The homotopy coincidence index

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Dedicated to Stephen Smale

Abstract. In a survey based on recent work of Koschorke, Klein and Williams, stable homotopy coincidence invariants are constructed using fibrewise methods generalizing the standard construction of the stable cohomotopy Euler class of a vector bundle.

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1. Introduction

Suppose that $e : B \to N$ and $f : B \to N$ are two maps from a compact ENR B to a smooth manifold N (without boundary). The *coincidence index* of e and f provides an algebraic count of the coincidence set

$$\mathrm{Coin}(e, f) = \{x \in B \mid e(x) = f(x)\}.$$

Classically, B and N were connected, oriented, closed manifolds of the same dimension and the coincidence index was defined as an integer. In a series of recent papers [16, 17, 18, 19, 20, 21, 22] Koschorke has studied, by differential-topological methods, when B is a closed manifold, rather than a general ENR, what we shall call the *homotopy coincidence index.* This index is an element of a certain stable homotopy group and in a range of dimensions ($\dim B < 2(\dim N - 1)$) it is the precise obstruction to the existence of a deformation of e and f to maps with empty coincidence set. The obstruction theory, which goes back to a paper of Hatcher and Quinn [12] in the 1970s, has just been reworked by Klein and Williams [14, 15]. Furthermore, their results, which use methods from fibrewise topology, have been linked in a paper of Ponto [23] to the construction in [5] of the homotopy fixed point index. In this survey we shall give a systematic account of the construction of the homotopy coincidence index in the light of these recent contributions to coincidence theory.

The construction uses fibrewise methods in an essential manner. There are two ways of setting the coincidence problem in a fibrewise framework.

1. *As a root problem.* Let $E \to B$ be the trivial bundle $B \times N \to B$. Choose e to define a preferred *null section* $z : B \to E$ mapping $x \in B$ to $(x, e(x)) \in E$, and let s be the section given by $s(x) = (x, f(x))$. Then $\mathrm{Coin}(e, f)$ is the *null-set*
$$\mathrm{Null}(s) = \{x \in B \mid s(x) = z(x)\}$$
of the section s.
2. *As an intersection problem.* Let $E \to B$ be the trivial bundle $B \times (N \times N) \to B$ with a *null sub-bundle* $Z = B \times \Delta(N) \to B$ (where Δ is the diagonal inclusion). Write s for the section $s(x) = (x, e(x), f(x))$. Then $\mathrm{Coin}(e, f)$ is the *null-set*
$$\mathrm{Null}(s) = \{x \in B \mid s(x) \in Z_x\}$$
of $s : B \to E$.

In Section 2 we shall develop the theory from the first point of view, following closely the classical theory of the stable cohomotopy Euler class of a vector bundle. Section 3 contains a number of examples, some well known in coincidence theory, but described in the fibrewise language. In Section 4, we restrict the base B to be a smooth manifold and relate the fibrewise homotopy-theoretic definition to Koschorke's construction through differential topology. Section 5 is a digression on fixed-point theory. The intersection theory, based on the work of Klein and Williams, is treated in Section 7, after a preparatory section on the *homotopy Pontrjagin-Thom map.* (The adjective 'homotopy' will be used systematically where objects, as here, are defined by a path-space construction. The usage may be unfamiliar in fixed-point theory, but is standard elsewhere, as in the term 'homotopy pull-back' employed in [12].)

The remainder of this Introduction establishes some basic notation. Given fibrewise pointed spaces $X \to B$ and $Y \to B$ over a compact ENR base B, we write
$$\omega_B^0\{X;\, Y\}$$
for the abelian group of stable fibrewise maps $X \to Y$ over B. (See, for example, [5] (Part II, Section 3).) More generally, if $A \subseteq B$ is a closed sub-ENR, the relative group
$$\omega_{(B,A)}^0\{X;\, Y\}$$
is defined in terms of homotopy classes of maps that are zero over the subspace A. We also need to consider fibrewise maps with compact supports. For an open subset $U \subseteq B$, we write
$${}_c\omega_U^0\{X_U;\, Y_U\}$$
for the group of fibrewise stable maps between the restrictions of X and Y to U that are zero outside a compact subspace of U. Using the fibrewise suspension Σ_B

over B, we extend the ω^0-theories to ω^i-cohomology theories indexed by $i \in \mathbb{Z}$, so that, for example, $\omega_B^i\{X;\, Y\}$ is identified with $\omega_B^0\{\Sigma_B^k X;\, \Sigma_B^{k+i} Y\}$ for k large. When $Y \to B$ is a trivial bundle $B \times S^i \to B$, there are natural identifications of the fibrewise groups with the reduced stable cohomotopy of an appropriate pointed space:

$$\omega_B^0\{X;\, B \times S^i\} = \tilde{\omega}^i(X/B), \quad \text{and} \quad \omega_{(B,A)}^0\{X;\, B \times S^i\} = \tilde{\omega}^i(X/(X_A \cup B)).$$

We make the following standing hypotheses. Throughout the paper, B will be a compact ENR (or finite complex) and $E \to B$ will be a locally trivial smooth fibrewise manifold over B. The fibres of $E \to B$ will be assumed to be manifolds without boundary (smooth, Hausdorff, with a countable basis, admitting a smooth partition of unity). See [5] (Part II, Section 11). We require each fibre E_x ($x \in B$) to be connected. The space A, when it appears, will be a compact sub-ENR of B.

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2. Roots

We develop the theory as an extension of the classical theory of the Euler class as an obstruction to the existence of a nowhere zero section of a real vector bundle. Notation will follow that used for the stable cohomotopy Euler class in, for example, [5, 6].

Suppose that the bundle $E \to B$ is equipped with a section $z : B \to E$, which we shall call the *null section*. (The letter 'z' is choesn to suggest 'zero'.) In a neighbourhood of any point of the base there is a local trivialization $E_U \cong U \times N$, where U is an open subspace of B and N is a smooth manifold, such that $z \,|\, U$ corresponds to a constant section $U \to U \times N$: $x \in U \mapsto (x, *)$, where $*$ is a basepoint of N. See, for example, [5] (Part II, Proposition 11.20).

Consider any section $s : B \to E$ of the fibrewise manifold $E \to B$.

Definition 2.1. We say that the section s is *null* at $x \in B$ if $s(x) = z(x)$ and write the *null-set* of s as

$$\mathrm{Null}(s) = \{x \in B \mid s(x) = z(x)\}.$$

We shall define, first, a basic obstruction $\gamma(s)$ to the existence of a homotopy from s to a section that is nowhere null. Let us write $\nu = z^* \tau_B E$ for the restriction of the fibrewise tangent bundle $\tau_B E$ to the null section B and equip ν with an inner product. Choose a fibrewise tubular neighbourhood $D(\nu) \hookrightarrow E$, over B, of the null section. Thus, in the fibre over $x \in B$ we have a closed disc $D(\nu_x) \hookrightarrow E_x$ centered on the point $z(x) \in E_x$. The Pontrjagin-Thom construction gives us a map $c_x : E_x \to \nu_x^+$ to the one-point compactification ν_x^+ of the vector space ν_x. To be precise, we collapse the complement of the disc to a point and identify the quotient $D(\nu_x)/S(\nu_x)$ of the disc modulo the sphere with ν_x^+ in the usual way by mapping a vector $v \in \nu_x$ with $\|v\| < 1$ to $(1 - \|v\|^2)^{-1/2} v$. These maps c_x for $x \in B$ give a global fibrewise map $c : E \to \nu_B^+$. to the fibrewise one-point compactification ν_B^+ of the vector bundle ν.

Definition 2.2. Any section $\sigma : B \to \nu_B^+$ determines a fibrewise pointed map

$$B \times S^0 \to \nu_B^+$$

mapping the basepoint $(x, 1)$ in the fibre at $x \in B$ to the basepoint at infinity and the point $(x, -1)$ to $\sigma(x)$. The stable class of the map determined in this way by the composition $c \circ s : B \to \nu_B^+$ will be called the *Euler index* of the section s, written as

$$\gamma(s) \in \omega_B^0\{B \times S^0;\, \nu_B^+\} = \tilde{\omega}^0(B^{-\nu})$$

in the reduced stable cohomotopy of the Thom space $B^{-\nu}$ of the virtual bundle $-\nu$. Elementary considerations show that the index is independent of the choices made.

The example of Nielsen fixed-point theory suggests the next definition.

Definition 2.3. The *homotopy null-set*, h-Null(s), of the section s is defined to be the space of pairs (x, α) where $x \in B$ and $\alpha : [0, 1] \to E_x$ is a continuous path in the fibre from $\alpha(0) = z(x)$ to $\alpha(1) = s(x)$, topologized as a subpace of the path-space of E. The space h-Null(s) is fibred over B by the projection π to the first factor, and the fibrewise space $\pi :$ h-Null$(s) \to B$ is locally fibre homotopy trivial. (To check local homotopy triviality, it is enough to look at a trivial bundle $E = U \times N \to U$ with $z(x) = (x, *)$ and $s(x) = (x, f(x))$, where $f : U \to N$ maps into a coordinate chart $\mathbb{R}^n \subseteq N$. A fibre homotopy trivialization is then easily constructed using straight line paths in $\mathbb{R}^n$ to join points in the chart.) There is an inclusion Null$(s) \hookrightarrow$ h-Null(s) mapping x to (x, α), where α is the constant path at $z(x) = s(x)$.

We now refine the construction of the Euler index by lifting from ν_B^+ to the fibrewise Thom space h-Null$(s)_B^{\pi^*\nu}$ of the pull-back of ν to h-Null(s). This fibrewise Thom space over B is pointed fibre homotopy trivial, with the fibre at $x \in B$ the Thom space of the pull-back of ν_x to the fibre of h-Null(s). It is the fibrewise smash product of ν_B^+ and h-Null$(s)_{+B}$ (the pointed fibrewise space h-Null$(s) \sqcup B$ obtained by adjoining a disjoint basepoint to each fibre).

Definition 2.4. The *homotopy Euler index*

$$\text{h-}\gamma(s) \in \omega_B^0\{B \times S^0;\, \text{h-Null}(s)_B^{\pi^*\nu}\} = \omega_B^0\{B \times S^0;\, \nu_B^+ \wedge_B \text{h-Null}(s)_{+B}\}$$

of the section s is defined as follows.

Recall that we have taken a fibrewise tubular neighbourhood $D(\nu) \hookrightarrow E$ of the null section. Choose an open neighbourhood W of Null(s) such that $\overline{W} \subseteq s^{-1}(D(\nu))$. By compactness there exists a real number ϵ with $0 < \epsilon < 1$ such that for $x \in \overline{W} - W$ the point $s(x)$ lies outside the closed disc bundle $D_\epsilon(\nu)$ of radius ϵ and centre 0 in $D(\nu)$. We can then define a section σ of h-Null$(s)_B^{\pi^*\nu}$ mapping

$x \in B$ to the basepoint in the fibre either if $x \notin W$ or if $x \in W$ and $s(x) \notin D_\epsilon(\nu)$, while if $s(x) \in D_\epsilon(\nu)$, say $s(x) = (z(x), v)$ with $v \in \nu_x$, we set $\sigma(x) = [\alpha, v/\epsilon]$, where $\alpha : [0,1] \to E_x$ is the path $\alpha(t) = (z(x), tv)$ from $z(x)$ to $s(x)$. The section σ extends to a fibrewise pointed map on $B \times S^0$, and this gives the fibrewise stable map $\text{h-}\gamma(s) : B \times S^0 \to \text{h-Null}(s)_B^{\pi^*\nu}$. Again it is straightforward to check that this class is independent of the choices made.

By construction $\text{h-}\gamma(s)$ lifts $\gamma(s)$.

Lemma 2.5. *The projection*

$$\pi_* : \omega_B^0\{B \times S^0;\, \text{h-Null}(s)_B^{\pi^*\nu}\} \to \omega_B^0\{B \times S^0;\, \nu_B^+\}$$

maps $\text{h-}\gamma(s)$ *to* $\gamma(s)$. □

It is also clear that the indices behave well under pull-backs.

Lemma 2.6. *Let $\phi : B' \to B$ be a continuous map from a compact ENR B' to B. The sections z and s lift to sections of the pull-back $\phi^* E \to B'$ and*

$$\text{h-}\gamma(\phi^* s) = \phi^* \text{h-}\gamma(s) \in \omega_{B'}^0\{B' \times S^0;\, \text{h-Null}(\phi^* s)_{B'}^{\pi^*\phi^*\nu}\},$$

where π is written, also, for the projection $\text{h-Null}(\phi^* s) \to B'$. □

We interrupt the generalities to define the coincidence invariant.

Definition 2.7. Let e, $f : B \to N$ be two maps, as in the Introduction, from the compact ENR B to a manifold N. Write $E \to B$ for the trivial bundle $B \times N \to B$, and let z and s be the sections given by $z(x) = (x, e(x))$ and $s(x) = (x, f(x))$ for $x \in B$. Then Null(s) is the *coincidence set* $\text{Coin}(e, f) = \{x \in B \mid e(x) = f(x)\}$ and h-Null(s) is naturally identified with the *homotopy coincidence set* h-Coin(e, f) consisting of the pairs (x, α), where $x \in B$ and $\alpha : [0,1] \to N$ is a continuous path from $e(x) = \alpha(0)$ to $f(x) = \alpha(1)$. We shall call

$$\text{h-}\gamma(s) \in \omega_B^0\{B \times S^0;\, (e^*\tau N)_B^+ \wedge_B \text{h-Coin}(e, f)_{+B}\}$$

the *homotopy coincidence index* of e and f. As it stands, this definition is not symmetric in e and f; we shall return to this point in Section 7.

Remark 2.8. Suppose, more generally, that the compact ENR B is the total space of a fibre bundle $B \to X$ over a compact ENR X and that $N \to X$ is a locally trivial fibrewise manifold. Consider two fibrewise maps e, $f : B \to N$ over X. Take E to be the pull-back $B \times_X N$ over B and lift e and f to sections z and s of $E \to B$. Then $\text{h-}\gamma(s)$ is the fibrewise homotopy coincidence index studied (under a different name) in [9].

The indices can be localized to a neighbourhood of the null-set.

Definition 2.9. Let $U \subseteq B$ be an open subspace such that the set $\text{Null}(s) \cap U$ is compact. By choosing the open set W in the construction (Definition 2.4) above such that $\overline{W} \subseteq U$, we obtain well-defined *localized Euler indices* in fibrewise stable homotopy with compact supports:

$$\gamma(s \,|\, U) \in {}_c\omega^0_U\{U \times S^0;\, \nu^+_U\}$$

and

$$\text{h-}\gamma(s \,|\, U) \in {}_c\omega^0_U\{U \times S^0;\, \text{h-Null}(s)^{\pi^*\nu}_U\}.$$

Let us write i_U for both of the natural (Gysin) maps

$${}_c\omega^0_U\{U \times S^0;\, \nu^+_U\} \to \omega^0_B\{B \times S^0;\, \nu^+_B\}$$

and

$${}_c\omega^0_U\{U \times S^0;\, \text{h-Null}(s)^{\pi^*\nu}_U\} \to \omega^0_B\{B \times S^0;\, \text{h-Null}(s)^{\pi^*\nu}_B\}$$

defined by extending a fibrewise map with compact supports in U to a map on the whole of B that is zero outside of U. The localization properties are summarized in the next lemma, proved by inspection of the definitions.

Lemma 2.10. (i) (Localization). *Suppose that U and V are open subsets of B with $U \subseteq V$ and $U \cap \text{Null}(s) = V \cap \text{Null}(s)$ compact. Then* $\text{h-}\gamma(s \,|\, U)$ *maps to* $\text{h-}\gamma(s \,|\, V)$ *under the Gysin map*

$${}_c\omega^0_U\{U \times S^0;\, \text{h-Null}(s)^{\pi^*\nu}_U\} \to {}_c\omega^0_V\{V \times S^0;\, \text{h-Null}(s)^{\pi^*\nu}_V\}.$$

In particular, if U is an open neighbourhood of $\text{Null}(s)$*, then*

$$\gamma(s) = i_U\gamma(s \,|\, U) \quad \textit{and} \quad \text{h-}\gamma(s) = i_U\text{h-}\gamma(s \,|\, U).$$

(ii) (Additivity), *Suppose that U_1 and U_2 are disjoint open subsets of B such that $U_1 \cap \text{Null}(s)$ and $U_2 \cap \text{Null}(s)$ are compact. Then*

$$i_{U_1 \cup U_2}\gamma(s \,|\, U_1 \cup U_2) = i_{U_1}\gamma(s \,|\, U_1) + +i_{U_2}\gamma(s \,|\, U_2)$$

and

$$i_{U_1 \cup U_2}\text{h-}\gamma(s \,|\, U_1 \cup U_2) = i_{U_1}\text{h-}\gamma(s \,|\, U_1) + +i_{U_2}\text{h-}\gamma(s \,|\, U_2). \qquad \square$$

Remark 2.11. For simplicity, we have assumed that the section s is globally defined on B. However, the indices $\gamma(s \,|\, U)$ and $\text{h-}\gamma(s \,|\, U)$ can evidently be defined for a section s that is defined only on the subspace U, provided that $\text{Null}(s) = \{x \in U \mid s(x) = z(x)\}$ is compact, with $\text{h-Null}(s)$ defined as the space, over U, of pairs (x, α) with $x \in U$ and α a path in E_x from $z(x)$ to $s(x)$.

In order to do obstruction theory we introduce, as a generalization of the classical relative Euler class, relative Euler indices.

Definition 2.12. Let $A \subseteq B$ be a closed sub-ENR. Suppose that $\text{Null}(s)$ is disjoint from A. By modifying the construction in Definition 2.4 so as to take W with $\overline{W} \subseteq B - A$, we obtain a section σ of $\text{h-Null}(s)^{\pi^*\nu}_B$ which is zero over A. The associated stable map defines the *relative Euler indices*

$$\text{h-}\gamma(s; A) \in \omega^0_{(B,A)}\{B \times S^0;\, \text{h-Null}(s)^{\pi^*\nu}_B\}.$$

and

$$\gamma(s; A) = \pi_*\text{h-}\gamma(s; A) \in \omega^0_{(B,A)}\{B \times S^0;\, \nu^+_B\} = \tilde{\omega}^0((B, A)^{-\nu})$$

(the stable cohomotopy group of the relative Thom space). Notice that these coincide with the localized indices $\gamma(s \mid B - A)$ and h-$\gamma(s \mid B - A)$.

Naturality follows directly from the definition.

Lemma 2.13. *Let $\phi : (B', A') \to (B, A)$ be a map of compact ENR pairs. Then, for a section s which is nowhere null on A, we have* h-$\gamma(\phi^* s; A') = \phi^*$h-$\gamma(s; A)$. □

The Euler indices depend only on the homotopy class of the section s in the sense that we now formulate.

Proposition 2.14. (Homotopy invariance). *Consider a closed sub-ENR A of B. Suppose that s_t, $0 \le t \le 1$, is a homotopy through sections of $E \to B$ such that* $\mathrm{Null}(s_t) \cap A = \emptyset$ *for each t. The homotopy determines, up to homotopy, a fibre homotopy equivalence* h-Null$(s_0) \to$ h-Null(s_1) *over B, inducing an isomorphism*

$$\omega^0_{(B,A)}\{B \times S^0;\ \text{h-Null}(s_0)^{\pi^*\nu}_B\} \xrightarrow{\cong} \omega^0_{(B,A)}\{B \times S^0;\ \text{h-Null}(s_1)^{\pi^*\nu}_B\}.$$

This isomorphism maps h-$\gamma(s_0; A)$ *to* h-$\gamma(s_1; A)$. *In particular,* $\gamma(s_0; A) = \gamma(s_1; A)$.

Proof. We shall establish a slightly more general result in which the null section, too, is allowed to vary through a homotopy. Suppose that we have a given null section z' of the product bundle $E' = [0,1] \times E$ over $B' = [0,1] \times B$. Consider a section s' of $E' \to B'$ such that $\mathrm{Null}(s') \cap ([0,1] \times A) = \emptyset$. The fibrewise space h-Null$(s') \to B'$ is a fibration and homotopically locally trivial, as noted in Definition 2.3. Write z_t and s_t for the restrictions of z' and s' to $\{t\} \times E \to \{t\} \times B$ (identified with $E \to B$) and $\nu_t = z_t^* \tau_B E$. Thus we obtain, up to homotopy, a fibre homotopy equivalence h-Null$(s_0) \to$ h-Null(s_1) and a bundle isomorphism $\nu_0 \to \nu_1$. The index

$$\text{h-}\gamma(s'; [0,1] \times A) \in \omega^0_{[0,1]\times(B,A)}\{[0,1] \times B \times S^0;\ \text{h-Null}(s')^{\nu'}_{B'}\},$$

where $\nu' = (z')^* \tau_{B'} E'$, restricts to both h-$\gamma(s_0; A)$ and h-$\gamma(s_1; A)$. And each restriction map from the stable homotopy group over $[0,1] \times (B, A)$ to the stable homotopy group over $\{t\} \times (B, A)$, for $0 \le t \le 1$, is an isomorphism.

By specializing to the case in which $z_t = z$ is constant, we deduce the proposition as stated. □

Remark 2.15. There is a corresponding local version. Suppose that U is an open subspace of B and that s_t is a homotopy through sections of $E \to B$ such that the intersection of $[0,1] \times U$ with $\{(t, x) \in [0,1] \times B \mid s_t(x) = z(x)\}$ is compact. Then h-$\gamma(s_0 \mid U)$ maps to h-$\gamma(s_1 \mid U)$ under the isomorphism given by the fibre homotopy equivalence h-Null$(s_0) \to$ h-Null(s_1) and $\gamma(s_t \mid U)$ is constant.

If a section s which is nowhere null on the sub-ENR A is homotopic through sections that coincide with s on A to a section that is nowhere null, then it follows from homotopy invariance, Lemma 2.14, and naturality, Lemma 2.13, that h-$\gamma(s; A) = 0$. In a stable range the homotopy Euler index is the precise obstruction.

Proposition 2.16. *Suppose that B is a finite complex of dimension $< 2(\dim \nu - 1)$. Then s is homotopic, relative to A, to a nowhere null section of $E \to B$ if and only if* $\text{h-}\gamma(s; A) = 0$.

Proof. This is a special case of Proposition 7.4 to appear later. □

The homotopy invariance of the Euler indices has an important consequence which we shall call *Jiang invariance*, by analogy with the corresponding result in fixed-point theory (see Section 5). Consider the fibrewise loop-space $\Omega_B E$, with fibre $\Omega(E_x, z(x))$ over $x \in B$. Concatenation of paths gives a fibrewise map

$$\Omega_B E \times_B \text{h-Null}(s) \to \text{h-Null}(s).$$

(A loop in E_x based at $z(x)$ is followed by a path from $z(x)$ to $s(x)$.) We also have, up to homotopy, a fibrewise (reversed) *monodromy* map

$$\mu : \Omega_B E \to \text{Aut}(\nu)$$

over B, where $\text{Aut}(\nu)$ is the automorphism bundle of the vector bundle ν over B. Indeed, the pull-back of the fibrewise tangent bundle $\tau_B E$ to $[0,1] \times \Omega_B E$ by the evaluation map $(t, \omega) \mapsto \omega(t) \in E$ is isomorphic to the pull-back of ν from B by a map which is the identity over $\{0\} \times \Omega_B E$. Fixing such an isomorphism, which is unique up to homotopy, we get maps $(\tau_B E)_{\omega(t)} \to \nu_{\omega(0)}$ for $0 \le t \le 1$. Taking $t = 1$, we get the monodromy map $\mu(\omega) : \nu_{\omega(0)} = (\tau_B E)_{\omega(1)} \to \nu_{\omega(0)}$ around the loop ω. Concatenation and the monodromy together give an action (on the left) of the group $\pi_0(\Gamma(\Omega_B E))$ of homotopy classes of sections of $\Omega_B E$ on the group $\omega^0_B\{B \times S^0;\, \text{h-Null}(s)^{\pi^*\nu}_B\}$.

Proposition 2.17. (Jiang invariance). *Consider a globally defined section s of $E \to B$. The homotopy Euler index*

$$\text{h-}\gamma(s) \in \omega^0_B\{B \times S^0;\, \text{h-Null}(s)^{\pi^*\nu}_B\}$$

is fixed under the action of the group $\pi_0(\Gamma(\Omega_B E))$ of homotopy classes of sections of $\Omega_B E$.

Proof. In view of the symmetry of the index construction in the sections s and z, discussed in connection with Lemma 7.9, the Jiang invariance is essentially equivalent to the homotopy invariance of the index as stated in Proposition 2.14 (when $A = \emptyset$). We shall deduce it here from the result obtained in the course of the proof of that proposition, with $A = \emptyset$.

Suppose that ω is a section of $\Omega_B E$. This time we specialize by defining the null section z' as $z'(t, x) = \omega(x)(t)$ for $(t, x) \in [0,1] \times B$ and the section s' as the constant lift of s: $s'(t, x) = s(x)$ The indices $\text{h-}\gamma(s_0)$ and $\text{h-}\gamma(s_1)$ correspond under the action of ω. But $s_0 = s = s_1$. □

The homotopy null-set $\text{h-Null}(s)$ is a disjoint union of path-components. We introduce notation for the corresponding decomposition of the Euler indices.

Definition 2.18. Let $\mathcal{R}(s)$ denote the set of path-components of h-Null(s), with a typical element written as $a \in \mathcal{R}(s)$. For clarity we also write h-Null$^a(s)$ for the component a when it appears as a space. Thus,

$$\text{h-Null}(s) = \bigsqcup_{a\in\mathcal{R}(s)} \text{h-Null}^a(s),$$

and we have a decomposition of the index h-$\gamma(s)$ as a sum of terms

$$\text{h-}\gamma^a(s) \in \omega^0_B\{B \times S^0;\ \text{h-Null}^a(s)^{\pi^*\nu}_B\},$$

indexed by $a \in \mathcal{R}(s)$, with h-$\gamma^a(s) = 0$ for all but finitely many a. We define

$$\gamma^a(s) = \pi_*\text{h-}\gamma^a(s) \in \omega^0_B\{B \times S^0;\ \nu^+_B\} = \tilde{\omega}^0(B^{-\nu}),$$

so that $\gamma^a(s)$ is also zero for all but finitely many $a \in \mathcal{R}(s)$ and

$$\gamma(s) = \sum_{a\in\mathcal{R}(s)} \gamma^a(s).$$

There are similar decompositions of the relative indices h-$\gamma(s;A)$ and $\gamma(s;A)$ if Null$(s) \cap A = \emptyset$.

The set $\mathcal{R}(s)$ can be described as follows. Suppose that B is connected and choose a basepoint $* \in B$. The fibre, F say, of h-Null(s) at $*$ is the space of paths $\alpha : [0,1] \to E_*$ from $z(*)$ to $s(*)$. As we have already observed in the global setting, concatenation of paths defines a free, transitive action of the fundamental group $\pi_1(E_*, z(*))$ on $\pi_0(F)$. There are also compatible monodromy actions of $\pi_1(B,*)$ on the group $\pi_1(E_*, z(*))$ and on the $\pi_1(E_*, z(*))$-set $\pi_0(F)$.

Lemma 2.19. *In the situation described in the text, $\mathcal{R}(s)$ is identified with the set of $\pi_1(B,*)$-orbits of $\pi_0(F)$. By choosing a class $[\alpha] \in \pi_0(F)$, we may express the action of $g \in \pi_1(B,*)$ on $\pi_0(F)$ as:*

$$q[\alpha] \mapsto ((g\cdot q)\sigma(g)^{-1})[\alpha] \quad \textit{for } q \in \pi_1(E_*, z(*)),$$

where $\sigma : \pi_1(B,*) \to \pi_1(E_*, z(*))$ is a 1-cocycle (satisfying $\sigma(gh) = \sigma(g)(g\cdot\sigma(h))$ for g, $h \in \pi_1(B,*)$). As $[\alpha]$ varies, the corresponding cocycles run through the homology class of σ. □

In terms of this description, $\pi_0(\Gamma(\Omega_B E))$ acts on $\mathcal{R}(s)$ through the evaluation homomorphism $\pi_0(\Gamma(\Omega_B E)) \to \pi_0(\Omega(E_*, z(*))) = \pi_1(E_*, z(*))$.

The indices behave well with respect to products. Given two sets of root problem data $E_i \to B_i$, z_i, $s_i : B_i \to E_I$, $i = 1, 2$, we can form the product $E = E_1 \times E_2 \to B = B_1 \times B_2$ with the null section $z = z_1 \times z_2$ and a section $s = s_1 \times s_2$. Then we can identify h-Null(s) with the product h-Null$(s_1)\times$h-Null(s_2) and the normal bundle ν with $\nu_1 \times \nu_2$, where $\nu_i = z_i^*\tau_{B_i}E_i$.

Lemma 2.20. *The homotopy Euler index*

$$\text{h-}\gamma(s) \in \omega^0_B\{B \times S^0;\ \text{h-Null}(s)^{\pi^*\nu}_B\}$$

is equal to the product h-$\gamma(s_1) \times$ h-$\gamma(s_2)$. *More precisely, we may make the identification: $\mathcal{R}(s) = \mathcal{R}(s_1) \times \mathcal{R}(s_2)$, and then* h-$\gamma^{a_1\times a_2}(s) =$ h-$\gamma^{a_1}(s_1) \times$ h-$\gamma^{a_2}(s_2)$. □

The construction of the basic Euler index $\gamma(s; A)$ for a section s that is nowhere null on a subspace $A \subseteq B$ follows closely the definition of the classical relative Euler class for a section of a vector bundle that is nowhere zero on a subspace. The precise relationship, stated below, is an immediate consequence of the definitions.

Proposition 2.21. *Suppose that E is the total space of a finite-dimensional real vector bundle ξ over B and that z is the zero section of the vector bundle, so that $\nu = \xi$.*

(i). *Suppose that $U \subseteq B$ is open and that s is a section defined on U with compact null-set. Then*

$$\gamma(s \,|\, U) \in {}_c\omega^0(U;\, -\nu)$$

is the stable cohomotopy Euler class with compact supports as defined in [6].

(ii). *Suppose that $A \subseteq B$ is a closed sub-ENR, that s is a section defined over B with* $\mathrm{Null}(s) \cap A = \emptyset$. *Then $\gamma(s; A)$ is equal to the relative Euler class $\gamma(\xi; s) \in \tilde{\omega}^0((B, A)^{-\xi})$ as defined in, for example,* [5]. *It depends only on the restriction of s to A, and, if $A = \emptyset$, then $\gamma(s)$ is equal to the (absolute) stable cohomotopy Euler class $\gamma(\xi)$ of the vector bundle ξ.* □

When E is a vector bundle as in Proposition 2.21, h-Null$(s) \to B$ is clearly fibre-homotopy equivalent to the trivial bundle $B \to B$. So h-$\gamma(s)$ contains no additional information in this case, and Proposition 2.16 specializes to the classical obstruction theory for the existence of a nowhere zero section of a vector bundle.

Any section of a vector bundle is homotopic to the zero section. In the general case, the Euler index of a section s which is homotopic to the null section z is given by a stable cohomotopy Euler class.

Proposition 2.22. *Consider a general bundle $E \to B$ with null section z and the section $s = z$. The Euler index $\gamma(z)$ is equal to the stable cohomotopy Euler class $\gamma(\nu)$ of the vector bundle ν and* h-$\gamma(z)$ *is its image under the injective map*

$$\omega^0_B\{B \times S^0;\, \nu^+_B\} \to \omega^0_B\{B \times S^0;\, (\Omega_B E)^{\pi^*\nu}_B\}$$

induced by the inclusion of the basepoint section in the fibrewise loop space $\Omega_B E =$ h-Null(z). *In particular, $\gamma^a(z) = 0$ if $a \in \mathcal{R}(z)$ is non-trivial.*

Proof. Following through the construction in Definition 2.4, we have Null$(s) = B$, $W = B$ and $\sigma(x)$, for $x \in B$, is equal to $[\alpha, 0]$, where α is the constant path at $z(x)$. □

3. Some examples

We begin by examining some consequences of the homotopy invariance of the homotopy Euler index. The hypotheses and notation of Section 2 are retained: z is a distinguished null section of the fibrewise manifold $E \to B$ and s is an arbitrary section. Recall that $\pi_0(\Gamma(\Omega_B E))$ acts on the set $\mathcal{R}(s)$ of components of h-Null(s).

Lemma 3.1. *Suppose that a, $b \in \mathcal{R}(s)$ lie in the same $\pi_0(\Gamma(\Omega_B E)))$-orbit. Then $\gamma^a(s) = u\gamma^b(s)$ for some unit $u \in \omega^0(B)$, and* h-$\gamma^a(s) = 0$ *if and only if* h-$\gamma^b(s) = 0$.

Proof. This follows directly from Jiang invariance (Proposition 2.17). The action on $\gamma(s)$ is given by the monodromy map μ: the induced automorphism of $\tilde{\omega}^0(B^{-\nu})$ is multiplication by an element in the image of the J-homomorphism from the real KO-group $K\mathrm{O}^{-1}(B)$ to the group of units in $\omega^0(B)$. □

The first application generalizes the main result (Theorem 1.4) of [11].

Proposition 3.2. *Suppose that, for each $x \in B$, the evaluation homomorphism $\pi_0(\Gamma(\Omega_B E))) \to \pi_1(E_x, z(x))$ is non-trivial. Then the stable cohomotopy Euler class, $\gamma(\nu)$, of $\nu = z^*\tau_B E$ vanishes and, hence,* h-$\gamma(z) = 0$.

Proof. There is no loss of generality in assuming that B is connected. We take $z = s$, and then the description of $\mathcal{R}(z)$ in Lemma 2.19 simplifies to identify $\mathcal{R}(s)$ with the set of orbits of the action of $\pi_1(B, *)$ on the group $\pi_1(E_*, z(*))$. The single element orbit $\{1\}$ corresponds to the trivial class, a say. By assumption, there is another class $b \neq a$ to which we may apply Lemma 3.1. But according to Proposition 2.22, we have $\gamma^a(a) = \gamma(\nu)$ and $\gamma^b(z) = 0$. Hence $\gamma^a(s) = 0$. □

The next result is an immediate corollary of Lemma 3.1.

Proposition 3.3. *Suppose that $\pi_0(\Gamma(\Omega_B E))$ acts transitively on the set of components $\mathcal{R}(s)$ of* h-Null(s)*. Then,*

(i) *either $\gamma^a(s) = 0$ for all $a \in \mathcal{R}(s)$ or $\gamma^a(s) \neq 0$ for all $a \in \mathcal{R}(s)$;*
(ii) *either* h-$\gamma^a(s) = 0$ *for all $a \in \mathcal{R}(s)$ or* h-$\gamma^a(s) \neq 0$ *for all $a \in \mathcal{R}(s)$.*

In particular, if $\mathcal{R}(s)$ is infinite, then h-$\gamma^a(s) = 0$ *and $\gamma^a(s) = 0$ for all $a \in \mathcal{R}(s)$.* □

We seek conditions for $\pi_0(\Gamma(\Omega_B E)))$ to act transitively on $\mathcal{R}(s)$.

Lemma 3.4. *Suppose that B is connected and the evaluation map $\pi_0(\Gamma(\Omega_B E)) \to \pi_1(E_x, z(x))$ is surjective for each $x \in B$. Then $\pi_0(\Gamma(\Omega_B E))$ acts transitively on $\mathcal{R}(s)$.* □

This is true, in particular, if $E \to B$ is trivial as a pointed bundle. In that case, we may refine Lemma 3.1 to the following more precise statement.

Proposition 3.5. *Let N be a connected closed manifold. Suppose that $E = B \times N \to B$ is the product bundle over a connected base and that $z(x) = (x, *)$, where $* \in N$ is a basepoint. Then $\pi_1(N, *)$ acts transitively on $\mathcal{R}(s)$ and there is a monodromy action*

$$w : \pi_1(N, *) \to \{1, -1\},$$

given by the orientation of loops. We have

$$\gamma^{\alpha \cdot a}(s) = w(\alpha)\gamma^a(s)$$

for $a \in \mathcal{R}(s)$, $\alpha \in \pi_1(N, *)$.

Proof. The fibrewise loop space $\Omega_B E$ is the product $B \times \Omega N$, and we have a split inclusion $\pi_1(N, *) = \pi_0(N) \to \pi_0(\Gamma(\Omega_B E))$ given by the constant sections. For a loop $\alpha \in \pi_1(N)$, the unit in $\omega^0(B)$ determined by the monodromy is $+1$ or -1 according as the bundle $\alpha^*\tau N$ over the circle is orientable or not. The homomorphism w, regarded as an element of $H^1(N; \mathbb{Z}/2\mathbb{Z})$ is the first Stiefel-Whitney class $w_1(\tau N)$. □

Consider more generally the case of classical coincidence theory in which E is a trivial bundle $B \times N \to B$, where N is a connected closed manifold, so that the null section z has the form $z(x) = (x, e(x))$ for some map $e : B \to N$, but that e is not necessarily constant. The evaluation $\pi_0(\Gamma(\Omega_B E)) \to \pi_1(E_*, z(*))$ at $* \in B$ is surjective, for any base B, if N is a so-called Jiang space. For the condition that N be a Jiang space can be formulated as precisely this property for the case that $B = N$ and z is the diagonal section $N \to N \times N$. One may then think of $\Omega_B E \to B$ as the evaluation at $1 \in \mathbb{T}$: $\mathrm{map}(\mathbb{T}, N) \to N$ on the free loop space of maps from the circle group $\mathbb{T}$ to N. If the base B is simply-connected, then a weaker condition will suffice, namely that the space of sections of the pull-back of $\mathrm{map}(\mathbb{T}, N) \to N$ to the universal cover $\tilde{N}$ of N should map onto $\pi_1(N, *)$. Homogeneous spaces and lens spaces provide well-known examples.

Example 3.6. *Suppose that G is a connected compact Lie group, $H \leq G$ is a connected closed subgroup, and $\Gamma \leq G$ is a finite subgroup that acts freely (on the left) on G/H. Let $N = \Gamma\backslash G/H$, and let $E = B \times N \to N$ be the trivial bundle equipped with the null section $z(x) = (x, e(x))$.*

Suppose that either (i) *Γ is a subgroup of some maximal torus T in G or* (ii) *$e : B \to N$ lifts to a map $\tilde{e} : B \to G/H$. Then the evaluation map $\pi_0(\Gamma(\Omega_B E)) \to \pi_1(E_x, z(x))$ is surjective for each $x \in B$.*

An example of type (ii) can be found in [22] Corollary 3.5.

Proof. (i). For $\gamma \in \Gamma$, let $\Omega_\gamma T$ be the space of paths from 1 to γ in T. We have a map $\Omega_\gamma T \to \Gamma(\Omega_B E)$ sending $\alpha \in \Omega_\gamma T$ to the section $x \mapsto (x, \alpha_x)$, where $\alpha_x : [0, 1] \to N$ is given by $\alpha_x(t) = \alpha(t)e(x)$. (Notice that T acts on N, $\alpha_x(0) = e(x)$ and $\alpha_x(1) = \gamma e(x) = e(x)$.) Since $\pi_1(T)$ maps onto $\pi_1(G)$, we see that $\pi_0(\Omega_\gamma T)$ maps onto the γ-coset of $\pi_1(G)$ in $\pi_1(N)$:

$$1 \to \pi_1(G) = \pi_1(G/H) \to \pi_1(N) \to \Gamma \to 1.$$

(If Γ is trivial, the argument is easier. We have a surjective bundle map

$$B \times G \to E = B \times G/H \quad : \quad (x, g) \mapsto (x, ge(x)),$$

which maps the constant section $x \mapsto (x, 1)$ to z. So we have a map $\Omega G \to \Gamma(\Omega_B E)$. Since H is connected, $\pi_1(G)$ maps onto $\pi_1(G/H)$, and $\pi_0(\Gamma(\Omega_B E))$ maps surjectively to $\pi_1(E_x, z(x))$ for any $x \in B$.)

(ii). For $\gamma \in \Gamma$, let $\Omega_\gamma G$ be the space of paths in G from 1 to γ. Now we have a map $\Omega_\gamma G \to \Gamma(\Omega_B E)$ sending α to the section $x \mapsto (x, \alpha_x)$, where $\alpha_x(t) = [\alpha(t)\tilde{e}(x)]$. □

The next result is related to Proposition 3.5 as the homotopy invariance of the Euler index is related to Jiang invariance.

Proposition 3.7. *Suppose that $E = B \times N \to B$ is the product bundle over a connected base B, with fibre a connected closed manifold N, equipped with a null section z and a section s that is constant, that is, $s(x) = (x, *)$ for some basepoint $* \in N$. Then $\gamma^a(s)$ is independent of $a \in \mathcal{R}(s)$.*

Proof. We interchange the rôles of z and s in the proof of Proposition 3.5. Let $\alpha : [0,1] \to N$ be a loop at $\alpha(0) = \alpha(1) = *$. It defines sections s_t of $E \to B$, for $0 \le t \le 1$, by $s_t(x) = (x, \alpha(t))$, with $s_0 = s_1 = s$. By Proposition 2.14, $\gamma(s)$ is fixed by the self-map $\text{h-Null}(s)_B^{\pi^*\nu}$ induced by concatenation of paths with α (on the right) and the identity on ν. □

Standard transfer techniques lead to the following generalization.

Corollary 3.8. *Suppose that there is a connected d-fold finite cover $p : \tilde{B} \to B$ such that there is a trivialization of the pull-back p^*E as $\tilde{B} \times N$ in which the section p^*s is constant. Then there exist positive integers $m_a \le d$, $(a \in \mathcal{R}(s))$, and a class $u \in \omega_B^0\{B \times S^0;\, \nu_B^+\}[1/d]$ such that*

$$\gamma^a(s) = m_a u \in \omega_B^0\{B \times S^0;\, \nu_B^+\}[1/d] \quad (a \in \mathcal{R}(s)).$$

Proof. Clearly $\text{h-Null}(p^*s) = p^*\text{h-Null}(s)$, $\text{h-}\gamma(p^*s) = p^*\text{h-}\gamma(s)$ by naturality,. and there is a surjection $\mathcal{R}(p^*s) \to \mathcal{R}(s)$. As well as the classical transfer map $p_! : \omega^0(\tilde{B}) \to \omega^0(B)$, we have a transfer

$$p_! : \omega_{\tilde{B}}^0\{\tilde{B} \times S^0;\, \text{h-Null}(p^*s)_{\tilde{B}}^{p^*\nu}\} \to \omega_B^0\{B \times S^0;\, \text{h-Null}(s)_B^{\nu}\}.$$

Now $p_!(\text{h-}\gamma(p^*s)) = p_!p^*\text{h-}\gamma(s) = p_!(1) \cdot \text{h-}\gamma(s)$.

The classes $\gamma^{\tilde{a}}(p^*s)$ for $\tilde{a} \in \mathcal{R}(p^*s)$ are all equal by Proposition 3.7; write

$$v \in \omega_{\tilde{B}}^0\{\tilde{B} \times S^0;\, (p^*\nu)_{\tilde{B}}^+\}$$

for their common value. Let m_a, for $a \in \mathcal{R}(s)$, denote the number of components of p^*a. Thus, $1 \le m_a \le d$, and

$$p_!(1) \cdot \gamma^a(s) = m_a p_!(v).$$

Since $p_!(1) - d \in \omega^0(B)$ is nilpotent, the class $p_!(1)$ is invertible in $\omega^0(B)[1/d]$ and we may set $u = (p_!(1))^{-1}p_!(v)$. This completes the proof. □

Remark 3.9. More generally, one may consider a connected fibrewise manifold $p : \tilde{B} \to B$ with fibres closed manifolds such that $p^*E \to \tilde{B}$ admits a trivialization in which p^*s is constant. Let d be the number of components and χ the Euler characteristic of the fibre. Assume that $\chi \neq 0$. Then there are positive integers $m_a \le d$ for $a \in \mathcal{R}(s)$ and a class $u \in \tilde{\omega}^0(B^{-\nu})[1/\chi]$ such that $\gamma^a(s) = m_a u \in \tilde{\omega}^0(B^{-\nu})[1/\chi]$.

Example 3.10. *Let $N = G/H$, where G is a connected Lie group and $H \leq G$ is a finite subgroup. Suppose that $E \to B$ is the trivial bundle $B \times N \to B$ over a connected base B. Then the conditions of Corollary* 3.8 *hold with d a divisor of $\# H$.*

For more information on this example the reader is referred to [26].

Proof. Writing the section s as $s(x) = (x, f(x))$, take $p : \tilde{B} \to B$ to be any connected component of the pull-back by $f : B \to G/H$ of the projection $G \to G/H$. If $\tilde{f} : \tilde{B} \to G$ is the lift of f, we have a trivialization $\tilde{B} \times G/H \to p^* E$: $(x, gH) \mapsto (x, \tilde{f}(x)gH)$ in which the constant section at $H \in G/H$ maps to $p^* s$. $\square$

Here is an explicit example in which the fibre is a homogeneous space, but only one of the indices $\gamma^a(s)$ is non-zero.

Example 3.11. Consider the maximal torus T of diagonal matrices in $G = \mathrm{SU}(l+1)$, where $l \geq 1$. The Weyl group $N_G(T)/T$ is the permutation group $\mathfrak{S}_{l+1}$. Let $H \leq N_G(T)$ be the inverse image of the alternating group $\mathfrak{A}_{l+1} \leq \mathfrak{S}_{l+1}$. Put $N = G/H$. Then $\pi_1(N) = \pi_0(H)$ is identified with $\mathfrak{A}_{l+1}$, N is orientable, and the Euler characteristic of N is equal to $(l+1)!/2$. Consider the case in which $B = N$, $E = B \times N \to B$ is the projection, and $z : B \to E$ is the diagonal map $z(x) = (x, x)$. Then $\pi_0(\text{h-Null}(z))$ is identified with the set of conjugacy classes in $\mathfrak{A}_{l+1}$, $\gamma^a(z)$ is non-zero if a is the trivial class and is zero otherwise, by Proposition 2.22.

Propositions 3.5 and 3.7 can be glued together in the following manner.

Proposition 3.12. *Consider a trivial bundle $E = B \times N \to B$ over a connected and simply connected base B, with* $\dim N > 1$. *Write $z(x) = (x, e(x))$ and $s(x) = (x, f(x))$, where e and f are maps $B \to N$. Suppose that there are non-empty open subsets U and V of B such that $B = U \cup V$, f is constant on U and e is constant on V, with* $\mathrm{Null}(s) \cap (U \cap V) = \emptyset$. *Choose a basepoint $* \in U \cap V$. Then $\pi_1(N, e(*))$ acts transitively and freely on $\mathcal{R}(s)$ and there is a class $u \in \tilde{\omega}^0(B^{-\nu})$ such that*

$$\gamma^{\alpha \cdot a}(s) - u = w(\alpha)(\gamma^a(s) - u) \quad \textit{for } a \in \mathcal{R}(s),\ \alpha \in \pi_1(N, e(*)).$$

Proof. We have a decomposition $\gamma^a(s) = i_U \gamma^a(s \,|\, U) + i_V \gamma^a(s \,|\, V)$, where i_U and i_V are the Gysin maps, by the additivity of the index. Consider a loop $\alpha : [0,1] \to N$ at $e(*)$ representing a class in $\pi_1(N, e(*))$. Since $\dim N > 1$, we can arrange that α does not pass through $f(*)$. From the homotopy invariance of the index, we deduce that $\gamma^{\alpha \cdot a}(s \,|\, V) = w(\alpha)\gamma^a(s \,|\, V)$ for $a \in \mathcal{R}(s)$. It follows similarly, by a relative version of Jiang invariance using loops at $f(*)$ which miss $e(*)$, that $\gamma^a(s \,|\, U)$ is independent of a. $\square$

As an application we have the following generalization of [21] Theorem 1.21.

Proposition 3.13. *Suppose that $B = S^0 * A$ is the unreduced suspension of a connected compact ENR A and that $e : B \to N$ is a map to a connected closed manifold N of dimension greater than* 1. *Let $E = B \times N \to B$ be the trivial bundle with the null section z: $x \mapsto (x, e(x))$. If N has a non-trivial orientation-preserving loop,* *then the stable cohomotopy Euler class* $\gamma(e^*\tau N)$ *is zero, that is* $\gamma(\nu) = 0$, *and hence* h-$\gamma(z) = 0$.

Proof. We can think of $S^0 * A$ as the quotient of $[-1,1]\times A$ in which $\{-1\}\times A$ and $\{1\}\times A$ are each collapsed to a point. Let U and V be the quotients of $[-1,1/2)\times A$ and $(-1/2,1]\times A$ respectively. Change z by a homotopy so that it is constant on V, and choose a section s homotopic to z and constant on U. The chosen homotopy from z to s picks out a component $a \in \mathcal{R}(s)$ with $\gamma^a(s) = \gamma(\nu)$. By assumption, there is a loop α with $w(\alpha) = 1$ such that $\alpha\cdot a \neq a$, which implies that $\gamma^{\alpha\cdot a}(s) = 0$. We deduce that $\gamma(\nu) = \gamma^a(s) = \gamma^{\alpha\cdot a}(s) = 0$. □

Consider a finite-dimensional real vector bundle η over B, and suppose that s_0 and s_1 are two sections of the sphere bundle $S(\eta)$ which coincide on the closed sub-ENR $A \subseteq B$. Let $p : B\times[0,1] \to B$ be the projection. Then we have a section σ of $\pi^*\eta$ given by $\sigma(x,t) = (1-t)s_0(x) + ts_1(x)$ ($x \in B$, $t \in [0,1]$). This section is non-zero on the subspace $(B\times\{0,\,1\}) \cup (A\times[0,1])$. We can, thus, construct the relative Euler class

$$\gamma(p^*\eta;\,\sigma) \in \tilde{\omega}^0((B\times[0,1],(B\times\{0,\,1\})\cup(A\times[0,1]))^{-p^*\eta}).$$

The corresponding class $\delta(s_0,s_1) \in \tilde{\omega}^{-1}((B,A)^{-\eta})$ under the suspension isomorphism is called the *difference class* of the two sections. (See, for example, [5].) It vanishes if the two sections are homotopic (relative to A) and in a stable range it is the precise obstruction to the existence of a homotopy. We now relate this difference class to the Euler index when E is the sphere bundle $E = S(\mathbb{R}\oplus\xi)$ on the direct sum of the trivial line bundle $B\times\mathbb{R}$ and a vector bundle ξ, with the null section z given by $z(x) = (1,0) \in S(\mathbb{R}\oplus\xi_x)$. The section $-z$ defined by $(-z)(x) = (-1,0)$ is nowhere null, and clearly any section s of $E\to B$ that is nowhere null is homotopic to $-z$. If B is a finite complex with $\dim B < 2\dim\xi - 1$, then, for a section s coinciding with $-z$ on a subcomplex A, $\delta(-z,s) = 0$ if and only if s is homotopic, through a homotopy constant on A, to $-z$.

Proposition 3.14. *Let* ξ *be a non-zero real vector bundle over* B, *with an inner product, and let* $E \to B$ *be the sphere bundle* $S(\mathbb{R}\oplus\xi)$. *Take* z *to be the section* $(1,0)$, *so that* $\nu = \xi$. *Consider a section* s *such that* Null(s) *is disjoint from the closed sub-ENR* $A \subseteq B$. *Up to homotopy we may assume that* s *coincides with* $-z$ *over* A *so that the difference class* $\delta(-z,s) \in \tilde{\omega}^{-1}((B,A)^{-\mathbb{R}\oplus\xi})$ *of the two sections of the sphere bundle is defined. Then, under the suspension isomorphism,*

$$\gamma(s;A) = \delta(-z,s) \in \tilde{\omega}^0((B,A)^{-\xi}).$$

In particular, when $A = \emptyset$ *we have*

$$\gamma(s) = \delta(-z,s) = \delta(z,s) + \gamma(\xi) \in \tilde{\omega}^0(B^{-\xi}).$$

Proof. This is established by direct comparison of the definitions of the Euler index and the difference class. The Euler index is represented by the fibrewise pointed map

$$f : B\times S^0 \to S(\mathbb{R}\oplus\xi)/_B(-z)B = \xi_B^+$$

determined by the section s of $S(\mathbb{R}\oplus\xi)$, and the differennce class by

$$g : B\times\mathbb{R}^+ = B\times([0,1]/\{0,1\}) \to D(\mathbb{R}\oplus\xi)/_B S(\mathbb{R}\oplus\xi) = (\mathbb{R}\oplus\xi)^+_B,$$

where $g(x,[t]) = [(1-t)(-z)(x)+ts(x)]$ ($t\in[0,1]$, $x\in B$). Here '$/_B$' denotes the fibrewise quotient: thus the fibre of $D(\mathbb{R}\oplus\xi)/_B S(\mathbb{R}\oplus\xi)$ is the pointed space $D(\xi_x)/S(\xi_x)$. Equality is used for natural identifications (up to homotopy). One checks by inspection that g is the fibrewise suspension of f.

When $A=\emptyset$, the difference class $\delta(-z,z)$ is equal to the stable cohomotopy Euler class $\gamma(\xi)$. The identity $\delta(-z,s)=\delta(-z,z)+\delta(z,s)$ is immediate, from the difference property: $\delta(s_0,s_1)+\delta(s_1,s_2)=\delta(s_0,s_2)$ for sections s_0, s_1, s_2. □

Remark 3.15. If $\dim\xi>1$, then the fibres of $E=S(\mathbb{R}\oplus\xi)$ are simply-connected. There is more to say when ξ is a line bundle. In this case, let us write $\xi=\lambda$, so that $E=S(\lambda)\times_{\mathbb{Z}/2}S(\mathbb{C})$ (where $\mathbb{Z}/2$ acts on $\mathbb{C}$ by conjugation and on the double cover $S(\lambda)\to B$ as ±1) is a bundle of circle groups. The group of homotopy classes of sections s is identified with the cohomology group $H^1(B;\,\mathbb{Z}(\lambda))$, where $\mathbb{Z}(\lambda)$ is the system of $\mathbb{Z}$-coefficients twisted by the bundle λ. Let us write

$$\eta : H^1(B;\,\mathbb{Z}(\lambda)) \to \tilde{\omega}^0(B^{-\lambda})$$

for the (not, in general, linear) map $[s]\mapsto\delta(z,s)$. The Hurewicz homomorphism

$$h : \tilde{\omega}^0(B^{-\lambda}) \to \tilde{H}^0(B^{-\lambda};\,\mathbb{Z}) = H^1(B;\,\mathbb{Z}(\lambda))$$

sends $\eta[s]$ to $[s]$.

Now consider a section s over B. The homotopy null-set h-Null(s) is homotopy equivalent to an infinite cyclic cover of B. For each $a\in\mathcal{R}(s)$ we can find a section s_a coinciding with s in a neighbourhood of the intersection of Null(s) with h-Null$^a(s)$ and nowhere null outside that neighbourhood, so that $\gamma^a(s)=\gamma(s_a)=\delta(z,s_a)+\gamma(\lambda)$. It follows that $\gamma^a(s)$ is determined by its Hurewicz image $x\in H^1(B;\,\mathbb{Z}(\lambda))$ as $\eta(x-e(\lambda))+\gamma(\lambda)$, where $e(\lambda)=h(\gamma(\lambda))$ is the cohomology Euler class of λ.

Suppose that B is connected with a basepoint $*$. Fix an orientation of the fibre λ_* and let $w:\pi_1(B,*)\to\{\pm1\}$ be first Stiefel-Whitney class of λ. The homotopy class $[s]\in H^1(B;\,\mathbb{Z}(\lambda))$ is represented by a 1-cocycle $\sigma:\pi_1(B,*)\to\mathbb{Z}$ satisfying $\sigma(gh)=\sigma(g)+w(g)\sigma(h)$ for g, $h\in\pi_1(B,*)$. The set of components $\mathcal{R}(s)$ is identified with the set of orbits of the action of $\pi_1(B,*)$ on $\mathbb{Z}$ by $g\cdot q=w(g)q-\sigma(g)$ ($q\in\mathbb{Z}$).

It is a routine exercise to describe $\mathcal{R}(s)$ in terms of σ and to calculate $h(\gamma^a(s))$ for $a\in\mathcal{R}(s)$.

Suppose first that λ is trivial. The group $H^1(B;\,\mathbb{Z})$ is torsion-free. If $\sigma=0$, then $\mathcal{R}(s)$ is identified with $\mathbb{Z}$ and each $\gamma^a(s)$ is zero. Otherwise, the homomorphism $\sigma:\pi_1(B,*)\to\mathbb{Z}$ has image $k\mathbb{Z}$ for $k\geq1$, $\mathcal{R}(s)$ is identified with $\mathbb{Z}/k\mathbb{Z}$, $\pi_0(\Gamma(\Omega_B E))=\mathbb{Z}$ acts transitively on $\mathcal{R}(s)$, and all the indices $h(\gamma^a(s))$ are equal to x, where $kx=[s]$.

Suppose that λ is non-trivial. The torsion subgroup of $H^1(B;\,\mathbb{Z}(\lambda))$ is cyclic of order 2, generated by $e(\lambda)$. Choose an element $g\in\pi_1(B,*)$ such that $w(g)=-1$

and write $r = \sigma(g)$. Let $\sigma(\ker w) = k\mathbb{Z}$, where $k \geq 0$. Then $\mathcal{R}(s)$ is identified with the quotient of $\mathbb{Z}/k\mathbb{Z}$ by the involution $\tau : q \mapsto -q + r$. The indices are as follows.

(i) $k = 0$, r even (that is, $[s] = [z]$). Then $h(\gamma^a(s)) = e(\lambda)$ for a corresponding to the unique fixed-point of τ; otherwise $h(\gamma^a(s)) = 0$.

(ii) $k = 0$, r odd (that is, $[s] = [-z]$). For all a we have $h(\gamma^a(s)) = 0$.

(iii) $k \geq 1$ odd. Then $\#\mathcal{R}(s) = (k+1)/2$, $h(\gamma^a(s)) = x + e(\lambda)$ for a corresponding to the unique fixed-point of τ and $h(\gamma^a(s)) = 2x$ for the remaining $(k-1)/2$ components, where $kx = [s]$.

(iv) k even, r even (so that $[s]$ is divisible by 2). The involution τ has two fixed-points and $\#\mathcal{R}(s) = k/2 + 1$. The indices $h(\gamma^a(s))$ for the components corresponding to the fixed-points are x and $x + e(\lambda)$, where $kx = [s]$, and the remaining $k/2 - 1$ components have index $2x$.

(v) k even, r odd (so that $[s] + e(\lambda)$ is divisible by 2). The involution τ has no fixed-points, $\#\mathcal{R}(s) = k/2$ and each component has index $h(\gamma^a(s)) = 2x$, where $kx = [s] + e(\lambda)$.

Specific examples are examined in more detail in [9].

4. Smooth fibre bundles

When the base B is a smooth manifold, the fibrewise stable homotopy groups in which the Euler indices lie can be expressed as ordinary stable homotopy groups. This simplification depends upon the following basic result from fibrewise stable homotopy theory; see, for example, [5, 4].

Proposition 4.1. *Let B be a compact manifold with boundary. Then there is a natural equivalence*

$$\lambda_B : \omega^*_{(B,\partial B)}\{B \times X;\, F\} \xrightarrow{\cong} \omega^*\{X;\, (F, B)^{-p^*\tau B}\},$$

for any finite pointed complex X and any pointed homotopy fibre bundle $p : F \to B$ (that is, a fibrewise pointed space which is locally fibre homotopy trivial) with fibres of the homotopy type of CW complexes. More generally, for an open subset $U \subseteq B - \partial B$ there is a natural equivalence

$$\lambda_U : {}_c\omega^*_U\{B \times X;\, F\} \xrightarrow{\cong} \omega^*\{X;\, (p^{-1}U, U)^{-p^*\tau U}\},$$

The more general statement follows easily from the basic result by expressing the stable homotopy with compact supports in U as a direct limit of relative homotopy groups over $(B', \partial B')$ indexed by the compact submanifolds B' of U. When the bundle is trivial, the equivalence λ_B reduces to the Poincaré-Atiyah duality between $B/\partial B$ and $B^{-\tau B}$.

Proposition 4.2. *For a vector bundle ν over B, the isomorphism*

$$\lambda_B : \omega^*_{(B,\partial B)}\{B \times S^0;\, \nu_B^+\} \xrightarrow{\cong} \omega^*\{S^0;\, (\nu_B^+, B)^{-p^*\tau B}\} = \omega^0\{S^0;\, B^{\nu-\tau B}\}$$

is precisely the duality isomorphism $\tilde{\omega}^*((B, \partial B)^\nu) \xrightarrow{\cong} \tilde{\omega}_{-*}(B^{\nu-\tau B})$.

Throughout this section $E \to B$ will be a locally trivial smooth fibre bundle over a compact manifold B, $z : B \to E$ will be a smooth null section and s will be a continuous section. Proposition 4.1 specializes to the following equivalence.

Lemma 4.3. *Suppose that B is a smooth manifold with boundary $A = \partial B$, that $E \to B$ is a smooth fibre bundle and z is a smooth section. Then we have the identification*

$$\omega^0_{(B,\partial B)}\{B \times S^0;\, \text{h-Null}(s)_B^{\pi^*\nu}\} = \tilde{\omega}_0(\text{h-Null}(s)^{\pi^*(\nu-\tau B)}). \qquad \square$$

This allows us to describe the groups quite explicitly when the base and fibre have the same dimension.

Proposition 4.4. *Suppose that the base B and the fibres of $E \to B$ have the same dimension. Then*

$$\omega^0_{(B,\partial B)}\{B \times S^0;\, \text{h-Null}(s)_B^{\pi^*\nu}\} = \bigoplus_{a\in\mathcal{R}(s)} H_0(\text{h-Null}^a(s);\, \mathbb{Z}(a)),$$

where $\mathbb{Z}(a)$ is the system of $\mathbb{Z}$-coefficients twisted by the orientation bundle of the lift of $\nu - \tau B$ to the a-component, so that $H_0(\text{h-Null}^a(s);\, \mathbb{Z}(a))$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ according as the lift is orientable or non-orientable. $\square$

In particular, if $\dim \nu = \dim B$ and ν and τB are both oriented, then

$$\omega^0_{(B,\partial B)}\{B \times S^0;\, \text{h-Null}(s)_B^{\pi^*\nu}\} = \mathbb{Z}[\mathcal{R}(s)]$$

is the free abelian group on $\mathcal{R}(s)$. When B is a closed n-manifold and $E = B \times N$ is the trivial bundle with fibre N, then h-$\gamma(s) \in \mathbb{Z}[\mathcal{R}(s)]$ is Staecker's coincidence Reidemeister trace [25], as is the local version i_Uh-$\gamma(s \,|\, U)$.

Now suppose that the null-set $\text{Null}(s)$ is a smooth submanifold of B with normal bundle ξ. As in Definition 2.4, let $D(\nu) \hookrightarrow E$ be a fibrewise tubular neighbourhood of the null section. Then we may choose a tubular neighbourhood $D(\xi) \hookrightarrow B - \partial B$ of $\text{Null}(s)$ so small that in the tubular neighbourhood the section s takes values in $D(\nu) \subseteq E$. This means that in the neighbourhood $D(\xi)$ we may think of z and s as sections of the disc bundle $D(\nu)$; the null section z is the zero section and $\text{Null}(s)$ is the zero-set of s. The relative Euler class

$$\gamma(\nu;\, s) \in \tilde{\omega}^0((D(\xi), S(\xi))^{-\nu}) = \tilde{\omega}^0(\text{Null}(s)^{\xi-\nu})$$

of s regarded as a nowhere zero section of ν over the subspace $S(\xi)$ of $D(\xi)$ determines the homotopy Euler index.

Lemma 4.5. *In the situation described above, the relative Euler class $\gamma(\nu;\, s)$ maps to the homotopy Euler index* h-$\gamma(s;\, \partial B)$ *under the composition*

$$\tilde{\omega}^0(\text{Null}(s)^{\xi-\nu}) \xrightarrow{\cong} \tilde{\omega}_0(\text{Null}(s)^{\nu-\tau B}) \to \tilde{\omega}_0(\text{h-Null}(s)^{\pi^*(\nu-\tau B)})$$

of the duality isomorphism and the homomorphism induced by the inclusion of $\text{Null}(s)$ *into* $\text{h-Null}(s)$.

Proof. By naturality, there is no loss of generality in assuming that $B = D(\xi)$, $A = S(\xi)$. Using Proposition 2.21(ii), we may identify the relative Euler class with the index

$$\gamma(s;\, A) \in \omega^0_{(B,A)}\{B \times S^0;\, \nu_B^+\}.$$

And from Definition 2.4, the homotopy index h-$\gamma(s;\, A)$ is the image of this class under the fibrewise map

$$\nu_B^+ \to \text{h-Null}(s)_B^{\pi^*\nu}$$

induced by the section $B \to$ h-Null(s) which assigns to a point $x \in B = D(\xi)$ the linear path from $0 = z(x)$ to $s(x)$ in $D(\nu_x) \subseteq E_x$.

The proof is completed by using the natural equivalence λ_B from Proposition 4.1 and then applying Proposition 4.2 (with the homotopy equivalence $B = D(\xi) \to \mathrm{Null}(s)$). □

As a first consequence, we deduce that the topologically defined indices agree with those defined by Koschorke in the differentiable case.

Proposition 4.6. *Suppose that s is a smooth section that is nowhere null on ∂B and is transverse to z. Then* Null(s) *is a submanifold of $B - \partial B$, and the normal bundle of* Null(s) *in B is identified with the restriction of ν, so that we have an induced map*

$$\tilde{\omega}_0(\mathrm{Null}(s)^{-\tau \mathrm{Null}(s)}) \to \tilde{\omega}_0(\text{h-Null}(s)^{\pi^*(\nu - \tau B)})$$

given by the inclusion of Null(s) *in* h-Null(s). *The homotopy Euler index* h-$\gamma(s; \partial B)$ *is the image, under this homomorphism, of the fundamental class of the null-set.*

Proof. In the notation of the lemma, we have $\xi = \nu$ (restricted to Null(s)), ν restricted to the tubular neighbourhood $D(\xi)$ is isomorphic to the pull-back of $\nu \,|\, \mathrm{Null}(s)$, and s on $S(\xi)$ is effectively the diagonal inclusion of $S(\xi) = S(\nu)$ in $D(\nu)$. The relative Euler class corresponds to the identity element in $\omega^0(\mathrm{Null}(s))$, which is dual to the fundamental class. □

Remark 4.7. More generally, we may consider an open subspace $U \subseteq B - \partial B$ such that $P = U \cap \mathrm{Null}(s)$ is compact. Suppose that s is transverse to z on U. Then P is a closed submanifold with normal bundle the restriction of ν. The image of h-$\gamma(s \,|\, U)$ in $\tilde{\omega}_0(\text{h-Null}(s)^{\pi^*(\nu - \tau B)})$ is represented by $P \hookrightarrow$ h-Null(s).

There are some special cases of this situation in which the inclusion Null$(s) \hookrightarrow$ h-Null(s) is a homotopy equivalence. The simplest example is: $B = \mathbb{T}$, E is the trivial bundle of groups $B \times \mathbb{T}$, z is the section $z(x) = (x, 1)$ and s is the section $s(x) = (x, x^d)$, where $d \neq 0$.

Example 4.8. Let Π and Γ be finitely-generated torsion-free nilpotent groups, and suppose that ϵ and σ are group homorphisms $\Pi \to \Gamma$. Recall (for example from [1]) that such a group Π is functorially embedded as a co-compact discrete subgroup of a contractible nilpotent Lie group $\Pi_{\mathbb{R}}$, and then $\Pi/\Pi_{\mathbb{R}}$ is a classifying space for Π. Take $B = \Pi_{\mathbb{R}}/\Pi$, $E = B \times N$ where $N = \Gamma_{\mathbb{R}}/\Gamma$, and let z and s be the sections determined by ϵ_* and σ_*: $\Pi_{\mathbb{R}}/\Pi \to \Gamma_{\mathbb{R}}/\Gamma$. Then $\mathcal{R}(s)$ can be identified with the set of Π-orbits of the action of Π on Γ by $\pi \cdot \gamma = \epsilon(\pi)\gamma\sigma(\pi)^{-1}$. If $\mathcal{R}(s)$ is finite, then z and s are transverse sections, the inclusion of $\mathrm{Null}(s)$ into h-$\mathrm{Null}(s)$ is a homotopy equivalence and h-$\gamma(s)$ corresponds to the fundamental class of $\mathrm{Null}(s)$ under the isomorphism:

$$\tilde{\omega}_0(\mathrm{Null}(s)^{-\tau\mathrm{Null}(s)}) \cong \tilde{\omega}_0(\text{h-Null}(s)^{\pi^*(\nu-\tau B)}).$$

More information on these examples can be found in [10].

We also have a Poincaré-Hopf theorem, extending the classical theorem for vector bundles, when the null-set is finite. At an isolated point x in $\mathrm{Null}(s)$, the relative Euler class construction of Lemma 4.5 determines a *local index*

$$\gamma_x(s) \in \tilde{\omega}^0((D(\tau_x B), S(\tau_x B))^{-\nu_x}) = \omega^0\{(\tau_x B)^+;\, (\nu_x)^+\} = \tilde{\omega}_0(\{x\}^{\nu_x - \tau_x B}).$$

Proposition 4.9. (Poincaré-Hopf theorem). *Let B be a compact manifold with boundary $A = \partial B$. Suppose that s is a section of $E \to B$ which is null at just a finite number of points (in $B - \partial B$). At each $x \in \mathrm{Null}(s)$, the relative Euler class determines a local index $\gamma_x(s) \in \tilde{\omega}_0(\{x\}^{\nu_x - \tau_x B})$. Then*

$$\text{h-}\gamma(s; \partial B) = \sum_{x \in \mathrm{Null}(s)} (i_x)_* \gamma_x(s) \in \tilde{\omega}_0(\text{h-Null}(s)^{\pi^*(\nu-\tau B)}),$$

where $i_x : \{x\} \to$ h-$\mathrm{Null}(s)$ is the inclusion.

Proof. This follows at once from Lemma 4.5. □

In the classical situation in which τB and ν are oriented and of the same dimension, the local indices can be regarded as integers and the Poincaré-Hopf formula computes h-$\gamma(s) \in \mathbb{Z}[\mathcal{R}(s)]$.

There is a natural generalization of Wecken's theorem for smooth manifolds. Versions very close to that stated below can be found in [21] Corollary 3.3 and [9]; see, also, [24].

Proposition 4.10. (The smooth Wecken theorem). *Suppose that B is a smooth manifold of dimension $n \geq 3$ with boundary $A = \partial B$ and that $E \to B$ is a smooth fibre bundle with fibres of the same dimension n and that z is a smooth section. Then any section which is nowhere null on ∂B is homotopic through sections that are constant on ∂B to a section s such that* $\mathrm{Null}(s)$ *is finite and contains precisely one point in each component $a \in \mathcal{R}(s)$ such that* h-$\gamma^a(s) \neq 0$ *and no point in a component a with* h-$\gamma^a(s) = 0$.

Proof. The proof follows closely the proof of the classical Wecken theorem, as described, for example, in [13]. We shall only outline the main steps.

By transversality, since $n = \dim B$ is equal to the fibre dimension, any section that is nowhere null on ∂B is homotopic, through sections constant on ∂B, to a section s with $\mathrm{Null}(s)$ finite.

One shows next that if $\mathrm{Null}(s)$ contains two points $x \neq y$ in the same component of h-$\mathrm{Null}(s)$, then we may replace s by a section with null-set of cardinality $\#\mathrm{Null}(s) - 1$. By assumption, there is a path $\alpha : [0,1] \to B$ with $\alpha(0) = x$, $\alpha(1) = y$, such that the pull-back $\alpha^* s$ over $[0,1]$ is homotopic, by a homotopy fixed at the end-points, to $\alpha^* z$. We can certainly arrange that the path lies in $B - \partial B$ and, since $\dim B \geq 3$, we may choose α to be a smooth embedding with $\alpha(t) \notin \mathrm{Null}(s)$ for $0 < t < 1$. Now take a smoothly embedded closed n-disc $D \hookrightarrow B - \partial B$ such that the path α lies in the interior of D and $D \cap \mathrm{Null}(s) = \{x, y\}$. There is a trivialization of $E \to B$ over D as $D \times N \to D$ with z given by $x \mapsto (x, *)$ for a basepoint $* \in N$. Since $\dim N \geq 3$, we may choose a homotopy s_u, $0 \leq u \leq 1$, between $s_0 = \alpha^* s$ and $s_1 = \alpha^* z$ such that s_u maps $(0,1)$ into $N - \{*\}$ for $0 \leq u < 1$. Using this homotopy one can deform s to a section s' coinciding with s outside the interior of D with $\mathrm{Null}(s') \cap D = \alpha[0,1]$. Finally, choose a coordinate chart $\mathbb{R}^n \hookrightarrow N$ centred at $*$ and an embedded closed n-disc $D' \hookrightarrow D$ such that $\alpha[0,1]$ lies in the interior of D' and s' maps D' into the coordinate chart. Then deform s' by radial extension to a section s'' coinciding with s' outside the interior of D' such that $\mathrm{Null}(s'') \cap D'$ consists of a single point, the centre of D'.

By iterating this procedure one reduces to the case in which $\mathrm{Null}(s)$ is finite and contains at most one point in each component of h-$\mathrm{Null}(s)$.

For $a \in \mathcal{R}(s)$ such that the pull-back of $\nu - \tau B$ to h-$\mathrm{Null}^a(s)$ is not orientable we can replace a single point x at which s is null by a circle. There is a path $\alpha : [0,1] \to B$ such that $\alpha(0) = x = \alpha(1)$, the pull-back $\alpha^* s$ is homotopic to $\alpha^* z$ relative to the end-points, and α reverses the orientation of $\nu_x - \tau_x B$. We may choose α so as to give a smoothly embedded circle $C = \alpha[0,1]$, and s may be modified in a small tubular neighbourhood of C to a section s' with $C = \mathrm{Null}(s') \cap \text{h-}\mathrm{Null}^a(s')$.

Finally, suppose that either (i) the pull-back of $\nu - \tau B$ to h-$\mathrm{Null}^a(s)$ is orientable and $\mathrm{Null}(s) \cap \text{h-}\mathrm{Null}^a(s)$ consists of a single point x or (ii) the pull-back is non-orientable and $\mathrm{Null}(s) \cap \text{h-}\mathrm{Null}^a(s)$ is a circle C as described above. Fix a fibrewise tubular neighbourhood $D(\nu) \hookrightarrow E$ of the null section. Then we may choose a tubular neighbourhood in $(B - \partial B) \cap s^{-1}(D(\nu))$ of (i) $\{x\}$ or (ii) C containing no other point of $\mathrm{Null}(s)$. The index h-$\gamma^a(s)$ is an integer invariant in case (i) and an element of $\mathbb{Z}/2\mathbb{Z}$ in case (ii). It is the precise obstruction to deforming s in the tubular neighbourhood, considered as a section of the disc bundle $D(\nu)$ $(\subseteq E)$, to a section such that $\mathrm{Null}(s) \cap \text{h-}\mathrm{Null}^a(s) = \emptyset$. □

5. A generalized fixed-point index

Consider, first, a continuous self-map $f : N \to N$ of a closed manifold N. Write $B = N$, and let $E \to B$ be the trivial bundle $B \times N \to B$ with the null section z given by the diagonal map $z(x) = (x, x)$. With the map f we associate the section $s(x) = (x, f(x))$.

The null-set $\mathrm{Null}(s)$ is the fixed-subspace $\mathrm{Fix}(f) = \{x \in B \mid f(x) = x\}$ of f and h-$\mathrm{Null}(s)$ is naturally identified with the homotopy fixed-point set h-$\mathrm{Fix}(f)$ (see [3, 5]) consisting of the paths $\alpha : [0,1] \to N$ such that $\alpha(1) = f(\alpha(0))$.

In [5] and [3] we defined, in a general fibrewise and equivariant setting, the *homotopy Lefschetz index*

$$\text{h-}L(f) \in \omega^0\{S^0;\, \text{h-Fix}(f)_+\} = \omega_0(\text{h-Fix}(f))$$

of f. (In [5] the term 'Nielsen-Reidemeister index' was used.) Here it is just the sum over the path-components of h-Fix(f) (that is, the Reidemeister classes) of the integral Lefschetz indices associated with each component.

Now the normal bundle ν over B is equal to τB, and so we have an isomorphism

$$\lambda_B : \omega^0_B\{B \times S^0;\, \text{h-Null}(s)^{\pi^*\nu}_B\} \xrightarrow{\cong} \omega_0(\text{h-Null}(s)).$$

Proposition 5.1. *In the situation described above, the homotopy Lefschetz index* h-$L(f)$ *of the map f corresponds to the homotopy Euler index* h-$\gamma(s)$ *of the associated section s.*

Proof. This will be proved as a special case of Proposition 5.4. □

Consider, next, a smooth fibre bundle $e : B \to N$, where B and N are closed manifolds, and suppose that $f : B \to N$ is a continuous map. As usual, we let $E = B \times N \to B$ be the trivial bundle with fibre N, equipped with the null section $z(x) = (x, e(x))$. The normal bundle is $\nu = e^*\tau N$, and associated with f there is a section $s(x) = (x, f(x))$. The homotopy Euler index h-$\gamma(s)$ lies in the group

$$\omega^0_B\{B \times S^0;\, \text{h-Null}(s)^{\pi^*\nu}_B\},$$

which we may identify, using λ_B, with

$$\omega^0\{S^0;\, \text{h-Null}(s)^{-\pi^*\tau(e)}\},$$

where $\tau(e)$ is the tangent bundle $(\tau_N B)$ along the fibres, because $\tau B \cong \tau(e) \oplus e^*\tau N$. We shall interpret the corresponding class as a generalized fixed-point index constructed, under weaker hypotheses, by an extension of Dold's method.

Definition 5.2. Let $e : B \to N$ be a smooth fibrewise manifold over a compact ENR N and let $f : U \to N$ be a continuous map defined on an open subset $U \subseteq B$ such that $\text{Fix}(f) = \{x \in U \mid f(x) = e(x)\}$ is compact. The *homotopy fixed-point set* of f is defined to be the set of pairs (x, α), where $x \in U$ and $\alpha : [0,1] \to N$ is a continuous path from $e(x) = \alpha(0)$ to $f(x) = \alpha(1)$, topologized in the usual way. Let $\pi : \text{h-Fix}(f) \to U$ be the projection to the first factor. We construct the generalized *homotopy Lefschetz index*

$$\text{h-}L(f, U) \in \omega^0\{S^0;\, \text{h-Fix}(f)^{-\pi^*\tau(e)}\} = \tilde{\omega}_0(\text{h-Fix}(f)^{-\pi^*\tau(e)})$$

as follows.

First we choose a fibrewise smooth embedding $j : B \hookrightarrow N \times \mathcal{V}$, over N, for some Euclidean space $\mathcal{V}$. The normal bundle $\nu(j)$ of j satisfies $\tau(e) \oplus \nu(j) = B \times \mathcal{V}$ (up to homotopy). So we can realize the stable homotopy of the virtual Thom space as

$$\omega^0\{S^0;\, \text{h-Fix}(f)^{-\pi^*\tau(e)}\} = \omega^0\{\mathcal{V}^+;\, \text{h-Fix}(f)^{\pi^*\nu(j)}\}.$$

Choose a fibrewise tubular neighbourhood of j over N: $D(\nu(j)) \hookrightarrow N \times \mathcal{V}$. We also need an embedding $i : N \hookrightarrow W \subseteq \mathcal{U}$ of the ENR N as a retract of an open subspace W of a Euclidean space $\mathcal{U}$, with a retraction $r : W \to N$. By the compactness of $\mathrm{Fix}(f)$, we may choose an open neighbourhood V with compact closure $\overline{V}$ such that $\mathrm{Fix}(f) \subseteq V \subseteq \overline{V} \subseteq U$ and $(1-t)ie(x) + tif(x) \in W$ for all $x \in \overline{V}$ and $0 \le t \le 1$, and then a real number $\epsilon > 0$ such that $\|if(x) - ie(x)\| \ge \epsilon$ for all $x \in \overline{V} - V$.

We use i and j to regard $D(\nu(j))$ as a subspace of $\mathcal{U} \times \mathcal{V}$. The homotopy Lefschetz index is then represented by the pointed map

$$\mathcal{U}^+ \wedge \mathcal{V}^+ \to \mathcal{U}^+ \wedge \text{h-Fix}(f)^{\pi^*\nu(j)|U} = (D(\mathcal{U})/S(\mathcal{U})) \wedge (D(\pi^*\nu(j)|U)/S(\pi^*\nu(j)|U))$$

sending a point v of $D(\nu(j)|V)$ in the fibre of $\nu(j)$ over $x \in B$ such that $\|if(x) - ie(x)\| < \epsilon$ to $[\epsilon^{-1}(if(x) - ie(x)), (x, \alpha, v)]$), where $\alpha(t) = r((1-t)ie(x) + tif(x))$, and collapsing the complement to the basepoint.

One must check, of course, that the class so constructed is independent of the choices made. It is implicit in the terminology that the index coincides when $e = 1 : B = N \to N$ with the standard fixed-point index. Indeed, we may take $\mathcal{V} = 0$ and $j = 1 : B \to N$. Then the construction reduces to that described in [5, 3].

Remark 5.3. A finite d-fold cover $e : B \to N$ is a fibrewise smooth manifold over N, but the construction is only slightly simpler in this special case. The fibrewise tangent bundle $\tau(e)$ is zero, and we can project from $\text{h-Fix}(f)$ to U by π:

$$\text{h-}L(f, U) \in \omega_0(\text{h-Fix}(f)) \to \omega_0(U),$$

to obtain a *Lefschetz index* $L(f, U) \in \omega_0(U) = \omega^0\{S^0;\ U_+\}$. A globally defined map $f : B \to N$ that is injective on each fibre can be regarded as a d-valued function from N to N with fixed-point set $e(\mathrm{Fix}(f)) \subseteq N$; see, for example, [2].

Proposition 5.4. *Suppose that $e : B \to N$ is a smooth fibre bundle, where B and N are closed manifolds. For a map $f : B \to N$, the homotopy Euler index*

$$\text{h-}\gamma(s) \in \omega^0_B\{B \times S^0; \text{h-Null}(s)^{\pi^*\nu}_B\}$$

of the associated section s of $E = B \times N \to B$ corresponds under the duality isomorphism λ_B to the homotopy Lefschetz index

$$\text{h-}L(f, B) \in \omega^0\{S^0; \text{h-Fix}(f)^{-\pi^*\tau(e)}\}.$$

Proof. We need an explicit description of the isomorphism λ_B. Choose a smooth embedding $k : B \hookrightarrow \mathcal{W}$ of B into some Euclidean space $\mathcal{W}$ with normal bundle $\nu(k)$ and fix a tubular neighbourhood $D(\nu(k)) \hookrightarrow \mathcal{W}$. In Definition 2.4, $\text{h-}\gamma(s)$ is defined by a section $\sigma : B \to \text{h-Null}(s)^{\pi^*\nu}_B$. The corresponding class $\lambda_B(\text{h-}\gamma(s))$ is represented by the pointed map

$$\mathcal{W}^+ \to \big(\text{h-Null}(s)^{\pi^*\nu}_B \wedge_B (D(\nu(k))/_B S(\nu(k))\big)/B = \text{h-Null}(s)^{\pi^*(\nu \oplus \nu(k))},$$

given by mapping a point $(x, v) \in D(\nu(k)) \subseteq \mathcal{W}$, where $x \in B$ and $v \in D(\nu(k)_x)$, to $[\sigma(x), (x, v)]$, and collapsing the complement to the basepoint. (Recall that '$/_B$' denotes the fibrewise quotient.)

The proof is completed by comparing the two definitions, taking i in Definition 5.2 to be a smooth embedding, W to be a tubular neighbourhood and $\mathcal{W}$ to be $\mathcal{U} \times \mathcal{V}$. □

Remark 5.5. There is a fibrewise version of the theory, in which the ENR N is generalized to be a fibrewise ENR over a compact ENR X and $f : U \to N$ is required to be a fibre-preserving map over X. See [9] for the case $B = N$.

6. The homotopy Pontrjagin-Thom construction

Consider a closed submanifold M of a manifold N without boundary. Let ν be the normal bundle of the embedding and choose a tubular neighbourhood $D(\nu) \hookrightarrow N$.

The Pontrjagin-Thom construction collapses the complement of the open disc bundle $B(\nu)$ to a point and identifies the quotient $D(\nu)/S(\nu)$ with the one-point compactification M^ν of the total space of ν to give a pointed map

$$N^+ \to M^\nu$$

from the one-point compactification of N to the Thom space of ν. By construction, we have a cofibre sequence

$$N - B(\nu) \hookrightarrow N \to M^\nu.$$

The homotopy Pontrjagin-Thom construction is less well known; see [5] (Part II, Section 12), where it is described as a 'refinement of the Gysin map'. Let $\mathcal{C} \to N$ be the space of continuous paths $\alpha : [0, 1] \to N$ such that $\alpha(0) \in M$, fibred over N by projection to the end-point $\alpha(1)$. We have a map $\pi : \mathcal{C} \to M$ given by $\pi(\alpha) = \alpha(0)$, and this lifts the normal bundle ν to $\pi^*\nu$ over $\mathcal{C}$. The *homotopy Pontrjagin-Thom map* is a fibrewise pointed map with compact support over N

$$N \times S^0 \to \mathcal{C}_N^{\pi^*\nu}$$

prescribed as follows by a section σ of the fibrewise Thom space of $\pi^*\nu$. Outside the tubular neighbourhood $B(\nu) \subseteq N$, the section σ maps to the basepoint in the fibre, so that the support lies within the compact subspace $D(\nu) \subseteq N$. For a point $(x, v) \in D(\nu) \subseteq N$, where $x \in M$, $v \in D(\nu_x)$, the fibre $\mathcal{C}_{(x,v)}$ is the space of paths $\beta : [0, 1] \to N$ with $\beta(0) \in M$ and $\beta(1) = (x, v)$. We define $\sigma(x, v) = [\alpha, v]$, where $\alpha : [0, 1] \to N$ is given by the radial path $\alpha(t) = (x, tv) \in D(\nu) \subseteq N$. The fibre of $\pi^*\nu$ at α is ν_x and v determines an element $[v] \in D(\nu_x)/S(\nu_x)$.

Let $\mathcal{B}$ be the space of continuous paths $\beta : [0, 1] \to N$ and let $\mathcal{A}$ be the subspace of paths β with $\beta(0) \in N - B(\nu)$, both fibred over N by $\beta \mapsto \beta(1)$. Clearly the path space $\mathcal{B} \to N$ is fibre homotopy equivalent, over N, to the identity map $N \to N$, and adjoining basepoints in each fibre gives a model $\mathcal{B}_{+N}$ for $N \times S^0$. The following analogue of the cofibre sequence for the Pontrjagin-Thom map is implicit in the work of Klein and Willioms [14].

Lemma 6.1. *The homotopy Pontrjagin-Thom map fits into a fibrewise (homotopy) cofibration sequence*

$$\mathcal{A}_{+N} \hookrightarrow \mathcal{B}_{+N} \,(\simeq N \times S^0)\, \to \mathcal{C}_N^{\pi^*\nu}$$

over N.

Notice that the fibrewise pointed spaces appearing in the statement are locally fibre homotopy trivial.

Proof. This is a proof by inspection. A path $\beta : [0,1] \to N$ in the complement $\mathcal{B} - \mathcal{A}$ starts at $\beta(0) \in B(\nu)$, say $\beta(0) = (x, v)$, where $x \in M$, $v \in B(\nu_x)$. We can extend β to a path $\alpha : [0,1] \to N$ with $\alpha(0) = x \in M$, $\alpha(1) = \beta(1)$, by

$$\alpha(t) = \begin{cases} (x, 2tv) & \text{if } 0 \le t \le \frac{1}{2}, \\ \beta(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

This constructs a map $\mathcal{B} - \mathcal{A} \to B(\pi^*\nu)$: $\beta \mapsto (\alpha, v)$. The routine checking that it induces a homotopy equivalence from the cofibre of $\mathcal{A}_{+N} \hookrightarrow \mathcal{B}_{+N}$ to the fibrewise Thom space of $\pi^*\nu$ over $\mathcal{C}$ is omitted. □

7. Intersections

Suppose that $Z \to B$ is a fibrewise smooth fibre bundle with closed (so compact) fibres embedded as a fibrewise submanifold in $E \to B$. Let ν, over Z, be the (fibrewise) normal bundle of the inclusion. Locally, such a bundle pair (E, Z) is of the form $(U \times N, U \times M)$, where $M \subseteq N$ is a closed submanifold (and U is an open subspace of B). For example, if $z : B \to E$ is a section, we may take Z to be $z(B) \subseteq E$; in that case, ν is the restriction of $\tau_B E$ to Z, so that Z is identified with B and ν with $z^*\tau_B E$. The definitions and constructions in Section 2 generalize easily.

Definition 7.1. Given a section s of $E \to B$ we say that s is *null* at a point $x \in B$ if $s(x) \in Z_x$ and write

$$\mathrm{Null}(s) = \{x \in B \mid s(x) \in Z_x\}$$

for the *null-set* of s. The *homotopy null-set*, h-Null(s) is defined as the set of pairs (x, α), with $x \in B$ and $\alpha : [0,1] \to E_x$ a continuous path in the fibre at x with $\alpha(0) \in Z_x$ and $\alpha(1) = s(x)$. (The set is topologized as a subspace of the fibrewise path-space.) It is fibred over B by projecting (x, α) to x. We include Null(s) in h-Null(s) as the space of constant paths. There is a map $\pi :$ h-Null$(s) \to Z$ over B, mapping (x, α) to $\alpha(0)$.

We shall rephrase the constructions of the Euler indices in the language of Section 6. The constructions there extend to fibre bundles. Thus, from the fibrewise submanifold $Z \subseteq E$ we obtain a *fibrewise Pontrjagin-Thom map* over B:

$$E_B^+ \to Z_B^\nu$$

from the fibrewise one-point compactification of E to the fibrewise Thom space.

Definition 7.2. Let s be a section of $E \to B$. It determines a fibrewise pointed map $B \times S^0 \to E_B^+$ mapping the basepoint $(x, 1)$ in the fibre at $x \in B$ to the basepoint at infinity and $(x, -1)$ to $s(x)$. The composition of this map with the fibrewise Pontrjagin-Thom map above is the *Euler index*

$$\gamma(s) \in \omega_B^0\{B \times S^0;\, Z_B^\nu\}.$$

More generally, if the null-set $\mathrm{Null}(s)$ is disjoint from a closed sub-ENR $A \subseteq B$, then we have a *relative Euler index*

$$\gamma(s;\, A) \in \omega_{(B,A)}^0\{B \times S^0;\, Z_B^\nu\},$$

which vanishes if s is homotopic, relative to A, to a section that is nowhere null.

Let $\mathcal{C}$ be the space of pairs (x, α), where $x \in B$ and $\alpha : [0, 1] \to E_x$ is a path with $\alpha(0) \in Z_x$. There is a projection $\mathcal{C} \to E$ mapping (x, α) to $\alpha(1)$ and a map $\pi : \mathcal{C} \to Z$ given by $\pi(x, \alpha) = \alpha(0)$. The *fibrewise homotopy Pontrjagin-Thom map* of the fibrewise submanifold $Z \subseteq E$ is a pointed map over E with compact support:

$$E \times S^0 \to \mathcal{C}_E^{\pi^*\nu}.$$

Definition 7.3. Let $s : B \to E$ be a section. The pull-back of $\mathcal{C} \to E$ by s is the homotopy null-set $\text{h-Null}(s) \to B$. We define the *homotopy Euler index*

$$\text{h-}\gamma(s) \in \omega_B^0\{B \times S^0;\, \text{h-Null}(s)_B^{\pi^* s}\}$$

to be the lift of the homotopy Pontrjagin-Thom map of the fibrewise embedding $Z \hookrightarrow E$.

There is again a relative version

$$\text{h-}\gamma(s;\, A) \in \omega_{(B,A)}^0\{B \times S^0;\, \text{h-Null}(s)_B^{\pi^* s}\}$$

if s is nowhere null on A, and $\gamma(s;\, A) = \pi_* \text{h-}\gamma(s;\, A)$. In the same way, if $U \subseteq B$ is an open subset such that $\mathrm{Null}(s) \cap U$ is compact, we have a local index

$$\text{h-}\gamma(s \,|\, U) \in {}_c\omega_U^0\{U \times S^0;\, \text{h-Null}(s)_U^{\pi^*\nu}\}.$$

The construction may be spelt out as in Definition 2.4. Having chosen a fibrewise tubular neighbourhood $D(\nu) \hookrightarrow E$ of the fibrewise submanifold $Z \subseteq E$ over B, we let W be an open neighbourhood of $\mathrm{Null}(s) \cap U$ such that $\overline{W} \subseteq U \cap s^{-1}(D(\nu))$. By compactness there exists ϵ with $0 < \epsilon < 1$ such that $s(x) \notin D_\epsilon(\nu)$ for $x \in \overline{W} - W$. We can then define a section of $\text{h-Null}(s)_B^{\pi^*\nu}$ mapping $x \in B$ to the basepoint in the fibre if $x \notin W$ or if $x \in W$ and $s(x) \notin D_\epsilon(\nu)$, while if $s(x) \in D_\epsilon(\nu)$, say $s(x) = (y, v)$ with $y \in Z_x$, $v \in \nu_y$, we map to $[\alpha, v/\epsilon]$, where $\alpha : [0, 1] \to E_x$ is the path $\alpha(t) = (y, tv)$ from $y \in Z_x$ to $s(x)$.

From its construction, the relative index $\text{h-}\gamma(s;\, A)$ is an obstruction to deforming s to a section with empty null-set. The next result, essentially due to Hatcher and Quinn [12], asserts that, in a range of dimensions, it is the exact obstruction. Our approach follows that taken by Klein and Williams in [14]. An application to immersion theory is described in [7].

Proposition 7.4. *Suppose that B is a finite complex of dimension $< 2(\dim \nu - 1)$. Let s be a section of $E \to B$ which is nowhere null on a subcomplex A. Then s is homotopic, relative to A, to a nowhere null section of $E \to B$ if and only if* h-$\gamma(s; A) = 0$.

Proof. Choose a fibrewise tubular neighbourhood $D(\nu) \hookrightarrow E$ over B such that $s(A) \cap D(\nu) = \emptyset$, and let $\mathcal{A} \to E$ be the space of pairs (x, β) where $x \in B$ and $\beta : [0,1] \to E_x$ is a continuous path with $\beta(0) \notin B(\nu)$, fibred over E by mapping to $\beta(1) \in E_x$. The pull-back $s^*\mathcal{A} \to B$ consists of the pairs (x, β), where $x \in B$ and β is a path in E_x with $\beta(0) \notin B(\nu)$ and $\beta(1) = s(x)$. Now we have a section σ of $s^*\mathcal{A} \to B$ over the subcomplex A, given by mapping $x \in A$ to the constant path at $s(x)$. And s is homotopic, through a homotopy constant on A, to a section that is nowhere null if and only if the section σ, defined on A, extends over B.

The rest is standard obstruction theory using Lemma 6.1 and the Blakers-Massey theorem (as formulated in [5], Part II, Proposition 2.18). Suppose that h-$\gamma(s;\, A) = 0$. To show that σ extends over B, one proceeds step by step over the cells of the complement. See, for example, the proof given in [5] (Part II, Proposition 4.9) in the special case that Z is the zero section in a real vector bundle E. □

Example 7.5. Suppose that $Z = B \times M$ and $E = B \times N$, where M is a submanifold of N, are fibred as trivial bundles over B. Then the section s is given by a map $f : B \to N$ such that $f(A) \cap M = \emptyset$. We are considering the problem of deforming f, relative to A, to a map that does not meet M.

Example 7.6. Suppoe that E is the total space of a real vector bundle over B and that $Z \to B$ is a finite cover. Let s be the zero section of ξ. Then $\pi :$ h-Null$(s) \to Z$ is a fibre homotopy equivalence. The index h-$\gamma(s)$ corresponds to the pull-back of the Euler class $\gamma(\xi)$ to Z. If B is a finite complex with $\dim B < 2(\dim \xi - 1)$, then there is a section of $E - Z \to B$ if and only if the stable cohomotopy Euler class of the lift of ξ to Z is zero.

There is another way of thinking of the Euler indices. By lifting from B to Z, we obtain an *associated root problem.* Write $\tilde{E} \to \tilde{B}$ for the pull-back $Z \times_B E \to Z$, and let $\tilde{z} : \tilde{B} \to \tilde{E}$ be the diagonal section: $\tilde{z}(y) = (y, y)$. A section s of $E \to B$ determines a section $\tilde{s} : \tilde{B} \to \tilde{E}$, namely, $\tilde{s}(y) = (y, s(x)) \in \tilde{E}_y$, where $x \in B$, $y \in Z_x$. The projection from Z to B induces homeomorphisms Null$(\tilde{s}) \to$ Null(s) and h-Null$(\tilde{s}) \to$ h-Null(s). The normal bundle $\tilde{\nu} = \tilde{z}^*\tau_{\tilde{B}}\tilde{E}$ is the restriction $\tau_B E | Z$ and splits, up to homotopy, as $\tau_B Z \oplus \nu$.

The equivalence described in Proposition 4.1 generalizes from a manifold to the fibrewise manifold $Z \to B$ to give compatible isomorphisms

$$\lambda_{Z\to B} : \omega^0_Z\{Z \times S^0;\, \text{h-Null}(s)_Z^{\pi^*(\tau_B E|Z)}\} \to \omega^0_B\{B \times S^0;\, \text{h-Null}(s)_B^{\pi^*\nu}\}.$$

and

$$\lambda_{Z\to B} : \omega^0_Z\{Z \times S^0;\, (\tau_B E|Z)^+_Z\} \to \omega^0_B\{B \times S^0;\, Z^\nu_B\}.$$

See [5] (Part II, Section 12).

Lemma 7.7. *Let s be a section of $E \to B$. Then the Euler indices* h-$\gamma(s)$ *and* $\gamma(s)$ *correspond, respectively, under the equivalence $\lambda_{Z\to B}$ above to the Euler indices* h-$\gamma(\tilde{s})$ *and* $\gamma(\tilde{s})$ *for the associated root problem.*

Proof. The inverse of the isomorphism

$$\lambda_{Z\to B} : \omega^0_Z\{Z \times S^0;\ (\tau_B Z)^+_Z \wedge_Z F\} \to \omega^0_B\{B \times S^0;\ F/_B Z\},$$

where $F \to Z$ is a pointed homotopy fibre bundle (with fibres of the homotopy type of CW complexes) is described explicitly in [5]. In the cases of interest here, F is $\mathrm{Null}(s)^{\pi^*\nu}_Z$ or ν^+_Z. The case in which $F = \nu^+_Z$ is easier to describe. The inverse of $\lambda_{Z\to B}$ is a composition of two maps:

$$\omega^0_B\{B \times S^0;\ Z^\nu_B\} \xrightarrow{Z\times_B} \omega^0_Z\{Z \times S^0;\ (Z\times_B Z)^\nu_Z\} \xrightarrow{\Delta^!} \omega^0_Z\{Z \times S^0;\ Z^{\tau_B Z\oplus\nu}_Z\}.$$

The first lifts spaces and maps over B to spaces and maps over Z by the taking the fibre product with Z and the identity map $Z \to Z$, respectively. We have identified $Z \times_B (Z^\nu_B)$ with the fibrewise Thom space of the pull-back of ν to the second factor of $Z \times Z$. The second is the fibrewise Pontrjagin-Thom map of the diagonal inclusion $\Delta : Z \to Z \times_B Z$; the normal bundle is identified with the tangent bundle $\tau_B Z$ of the second factor Z.

Now the Euler indices $\gamma(s)$ and $\gamma(\tilde{s})$ are constructed from fibrewise Pontrjagin-Thom maps

$$E^+_B \xrightarrow{i^!} Z^\nu_B \quad \text{and} \quad (Z\times_B E)^+_Z \xrightarrow{\tilde{i}^!} (\tau_B Z \oplus \nu)^+_Z.$$

One checks that $\tilde{i}^!$ is the composition of $Z \times_B i^!$ and the diagonal Pontrjagin-Thom map:

$$(Z\times_B E)^+_Z \xrightarrow{Z\times_Z i^!} (Z\times_B Z)^\nu_Z \xrightarrow{\Delta^!} (\tau_B Z \oplus \nu)^+_Z.$$

Pulling back by the section s, one obtains the identity $\gamma(\tilde{s}) = \lambda^{-1}_{Z\to B}(\gamma(s))$.

The case of the homotopy indices is similar; we omit the details. □

Notice, however, that, although the indices of s and $\tilde{s}$ are the same, the obstruction theory is different. If s is homotopic to a nowhere null section, then so is $\tilde{s}$. But the converse is, in general, false.

When the base B is a smooth manifold, we may use the equivalence λ_B from Proposition 4.1 to write

$$\omega^0_{(B,\partial B)}\{B \times S^0;\ \text{h-Null}(s)^{\pi^*\nu}_B\} = \tilde{\omega}_0(\text{h-Null}(s)^{\pi^*(\nu - p^*\tau B)}),$$

where $p : Z \to B$ is the projection. The differential-topological interpretation of the indices in Section 4 then has the following generalization.

Proposition 7.8. *Suppose that $E \to B$ is a smooth fibre bundle over a manifold B and that $Z \to B$ a smooth sub-bundle. Consider a smooth section s which is nowhere null on ∂B and is transverse to Z. Then* $\mathrm{Null}(s)$ *is a submanifold of $B - \partial B$ with normal bundle $s^*\nu$. This manifold, with the inclusion*

$\mathrm{Null}(s) \to \text{h-Null}(s)$ *and the stable isomorphism between the tangent bundle and the restriction of* $-\pi^*(\nu - p^*\tau B)$*, represents the homotopy Euler index*

$$\text{h-}\gamma(s;\, \partial B) \in \tilde{\omega}_0(\text{h-Null}(s)^{\pi^*(\nu - p^*\tau B)}).$$

Proof. Using the functoriality of the λ-equivalences: $\lambda_Z = \lambda_B \circ \lambda_{Z \to B}$, we can deduce this from Proposition 4.6 and a relative version of Lemma 7.7. □

We return to the discussion of coincidence invariants. Consider, as in Section 2, a root problem given by a bundle $E \to B$ and sections z and s, where E is compact. There are two ways in which this may be fitted into the intersection framework. We may simply take $Z = z(B) \subseteq E$ with the given section s. Alternatively, treating the sections z and s symmetrically, we may take the intersection problem specified by the bundle $E' = E \times_B E \to B$ and the null sub-bundle $Z' = \Delta(E) \subseteq E'$ with the section $s' = (z, s)$. The null-sets $\mathrm{Null}(s)$ and $\mathrm{Null}(s')$ are equal. If s is homotopic to a nowhere null section, so also is s'. If s' is homotopic to a nowhere null section then z and s may be deformed to nowhere coincident sections.

The homotopy null-sets $\mathrm{Null}(s')$ and $\text{h-Null}(s)$ are homeomorphic as spaces over B. Indeed, we may think of elements of $\text{h-Null}(s')$ as triples (x, α_1, α_2) where $x \in B$ and α_1, $\alpha_2 : [0,1] \to E_x$ are paths such that $\alpha_1(0) = \alpha_2(0)$, $\alpha_1(1) = z(x)$ and $\alpha_2(x) = s(x)$. Such a triple corresponds to $(x, \alpha) \in \text{h-Null}(s)$, where $\alpha : [0,1] \to E_x$ is the path:

$$\alpha(t) = \begin{cases} \alpha_1(1-2t) & \text{if } 0 \le t \le 1/2, \\ \alpha_2(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

For $0 \le t \le 1$, let $\pi_t : \text{h-Null}(s) \to E$ map (x, α) to $\alpha(t) \in E_x$. Then the projection $\pi' : \text{h-Null}(s') \to E = Z'$, corresponding to $\pi_{1/2}$, is homotopic to π_0. The homotopy provides, up to homotopy, an isomorphism between $(\pi')^*\tau_B E$ and $\pi^*(\tau_B E | Z)$. This allows us to compare the Euler indices of s and s'.

Lemma 7.9. *The homotopy Euler indices* $\text{h-Null}(s)$ *and* $\text{h-Null}(s')$ *coincide under the isomorphism*

$$\omega_B^0\{B \times S^0;\, \text{h-Null}(s)_B^{\pi^*(\tau_B E|Z)}\} = \omega_B^0\{B \times S^0;\, \text{h-Null}(s')_B^{(\pi')^*(\tau_B E)}\}$$

induced by the fibre homotopy equivalence described above. The Euler index $\gamma(s')$ *is the image of* $\gamma(s)$ *under the homomorphism*

$$\omega_B^0\{B \times S^0;\, (z^*\tau_B E)_B^+\} \to \omega_B^0\{B \times S^0;\, E_B^{\tau_B E}\}$$

induced by the section $z \colon B \to E$ *considered as a map over* B. □

Remark 7.10. These considerations suggest that the definition of the homotopy coincidence index in Definition 2.7 might be modified to define the index as an element of

$$\omega_B^0\{B \times S^0;\, \text{h-Coin}(e, f)_B^{\pi^*\tau N}\}$$

where $\pi : \text{h-Coin}(e, f) \to N$ maps (x, α) symmetrically to $\alpha(1/2)$. Its image under π in

$$\omega^0_B\{B \times S^0;\ (B \times N)^{\tau N}_B\} = \omega^0\{B;\ N^{\tau N}\}$$

might then be called the *coincidence index* of e and f.

We conclude with a result that has applications to linking invariants. Returning to a general fibrewise submanifold $Z \subseteq E$ and section s, suppose that $\phi : B \to \mathbb{R}$ is a continuous map such that $\phi(x) \neq 0$ for all $x \in \text{Null}(s)$. Put $U_+ = \{x \in B \mid \phi(x) > 0\}$ and $U_- = \{x \in B \mid \phi(x) < 0\}$.

Definition 7.11. We write

$$\text{h-}\gamma_\pm(s, \phi) = i_{U_\pm}(\text{h-}\gamma(s \,|\, U_\pm)) \in \omega^0_B\{B \times S^0;\ \text{h-Null}(s)^{\pi^*\nu}_B\},$$

where $i_{U_\pm}$ is the inclusion map.

Thus $\text{h-}\gamma_+(s, \phi)$ is zero if $\phi(x) < 0$ for all $x \in \text{Null}(s)$ and, in general, this class is an obstruction to deforming s and ϕ to a section and function with this property. By Lemma 2.10 (ii),

$$\text{h-}\gamma(s) = \text{h-}\gamma_+(s, \phi) + \text{h-}\gamma_-(s, \phi).$$

Proposition 7.12. *Suppose that B is a finite complex with $\dim B \leq 2(\dim \nu - 1)$. Then there is a homotopy (s_t, ϕ_t), $0 \leq t \leq 1$, through sections s_t and functions ϕ_t with $\phi_t(x) \neq 0$ for all $x \in \text{Null}(s_t)$, such that $(s_0, \phi_0) = (s, \phi)$ and $\phi_1(x) < 0$ for all $x \in \text{Null}(s_1)$ if and only if $\text{h-}\gamma_+(s, \phi) = 0$.*

Proof. Choose $m > 0$ such that $m > \phi(x)$ for all $x \in B$.

Consider the product bundle $E' = [0, 1] \times E \times \mathbb{R}$ over $B' = [0, 1] \times B$ with $Z' = [0, 1] \times Z \times \{0\}$ and section $s'(t, x) = (t, s(x), \phi(x) - tm)$. Then

$$\text{Null}(s') = \{(t, x) \mid s(x) \in Z_x,\ \phi(x) = tm\}$$

is homeomorphic to $\text{Null}(s) \cap U_+$ by projection to B and is disjoint from $A' = \{0, 1\} \times B$.

Now

$$\text{h-}\gamma(s'; A') \in \omega^0_{(B', A')}\{B' \times S^0;\ \text{h-Null}(s')^{\pi^*(\nu \oplus \mathbb{R})}_{B'}\}$$

corresponds under the suspension isomorphism and the homotopy equivalence between $\text{h-Null}(s')$ and the pull-back of $\text{h-Null}(s)$ to $\text{h-}\gamma_+(s)$.

The result follows from Proposition 7.4. □

Example 7.13. Consider two closed manifolds P_1, P_2 and maps $f_i : P_i \to Q$ to a manifold Q (without boundary). Take $B = P_1 \times P_2$, $E = B \times (Q \times Q)$, $Z = B \times \Delta(Q)$, fibred by projection to B, and $s(x_1, x_2) = (x_1, x_2, f_1(x_1), f_2(x_2))$. Suppose that $\phi_1 : P_1 \to \mathbb{R}$ and $\phi_2 : P_2 \to \mathbb{R}$ are maps such that $\phi_1(x_1) \neq \phi_2(x_2)$ if $f_1(x_1) = f_2(x_2)$. Put $\phi(x_1, x_2) = \phi_1(x_1) - \phi_2(x_2)$. Then $\text{h-}\gamma_+(s, \phi)$ is Koschorke's linking obstruction introduced in [17].

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