k, X^K) = \sum_{l \mid ke} \rho_{(\alpha_l)}\mathcal{D}^l(\phi, X) \in \mathbb{Z},$$

where $\rho_{(\alpha_l)} : B^{(l)}(G) \to \mathbb{Z}$ is the fixed-point map defined by the homomorphism $\alpha_l : H \to \mathbb{Z}/l\mathbb{Z}$ mapping K to 0 and g to k. □

In the opposite direction, it follows that the equivariant Dold indices are determined by the integers $L(g^{-1}\phi^k, X^K)$.

Corollary 2.15. *The equivariant Dold index $\mathcal{D}^m(\phi, X) \in B^{(m)}(G)$ is determined by the non-equivariant fixed-point indices $L(g^{-1}\phi^k, X^K) \in \mathbb{Z}$ for the various subgroups $K \leq G$, elements $g \in N_G(K)$, and integers $k \geq 1$ such that ke divides m, where e is the order of gK in $W_G(K)$.*

Proof. Consider a subgroup $H \leq G$ and a homomorphism $\alpha : H \to \mathbb{Z}/m\mathbb{Z}$. It suffices to show that $\rho_{(\alpha)}\mathcal{D}^m(\phi, X)$ is determined by the Lefschetz indices. Let K be the kernel of α and choose an element $g \in H$ such that $\alpha(g) = m/e$, where e is the order of the image of α. Applying Proposition 2.14 with $k = m/e$, we see that $\rho_{(\alpha)}\mathcal{D}^m(\phi, X)$ is determined by $L(g^{-1}\phi^k, X^K)$ and the indices $\mathcal{D}^l(\phi, X)$ for the strict divisors $l < m$ of m.

The proof is completed by induction on m. □

Remark 2.16. In Proposition 2.14, we have discussed only the non-equivariant index $L(g^{-1}\phi^k, X^K) \in \mathbb{Z}$. The group $W_G(K)$ acts on X^K and the map $g^{-1}\phi^k : X^K \to X^K$ is equivariant with respect to the action of the centralizer Z of gK in $W_G(K)$. We thus have a Z-equivariant fixed-point index $L(g^{-1}\phi^k, X^K) \in A(Z)$. This, too, is determined by the Dold indices $\mathcal{D}^l(\phi, X)$ for the divisors l of ke.

3. Equivariant stable homotopy groups

Restricting attention to metrizable ANRs, we shall call a G-space X a G-*ANR* if it can be embedded as a G-retract of an open subset of a normed G-vector space. Thus, we have a normed vector space E, on which G acts by a homomorphism $G \to \mathrm{GL}(E)$ (where $\mathrm{GL}(E)$ is the group of invertible continuous linear operators on E), an open G-subset U of E, and continuous G-maps $i : X \to U$ and $r : U \to X$ such that $r \circ i = 1_X : X \to X$.

We shall use the notation $\omega_i^G(X)$, where $i \in \mathbb{Z}$, for the ith unreduced G-equivariant stable homotopy group of X. (This is the group of stable maps from the sphere S^i, for $i \geq 0$, to the pointed space obtained by adding a disjoint basepoint to X.) A good reference for the basic theory is the textbook [6], with the caution that this notation is used there for the reduced stable homotopy of a pointed space.

In particular, the equivariant stable homotopy ring $\omega_0^G(*)$ of a point $*$ is naturally identified with the Burnside ring $A(G)$, described in Section 2 as a free abelian group on the set of conjugacy classes of subgroups. This identification may be regarded as a special case of the decomposition theorem of tom Dieck et al.:

$$\omega_i^G(X) = \bigoplus_{(H)} \omega_i(\mathbf{E}W_G(H) \times_{W_G(H)} X^H)$$

expressing the G-equivariant stable homotopy group as a sum of components indexed by the conjugacy classes of subgroups H of G. Here the quotient $W_G(H) = N_G(H)/H$ acts on the fixed subspace X^H and $\mathbf{E}W_G(H)$ is the universal free $W_G(H)$-space. In dimension 0 this gives an elementary description of the group.

Lemma 3.1. *Let X be a G-ANR. Then*

$$\omega_0^G(X) = \bigoplus_{(H)} \mathbb{Z}[\pi_0(X^H)/W_G(H)],$$

where $\pi_0(X^H)/W_G(H)$ is the set of $W_G(H)$-orbits of the action of $W_G(H) = N_G(H)/H$ on the set of path-components of the subspace X^H.

Proof. The 0th stable homotopy group of a space is free abelian on the set of path-components of the space. The path-components of the homotopy-orbit space $\mathbf{E}W_G(H) \times_{W_G(H)} X^H$ correspond to orbits of the action of $W_G(H)$ on X^H. □

Remark 3.2. This result allows us to interpret $\omega_0^G(X)$ as a Grothendieck group of G-maps from a finite G-set to X. Two such G-maps $f : S \to X$ and $f' : S' \to X$ are to be regarded as equivalent if there is a G-isomorphism $t : S \to S'$ such that f is homotopic to $f' \circ t$. The monoid structure is provided by disjoint union.

For each integer $k \geq 1$, the k-fold product X^k is a $G \times \mathbb{Z}/k\mathbb{Z}$-space, with $\mathbb{Z}/k\mathbb{Z}$ acting, according to the interpretation of X^k as $\mathrm{map}(\mathbb{Z}/k\mathbb{Z}, X)$, by translation as described in Section 2. The splitting of the Burnside ring $A(G \times \mathbb{Z}/k\mathbb{Z})$ in Lemma 2.9 generalizes to a decomposition of the $G \times \mathbb{Z}/k\mathbb{Z}$-equivariant stable homotopy groups of X^k.

Lemma 3.3. *There is a natural decomposition*

$$\omega_i^{G\times\mathbb{Z}/k\mathbb{Z}}(X^k) = \bigoplus_{l\,|\,k} \omega_i^G(\mathbf{E}_G(\mathbb{Z}/l\mathbb{Z}) \times_{\mathbb{Z}/l\mathbb{Z}} X^l),$$

where $\mathbf{E}_G(\mathbb{Z}/l\mathbb{Z}) \to \mathbf{B}_G(\mathbb{Z}/l\mathbb{Z})$ *is a universal* G*-equivariant principal* $\mathbb{Z}/l\mathbb{Z}$*-bundle.*

Proof. This may be established by the methods of tom Dieck [6], who writes $E(G,\mathbb{Z}/l\mathbb{Z})$ rather than $\mathbf{E}_G(\mathbb{Z}/l\mathbb{Z})$. It is a special case of results in [19], where the notation $E\mathcal{F}(\mathbb{Z}/l\mathbb{Z}; G\times\mathbb{Z}/l\mathbb{Z})$ is used. □

Remark 3.4. As a concrete realisation of $\mathbf{E}_G(\mathbb{Z}/k\mathbb{Z})$ we may take a direct limit of spheres $S(E)$ on finite-dimensional complex G-modules E with $\mathbb{Z}/k\mathbb{Z}$ acting as complex multiplication by kth roots of unity.

The fact that Lemma 2.9 is a specialization of Lemma 3.3 is a consequence of:

Lemma 3.5. *There is a natural identification*

$$B^{(k)}(G) = \omega_0^G(\mathbf{B}_G(\mathbb{Z}/k\mathbb{Z})).$$

Proof. This may be seen by relating the description of $A(G\times\mathbb{Z}/k\mathbb{Z})$ in terms of the conjugacy classes of subgroups of $G\times\mathbb{Z}/k\mathbb{Z}$, described in Section 2, to the decomposition in Lemma 2.9.

Alternatively, we may apply Lemma 3.1 to the G-space $\mathbf{B}_G(\mathbb{Z}/k\mathbb{Z})$. Consider a fixed subgroup $H \le G$. The components of the subspace of $\mathbf{B}_G(\mathbb{Z}/k\mathbb{Z})$ fixed by H are indexed by the homomorphisms $\alpha : H \to \mathbb{Z}/k\mathbb{Z}$. (See [3] for the corresponding statement for $B(G)$.) The stable homotopy group is thus identified with the free abelian group on generators indexed by $(\alpha) \in \Xi^{(k)}(G)/G$. This coincides with the description of $B^{(k)}(G)$ given in Lemma 2.3. □

Remark 3.6. From this point of view, the ring structure on $B^{(k)}(G)$ described in Remark 2.8 appears as the Pontryagin ring structure on the stable homotopy of the Hopf space $\mathbf{B}_G(\mathbb{Z}/k\mathbb{Z})$.

The forgetful mapping $\sigma^{(k)} : B^{(k)}(G) \to A(G)$ admits a straightforward generalization.

Definition 3.7. For $k \ge 1$, we define

$$\sigma^{(k)} : \omega_i^G(\mathbf{E}_G(\mathbb{Z}/k\mathbb{Z}) \times_{\mathbb{Z}/k\mathbb{Z}} X^k) \to \omega_i^G(X)$$

to be the composition of the inclusion of the top summand in the splitting of $\omega_i^{G\times\mathbb{Z}/k\mathbb{Z}}(X^k)$ in Lemma 3.3, the restriction map to $\omega_i^G(X^k)$ (forgetting the action of $\mathbb{Z}/k\mathbb{Z}$) and the map to $\omega_i^G(X)$ induced by the projection $X^k \to X$ to any factor. (Because we started with invariance under the cyclic action of $\mathbb{Z}/k\mathbb{Z}$, it does not matter which factor we choose.)

To complete this preparation for the definition of the Dold indices we need to extend Lemma 2.12. Suppose that l divides k. Then by taking fixed-points with respect to the action of the subgroup $G \times l\mathbb{Z}/k\mathbb{Z}$ of $G \times \mathbb{Z}/k\mathbb{Z}$ we obtain a homomorphism $\rho_{l\mathbb{Z}/k\mathbb{Z}}$:

$$\omega_i^{G\times\mathbb{Z}/k\mathbb{Z}}(X^k) = \bigoplus_{m\,|\,k} \omega_i^G(\mathbf{E}_G(\mathbb{Z}/m\mathbb{Z}) \times_{\mathbb{Z}/m\mathbb{Z}} X^m)$$
$$\to \omega_i^{G\times\mathbb{Z}/l\mathbb{Z}}(X^l) = \bigoplus_{m\,|\,l} \omega_i^G(\mathbf{E}_G(\mathbb{Z}/m\mathbb{Z}) \times_{\mathbb{Z}/m\mathbb{Z}} X^m).$$

Lemma 3.8. *The homomorphism $\rho_{l\mathbb{Z}/k\mathbb{Z}}$ described above maps a summand indexed by a divisor m of k to 0 if m does not divide l, and identically to the corresponding summand if m does divide l.*

Proof. This is a feature of the naturality of the decomposition in Lemma 3.3. □

4. Topological fixed-point indices

We recall, first, the notation for the Lefschetz fixed-point index as used in [5]. (See, also, [4], where the emphasis is on the parametrized theory.)

Suppose that X is a G-ANR, $U \subseteq X$ is an open G-subspace, and $\phi : U \to X$ is a (continuous) G-map which is *compactly fixed* in the sense that the fixed-subspace $\mathrm{Fix}(\phi)$ is compact and there is an open G-neighbourhood V of $\mathrm{Fix}(\phi)$ such that $\phi(V)$ has compact closure in X. The *Lefschetz fixed-point index* is an element

$$L(\phi, U) \in \omega_0^G(*) = A(G)$$

of the equivariant stable homotopy ring of a point, $\omega_0^G(*)$. More precisely, we have a *Lefschetz–Hopf fixed-point index*

$$\tilde{L}(\phi, U) \in \omega_0^G(U),$$

lifting $L(\phi, U)$ and localizing the fixed-point information to the neighbourhood U of the fixed-subspace.

Having established notation, we turn to the definition of the Dold indices for iterated maps. For an integer $k \geq 1$, we write

$$\pi_k(\phi) : U^k \to X^k$$

as in the discrete case, for the $G \times \mathbb{Z}/k\mathbb{Z}$-equivariant map

$$(x_1, \dots, x_k) \mapsto (\phi(x_k), \phi(x_1), \dots, \phi(x_{k-1})).$$

When $\pi_k(\phi)$ is compactly fixed, we thus have fixed-point indices

$$L(\pi_k(\phi), U^k) \in \omega_0^{G\times\mathbb{Z}/k\mathbb{Z}}(*) = A(G \times \mathbb{Z}/k\mathbb{Z})$$

and

$$\tilde{L}(\pi_k(\phi), U^k) \in \omega_0^{G\times\mathbb{Z}/k\mathbb{Z}}(U^k).$$

The condition that $\pi_k(\phi)$ be compactly fixed is easily expressed in terms of the kth iterate ϕ^k.

Lemma 4.1. *The fixed-point set* $\mathrm{Fix}(\pi_k(\phi))$ *is naturally identified with the fixed-point set of the kth iterate* $\phi^k : (\phi^{-1})^{k-1}(U) \to X$, *and* $\pi_k(\phi)$ *is compactly fixed if and only if* $\mathrm{Fix}(\phi^k)$ *is compact and there is a G-neighbourhood* V *of* $\mathrm{Fix}(\phi^k)$ *in* $(\phi^{-1})^{k-1}(U)$ *such that* $\phi(V)$ *has compact closure in* X. □

In what follows, we shall assume that the map ϕ satisfies the compactly fixed properties required to define the various indices considered.

The compatibility of the Lefschetz indices of $\pi_k(\phi)$ and $\pi_l(\phi)$, as expressed in the next lemma, is proved just as in the non-equivariant case, [5, Lemma 3.8].

Lemma 4.2. *Suppose that* l *divides* k. *Then* $L(\pi_l(\phi), U^l)$ *and* $\tilde{L}(\pi_l(\phi), U^l)$ *are the images of* $L(\pi_k(\phi), U^k)$ *and* $\tilde{L}(\pi_k(\phi), U^k)$ *respectively under the fixed-point homomorphisms* $\rho_{l\mathbb{Z}/k\mathbb{Z}}$:

$$A(G \times \mathbb{Z}/k\mathbb{Z}) \to A(G \times \mathbb{Z}/l\mathbb{Z}), \qquad \omega_0^{G\times\mathbb{Z}/k\mathbb{Z}}(U^k) \to \omega_0^{G\times\mathbb{Z}/l\mathbb{Z}}(U^l). \qquad \square$$

We can now extend Definition 2.10 from the discrete to the topological situation.

Definition 4.3. Let $\phi : U \to X$ be a G-map such that $\pi_k(\phi) : U^k \to X^k$ is compactly fixed. The *G-equivariant Dold indices*

$$\mathcal{D}^k(\phi, U) \in B^{(k)}(G) = \omega_0^G(\mathbf{B}_G(\mathbb{Z}/k\mathbb{Z}))$$

and

$$\tilde{\mathcal{D}}^k(\phi, U) \in \omega_0^G(\mathbf{E}_G(\mathbb{Z}/k\mathbb{Z}) \times_{\mathbb{Z}/k\mathbb{Z}} U^k)$$

are defined to be the respective top components of the $G \times \mathbb{Z}/k\mathbb{Z}$-equivariant Lefschetz indices $L(\pi_k(\phi), U^k) \in A(G \times \mathbb{Z}/k\mathbb{Z})$ and $\tilde{L}(\pi_k(\phi), U^k) \in \omega_0^{G\times\mathbb{Z}/k\mathbb{Z}}(U^k)$ in the decompositions given by (Lemma 2.9 and) Lemma 3.3.

It now follows from Lemmas 4.2 and 3.8, exactly as in the discrete case, that

$$L(\pi_k(\phi), U^k) = \sum_{l\,|\,k} \mathcal{D}^l(\phi, U) \in \bigoplus_{l\,|\,k} B^{(l)}(G) = A(G \times \mathbb{Z}/k\mathbb{Z})$$

and

$$\tilde{L}(\pi_k(\phi), U^k) = \sum_{l\,|\,k} \tilde{\mathcal{D}}^l(\phi, U) \in \bigoplus_{l\,|\,k} \omega_0^G(\mathbf{E}_G(\mathbb{Z}/l\mathbb{Z}) \times_{\mathbb{Z}/l\mathbb{Z}} U^l) = \omega_0^{G\times\mathbb{Z}/k\mathbb{Z}}(U^k).$$

Lemma 4.4. *The Lefschetz index* $\tilde{L}(\pi_k(\phi), U^k) \in \omega_0^{G\times\mathbb{Z}/k\mathbb{Z}}(U^k)$ *maps by restriction of groups from* $G \times \mathbb{Z}/k\mathbb{Z}$ *to* G *and projection from* U^k *to any factor* U *to an element of* $\omega_0^G(U)$. *This element is the image of* $\tilde{L}(\phi^k, V) \in \omega_0^G(V)$, *where* $V = (\phi^{-1})^{k-1}(U)$, *under the inclusion* $V \hookrightarrow U$.

Proof. See the proof of [5, Lemma 3.9], which is essentially a classical proof of the commutativity property of the Lefschetz index. □

The corresponding statement for $L(\phi^k, V) \in A(G)$ follows by projecting U and V to a point. Expressing the Lefschetz indices of $\pi_k(\phi)$ in terms of the Dold indices and writing the forgetful maps on the components as $\sigma^{(l)}$ (Definition 3.7), we obtain:

Theorem 4.5. *Let $\phi : U \to X$ be a G-map such that $\pi_k(\phi)$, for some fixed $k \geq 1$, is compactly fixed. Then the Lefschetz index of the iterate ϕ^k, defined on $V = (\phi^{-1})^{k-1}(U)$, is given by*

$$L(\phi^k, V) = \sum_{l\,|\,k} \sigma^{(l)} \mathcal{D}^l(\phi, U) \in \omega_0^G(*) = A(G),$$

and $\tilde{L}(\phi^k, V) \in \omega_0^G(V)$ maps, by the inclusion of V in U, to

$$\sum_{l\,|\,k} \sigma^{(l)} \tilde{\mathcal{D}}^l(\phi, U) \in \omega_0^G(U).$$

□

Proposition 2.14 also extends to the topological case.

Theorem 4.6. *Let $\phi : U \to X$ be a G-map such that $\pi_k(\phi)$, for some fixed $k \geq 1$, is compactly fixed. Consider a subgroup $K \leq G$ and an element $g \in N_G(K)$. Then the non-equivariant Lefschetz index of $g^{-1}\phi^k$ is given by*

$$L(g^{-1}\phi^k, (\phi^{-1})^{k-1}(U)) = \sum_{l\,|\,ke} \rho_{(\alpha_l)} \mathcal{D}^l(\phi, U) \in \mathbb{Z},$$

where e is the order of gK in $W_G(K)$ and $\alpha_l : \langle K, g\rangle \to \mathbb{Z}/l\mathbb{Z}$ is the homomorphism mapping K to 0 and g to k.

Proof. The main ingredients, including the description of $\rho_{(\alpha_l)} : B^{(l)}(G) \to \mathbb{Z}$, are already in Section 2. Let $T_{ke}(\alpha_k)$ be the subgroup of $G \times \mathbb{Z}/ke\mathbb{Z}$ associated with the homomorphism α_k, as in Section 2. We apply the fixed-point construction for this subgroup to the Lefschetz index $L(\pi_{ke}(\phi), U^{ke}) \in B^{(ke)}(G)$. The result is the integer described in the statement as a linear combination of Dold indices. On the other hand, it is the Lefschetz fixed-point index of the restricted map

$$\pi_{ke}(\phi)| : (U^{ke})^{T_{ke}(\alpha_k)} \to (X^{ke})^{T_{ke}(\alpha_k)}.$$

Following the proof of Proposition 2.14, we may identify this restriction with the map

$$(U^K)^k \to (X^K)^k, \qquad (x_1, \ldots, x_k) \mapsto (g^{-1}\phi(x_k), \phi(x_1), \ldots, \phi(x_{k-1})).$$

The fixed-points of this map clearly correspond to the fixed-points of $g^{-1}\phi^k$, and the classical homotopy argument shows that the Lefschetz numbers of the two maps coincide. □

For future use we record some properties of the Dold index that follow immediately from the corresponding properties of the Lefschetz index.

Proposition 4.7. *Let $\phi : U \to X$ be a G-map such that $\pi_k(\phi) : U^k \to X^k$ is compactly fixed for a given integer $k \geq 1$.*

(i) (Localization). *Suppose that* $U' \subseteq U$ *is an open* G*-subset containing* $\mathrm{Fix}(\phi^k)$*. Then* $\tilde{\mathcal{D}}^k(\phi|U', U')$ *maps to* $\tilde{\mathcal{D}}^k(\phi, U)$ *under the inclusion map*

$$\omega_0^G(\mathbf{E}_G(\mathbb{Z}/k\mathbb{Z}) \times_{\mathbb{Z}/k\mathbb{Z}} (U')^k) \to \omega_0^G(\mathbf{E}_G(\mathbb{Z}/k\mathbb{Z}) \times_{\mathbb{Z}/k\mathbb{Z}} U^k)$$

and $\mathcal{D}^k(\phi|U', U') = \mathcal{D}^k(\phi, U) \in B^{(k)}(G)$*.*

(ii) (Additivity). *Suppose that* U *is the disjoint union of open* G*-subsets* U_1 *and* U_2*. Then*

$$\mathcal{D}^k(\phi, U) = \mathcal{D}^k(\phi|U_1, U_1) + \mathcal{D}^k(\phi|U_2, U_2).$$

□

5. Isolated fixed orbits

Continuing to assume that X is a G-ANR and that $\phi : U \to X$ is a G-map defined on an open G-subspace $U \subseteq X$, we first review some results from [5].

Suppose that the fixed-subspace $\mathrm{Fix}(\phi)$ of ϕ consists of just one G-orbit P (and that there is an open G-neighbourhood V of P in U such that $\phi(V)$ has compact closure in X). By the localization property, the Lefschetz index $L(\phi, U) \in A(G)$ is unchanged if we replace U by a smaller neighbourhood of P. It depends only on the germ of ϕ at P, and we denote it by

$$L(\phi, P) \in A(G).$$

Choose a point $x \in P$, with stabilizer subgroup $H \leq G$.

Lemma 5.1. *The* G*-equivariant index* $L(\phi, P) \in A(G)$ *is equal to the image of the* H*-equivariant index* $L(\phi, x) \in A(H)$ *under the induction homomorphism*

$$A(H) \to A(G), \quad [S] \mapsto [G \times_H S].$$

Proof. Choose an H-invariant open neighbourhood $Y \subseteq X$ of x such that the translates gY, $gH \in G/H$, are disjoint. (Since X is a G-ANR, it admits a G-invariant metric. We may take Y to be an open ball, of centre x, with small radius.) Put $V = \phi^{-1}(Y)$. The restriction of ϕ to the union of the disjoint open sets gV, $gH \in G/H$, reduces to the union

$$G \times_H V \to G \times_H Y$$

of the translates of the restriction $\phi|V : V \to Y$. The assertion follows from the geometric construction of the induction map as '$G \times_H -$'. □

When X is a finite-dimensional (smooth) G-manifold, the fixed-point index of an isolated fixed-G-orbit P can be computed as a local degree. Without loss of generality, we may suppose that $X = E$ is a Euclidean G-module and, in view of Lemma 5.1, consider the case in which P consists of the single point $0 \in U \subseteq E$. We may also assume that U contains the closed unit disc $D(E)$ of radius 1 and centre 0. The vector field $v : U \to E$ defined by $v(x) = x - \phi(x)$ has an isolated zero at 0, and $L(\phi, a) \in A(G)$ is equal to the equivariant degree of the self-map

$$a : x \mapsto \|v(x)\|^{-1} v(x), \quad S(E) \to S(E)$$

of the unit sphere $S(E)$ in E.

We can generalize this description of the fixed-point index as a degree, by allowing E to be a (real) Banach G-module. Suppose further that $\phi : U \to E$ is continuously differentiable with derivative $A = D\phi(0)$ at the fixed-point 0. To define the index $L(\phi, 0)$, we require that $\{0\}$ be a compactly fixed subspace, and this implies that A is a compact operator. If $1-A$ is non-singular, then the index is computed by real K-theory, as we now explain. The endomorphism $1-A : E \to E$ defines an element of the real K-group $K\mathrm{O}_G^{-1}(*)$, and the local degree is the image of $[1-A]$ under the equivariant J-homomorphism

$$J : K\mathrm{O}_G^{-1}(*) \to A(G)^\times \ (\subseteq A(G))$$

to the group of units in the Burnside ring. Let $\hat{G}$ be the set of isomorphism classes of irreducible real G-modules and for each $\alpha \in \hat{G}$ choose a representative G-module V_α. The KO-group is a direct sum

$$K\mathrm{O}_G^{-1}(*) = \bigoplus_{\alpha \in \hat{G} \text{ real}} (\mathbb{Z}/2\mathbb{Z})\eta[V_\alpha],$$

where η is the generator of $K\mathrm{O}^{-1}(*) = \mathbb{Z}/2\mathbb{Z}$ and the summation runs over the isomorphism classes of irreducible real representations α of G whose endomorphism ring is equal to $\mathbb{R}$. The G-module E splits as a direct sum $\bigoplus_{\alpha\in\hat{G}} E_\alpha$, where

$$E_\alpha = \mathrm{Hom}^G(V_\alpha, E) \otimes_{K_\alpha} V_\alpha, \quad K_\alpha = \mathrm{End}^G(V_\alpha),$$

and A splits, correspondingly, as a sum of endomorphisms $A_\alpha \in \mathrm{End}(E_\alpha)$. Let $t_\alpha = J(\eta[V_\alpha]) \in A(G)$, so that $t_\alpha^2 = 1$.

Lemma 5.2. *Suppose that E is finite-dimensional. Then the local degree is computed K-theoretically as*

$$J([1-A]) = \prod_{\alpha\in\hat{G} \text{ real}} t_\alpha^{\mathrm{sign}(\det(1-A_\alpha))} \in A(G)^\times.$$

Proof. The α-component of $[1-A]$ is equal to 0 if $\det(1-A_\alpha) > 0$, and $\eta[V_\alpha]$ if $\det(1-A_\alpha) < 0$. In the infinite-dimensional case the determinant is not, in general, defined, but one can still make sense of the sign as a topological index. □

We shall need the following technical lemma from [5, Lemma 3.17].

Lemma 5.3. *Let E be a Banach G-module and let $\phi : U \to E$ be a C^1 map with an isolated (and compactly fixed) fixed-point at 0. Suppose that $E = F' \oplus F''$ splits as a direct sum of Banach G-modules such that $\phi(U \cap F') \subseteq F'$. Write $\phi' : U' \to F'$ for the restriction of ϕ to the first factor $U' = U \cap F'$ and $A'' : F'' \to F''$ for the $(2,2)$ component of $A = D\phi(0)$. Suppose that $1 - A''$ is non-singular. Then*

$$L(\phi, 0) = L(\phi', 0) \cdot L(A'', 0) = L(\phi', 0) \cdot J([1 - A'']) \in A(G). \qquad \square$$

This completes the preparatory material, and we turn to consideration of the iterates of a G-map $\phi : U \to X$ defined on an open subspace of a G-ANR.

Definition 5.4. Consider a G-orbit $P \subseteq U$ such that $\phi(P) = P$. We assume that there is a G-neighbourhood V of P in U such that $\phi(V)$ has compact closure in X. Suppose that P is an isolated fixed-subset of ϕ^k, where $k \geq 1$. Choose an open neighbourhood U' of P in U such that the points of P are the only fixed-points of ϕ^k in U'. By the localization property (Proposition 4.7(i)), the indices $\mathcal{D}^l(\phi, U')$, for $l \,|\, k$, depend only on P, not on the open neighbourhood U'. We write them as the *local Dold indices*

$$\mathcal{D}^l(\phi, P) \in B^{(l)}(G).$$

Our first result is an equivariant version of the theorem of Shub and Sullivan [22].

Theorem 5.5. *Let E be a finite-dimensional G-module, and let $\phi : U \to E$ be a continuously differentiable G-map defined on an open G-neighbourhood U of 0. Suppose that 0 is an isolated fixed-point of ϕ^k for each $k \geq 1$. Then $\mathcal{D}^k(\phi, 0)$ vanishes for all but finitely many k.*

This will be proved in the more precise form, stated below as Theorem 5.6, due to Chow, Mallet-Paret and Yorke [2]. (See also the accounts in [15] and [16].) Our exposition will follow closely that in [5] (but sharpened to include the full result of Chow et al.).

Theorem 5.6. *Let U be an open G-neighbourhood of 0 in a Banach G-module E and let $\phi : U \to E$ be a continuously differentiable mapping such that $\phi(0) = 0$ and $\phi(U)$ has compact closure in E. Suppose that 0 is an isolated fixed-point of ϕ^k for some fixed $k \geq 1$.*

Let $\Lambda \subseteq \mathbb{N}$ be the smallest set of natural numbers that is closed under formation of least common multiples and contains 1 and the order of each eigenvalue of $(D\phi)(0)$ that is a root of unity.

Suppose that $k \notin \Lambda$. Then $\mathcal{D}^k(\phi, 0) = 0$ unless $k = 2l$, where $l \in \Lambda$ is odd, and in that case $\mathcal{D}^k(\phi, 0) = \epsilon \cdot \mathcal{D}^l(\phi, 0)$, where $\epsilon \in B^{(2)}(G)$ is defined in the text and the product is formed in $A(G \times \mathbb{Z}/2l\mathbb{Z})$.

Proof. We again write A for the compact operator $D\phi(0)$.

(1) Suppose that 1 is not an eigenvalue of A. Then, since $\mathcal{D}^1(\phi, 0) = L(\phi, 0)$, we have already shown that

$$\mathcal{D}^1(\phi, 0) = J([1 - A]) \in A(G) = B^{(1)}(G),$$

where

$$J : KO_G^{-1}(*) \to A(G)^{\times}$$

is the J-homomorphism.

(2) Suppose that k is even and that -1 is not an eigenvalue of A. Then, writing L for the representation $\mathbb{R}$ of $\mathbb{Z}/2\mathbb{Z}$ on which the generator acts as -1, we have a class

$$J([(1 + A) \otimes L]) \in A(G \times \mathbb{Z}/2\mathbb{Z}) = B^{(1)}(G) \oplus B^{(2)}(G).$$

The first component is $1 \in A(G)$, because taking fixed-points with respect to $\mathbb{Z}/2\mathbb{Z}$ sends $[(1+A)\otimes L]$ to 0. Let us write the second component as $\epsilon \in B^{(2)}(G)$.

(3) Now suppose that l ($l \neq k$) is a proper divisor of k and that every kth root of unity which is an eigenvalue of A is an lth root of unity.

We examine the index of $\pi_k(\phi) : U^k \to E^k$. As a $\mathbb{Z}/k\mathbb{Z}$-module, $E^k = E \otimes V_k$, where V_k is the representation $\mathbb{R}^k$ on which $\mathbb{Z}/k\mathbb{Z}$ acts, in the usual way, by cyclic permutation. The derivative of $\pi_k(\phi)$ at 0 is $\pi_k(A) = A \otimes S$, where $S : V_k \to V_k$ is given by the action of the generator $1 \in \mathbb{Z}/k\mathbb{Z}$ (so the cyclic shift by one step).

Now

$$V_k = \begin{cases} \mathbb{R} \oplus (\bigoplus_{1\le r\le (k-1)/2} L_r) & \text{if } k \text{ is odd,} \\ \mathbb{R} \oplus (\bigoplus_{1\le r< k/2} L_r) \oplus L & \text{if } k \text{ is even,} \end{cases}$$

where L_r is the 1-dimensional complex representation $\mathbb{C}$ on which S acts as multiplication by $e^{2\pi i r/k}$, and L, if k is even, is the 1-dimensional real representation $\mathbb{R}$ on which S acts as -1 (as in the case $k = 2$ already considered).

We split $E^k = E \otimes V_k$ as the direct sum $F' \oplus F''$ of $F' = E \otimes V_l$ and $F'' = E \otimes V_l^{\perp}$, where V_l is the submodule fixed by $l\mathbb{Z}/k\mathbb{Z}$, and $V_l^{\perp}$ is the direct sum of the complex lines L_r for r not divisible by l and, in the special case that k is even and l is odd, the real line L. Observe that, in this special case, -1 is not an eigenvalue of A.

The proof will be completed by applying Lemma 5.3 to the $G \times \mathbb{Z}/k\mathbb{Z}$-equivariant map $\pi_k(\phi) : U^k \to F' \oplus F''$. In the notation employed there, $1 - A''$ is non-singular and the index $L(A'', 0)$ is determined by the class of $1 - A''$ in

$$KO^{-1}_{G\times\mathbb{Z}/k\mathbb{Z}}(*) = \begin{cases} KO^{-1}_G(*) & \text{if } k \text{ is odd,} \\ KO^{-1}_G(*) \oplus KO^{-1}_G(*)\cdot[L] & \text{if } k \text{ is even.} \end{cases}$$

Apart from the special case, A'' is $\mathbb{C}$-linear and the KO-theory class is trivial, because the complex K-group $K^{-1}_{G\times\mathbb{Z}/k\mathbb{Z}}(*)$ is zero. Hence the index $L(A'', 0) \in A(G \times \mathbb{Z}/k\mathbb{Z})$ is equal to 1, except in the case that k is even and l is odd. In that case, the class is the pullback of $1 + \epsilon \in A(G \times \mathbb{Z}/2\mathbb{Z})$ to $A(G \times \mathbb{Z}/k\mathbb{Z})$.

The index of the restriction of $\pi_k(\phi)$ to the subspace $U^k \cap F'$ fixed by $l\mathbb{Z}/k\mathbb{Z}$ is equal to $p^*L(\pi_l(\phi), 0)$, where

$$p^* : A(G \times \mathbb{Z}/l\mathbb{Z}) \to A(G \times \mathbb{Z}/k\mathbb{Z})$$

is induced by the projection $p : \mathbb{Z}/k\mathbb{Z} \to \mathbb{Z}/l\mathbb{Z}$. (The homomorphism p^* splits the fixed-point homomorphism $\rho_{l\mathbb{Z}/k\mathbb{Z}}$.)

By Lemma 5.3, $L(\pi_k(\phi), 0)$ is the product $p^*L(\pi_l(\phi)) \cdot L(A'', 0)$. It follows that $\mathcal{D}^k(\phi, 0) = 0$, except in the special case. When $k = 2l$ and l is odd, then $\mathcal{D}^{2l}(\phi, 0) \in B^{(2l)}(G)$ is the product in $A(G\times\mathbb{Z}/2l\mathbb{Z})$ of $\epsilon \in B^{(2)}(G)$ and $\mathcal{D}^l(\phi, 0) \in B^{(l)}(G)$. □

Consider again a general G-map $\phi : U \to X$ defined on an open G-subspace of a G-ANR. Suppose that $x \in U$ is a periodic point of ϕ, that is, $\phi^m(x)$ is defined and equal to x for some $m \geq 1$. The least such integer m, the *minimal period* of x, will be denoted by $m(x)$. Clearly every point in the orbit P of x is periodic, with the same minimal period, which we may write as $m(P)$. Let $[P]$ be the union of the sets $\phi^i(P)$, $i \geq 0$. There is a free action of the group $\mathbb{Z}/m(P)\mathbb{Z}$ on $[P]$ with 1 acting as ϕ, and $[P]$ thus becomes a transitive $G \times \mathbb{Z}/m(P)\mathbb{Z}$-set determining an element $(\alpha) \in \Xi^{(m(P))}(G)/G$, where $\alpha : H \to \mathbb{Z}/m(P)\mathbb{Z}$ is the homomorphism defined on the subgroup H of elements $g \in G$ such that $g(x) = \phi^i(x)$ for some $i \geq 0$ by $\alpha(g) = i \pmod{m(P)}$. We denote by $e(P)$ the least positive integer such that $\phi^{e(P)}(P) \subseteq P$; in other words, $e(P)$ is the order of the image of α.

Suppose that for some integer $k \geq 1$, necessarily a multiple of $m(P)$, the subset $[P]$ is an isolated fixed-subset of ϕ^k. Then we may choose an open G-neighbourhood $V \subseteq U$ of $[P]$ in which the only fixed-points of ϕ^k are the points of $[P]$. In order to define the fixed-point indices, we assume that V may be chosen so that $\phi(V)$ has compact closure in X.

Recall from Lemma 2.7 that for a divisor l of k we may regard $B^{(l)}(G)$ as an $A(G)$-submodule of $B^{(k)}(G)$.

Lemma 5.7. *In the situation described above, let $l = k/m(P)$ and let $K \leq G$ be the stabilizer subgroup,* $\ker \alpha$*, of the point x. Then*

(i) $\mathcal{D}^k(\phi|V, V) = \mathcal{D}^{k/e(P)}(\phi^{e(P)}, P) \in B^{(k/e(P))}(G) \subseteq B^{(k)}(G)$;
(ii) $\mathcal{D}^{k/e(P)}(\phi^{e(P)}, P) = \mathcal{D}^l(\phi^{m(P)}, P) \in B^{(l)}(G) \subseteq B^{(k/e(P))}(G)$;
(iii) $\mathcal{D}^l(\phi^{m(P)}, P) \in B^{(l)}(G)$ *is the image of* $\mathcal{D}^l(\phi^{m(P)}, x) \in B^{(l)}(K)$ *under the induction homomorphism* $B^{(l)}(K) \to B^{(l)}(G)$.

Proof. Choose an open G-neighbourhood $W \subseteq V$ of P containing no other points of $[P]$. The argument used to establish Proposition 3.13 of [5] then shows that $\mathcal{D}^k(\phi|V, V) \in B^{(k)}(G)$ is equal to the image of $\mathcal{D}^{k/e(P)}(\phi^{e(P)}|W, W) \in B^{k/e(P)}(G)$ under the inclusion map, and so proves (i).

Lemma 3.9 in [5], with the formal introduction of G-equivariance, verifies (ii). (Like Lemma 4.4, this is essentially the classical argument used to prove the commutativity formula for the Lefschetz index.)

The statement (iii) follows immediately from Lemma 5.1. □

The next global result extends a theorem of Franks [12] to the equivariant case.

Theorem 5.8. *Suppose that $\phi : X \to X$ is a self-map of a compact G-ANR such that the kth power ϕ^k, for some fixed $k \geq 1$, has only finitely many fixed-points. Then*

$$\mathcal{D}^k(\phi, X) = \sum_{[P]} \mathcal{D}^{k/m(P)}(\phi^{m(P)}, P) \in B^{(k)}(G),$$

where P runs over the set of orbits of periodic points fixed by ϕ^k, $m(P)$ is the least integer $m \geq 1$ such that ϕ^m fixes each point of P, and $[P]$ is the union of the translates $\phi^i(P)$ of P.

Proof. This now follows easily from Lemma 5.7(ii) and the additivity and localization properties (Proposition 4.7) of the Dold index. □

Example 5.9. Tori provide the simplest examples. Let $E_{\mathbb{Z}}$ be a finitely generated free $\mathbb{Z}$-module with a (linear) action of G and let $A_{\mathbb{Z}} : E_{\mathbb{Z}} \to E_{\mathbb{Z}}$ be a G-endomorphism. Write $E = \mathbb{R} \otimes E_{\mathbb{Z}}$ and $A = 1 \otimes A_{\mathbb{Z}} : E \to E$. Then A induces a G-equivariant self-map $\phi : X \to X$ of the torus $X = E/E_{\mathbb{Z}}$. The fixed-subspace is $\mathrm{Fix}(\phi) = (1 - A)^{-1}(E_{\mathbb{Z}})/E_{\mathbb{Z}}$.

Suppose that, for each $k \geq 1$, the endomorphism $1 - A^k$ is non-singular and, in the notation of Lemma 5.2, that $\det(1 - A^k_\alpha) > 0$ for each irreducible real representation V_α of G with $\mathrm{End}_G V_\alpha = \mathbb{R}$. Then the set, D_k say, of periodic points with minimal period k is a finite G-set with a free action of $\mathbb{Z}/k\mathbb{Z}$, via ϕ, and

$$\mathcal{D}^k(\phi, X) = [D_k] \in B^{(k)}(G).$$

We conclude this section with an equivariant form of a theorem of Matsuoka and Shiraki [20], [21].

Corollary 5.10. *Suppose that $\phi : X \to X$ is a G-equivariant C^1-self-map of a closed (smooth, finite-dimensional) G-manifold X such that each power ϕ^r has only isolated fixed-points. Let $p > 2$ be an odd prime. Suppose that for each periodic point x of ϕ the minimal period $m(x)$ is not divisible by p and no eigenvalue of $D\phi^{m(x)}(x)$ is a root of unity with order divisible by p. If p divides k then $\mathcal{D}^k(\phi, X) = 0 \in B^{(k)}(G)$.*

Proof. Suppose that k is a multiple of p. Consider a periodic point x with period $m(x)$ dividing k. Using the notation of Lemma 5.7, we shall show that $\mathcal{D}^l(\phi^{m(P)}, x) \in B^{(l)}(K)$ is zero. It will then follow from part (i) of Lemma 5.7 and Theorem 5.8 that $\mathcal{D}^k(\phi, X)$ is zero. By hypothesis, p does not divide $m(P)$, and so p divides l. We now apply Theorem 5.6 to the isolated fixed-point x of the map $\phi^{m(P)}$. Defining $\Lambda \subseteq \mathbb{N}$ as in that theorem, we see that $l \notin \Lambda$ and, if l is even, $l/2 \notin \Lambda$. Hence $\mathcal{D}^l(\phi^{m(P)}, x) = 0$. □

Further results on isolated periodic points can be found in [1], [11] and [16].

6. Homotopy fixed-point indices

We begin with a survey of the equivariant Nielsen–Reidemeister index, adapting the fibrewise theory described in [4]; see [14] for a more conventional account. Throughout this section, X will be a G-ANR and $\phi : X \to X$ will be a G-map. The *homotopy fixed-point set* of ϕ is defined to be the G-space

$$\text{h-Fix}(\phi) = \{\gamma : [0, 1] \to X \mid \gamma(1) = \phi(\gamma(0))\}$$

(of continuous paths). For example, the homotopy fixed-point set of the identity map is the free loop space of X. There is a projection map $\text{h-Fix}(\phi) \to X$ given by evaluation at 0 and an inclusion of the fixed-subspace $\mathrm{Fix}(\phi) \hookrightarrow \text{h-Fix}(\phi)$ as the space of constant paths.

Lemma 6.1 (Properties of the homotopy fixed-point set).

(i) *The homotopy fixed-point set* h-Fix(ϕ) *is a G-ANR.*

(ii) *The map* $\phi_* : \text{h-Fix}(\phi) \to \text{h-Fix}(\phi)$ *mapping* γ *to* $\phi \circ \gamma$ *is homotopic to the identity.*

(iii) *A* G*-homotopy* $\phi_t : X \to X$*,* $0 \le t \le 1$*, determines (up to homotopy) a* G*-homotopy equivalence* $\text{h-Fix}(\phi_0) \to \text{h-Fix}(\phi_1)$*.*

Proof. (i) Let $U \subseteq E$ be an open G-subspace of a normed G-vector space E and let $i : X \to U$ and $r : U \to X$ be G-maps such that $r \circ i = 1$. To simplify the notation we may assume that i is the inclusion of a subspace $X \subseteq U$. The homotopy fixed-point set is thus included as a subspace of the normed G-vector space of continuous paths $[0,1] \to E$. Let W be the set of paths $\gamma : [0,1] \to U \subseteq E$ such that $\gamma'(t) = \gamma(t) + t((\phi \circ r)(\gamma(0)) - \gamma(1)) \in U$ for $0 \le t \le 1$. Then W is open in the normed vector space $C([0,1]; E)$ and retracts onto h-Fix(ϕ) by the mapping $\gamma \mapsto r \circ \gamma'$.

(ii) An explicit homotopy between γ and $\phi \circ \gamma$ is given by

$$A_s(t) = \begin{cases} \gamma(s+t) & \text{if } 0 \le t \le 1-s, \\ \phi(\gamma(s+t-1))) & \text{if } 1-s \le t \le 1, \end{cases} \qquad (0 \le s \le 1).$$

We have $A_0 = \gamma$ and $A_1 = \phi \circ \gamma$.

(iii) This follows from the description of the homotopy fixed-point set of ϕ as the pullback of the fibration

$$\text{map}([0,1], X) \to X \times X, \qquad \gamma \mapsto (\gamma(0), \gamma(1)),$$

under the map $x \mapsto (x, \phi(x)) : X \to X \times X$. □

Example 6.2. Consider again the self-map $\phi : X \to X$ of the torus $E/E_{\mathbb{Z}}$ introduced in Example 5.9. The homotopy fixed-point set is G-homotopy equivalent to the discrete space $E_{\mathbb{Z}}/(1-A)E_{\mathbb{Z}}$. If $1 - A$ is non-singular, the inclusion $\text{Fix}(\phi) \hookrightarrow \text{h-Fix}(\phi)$ is an equivariant homotopy equivalence.

Now suppose that $\phi : X \to X$ is compactly fixed. Then the Lefschetz index $\tilde{L}(\phi, X) \in \omega_0^G(X)$ lifts to the *homotopy Lefschetz index*

$$\text{h-}L(\phi, X) \in \omega_0^G(\text{h-Fix}(\phi)).$$

(Although this notation for the Nielsen–Reidemeister index is not standard, it is systematic, and better than the confusing notation 'N' used in [4].) The index is constructed in [4] by extending the inclusion $\text{Fix}(\phi) \hookrightarrow \text{h-Fix}(\phi)$ to a G-neighbourhood U of Fix(ϕ) using the equivariant uniform local contractibility of X and defining h-$L(\phi, X)$ to be the image of $\tilde{L}(\phi|U, U) \in \omega_0^G(U)$. (If X is an open subspace of a normed G-vector space E, we may take U to be the set of points $x \in X$ such that $\gamma_x(t) = (1-t)x + t\phi(x) \in X$ for all $t \in [0,1]$ and map $x \in U$ to $\gamma_x \in \text{h-Fix}(\phi)$.) By construction, if ϕ has no fixed-points, then h-$L(\phi, X) = 0$.

The subspace of h-Fix(ϕ) fixed by a subgroup $H \le G$ is clearly just the homotopy fixed-point set of the restricted map $\phi^H : X^H \to X^H$ and its components are the Nielsen–Reidemeister classes of ϕ^H. Hence $\omega_0^G(\text{h-Fix}(\phi))$ can be computed as a free abelian group by Lemma 3.1.

We shall now introduce the corresponding *homotopy Dold indices*.

Definition 6.3. Let $\phi : X \to X$ be a G-map such that, for a given $k \geq 1$, the $G \times \mathbb{Z}/k\mathbb{Z}$-equivariant map

$$\pi_k(\phi) : X^k \to X^k$$

is compactly fixed. We define the *homotopy Dold index*

$$\text{h-}\mathcal{D}^k(\phi, X) \in \omega_0^G(\mathbf{E}_G(\mathbb{Z}/k\mathbb{Z}) \times_{\mathbb{Z}/k\mathbb{Z}} \text{h-Fix}(\pi_k\phi))$$

to be the top component of the homotopy Lefschetz index

$$\text{h-}L(\pi_k(\phi), X^k) \in \omega_0^{G\times\mathbb{Z}/k\mathbb{Z}}(\text{h-Fix}(\pi_k(\phi)))$$

in the canonical decomposition of the equivariant stable homotopy of h-Fix$(\pi_k(\phi))$ (as given in Lemma 3.3 for the space X^k).

Various properties follow at once from the definition.

Lemma 6.4 (Properties of the homotopy Dold indices).

(i) *Under the homomorphism*

$$\omega_0^G(\mathbf{E}_G(\mathbb{Z}/k\mathbb{Z}) \times_{\mathbb{Z}/k\mathbb{Z}} \text{h-Fix}(\pi_k(\phi))) \to \omega_0^G(\mathbf{E}_G(\mathbb{Z}/k\mathbb{Z}) \times_{\mathbb{Z}/k\mathbb{Z}} X^k)$$

induced by the projection h-Fix$(\pi_k(\phi)) \to X^k$ *the homotopy index* h-$\mathcal{D}^k(\phi, X)$ *maps to* $\tilde{\mathcal{D}}^k(\phi, X)$.

(ii) *If* ϕ^k *has no fixed-points, then* h-$\mathcal{D}^k(\phi, X) = 0$.

(iii) *Suppose that* $\phi_t : X \to X$, $0 \leq t \leq 1$, *is a* G*-homotopy satisfying the appropriate compactness condition (that there exists an open neighbourhood of the compact set* $\{(t, x) \in [0, 1] \times X \mid \phi_t^k(x) = x\}$ *mapping under the homotopy* $(t, x) \mapsto \phi_t x$ *into a compact subset of* X*). Then* h-$\mathcal{D}^k(\phi_0, X)$ *is mapped to* h-$\mathcal{D}^k(\phi_1, X)$ *under the* $G\times\mathbb{Z}/k\mathbb{Z}$*-equivalence* h-Fix$(\pi_k(\phi_0)) \to$ h-Fix$(\pi_k(\phi_1))$ *determined by the homotopy.* □

Remark 6.5. Part (iii) of Lemma 6.4 gives, in particular, the Jiang-invariance of the homotopy Dold indices. The various G-homotopies $\phi_t : X \to X$ (satisfying the compactness condition) with $\phi_0 = \phi = \phi_1$ determine a group of automorphisms of $\omega_0^G(\mathbf{E}_G(\mathbb{Z}/k\mathbb{Z}) \times_{\mathbb{Z}/k\mathbb{Z}} X^k)$ which fixes h-$\mathcal{D}^k(\phi, X)$.

Suppose that l divides k. Then the subspace of h-Fix$(\pi_k(\phi))$ fixed by the subgroup of index l in $\mathbb{Z}/k\mathbb{Z}$ is naturally identified, $G \times \mathbb{Z}/l\mathbb{Z}$-equivariantly, with h-Fix$(\pi_l(\phi))$. So we may write the decomposition as

$$\omega_0^{G\times\mathbb{Z}/k\mathbb{Z}}(\text{h-Fix}(\pi_k(\phi))) = \bigoplus_{l \mid k} \omega_0^G(\mathbf{E}_G(\mathbb{Z}/l\mathbb{Z}) \times_{\mathbb{Z}/l\mathbb{Z}} \text{h-Fix}(\pi_l(\phi))).$$

Following the argument used in the topological case, we see that the components of h-$L(\pi_k(\phi), X^k)$ are the Dold indices h-$\mathcal{D}^l(\phi, X)$.

Lemma 6.6. *The homotopy fixed-point set* h-Fix$(\pi_k(\phi))$ *of* $\pi_k(\phi)$ *is* G*-homotopy equivalent to the homotopy fixed-point set* h-Fix(ϕ^k) *of the iterate* $\phi^k : X \to X$. *Moreover, the action of the generator* 1 *of* $\mathbb{Z}/k\mathbb{Z}$ *on* h-Fix$(\pi_k(\phi))$ *corresponds to the operation of* ϕ_**, as a* G*-homotopy equivalence, on* h-Fix(ϕ^k).

Proof. We can think of an element of h-Fix$(\pi_k(\phi))$ as a k-tuple $(\gamma_1,\ldots,\gamma_k)$ of paths $\gamma_i : [0,1] \to X$ such that $\gamma_i(1) = \phi(\gamma_{i-1}(0))$ (for $i \in \mathbb{Z}/k\mathbb{Z}$). Concatenation of the paths γ_k, $\phi\circ\gamma_{k-1},\ldots,\phi^{k-1}\circ\gamma_1$ gives a path from $\gamma_k(0)$ to $\phi^k(\gamma_k(0))$, so an element of h-Fix(ϕ^k). This defines a map $\lambda :$ h-Fix$(\pi_k(\phi)) \to$ h-Fix(ϕ^k).

In the opposite direction, given $\gamma \in$ h-Fix(ϕ^k), we may write it as a similar concatenation of paths $\delta_k,\delta_{k-1},\ldots,\delta_1$. Put $\gamma_k = \phi^k\circ\delta_k$, $\gamma_{k-1} = \phi^{k-1}\circ\delta_{k-1}$, $\ldots,\gamma_1 = \phi\circ\delta_1$. This defines an element of h-Fix$(\pi_k(\phi))$ and we have thus constructed a map $\mu :$ h-Fix$(\phi^k) \to$ h-Fix$(\pi_k(\phi))$.

Now the composition $\lambda\circ\mu :$ h-Fix$(\phi^k) \to$ h-Fix(ϕ^k) is induced by ϕ^k, and $\mu\circ\lambda :$ h-Fix$(\pi_k(\phi)) \to$ h-Fix$(\pi_k(\phi))$ is induced by $(\pi_k(\phi))^k$. Both compositions are homotopic to the identity, by Lemma 6.1(ii). □

This leads us to a homotopy version of the maps $\sigma^{(k)}$ in Definition 3.7.

Definition 6.7. For $k \geq 1$,

$$\sigma^{(k)} : \omega_0^G(\mathbf{E}_G(\mathbb{Z}/k\mathbb{Z}) \times_{\mathbb{Z}/k\mathbb{Z}} \text{h-Fix}(\pi_k(\phi))) \to \omega_0^G(\text{h-Fix}(\phi^k))$$

is defined to be the composition of the inclusion in $\omega_0^{G\times\mathbb{Z}/k\mathbb{Z}}$(h-Fix$(\pi_k(\phi))$) of the top summand and the forgetful map to ω_0^G(h-Fix$(\pi_k(\phi))$) using the equivalence from Lemma 6.6.

Proceeding as in the case of the topological indices, we can express the homotopy Lefschetz index of ϕ^k in terms of the Dold indices.

Theorem 6.8. *Let $\phi : X \to X$ be an equivariant self-map of a G-ANR X such that, for a given $k \geq 1$, there is an open neighbourhood of* Fix(ϕ^k) *whose image under ϕ has compact closure in X. Then*

$$\text{h-}L(\phi^k, X) = \sum_{l\,|\,k} (j_l)_* \sigma^{(l)} \text{h-}\mathcal{D}^l(\phi, X) \in \omega_0^G(\text{h-Fix}(\phi^k)),$$

where $j_l :$ h-Fix$(\phi^l) \hookrightarrow$ h-Fix(ϕ^k) *is the natural inclusion.* □

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