FIBREWISE HOPF STRUCTURES ON SPHERE-BUNDLES

A. L. COOK AND M. C. CRABB

1. *Introduction*

According to a celebrated theorem of Adams the sphere $S^n, n > 0$, admits a Hopf structure precisely when $n = 1, 3$ or 7. The spheres S^1 and S^3 are groups, as the unit complex numbers and quaternions; S^7 has a product given by the Cayley multiplication. We investigate the symmetry of these and the other non-standard Hopf structures on spheres, looking mainly at families of Hopf structures parametrized by the points of a base space or, more precisely, fibrewise Hopf spaces, but also at equivariant multiplications.

Let us recall briefly the definitions of equivariant and fibrewise Hopf space. Consider first a space X with a basepoint which we shall write as 1. Given a pointed map

$$\mu: X \times X \longrightarrow X$$

we write $\mu_L: X \to X$, $\mu_R: X \to X$ for the maps $x \mapsto \mu(x, 1)$, $x \mapsto \mu(1, x)$ respectively. The multiplication μ is a *Hopf structure* if μ_L and μ_R are homotopic (through pointed maps) to the identity. The definition of a G-equivariant Hopf structure, where G is a compact Lie group, is similar: the group G acts on X, preserving the basepoint 1, and the multiplication μ and the homotopies are required to be equivariant. For the fibrewise theory we work over a finite complex B. We think of a fibrewise pointed space over B as a family of pointed spaces parametrized by the points of the base. Let $X \to B$ now be such a fibrewise pointed space over B and

$$\mu: X \times_B X \longrightarrow X$$

a fibrewise pointed map. Maps μ_L and $\mu_R: X \to X$ are then defined by

$$\mu_L(x) = \mu(x, 1_b), \quad \mu_R(x) = \mu(1_b, x),$$

where x lies in the fibre over $b \in B$ and 1_b denotes the basepoint in that fibre. We say that μ is a *fibrewise Hopf structure* if μ_L and μ_R are homotopic (through fibrewise pointed maps) to the identity on X. (Here and elsewhere, when discussing pointed spaces we assume that maps and homotopies are pointed and, when appropriate, G-equivariant; in the context of fibrewise pointed spaces, maps and homotopies are assumed to be fibrewise pointed.)

Before giving an account of our theorems we introduce some notation which will be used throughout the paper. The letter n is reserved for an odd integer. We write V for an n-dimensional real vector with inner product, and ζ for a real vector bundle of (odd) dimension n over a finite complex B. When required we shall assume that ζ is equipped with an inner product. We always take the base space B to be connected.

It will often be convenient to identify the unit n-sphere $S(\mathbb{R} \oplus V)$, equipped with

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the basepoint $1 = (1,0)$, with V^+, the 1-point compactification of V, by stereographic projection (mapping 1 to ∞). Similarly, we identify the n-sphere-bundle $S(\mathbb{R} \oplus \zeta)$ over B, with basepoint (1,0) in each fibre, with the fibrewise 1-point compactification ζ_B^+.

We are able now to formulate precisely the problem with which this paper is concerned: given a real vector bundle ζ (of odd dimension n over a connected finite complex B), does there exist a fibrewise Hopf structure

$$\mu\colon S(\mathbb{R} \oplus \zeta) \times_B S(\mathbb{R} \oplus \zeta) \longrightarrow S(\mathbb{R} \oplus \zeta)? \tag{1.1}$$

It is sensible to refine the question slightly. Up to homotopy, there is just one Hopf structure on S^1, but there are 12 distinct Hopf structures on S^3 and 120 on S^7. If we choose a point of B and fix a Hopf structure on the fibre of ζ_B^+ at that point, then we can ask whether this Hopf structure extends to a fibrewise Hopf structure over B. We think of such an extension as reflecting a sort of symmetry of the Hopf structure on the fibre. To explain this, we go to the equivariant theory.

Suppose that a compact Lie group G acts orthogonally on a vector space V and that there is a G-equivariant Hopf structure

$$\mu\colon S(\mathbb{R} \oplus V) \times S(\mathbb{R} \oplus V) \longrightarrow S(\mathbb{R} \oplus V). \tag{1.2}$$

If $P \to B$ is a principal G-bundle over a finite complex B, we can form the real vector bundle $\zeta := P \times_G V$, and the sphere-bundle $S(\mathbb{R} \oplus \zeta) = P \times_G S(\mathbb{R} \oplus V)$, or fibrewise 1-point compactification $\zeta_B^+ = P \times_B V^+$, then acquires a fibrewise Hopf structure as in (1.1).

This principal bundle construction supplies the basic examples of fibrewise Hopf structures. We shall also use it in the opposite direction to establish non-existence of equivariant Hopf structures: if there is no fibrewise Hopf structure on $P \times_B V^+$, then there can be equivariant Hopf structure on V^+.

Our first theorem gives a complete answer for $n = 1$ and $n = 3$. In its statement the term *fibre type*, explained in Section 2, refers to the Hopf structure on a fibre: existence of Hopf structures of each fibre type means that any Hopf structure on a chosen fibre extends to a fibrewise Hopf structure.

THEOREM 1.3. *Let ζ be an n-dimensional real vector bunble over a finite complex B.*

(i) *If $n = 1$, then ζ_B^+ admits a fibrewise Hopf structure.*

(ii) *If $n = 3$, then ζ_B^+ admits a fibrewise Hopf structure if and only if ζ is orientable. In that case ζ_B^+ admits fibrewise Hopf structures of each fibre type.*

(iii) *If $n = 7$, a sufficient condition for ζ_B^+ to admit a fibrewise Hopf structure is that the structure group of ζ reduce from $O(7)$ to the subgroup G_2, and then it admits structures of each fibre type. A necessary condition is that ζ admit a spin structure.*

Our effort, therefore, will be concentrated on the case $n = 7$. Obstruction theory shows that if the dimension of B is less than 8, then every 7-dimensional spin bundle ζ has a G_2-structure; it follows, by (1.3)(iii), that ζ_B^+ admits a fibrewise Hopf structure. The first interesting example (for $n = 7$) is therefore the case $B = S^8$.

THEOREM 1.4. *Let ζ over $B = S^8$ be classified by the integer $m \in \mathbb{Z} = \pi_7(O(7))$. Then ζ_B^+ admits a fibrewise Hopf structure if and only if $m \equiv 0 \pmod 8$. In that case there exist fibrewise Hopf structures of each fibre type.*

This result has also been obtained by James [16], using direct computation of the obstruction to existence of a multiplication. In view of (1.4) there exist both sphere-bundles which admit a spin structure but no fibrewise Hopf multiplication and sphere-bundles which possess a fibrewise Hopf structure but no G_2-structure. Our second example, concerning bundles over lens spaces, and the related equivariant problem were suggested by the paper [13] of Ishikawa.

THEOREM 1.5. *Let the cyclic group* $\mathbb{Z}/2^r$ *act freely on the sphere* $S^{2m-1} = S(\mathbb{C}^m)$ *as multiplication by roots of unity, and write* λ *for the standard complex line bundle (lifting the Hopf line bundle on* $\mathbb{C}P^{m-1}$*) over the lens space* $B = S^{2m-1}/(\mathbb{Z}/2^r)$*. Assume that* $m > r2^{r-1}$*. Let a, b and c be non-zero integers with* $v_2(a) \leqslant v_2(b) \leqslant v_2(c)$ *and* $v_2(a) < r$*, and consider the vector bundle* $\zeta := \mathbb{R} \oplus \lambda^a \oplus \lambda^b \oplus \lambda^c$*, where* λ^k *denotes the k-fold tensor product of the complex line bundle* λ*.*

Then ζ_B^+ *admits a fibrewise Hopf structure if and only if* $v_2(a) = v_2(b) < v_2(c)$*, and in that case there exist structures of each fibre type.*

The associated equivariant result is contained, as the case $p = 2$, in our final theorem.

THEOREM 1.6. *We take* $G = \mathbb{Z}/p^r$*, where* $p > 1$ *is prime. Write E for the standard one-dimensional complex representation of G and* E^k *for its* kth *tensor power. Let V be a non-trivial 7-dimensional real representation of G.*

Then V^+ *admits a G-equivariant Hopf structure if and only if V is isomorphic to* $\mathbb{R} \oplus E^a \oplus E^b \oplus E^c$*, for some non-zero integers a, b, c with* $v_2(a) = v_2(b) < v_2(c) \leqslant r$ *if* $p = 2$*,* $v_p(a) = v_p(b) \leqslant v_p(c) \leqslant r$ *if p is odd (and* $v_p(a) < r$ *since V is non-trivial).*

Constructions of Hopf structures are described in Sections 2 and 3. In order to obtain obstructions to the existence of fibrewise Hopf structures we generalize an appropriate version of Adams's proof for spheres. The general method is described in Section 4. In Section 5 we carry through the computations in mod 2 homology, generalizing the classical proof that S^n can only admit a Hopf structure if $n+1$ is a power of 2, and obtain the necessary conditions of (1.3). Section 6 describes the KO-theory (which for spheres gives Adams's result). Sections 7 and 8 contain the specific KO-theory computations needed to establish (1.4) and (1.5).

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2. *Elementary considerations*

We look first at the symmetry of the classical Hopf structures on S^1, the complex numbers of modulus 1, on S^3, the quaternions of norm 1, and on S^7, the Cayley numbers of norm 1, given by multiplication in $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. These are clearly equivariant with respect to the action of the automorphism groups $O(1)$ of $\mathbb{C}$, $SO(3)$ of $\mathbb{H}$ and G_2 of $\mathbb{O}$. We identify the purely imaginary complex numbers, quaternions, Cayley numbers with $\mathbb{R}^n$, $n = 1, 3, 7$ in the usual way.

PROPOSITION 2.1. *Let $V = \mathbb{R}^n$, where $n = 1$, 3 or 7, with G the subgroup $O(1)$, $SO(3)$ or G_2 of $O(n)$ respectively. Then multiplication in $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ defines a G-equivariant Hopf structure on $S(\mathbb{R} \oplus V)$. In the first two cases this is actually a group multiplication.*

We have already, in the Introduction, pointed out the relationship between equivariant and fibrewise Hopf spaces: the principal bundle construction gives immediately the existence of fibrewise Hopf structures on the sphere-bundles considered in (1.3).

REMARK 2.2. A special case will be useful. Because $SU(3) \subset G_2$, we have:

(i) if U is a 3-dimensional complex representation of a group G, with $\Lambda^3 U$ a trivial G-module, then $S(\mathbb{C} \oplus U)$ has a G-equivariant Hopf structure;

(ii) if η is a 3-dimensional complex vector bundle over B with $c_1(\eta) = 0$, then $(\mathbb{R} \oplus \eta)^+_B$ has a fibrewise Hopf structure.

Now we turn to the non-standard multiplications; for details we refer to [**14**, **4**]. The Hopf structures on S^n are classified, up to homotopy, by $\pi_{2n}(S^n) = [S^n \wedge S^n; S^n]$. Remembering that any two elements of $\mathbb{O}$ generate an associative sub-algebra, we can write down explicit Hopf structures μ_{2s+1}, indexed by odd integers $2s+1$, on $S(\mathbb{R} \oplus V)$, for $n = 3$ and 7, in terms of quaternionic and Cayley multiplication, as

$$\mu_{2s+1}(v, w) := [v, w]^s (vw). \tag{2.3}$$

Here $[v, w]$ is the commutator $vwv^{-1}w^{-1}$. The commutator map (or Samelson product $\langle \iota, \iota \rangle$) $S^n \wedge S^n \to S^n$ generates the cyclic group $\pi_{2n}(S^n)$ and the multiplication μ_{2s+1} is determined (up to homotopy) by $2s+1 \pmod{24}$ if $n = 3$, by $2s+1 \pmod{240}$ if $n = 7$. (We are, in fact, indexing the multiplication by the stable class of the Hopf construction.) We shall say that a Hopf structure on S^n homotopic to μ_{2s+1} has *type* $2s+1$.

REMARK 2.4. More generally, we can look at the multiplication μ given by

$$\mu(v, w) := (v^{a_1} w^{b_1} \ldots v^{a_r} w^{b_r})(vw),$$

where the a_i and the b_i are integers summing, separately, to zero. Since addition in $\pi_{2n}(S^n)$ can be described in terms of either the co-Hopf structure on S^{2n} or the Hopf structure on S^n, we see that the maps $(v, w) \mapsto [v^a, w^b]$ and $(v, w) \mapsto [v, w]^{ab}$: $S^n \wedge S^n \to S^n$ are homotopic. It follows, as an elementary exercise, that μ has type $2s+1$ with $s = \sum_{i \leqslant j} a_i b_j$.

DEFINITION 2.5. Suppose that $n = 3$ or $n = 7$ and that ζ^+_B has a fibrewise Hopf structure μ. If we choose a point $b \in B$ and an isomorphism of the fibre ζ_b with $\mathbb{R}^n$, then we get an induced multiplication on S^n of type $2s+1$, say. An orientation-reversing self-equivalence of S^n is easily seen, by (2.4), to transform a multiplication of type $2s+1$ into one of type $-(2s+1)$. Hence, because B is connected, the type is well-defined up to sign. We shall call $\pm(2s+1)$ the *fibre type* of μ. Notice that the self-map $-1 : \zeta \to \zeta$ reverses orientation in fibres. So there is another fibrewise Hopf structure on ζ^+_B which induces a multiplication of type $-(2s+1)$ on S^n.

Since the maps μ_{2s+1}, (2.3), are manifestly G-equivariant, we can use them to construct fibrewise multiplications, thus establishing the sufficient conditions of (1.3), which are collected in the next proposition.

PROPOSITION 2.6. *If $n = 1$, or $n = 3$ and ζ is orientable, or $n = 7$ and the structure group of ζ reduces to G_2, then ζ_B^+ admits fibrewise Hopf structures of each fibre type.*

The case $n = 3$ is well understood; more needs to be said when $n = 7$.

LEMMA 2.7. *Suppose that $n = 7$ and that ζ_B^+ admits a fibrewise Hopf structure of type $\pm(2s+1)$. Then it admits a fibrewise Hopf structure of type $\pm(2r+1)(2s+1)$ for each $r \in \mathbb{Z}$.*

The fibre types $\pm(2s+1) (\mathrm{mod}\, 240)$ thus split into classes indexed by the highest common factor $(2s+1, 15)$: 1, 3, 5, 15.

Proof. We use notation of the form $[-;-]_B$ for a set of fibrewise pointed homotopy classes over B. Suppose that ζ_B^+ has a fibrewise Hopf structure μ of fibre type $\pm(2s+1)$. This determines a product, which we write as $*$, on the set $\mathscr{L} := [\zeta_B^+ \times_B \zeta_B^+; \zeta_B^+]_B$. Then, just as when B reduces to a point, every element of $\mathscr{L}$ can be expressed uniquely as $a * [\mu]$ with $a \in \mathscr{L}$. The set $[\zeta_B^+ \wedge_B \zeta_B^+; \zeta_B^+]_B$ embeds in $\mathscr{L}$, and $a * [\mu]$ yields a fibrewise Hopf structure precisely when a lies in this subset.

In particular, the opposite multiplication $((x, y) \mapsto \mu(y, x))$ corresponds to an element c, say, in $[\zeta_B^+ \wedge_B \zeta_B^+; \zeta_B^+]_B$. We define further Hopf structures $\mu^{(r)}$ for $r \geqslant 1$ by

$$\mu^{(0)} := \mu, \quad [\mu^{(r)}] := c * [\mu^{(r-1)}] \in \mathscr{L}.$$

It is a straightforward computation using (2.4) to check that $\mu^{(r)}$ has fibre type $\pm(2r+1)(2s+1)$.

This is an appropriate point at which to recollect the notions of *fibre degree* and *bi-degree*. If ζ and η are real vector bundles of the same dimension n over (connected) B and $f\colon \zeta_B^+ \to \eta_B^+$ is a fibrewise pointed map, then the fibre degree is defined up to sign as follows. Choose a point in B and trivializations of the fibres of ζ and η at that point to get a map $S^n \to S^n$ of some degree k. We say that f has degree $\pm k$. For a self-map of ζ_B^+ there is no ambiguity of sign. The bi-degree of a multiplication μ on ζ_B^+ is $(\deg \mu_L, \deg \mu_R) \in \mathbb{Z} \times \mathbb{Z}$.

We recall next a construction due to Noakes, **[21, 22]**.

DEFINITION 2.8. For an integer k, we define the kth power map

$$\phi_k \colon S(\mathbb{R} \oplus V) \longrightarrow S(\mathbb{R} \oplus V)$$

as follows. Notice first that for any 1-dimensional subspace L of V, $S(\mathbb{R} \oplus L)$ has a canonical group structure given by (2.1), $n = 1$. We now define ϕ_k to be that unique self-map of $S(\mathbb{R} \oplus V)$ which, when restricted to each $S(\mathbb{R} \oplus L)$, gives the kth power $z \mapsto z^k$ in that group. It is clear from the preceding description that ϕ_k is equivariant with respect to the action of the orthogonal group $O(V)$ of V. We also write

$$\phi_k \colon \zeta_B^+ = S(\mathbb{R} \oplus \zeta) \longrightarrow S(\mathbb{R} \oplus \zeta) = \zeta_B^+$$

for the associated self-map of a sphere bundle.

On S^1, S^3 and S^7, ϕ_k thus coincides with the kth power in complex, quaternionic or Cayley multiplication.

LEMMA 2.9. *The map ϕ_k has degree k.*

Proof. This is proved in [22]. The result can also be seen as follows. Write V as $\mathbb{R} \oplus \mathbb{C}^m$, and let the group $\mathbb{T}$ of complex numbers of modulus 1 act on V as complex multiplication on the summand $\mathbb{C}^m$. Then the degree of ϕ_k on $S(\mathbb{R} \oplus V)$ is equal to the degree of the restriction to the fixed subspace $S(\mathbb{R} \oplus V)^{\mathbb{T}} = S(\mathbb{R} \oplus \mathbb{R})$, and this is k by definition.

The next construction is classical and will be used to show that the existence of fibrewise Hopf structures is a 2-local problem.

DEFINITION 2.10. For $v \in S(\mathbb{R} \oplus V)$ we define the reflexion map

$$R_v : \mathbb{R} \oplus V \to \mathbb{R} \oplus V$$

by $R_v w = w - 2\langle v, w\rangle v$. Now define an $O(V)$-equivariant map

$$\rho : S(\mathbb{R} \oplus V) \times S(\mathbb{R} \oplus V) \longrightarrow S(\mathbb{R} \oplus V)$$

by $\rho(v, w) = R_1(R_v w)$. We also write ρ for any associated bundle map.

On S^1, S^3 and S^7 it is easy to see that ρ is given in terms of complex, quaternionic or Cayley multiplication by the formula $\rho(v, w) = v^{-1} w v^{-1}$. We deduce the following lemma, which shows, in particular, that ρ has bi-degree $(-2, 1)$.

LEMMA 2.11. *For $k, l \in \mathbb{Z}$, we have $\rho(\phi_k v, \phi_l v) = \phi_{-2k+l} v$.*

3. *Constructions*

In constructing fibrewise Hopf structures we shall make use of the following observation. If $\mu : \zeta_B^+ \times_B \zeta_B^+ \to \zeta_B^+$ is a multiplication on ζ_B^+ of bi-degree $(1, 1)$, so that, by Dold's theorem (for fibrewise pointed spaces), μ_L and μ_R are fibre-homotopy equivalences, then we can define a fibrewise Hopf structure μ' on ζ_B^+ by $\mu' = \mu \circ (\sigma_L \times \sigma_R)$, where σ_L and σ_R are homotopy inverses of μ_L and μ_R.

We begin this section by discussing the local Dold theorem for sphere bundles. The form that we use can be stated without using fibrewise localization theory (in the spirit of Adams's early paper [1] on local spheres).

LEMMA 3.1. *Suppose that ζ and η are real vector bundles over B of the same (odd) dimension n and $f : \zeta_B^+ \to \eta_B^+$ is a (fibrewise pointed) map over B with non-zero fibre degree $\pm k$. Then there is a (fibrewise pointed) map $g : \eta_B^+ \to \zeta_B^+$ with fibre degree $\pm l$, l a power of k, such that*

$$f \circ g \simeq \phi_{kl} : \eta_B^+ \longrightarrow \eta_B^+ \quad \text{and} \quad g \circ f \simeq \phi_{lk} : \zeta_B^+ \longrightarrow \zeta_B^+$$

(homotopic through fibrewise pointed maps).

The proof of (3.1) depends upon the following lemma.

LEMMA 3.2. *Suppose that $f : \zeta_B^+ \to \zeta_B^+$ is a self-map with non-zero fibre degree k. Then for some $q \geq 1$ the qth power f^q is homotopic (through fibrewise pointed maps) to ϕ_{k^q}.*

Proof. We work by induction over the cells of B. Let us assume, therefore, that B is obtained by adjoining a j-cell to a subcomplex A and that f coincides over A with ϕ_k. We shall show that, for some positive integer q, f^q is fibrewise homotopic to ϕ_{k^q}. We may suppose that $(B, A) = (D^j, S^{j-1})$. This reduces us to the situation in which $B = S^j$ and $\zeta = B \times \mathbb{R}^n$ is trivial.

In this case, the set of homotopy classes of self-maps of ζ_B^+ may be naturally identified with $\mathbb{Z} \oplus \pi_j(\Omega^n S^n) = \mathbb{Z} \oplus \pi_{j+n}(S^n)$. Composition is given by the formula

$$(k, x) \circ (l, y) = (kl, lx + m_k y), \tag{3.3}$$

in which m_k denotes composition on the left with a degree k self-map of S^n. Then, by induction, we obtain the qth power

$$(k, x)^q = (k^q, (k^{q-1} + k^{q-2} m_k + \ldots + m_k^{q-1})\, x). \tag{3.4}$$

Now recall one of the first results of Sullivan's localization theory [**23**, p. 19]:

if $k \neq 0$, the map

$$\pi_{j+n}(S^n) \longrightarrow \varinjlim \{\pi_{j+n}(S^n) \xrightarrow{m_k} \pi_{j+n}(S^n) \xrightarrow{m_k} \ldots\} \tag{3.5}$$

is localization: $\pi_{j+n}(S^n) \to \pi_{j+n}(S^n)[1/k]$.

In other words, m_k is nilpotent on the p-torsion summand of the finite abelian group $\pi_{j+n}(S^n)$ for primes p dividing k and an isomorphism for $(p, k) = 1$.

The rest is elementary algebra. (We want $(k, x)^q = (k^q, 0)$. By (3.5), there is a positive integer N such that $(m_k/k)^N = 1$ on $\pi_{j+n}(S^n)[1/k]$. Writing M for the order of $\pi_{j+n}(S^n)[1/k]$, one checks easily that $(k, x)^{MN} = (k^{MN}, 0)$ whenever x has order prime to k. If x is k-torsion any large integer q will do. In general, one can take q to be a large multiple of MN.)

Proof of Lemma 3.1. Again we work by induction over the cells and assume that B is obtained by adjoining a j-cell to a subcomplex A. Given a map h on A with $f \circ h \simeq \phi_{km}$, we shall show that $h \circ \phi_s$ extends to B for some power s of k. The obstruction to extending h itself is an element, say x, of $\pi_{j+n-1}(S^n)$. Then the obstruction to extending $f \circ h$ is $m_k x$ and is zero. The obstruction to extending $h \circ \phi_s$ is sx. By (3.5) above we can find s as required such that $h \circ \phi_s$ has an extension, say h', to B.

By (3.2) we now have $(f \circ h')^q \simeq \phi_{kl}$ and $(h' \circ f)^q \simeq \phi_{kl}$ for some q. Take $g = h' \circ (f \circ h')^{q-1}$. Then $f \circ g \simeq \phi_{kl}$ and $g \circ f \simeq \phi_{kl}$.

PROPOSITION 3.6. *Suppose that there is a (fibrewise pointed) map $\mu\colon \zeta_B^+ \times_B \zeta_B^+ \to \zeta_B^+$ with bi-degree $(2k+1, 2l+1)$. Then ζ_B^+ admits a fibrewise Hopf structure.*

Proof. Define a new multiplication μ' by

$$\mu'(x, y) = \rho(\phi_k x, \rho(\phi_l y, \mu(x, y))).$$

By (2.11), μ' has bi-degree $(1, 1)$, and it follows (as noted at the beginning of this section) that ζ_B^+ admits a fibrewise Hopf structure.

COROLLARY 3.7. *Suppose that ζ and η are two n-dimensional real vector bundles such that there is a (fibrewise pointed) map $f\colon \zeta_B^+ \to \eta_B^+$ of odd degree. Then ζ_B^+ admits a fibrewise Hopf structure if and only if η_B^+ does.*

Proof. Let f have fibre degree $\pm(2k+1)$. Then, according to (3.1), there is a map $g: \eta_B^+ \to \zeta_B^+$ of fibre degree $\pm(2l+1)$ for some power $2l+1$ of $2k+1$. Suppose that ν is a fibrewise Hopf structure on η_B^+. Then $\mu = f \circ \nu \circ (g \times g)$ has bi-degree $\pm((2k+1)(2l+1), (2k+1)(2l+1))$. Now construct a Hopf structure on ζ_B^+ as in (3.6).

In the case $n = 7$ we can make the following more precise statement about fibre types. The proof is just a matter of following through the construction for (3.7) when B is a point, with the help of (2.4), and then using (2.7).

PROPOSITION 3.8. *Consider the setting of* (3.7) *in the case* $n = 7$. *Suppose that* f *has fibre degree* $\pm(2k+1)$ *and that* η_B^+ *admits a fibrewise Hopf structure of fibre type* $\pm(2s+1)$. *Then* ζ_B^+ *admits fibrewise Hopf structures of all types divisible by* $((2k+1)(2s+1), 15)$.

In the remainder of this section we describe the specific constructions needed to establish Theorems 1.4, 1.5 and 1.6.

PROPOSITION 3.9. *Suppose that* ζ *is a 7-dimensional real vector bundle over a sphere* $B = S^j$, *classified by an element* $v \in \pi_{j-1}(O(7))$ *whose image in* $\pi_{j+6}(S^7)$ *under the Whitehead J-map is of odd order. Then* ζ_B^+ *admits a fibrewise Hopf structure. In the special case* $j = 8$, *there exist Hopf structures of each fibre type.*

We shall derive the proof with the aid of a general lemma.

LEMMA 3.10. *Let* ξ *and* η *be two real vector bundles of dimension* k *and* l *over* $B = S^j$ *classified by elements* $u \in \pi_{j-1}(O(k))$, $v \in \pi_{j-1}(O(l))$. *Consider the restriction map*

$$[\xi_B^+; \eta_B^+]_B \longrightarrow [S^k; S^l] = \pi_k(S^l)$$

from the set of fibrewise homotopy classes over S^j *to the maps of fibres at the basepoint in* S^j. *Then a class* $x \in \pi_k(S^l)$ *extends to a fibrewise map over* S^j *if and only if*

$$x \circ Ju = Jv \circ \Sigma^{j-1} x \text{ in } \pi_{j-1+k}(S^l).$$

Proof. This is standard obstruction theory. We outline what is essentially a Mayer–Vietoris argument. Decompose the sphere S^j as the union of two hemispheres with intersection S^{j-1}. Write S_+^{j-1} for the sphere with an added disjoint basepoint. The clutching map for ξ_B^+ is

$$(1, Ju) \in [S_+^{j-1}; \Omega^k S^k] = \pi_k(S^k) \oplus \pi_{j-1+k}(S^k),$$

and for η_B^+ it is $(1, Jv)$. The condition that classes x and y in $\pi_k(S^l)$ at the two poles patch together is that

$$(x, 0) \circ (1, Ju) = (1, Jv) \circ (y, 0) \text{ in } [S_+^{j-1} : \Omega^k S^l] = \pi_k(S^l) \oplus \pi_{j-1+k}(S^l).$$

Proof of Proposition 3.9. From (3.10) with $\xi = B \times \mathbb{R}^7$ and $\eta = \zeta$, a (pointed) map $S^7 \to S^7$ of degree d extends to a (fibrewise pointed) map $B \times S^7 \to \zeta_B^+$ if and only if $d \cdot Jv = 0$. So if Jv has odd order there is a (fibrewise pointed) map $B \times S^7 \to \zeta_B^+$ with odd fibre degree; the existence of a fibrewise Hopf structure follows from (3.7).

We now consider the case $j = 8$ in detail. Recall that $\pi_{14}(S^7)$ is cyclic of order 120, with generator τ say. We also need to know that $\pi_{10}(S^3)$ is equal to $\mathbb{Z}/3\alpha$, where $\Sigma^4\alpha = 8\tau$. Write $Jv = m\tau$. The order of Jv is thus a divisor of 15 and we cannot apply (3.8).

As noted in the proof of (2.7), fibrewise Hopf structures are classified by $[\zeta_B^+ \wedge_B \zeta_B^+; \zeta_B^+]_B$. So we have to show that this set maps, by restriction to a fibre, onto $\pi_{14}(S^7)$. According to (3.10) with $\xi = \zeta \oplus \zeta$ and $\eta = \zeta$ and hence $Ju = 2\Sigma^7 Jv = 2m\tau$, the obstruction to lifting the generator τ of $\pi_{14}(S^7)$ is $\tau \circ \Sigma^7(2m\tau) - (m\tau) \circ \Sigma^7\tau = m(\tau \circ \Sigma^7\tau)$ in $\pi_{21}(S^7)$. (Remember that S^7 is a Hopf space.). Now this obstruction has odd order. On the other hand, $64(\tau \circ \Sigma^7\tau) = \Sigma^4\alpha \circ \Sigma^{11}\alpha$. But $\Sigma^3\alpha \circ \Sigma^{10}\alpha = \alpha \wedge \alpha$, which, by symmetry, is equal to $(-1) \circ (\alpha \wedge \alpha)$. Hence $2(\Sigma^4\alpha \circ \Sigma^{11}\alpha) = 0$, and the obstruction vanishes, as required.

REMARK 3.11. The results above may be expressed more elegantly, if at a less elementary level, using fibrewise localization theory. In (3.7) the fibrewise 2-localizations of ζ_B^+ and η_B^+ are homotopy equivalent. The content of (3.6) is that the existence of a fibrewise Hopf structure depends only on the 2-local homotopy type of the sphere-bundle. And this fact lies behind (3.9), for there the fibrewise 2-localization of ζ_B^+ is fibre-homotopy trivial. Compare the results of Kahn [**17**, Theorem 2.1] and May [**18**, Proposition 5.5].

PROPOSITION 3.12. *Suppose that λ is a complex line bundle over B. Set $\zeta = \mathbb{R} \oplus \lambda^a \oplus \lambda^b \oplus \lambda^c$, where a, b, c are non-zero integers with $\nu_2(a) = \nu_2(b) < \nu_2(c)$. Let d be the highest common factor of a, b and c. Then ζ_B^+ admits fibrewise Hopf structures of each fibre type divisible by the highest common factor $(abc/d^3, 15)$.*

Proof. Recall that for any complex line bundle λ and integer k we have a map

$$\mathfrak{a}_k \colon S(\lambda) \longrightarrow S(\lambda^k), \tag{3.13}$$

(which is the solution of the Adams conjecture for line bundles) given by the kth tensor power $z \mapsto z^k$. It has fibre degree $(\pm)k$.

We shall show below that there exist odd integers i, j, k such that $ai + bj + ck = 0$ and $(ijk, 15) = (abc/d^3, 15)$. Put $\eta = \mathbb{R} \oplus \lambda^{ai} \oplus \lambda^{bj} \oplus \lambda^{ck}$. The fibrewise join

$$S(\mathbb{C}) *_B S(\lambda^a) *_B S(\lambda^b) *_B S(\lambda^c) \xrightarrow{1 * \mathfrak{a}_i * \mathfrak{a}_j * \mathfrak{a}_k} S(\mathbb{C}) *_B S(\lambda^{ai}) *_B S(\lambda^{bj}) *_B S(\lambda^{ck}), \tag{3.14}$$

gives a (fibrewise pointed) map $\zeta_B^+ \to \eta_B^+$ of odd fibre degree $(\pm)\, ijk$.

By (2.2)(ii), η_B^+ admits a fibrewise Hopf structure of fibre type ± 1. The result follows from (3.7) and (3.8).

It remains to show that there exist integers i, j, k with the property claimed. Write $\delta = \nu_2(c) - \nu_2(a)$, and let e be an odd integer such that $e \cdot 2^\delta \equiv 2 \pmod{15}$. Set $a = a'd$, $b = b'd$, $c = c'2^\delta d$. Then we may take $i = -b'c'$, $j = (1 - e2^\delta)a'c'$, $k = a'b'e$.

We conclude this section by constructing the equivariant Hopf structures required by Theorem 1.6. Fix then a prime $p > 1$, let G be the group $\mathbb{Z}/p^r$, and let E be the standard one-dimensional complex representation $\mathbb{C}$ with the generator acting as multiplication by $e^{2\pi i/p^r}$. We identify E^a with $\mathbb{C}$ again in the natural way using the ath power and this allows us to write $E^a = E^{a+p^r}$. The kth power thus gives an equivariant map, of degree k, $\mathfrak{a}_k \colon S(E^a) \to S(E^{a'})$ for integers a and a' satisfying the congruence $a' \equiv ka \pmod{p^r}$.

Let a, b, c be non-zero integers with $v_p(a) \leqslant v_p(b) \leqslant v_p(c)$, and $v_p(a) < r$; set $V = \mathbb{R} \oplus E^a \oplus E^b \oplus E^c$. We write $\alpha = v_p(a)$, $\beta = v_p(b)$, $\gamma = v_p(c)$.

LEMMA 3.15. *If V^+ admits a G-equivariant Hopf structure, then $v_p(a) = v_p(b)$.*

Proof. Consider the subspace of V^+ fixed by the subgroup $\mathbb{Z}/p^\beta$. This is a sphere with a Hopf structure, and is, therefore, of dimension 1, 3 or 7. Only the last is possible, and then $v_p(a) = v_p(b)$.

Suppose now that $\alpha = \beta \leqslant \gamma$, and, if $p = 2$, that $\alpha < \gamma$. We shall show that for such a representation V the sphere V^+ does admit a G-Hopf structure. The argument is similar to that used in the proof of (3.12). We construct G-maps of the type $\mathfrak{a}_k$ with k odd and prime to p as indicated below:

$$\begin{gathered} S(E^a) \longrightarrow S(E^{p^\alpha}) \longrightarrow S(E^a), \\ S(E^b) \longrightarrow S(E^{-p^\alpha - p^\gamma}) \longrightarrow S(E^b), \\ S(E^c) \longrightarrow S(E^{p^\gamma}) \longrightarrow S(E^c). \end{gathered} \tag{3.16}$$

(Note that, if p is odd, then one of $\mathfrak{a}_k$ and $\mathfrak{a}_{k+p^r}$ has odd degree.) According to (2.2)(i), the G-sphere

$$(\mathbb{R} \oplus E^{p^\alpha} \oplus E^{-p^\alpha - p^\gamma} \oplus E^{p^\gamma})^+$$

admits an equivariant Hopf structure. Using it we may construct a multiplication $\mu \colon V^+ \times V^+ \to V^+$ such that $\mu_L = \mu_R$ is the join $h \colon V^+ \to V^+$ of the three maps listed at (3.16) above together with $1 \colon S(\mathbb{C}) \to S(\mathbb{C})$.

We must now analyse $[V^+; V^+]^G$ and we do this in the standard way by looking at degrees on fixed subspaces. We write

$$d = (d_0, \ldots, d_r) \colon [V^+; V^+]^G \longrightarrow \prod_{0 \leqslant i \leqslant r} \mathbb{Z}, \tag{3.17}$$

where $d_i(f)$ is the degree of the restriction of a G-map $f \colon V^+ \to V^+$ to the subspace, a sphere, fixed by the subgroup $\mathbb{Z}/p^i$ of order p^i. From consideration of fixed points, we see that $d_i(f)$ is equal to $d_\alpha(f)$ if $i \leqslant \alpha$, $d_\gamma(f)$ if $\alpha < i \leqslant \gamma$, $d_r(f)$ if $\gamma < i \leqslant r$.

To complete the proof it suffices to find a map $g \colon V^+ \to V^+$ such that

$$d(g \circ h) = (2k+1)(1, \ldots, 1)$$

for some k. For then the construction of (3.6) applied to $g \circ \mu$ gives a multiplication on V^+ which on each factor of $V^+ \times V^+$ is a map f with $d(f) = (1, \ldots, 1)$. It follows that f is a G-homotopy equivalence, and, therefore (compare the opening paragraph of this section), that V^+ admits a G-Hopf structure.

We proceed to find g. Let R be the subring of $\prod_{0 \leqslant i \leqslant r} \mathbb{Z}$ consisting of all $(s_0, \ldots, s_r)$ with: $s_i = s_\alpha$ if $i \leqslant \alpha$, $s_i = s_\gamma$ if $\alpha < i \leqslant \gamma$, $s_i = s_r$ if $\gamma < i \leqslant r$, and $s_{i+1} \equiv s_i \pmod{p^{r-i}}$ for $0 \leqslant i < r$. Because V^+ is a suspension, the image of d, (3.17), is a subring. By looking at self-maps of $S(E^t)$, $t = a$, b or c, joined with the identity on the other three factors, one sees that $R \subseteq \operatorname{im} d$. It is clear, from the definition of h above, that $d(h) \in R$. (In fact, by looking at the Burnside ring one can see that $\operatorname{im} d$ is precisely R. See tom Dieck [11, II(5.17)] for a general result.) Moreover, each $d_i(h)$ is odd.

By elementary algebra one can show that there is a map $g \colon V^+ \to V^+$ with $d(g \circ h)$ as required. (The ring $R_{(p)}$ is local and $R[1/p]$ is a product of factors $\mathbb{Z}[1/p]$.)

REMARK 3.18. If p is odd and ζ is the bundle $\mathbb{R} \oplus \lambda^a \oplus \lambda^b \oplus \lambda^c$, over a mod p^r lens space B, then by (3.7) ζ_B^+ has a fibrewise Hopf structure, whatever the integers a, b, c. For $p > 5$ there will be Hopf structures of each fibre type, by (3.8). We have not determined the possible fibre types for $p = 3$ and $p = 5$.

4. *Obstructions to the existence of fibrewise Hopf structures*

Our strategy is a fibrewise generalization of one version of the standard homology and K-theory proof, mostly contained in Adams's work **[2]** on the vector field theorem, that the sphere S^n does not admit a Hopf structure unless $n = 1, 3$ or 7, and we begin by reviewing that argument.

We shall use ω for unreduced stable homotopy theory. Following **[10**, Section 4**]**, we shall often use the local coefficient notation $\omega^*(X;\alpha)$ for the stable cohomotopy $\tilde{\omega}^*(X^\alpha)$ of the Thom space of a virtual bundle α over a finite complex X. A subscript '+' will denote adjunction of a disjoint basepoint to a space. We write $P(V)$ for the real projective space on a vector space V, and $P(\zeta)$ for the bundle of projective spaces over B associated to the vector bundle ζ.

Recall first the stable Hopf invariant

$$H_k : \pi_{k+q}(S^{k+r}) \longrightarrow \tilde{\omega}_{q-r}(P(\mathbb{R}^{k+r})/P(\mathbb{R}^r)),$$

defined for $k \geqslant 1, q, r \geqslant 0$. (See, for example, **[15**, Chapter 14, or **19]**.) In the range $q < 3r-2$ this map is the 'H' of the James–Toda EHP-sequence

$$\cdots \longrightarrow \pi_q(S^r) \xrightarrow{E} \pi_{k+q}(S^{k+r}) \xrightarrow{H} \tilde{\omega}_{q-r}(P(\mathbb{R}^{k+r})/P(\mathbb{R}^r)) \xrightarrow{P} \cdots.$$

In any case $H_k x$ is an obstruction to k-fold desuspension of a class $x \in \pi_{k+q}(S^{k+r})$ to $\pi_q(S^r)$. The obstruction H_1 to a single desuspension is the classical stable Hopf invariant. The maps H_k for different k are compatible; to be precise, for $j < k$ we have a commutative square

$$\begin{array}{ccc} \pi_{k+q}(S^{k+r}) & \xrightarrow{H_k} & \tilde{\omega}_{q-r}(P(\mathbb{R}^{k+r})/P(\mathbb{R}^r)) \\ \downarrow = & & \downarrow \pi_* \\ \pi_{(k+q-j)+j}(S^{(k+r-j)+j}) & \xrightarrow[H_j]{} & \tilde{\omega}_{q-r}(P(\mathbb{R}^{k+r})/P(\mathbb{R}^{k+r-j})) \end{array} \qquad (4.1)$$

in which π is the projection map.

Given a Hopf structure $\mu\colon S^n \times S^n \to S^n$ on S^n, the Hopf construction gives a map $h_\mu\colon S^{2n+1} \to S^{n+1}$ with classical Hopf invariant, $H_1(h_\mu)$, equal to $(\pm)1$. It follows from (4.1) that the element

$$H_{n+1}(h_\mu) \in \tilde{\omega}_n(P(\mathbb{R}^{n+1})_+) \text{ is mapped to } (\pm)1 \in \tilde{\omega}_n(P(\mathbb{R}^{n+1})/P(\mathbb{R}^n)) = \mathbb{Z}.$$

The first step of the argument uses mod 2 homology H. If we apply the Hurewicz map $\tilde{\omega}_n(P(\mathbb{R}^{n+1})_+) \to \tilde{H}_n(P(\mathbb{R}^{n+1})_+)$ to $H_{n+1}(h_\mu)$ we obtain a class in $\tilde{H}_n(P(\mathbb{R}^{n+1})_+)$ which is fixed by the total Steenrod square Sq and which maps (under π^*) to $1 \in \tilde{H}_n(P(\mathbb{R}^{n+1})/P(\mathbb{R}^n)) = \mathbb{F}_2$. From this one deduces easily that $n+1$ is necessarily a power of 2.

In the second step we replace mod 2 homology by real KO-theory. Using the Hurewicz map to $KO_{(2)}$, we obtain an element of $\widetilde{KO}_n(P(\mathbb{R}^{n+1})_+)_{(2)}$ fixed by the Adams operation ψ^3 and mapping to $1 \in \widetilde{KO}_n(P(\mathbb{R}^{n+1})/P(\mathbb{R}^n))_{(2)} = \mathbb{Z}_{(2)}$. If $n+1 \equiv 0 \pmod 8$, it follows readily that $n = 7$.

We shall repeat the above argument *over the base B*. The definition of the stable Hopf invariant outlined in [7, pp. 60–63], extends easily to the theory over B. Details of the construction and of the properties which we need can be found in [6]. Let κ and η be real vector bundles over B, and $Z \to B$ a (nice) fibrewise pointed space over B. If $X \to B$ and $Y \to B$ are (nice) pointed spaces over B, we write $[X; Y]_B$ for the set of homotopy classes of fibrewise pointed maps from X to Y, and $\omega_B^0\{X; Y\}$ for the group of stable homotopy classes over B. Then we have a stable Hopf invariant

$$H_k\colon [\kappa_B^+ \wedge_B Z; (\kappa \oplus \eta)_B^+]_B \longrightarrow \omega_B^0\{Z; (P(\kappa \oplus \eta)/_B P(\eta)) \wedge_B \eta_B^+\}, \tag{4.2}$$

which is an obstruction to fibrewise desuspension to $[Z; \eta_B^+]_B$.

Suppose now that $\mu\colon \zeta_B^+ \times_B \zeta_B^+ \to \zeta_B^+$ is a fibrewise Hopf structure. Let

$$h_\mu \in [(\mathbb{R} \oplus \zeta \oplus \zeta)_B^+; (\mathbb{R} \oplus \zeta)_B^+]_B = [\Sigma_B(\zeta_B^+ \wedge_B \zeta_B^+); \Sigma_B \zeta_B^+]_B$$

denote the class given by the fibrewise Hopf construction $\zeta_B^+ *_B \zeta_B^+ \to \Sigma_B \zeta_B^+$. Recall that the projection π from the Cartesian to the smash product induces an inclusion

$$[\Sigma_B(\zeta_B^+ \wedge_B \zeta_B^+); \Sigma_B \zeta_B^+]_B \longrightarrow [\Sigma_B(\zeta_B^+ \times_B \zeta_B^+); \Sigma_B \zeta_B^+]_B$$

under which h_μ maps to

$$\pi^* h_\mu = -[\Sigma \mu_L \pi_L] + [\Sigma \mu] - [\Sigma \mu_R \pi_R], \tag{4.3}$$

where π_L and π_R are the projections onto the left- and right-hand factors. (Although this is well known we have been unable to find a reference. For completeness we include an outline proof in an appendix to this section.)

We next compute the Hopf invariant $H_{\mathbf{R}}(h_\mu)$. The quotient $P(\mathbb{R} \oplus \zeta)/_B P(\zeta)$ is identified in the usual way with ζ_B^+. We have a suspension isomorphism

$$\omega^0(B) = \omega_B^0\{0_B^+; 0_B^+\} \longrightarrow \omega_B^0\{\zeta_B^+; \zeta_B^+\},$$

which then allows us to write $\omega_B^0\{\zeta_B^+; P(\mathbb{R} \oplus \zeta)/_B P(\zeta)\}$ as $\omega^0(B)$. The stabilization $\mathbf{h} \in \omega_B^0\{\zeta_B^+ \wedge_B \zeta_B^+; \zeta_B^+\}$ of h_μ can be regarded by suspension as an element of the group $\omega_B^0\{\zeta_B^+; 0_B^+\}$, which is naturally identified with the stable homotopy $\tilde{\omega}^0(B^\zeta)$ of the Thom space of ζ. Making these identifications and using the local coefficient notation, we have a class $\mathbf{h} \in \omega^0(B; \zeta)$. We also need the stable cohomotopy Euler class $\gamma(\zeta) \in \omega^0(B; -\zeta)$. Again see [10].

LEMMA 4.4. *The stable Hopf invariant*

$$H_{\mathbf{R}}(h_\mu) \in \omega^0(B) = \omega_B^0\{\zeta_B^+; P(\mathbb{R} \oplus \zeta)/_B P(\zeta)\}$$

is equal to $-(1 + 2\gamma(\zeta) \cdot \mathbf{h})$, *where* $\mathbf{h}$ *is the stabilization of* h_μ *and* $\gamma(\zeta)$ *is the stable cohomotopy Euler class.*

Proof. We lift to $\zeta_B^+ \times_B \zeta_B^+$. The Hopf invariant $H_{\mathbf{R}}$ above extends to

$$H_{\mathbf{R}}\colon [\Sigma_B(\zeta_B^+ \times_B \zeta_B^+); \Sigma_B \zeta_B^+]_B \longrightarrow \omega_B^0\{\zeta_B^+ \times_B \zeta_B^+; \zeta_B^+ \wedge_B \zeta_B^+\}.$$

This obeys the usual additivity formula for the classical Hopf invariant, namely

$$H_{\mathbf{R}}(x + y) = H_{\mathbf{R}}(x) + H_{\mathbf{R}}(y) + \bar{x} \wedge \bar{y},$$

for classes x, y stabilizing to $\bar{x}, \bar{y}$. (See, for example, [6].) We use this repeatedly, together with the fact that $H_{\mathbf{R}}(x) = 0$ if x is a suspension. In particular, $H_{\mathbf{R}}(-y) = y \wedge y - H_{\mathbf{R}}(y)$.

One obtains

$$-\pi^* H_{\mathbb{R}}(h_\mu) = [\mu_L \pi_L] \wedge [\mu_R \pi_R] + [\mu_L \pi_L] \wedge \pi^*\mathbf{h} + \pi^*\mathbf{h} \wedge [\mu_R \pi_R],$$

where $\pi^*\mathbf{h} = [\mu] - [\mu_L \pi_L] - [\mu_R \pi_R]$. We look at the three terms on the right in turn. The first is π^*1. The second is the lift of the stable composition

$$\zeta_B^+ \wedge_B \zeta_B^+ \xrightarrow{\Delta \wedge 1} \zeta_B^+ \wedge_B \zeta_B \wedge_B \zeta_B^+ \xrightarrow{1 \wedge \mathbf{h}} \zeta_B^+ \wedge_B \zeta_B^+,$$

where Δ is the diagonal inclusion. But $\Delta\colon \zeta_B^+ \to \zeta_B^+ \wedge_B \zeta_B^+$ is fibrewise homotopic to the product $z \wedge 1\colon 0_B^+ \wedge_B \zeta_B^+ \to \zeta_B^+ \wedge_B \zeta_B^+$ of the inclusion z of the zero-section and the identity, or, in the opposite order, to $1 \wedge z$. By definition, z represents the Euler class $\gamma(\zeta) \in \omega^0(B; -\zeta) = \omega_B^0\{0_B^+; \zeta_B^+\}$. (See, for example, [9, Section 1].) Hence the second term is $\pi^*(\gamma(\zeta)\cdot\mathbf{h}) = \pi^*(\mathbf{h}\cdot\gamma(\zeta))$; the third is the same.

Since the stable Hopf invariant $H_{\mathbb{R}\oplus\zeta}(h_\mu) \in \omega_B^0\{\zeta_B^+;\ P(\mathbb{R}\oplus\zeta)_{+B}\}$ maps to $H_{\mathbb{R}}(h_\mu)$, we deduce the following necessary condition for the existence of a fibrewise Hopf structure.

PROPOSITION 4.5. *If ζ_B^+ admits a fibrewise Hopf structure, then there is a class $\mathbf{H} \in \omega_B^0\{\zeta_B^+; P(\mathbb{R}\oplus\zeta)_{+B}\}$ mapping to $1 + 2\gamma(\zeta)\cdot\mathbf{h}$ in $\omega^0(B) = \omega_B^0\{\zeta_B^+; P(\mathbb{R}\oplus\zeta)/_B P(\zeta)\}$, for some class $\mathbf{h}$ in $\omega^0(B;\zeta)$.*

The group $\omega_B^0\{\zeta_B^+; P(\mathbb{R}\oplus\zeta)_B^+\}$ can be identified by duality over B with

$$\omega_B^0\{\zeta_B^+ \wedge_B P(\mathbb{R}\oplus\zeta)_B^{\mathbb{R}-H\otimes(\mathbb{R}\oplus\zeta)}; 0_B^+\} = \omega^0(P(\mathbb{R}\oplus\zeta); \mathbb{R}\oplus\zeta - H\otimes(\mathbb{R}\oplus\zeta)).$$

In fact this duality is essential to the construction in [6] of the Hopf invariant, and it is the dual form which we shall use in our calculations. So we have a class $\mathbf{H} \in \omega^0(P(\mathbb{R}\oplus\zeta); \mathbb{R}\oplus\zeta - H\otimes(\mathbb{R}\oplus\zeta))$ restricting to $1 + 2\gamma(\zeta)\cdot\mathbf{h}$ in

$$\omega^0(B) = \omega^0(B \times P(\mathbb{R}); \mathbb{R}\oplus\zeta - H\otimes(\mathbb{R}\oplus\zeta)).$$

We shall compute the cohomology of the projective bundle using $\mathbb{Z}/2$-equivariant methods. Let L denote the real representation $\mathbb{R}$ of $\mathbb{Z}/2$ with the involution -1. Then the projective bundle $P(\mathbb{R}\oplus\zeta)$ is the quotient of the sphere-bundle $S(L\otimes(\mathbb{R}\oplus\zeta))$ by the free involution, and the Hopf line bundle H corresponds to the trivial bundle L. We have

$$\omega^0(P(\mathbb{R}\oplus\zeta); \mathbb{R}\oplus\zeta - H\otimes(\mathbb{R}\oplus\zeta)) = \omega_{\mathbb{Z}/2}^0(S(L\otimes(\mathbb{R}\oplus\zeta)); (\mathbb{R}\oplus\zeta) - L\otimes(\mathbb{R}\oplus\zeta)),$$

and this last group can be calculated from the (Gysin) exact sequence

$$\begin{aligned} \ldots \longrightarrow \omega_{\mathbb{Z}/2}^0(B; \mathbb{R}\oplus\zeta) &\xrightarrow{\cdot\gamma(L\otimes(\mathbb{R}\oplus\zeta))} \omega_{\mathbb{Z}/2}^0(B; \mathbb{R}\oplus\zeta - L\otimes(\mathbb{R}\oplus\zeta)) \\ &\xrightarrow{\delta} \omega^0(P(\mathbb{R}\oplus\zeta); \mathbb{R}\oplus\zeta - H\otimes(\mathbb{R}\oplus\zeta)) \longrightarrow \omega_{\mathbb{Z}/2}^1(B; \mathbb{R}\oplus\zeta) \longrightarrow \ldots \end{aligned} \tag{4.6}$$

of the disc modulo the sphere. See [8, Section 1].

Appendix: the Hopf construction

Let X, Y and Z all be pointed compact ENR, $\mu\colon X \times Y \to Z$ a pointed map. The Hopf construction

$$h_\mu\colon X * Y \longrightarrow \Sigma Z$$

is defined on the (reduced) join by $[x, t, y] \mapsto [t, \mu(x, y)]$. Write $f\colon X * Y \to \Sigma(X \times Y)$ for $[x, t, y] \mapsto [t, (x, y)]$ and let $\pi\colon X \times Y \to X \wedge Y$ denote the projection. Then $\Sigma\pi \circ f$ is a homotopy equivalence.

LEMMA 4.7. *In the group* $[\Sigma(X\times Y);\Sigma Z]$ *we have*

$$h_\mu \circ (\Sigma\pi\circ f)^{-1}\circ \Sigma\pi = (\Sigma\mu)\circ(-\Sigma l+\Sigma 1-\Sigma r),$$

where $l(x,y)=(x,*)$, 1 *is the identity map on* $X\times Y$, *and* $r(x,y)=(*,y)$.

Proof. Define $g\colon \Sigma(X\times Y)\to X*Y$ by

$$[t,(x,y)]\longmapsto \begin{cases} [x,1-3t,*] & \text{if } 0\leqslant t\leqslant \frac{1}{3},\\ [x,3t-1,y] & \text{if } \frac{1}{3}\leqslant t\leqslant \frac{2}{3},\\ [*,3-3t,y] & \text{if } \frac{2}{3}\leqslant t\leqslant 1.\end{cases}$$

Then $f\circ g$ is equal to $-\Sigma l+\Sigma 1-\Sigma r$ in $[\Sigma(X\times Y);\Sigma(X\times Y)]$ and $\Sigma\pi\circ(f\circ g)$ is homotopic to $\Sigma\pi$. The result follows.

The proof of the fibrewise version used above is a routine extension using Dold's theorem.

5. *Homology computations*

In this section H^* will denote cohomology with $\mathbb{F}_2$-coefficients. We use notation for H^* corresponding exactly with that for stable cohomotopy ω^*. To avoid cumbersome formulae we shall often use the same symbol for a stable cohomotopy class and its Hurewicz image in cohomology.

We apply the Hurewicz homomorphism from stable cohomotopy to mod 2 cohomology to the criterion (4.5). The argument is conveniently illustrated by the diagram

$$\begin{array}{ccc} \omega^0_B\{\zeta^+_B;\ P(\mathbb{R}\oplus\zeta)_{+B}\} & \longrightarrow & \omega^0(B)\\ \Big\downarrow{\scriptstyle \text{Hurewicz}} & & \Big\downarrow{\scriptstyle \text{Hurewicz}}\\ H^0_B\{\zeta^+_B;\ P(\mathbb{R}\oplus\zeta)_{+B}\} & \longrightarrow & H^0(B)\end{array}$$

Here H^*_B denotes the cohomology over B, defined, for example, in terms of spectra over B: if $X\to B$ and $Y\to B$ are (nice) fibrewise pointed spaces over B then $H^0_B\{X;Y\}$ is the group $[\mathbf{X};\mathbf{Y}\wedge_B(B\times\mathbf{K})]_B$, where $\mathbf{X}$ and $\mathbf{Y}$ are the suspension spectra of X and Y over B and $\mathbf{K}$ is the mod 2 Eilenberg–MacLane spectrum.

The Hurewicz image of the class $\mathbf{H}$ in (4.5), which we denote by the same letter, is an element $\mathbf{H}\in H^0_B\{\zeta^+_B;P(\mathbb{R}\oplus\zeta)_{+B}\}$, fixed (since it is spherical) by the total Steenrod square Sq, and restricting to $1\in H^0(B)$. This leads to the following condition.

PROPOSITION 5.1. *A necessary condition for* ζ^+_B *to admit a fibrewise Hopf structure is that*

$$w_i(\zeta)=0 \text{ if } n+1-i \text{ is not a power of } 2.$$

In particular, if $n=3$, then $w_1\zeta=0$; if $n=7$, then $w_1\zeta=0$, $w_2\zeta=0$, so that ζ has a spin structure, (and then, necessarily, $w_3=Sq^1w_2+w_1w_2$ and $w_5=Sq^2w_3+w_1w_4+w_2w_3$ vanish). We also obtain the classical restriction that $n+1$ be a power of 2. This establishes the necessary conditions of (1.3).

REMARK 5.2. The original proof of these conditions on the Stiefel–Whitney classes, given in [5], used computations in the cohomology of the fibrewise projective plane.

For the proof of (5.1) we replace $H_B^0\{\zeta_B^+; P(\mathbb{R}\oplus\zeta)_{+B}\}$ by

$$H^0(P(\mathbb{R}\oplus\zeta); \mathbb{R}\oplus\zeta - H\otimes(\mathbb{R}\oplus\zeta))$$

and use the cohomology version

$$\ldots \longrightarrow H^0_{\mathbb{Z}/2}(B;\mathbb{R}\oplus\zeta) \xrightarrow{\cdot\gamma(L\otimes(\mathbb{R}\oplus\zeta))} H^0_{\mathbb{Z}/2}(B;\mathbb{R}\oplus\zeta - L\otimes(\mathbb{R}\oplus\zeta))$$
$$\xrightarrow{\delta} H^0(P(\mathbb{R}\oplus\zeta);\mathbb{R}\oplus\zeta - H\otimes(\mathbb{R}\oplus\zeta)) \longrightarrow H^1_{\mathbb{Z}/2}(B;\mathbb{R}\oplus\zeta)\longrightarrow\ldots \quad (5.3)$$

of the Gysin sequence (4.6). Here by $H^*_{\mathbb{Z}/2}$ we mean Borel cohomology, which has coefficient ring $H^*_{\mathbb{Z}/2}(*) = \mathbb{F}_2[[T]]$, where $T\in H^1_{\mathbb{Z}/2}(*)$ is the Euler class of L.

Now we have Thom classes $u\in H^{n+1}(B;\mathbb{R}\oplus\zeta)$ of $\mathbb{R}\oplus\zeta$ and

$$\tilde{u}\in H^{n+1}_{\mathbb{Z}/2}(B; L\otimes(\mathbb{R}\oplus\zeta))$$

of $L\otimes(\mathbb{R}\oplus\zeta)$. Recollect that the total Steenrod square acts on a Thom class as multiplication by the total Stiefel–Whitney class.

Consider the class **H**, given by (4.5), in $H^0(P(\mathbb{R}\oplus\zeta);\mathbb{R}\oplus\zeta-H\otimes(\mathbb{R}\oplus\zeta))$. Because **H** maps to 1 in $H^0(B)$, it must be the image in (5.3) of the generator $u\cdot\tilde{u}^{-1}$ of $H^0_{\mathbb{Z}/2}(B;\mathbb{R}\oplus\zeta-L\otimes(\mathbb{R}\oplus\zeta))$. Because **H** is fixed by Sq, we see, from an examination of the sequence (5.3), that there is an element y of $H^0_{\mathbb{Z}/2}(B;\mathbb{R}\oplus\zeta)$ such that

$$Sq(u\cdot\tilde{u}^{-1}) - u\cdot\tilde{u}^{-1} = y\cdot\gamma(L\otimes(\mathbb{R}\oplus\zeta))\in H^*_{\mathbb{Z}/2}(B;\mathbb{R}\oplus\zeta-L\otimes(\mathbb{R}\oplus\zeta)),$$

or, more usefully, that

$$u^{-1}Sq(u) - \tilde{u}^{-1}Sq(\tilde{u}) = y'\cdot\tilde{u}\cdot\gamma(L\otimes(\mathbb{R}\oplus\zeta))\in H^*_{\mathbb{Z}/2}(B)$$

for some y' in $H^*_{\mathbb{Z}/2}(B)$. This statement is easily rewritten as

$$e(L\otimes(\mathbb{R}\oplus\zeta)) \text{ divides } w(\mathbb{R}\oplus\zeta) - w(L\otimes(\mathbb{R}\oplus\zeta)), \quad (5.4)$$

where w denotes the total Stiefel–Whitney class and $e(L\otimes(\mathbb{R}\oplus\zeta))$ is the classical cohomology Euler class $\tilde{u}\cdot\gamma(L\otimes(\mathbb{R}\oplus\zeta))$. (The relation between γ and the classical Euler class e is described, for example, in [9, Lemma 1.1].) The criterion (5.4) readily reduces to

$$\sum_{i=0}^{n}((1+T)^{n+1-i} - (1+T^{n+1-i}))\, w_i\zeta = 0 \quad (5.5)$$

in $H^*(B)[[T]]$, which immediately gives the condition of the proposition.

6. *K-theory computations*

In this section we map (4.5) to KO-theory. For notation we follow that used for cohomology in the preceding section; the method, too, runs parallel to that in Section 5. We may assume that $n\equiv 7 \pmod 8$ and that ζ admits a spin structure.

Suppose first that q is an odd integer and ξ is a real vector bundle over B of dimension $0 \pmod 8$ which admits a spin structure. A choice of spin structure determines a Bott class $u\in KO^0(B;\xi)$. (A different choice will multiply u by $\pm$ the class of a real line bundle.) The Bott cannibalistic class $\rho^q(\xi)$ in $KO^0(B)$ is defined by the equation $\psi^q(u) = \rho^q(\xi)\cdot u$; it is independent of the choice of spin structure and orientation. We shall also need a $\mathbb{Z}/2$-equivariant version. Write $KO^0_{\mathbb{Z}/2}(*) = \mathbb{Z}1\oplus\mathbb{Z}t$, where $t = [L]$. The bundle $L \otimes \xi$ over B admits a $\mathbb{Z}/2$-equivariant spin structure and Bott class $\tilde{u} \in KO^0_{\mathbb{Z}/2}(B)$ lifting u. We now define classes $\rho^q_+(\xi), \rho^q_-(\xi)$ in $KO^0(B)$ by the equation

$$\psi^q(\tilde{u}) = (\rho^q_+(\xi) + \rho^q_-(\xi)\, t)\, \tilde{u}.$$

(For more information the reader is referred to [8, Section 2].) In what follows ξ will be $\mathbb{R} \oplus \zeta$.

Consider next the KO-theory Euler class

$$\gamma(L \otimes (\mathbb{R} \oplus \zeta)) \text{ in } KO^0_{\mathbb{Z}/2}(B; -L \otimes (\mathbb{R} \oplus \zeta)).$$

Given a choice of spin structure as above, we have the classical KO-Euler class $\tilde{u} \cdot \gamma(L \otimes (\mathbb{R} \oplus \zeta))$ in $KO^0_{\mathbb{Z}/2}(B) = KO^0(B) \oplus KO^0(B)\, t$, and it must be of the form $(1-t)\mathbf{e}$ for some $\mathbf{e} \in KO^0(B)$, since $\mathbb{R} \oplus \zeta$ has a non-zero section and the non-equivariant Euler class obtained by replacing t by 1 must vanish.

To obtain a manageable criterion from the KO-Hurewicz image of (4.5) it is convenient to assume that $KO^1(B)$ has no 2-torsion.

PROPOSITION 6.1. *Suppose that $n \equiv 7 \pmod 8$ and that ζ^+_B admits a fibrewise Hopf structure. Write the classical KO-Euler class of $L \otimes (\mathbb{R} \oplus \zeta)$ for some choice of equivariant spin structure as $(1-t)\mathbf{e}$, $\mathbf{e} \in KO^0(B)$. Assume that $KO^1(B)_{(2)}$ is torsion-free. Then there exists a class $\mathbf{f} \in KO^0(B)_{(2)}$ such that*

$$\mathbf{e} \text{ divides } (1 + 2\mathbf{f})\, \rho^q_-(\mathbb{R} \oplus \zeta) + (\psi^q \mathbf{f} - \mathbf{f})\, \rho^q(\mathbb{R} \oplus \zeta)$$

in $KO^0(B)_{(2)}$.

Proof. The criterion (4.5) mapped to 2-local KO-theory gives spherical classes $\mathbf{h} \in KO^0(B; \zeta)_{(2)}$ and

$$\mathbf{H} \in KO^0_B\{\zeta^+_B; P(\mathbb{R} \oplus \zeta)_{+B}\}_{(2)} = KO^0(P(\mathbb{R} \oplus \zeta); \mathbb{R} \oplus \zeta - H \otimes (\mathbb{R} \oplus \zeta))_{(2)}$$

such that $\mathbf{H}$ restricts to $1 + 2\gamma(\zeta) \cdot \mathbf{h}$ in $KO^0(B)_{(2)}$.

First, we shall show that $\mathbf{h}$ is zero. The group $KO^0(B; \zeta)_{(2)}$ is isomorphic to $KO^1(B)_{(2)}$ by Bott periodicity, so is torsion-free; but $\omega^0(B; \zeta) \otimes \mathbb{Q}$ is zero, because rational cohomology $H^{-n}(B; \mathbb{Q})$ is zero. Hence $\mathbf{h} = 0$.

We look next at the $KO_{(2)}$-Gysin sequence (4.6). A similar argument shows that only the zero element of $KO^1_{\mathbb{Z}/2}(B; \mathbb{R} \oplus \zeta)_{(2)}$ is fixed by ψ^q. (The subspace of $KO^1(B; \mathbb{R} \oplus \zeta) \otimes \mathbb{Q}$ fixed by ψ^q is isomorphic to $H^{-n}(B; \mathbb{Q}) = 0$ again. Because ψ^q fixes t it respects the decomposition of $KO^1_{\mathbb{Z}/2}(B; \mathbb{R} \oplus \zeta)_{(2)}$ as

$$KO^1(B; \mathbb{R} \oplus \zeta)_{(2)} \oplus KO^1(B; \mathbb{R} \oplus \zeta)_{(2)}\, t.$$

The assertion follows.)

This means that $\mathbf{H}$ lifts to a class in

$$KO^0_{\mathbb{Z}/2}(B; \mathbb{R} \oplus \zeta - L \otimes (\mathbb{R} \oplus \zeta))_{(2)}$$

(isomorphic under Bott periodicity to $KO^0_{\mathbb{Z}/2}(B)_{(2)} = KO^0(B)_{(2)} \oplus KO^0(B)_{(2)}\, t$) of the form $u \cdot \tilde{u}^{-1}(1 + (1-t)\mathbf{f})$. The assertion that $\mathbf{H}$ is fixed by ψ^q becomes the statement that $(1-t)\mathbf{e}$ divides $\rho^q(\xi) \cdot (\rho^q_+(\xi) + \rho^q_-(\xi) \cdot t)^{-1}(1 + (1-t)\, \psi^q \mathbf{f}) - (1 + (1-t)\mathbf{f})$, which quickly reduces to the form given.

The classical theorem of Adams for B a point now follows easily. Write $n = 8m - 1$. Then $\mathbf{e} = 2^{4m-1}$ and $\rho^3_-(\mathbb{R}^{8m}) = \frac{1}{2}(3^{4m} - 1)$ in $\mathbb{Z} = KO^0(*)$. (See Section 8 for similar, more complicated calculations.) The condition (6.1) becomes: $2^{4m-1} \mid \frac{1}{2}(3^{4m} - 1)$, whence $m = 1$.

7. *Proof of Theorem* 1.4

In this section we take $B = S^8$ and $n = 7$. Recall that the group $\pi_7(O(7))$ is infinite cyclic and maps under stabilization onto $2\mathbb{Z} \subseteq \mathbb{Z} = \pi_7(O(\infty))$. The vector bundle ζ is classified by m times a generator of $\pi_7(O(7))$, and if we write v for a generator of $\widetilde{KO}^0(S^8)$, we have $[\zeta] = 7+2mv \in KO^0(B) = \mathbb{Z}[v]/(v^2)$.

Recall, too, that under the J-map, $\pi_7(O(7))$ maps onto

$$\pi_7(\Omega^7 S^7) = \pi_{14}(S^7) = \mathbb{Z}/120.$$

So if m is divisible by 8 we deduce from (3.9) that ζ_B^+ admits fibrewise Hopf structures of each fibre type. Notice here that $\pi_7(G_2)$ is zero [20] so that only if ζ is trivial does the structure group reduce to G_2.

Suppose then that ζ_B^+ does admit a Hopf structure. It remains to show that $8 \mid m$. We apply (6.1).

LEMMA 7.1. *We have*
(i) $\rho^3(\mathbb{R} \oplus \zeta) = 81-54mv$,
(ii) $\rho^3_-(\mathbb{R} \oplus \zeta) = 40-22mv$.

Proof. Part (i) is due to Adams [3]. For (ii), we may use the relation

$$\rho^q_+(\xi)-\rho^q_-(\xi) = \rho^q(\xi)^{-1}\cdot\psi^2(\rho^q(\xi)). \tag{7.2}$$

This is proved in [8, Lemma 2.6], for the complex theory; the same proof works in the real theory for spin classes.

LEMMA 7.3. *The classical KO-Euler class is given by*

$$\tilde{u}\cdot\gamma(L\otimes(\mathbb{R}\oplus\zeta)) = \pm(8-mv)(1-t).$$

Proof. The vector bundle ζ is stably complex; indeed, because $[\zeta]-7$ is divisible by 2 in $KO^0(B)$, there is an isomorphism $\mathbb{R}^9 \oplus \zeta \cong \mathbb{C}^4 \oplus \eta$ for some complex 4-dimensional bundle η. Also, complexification $KO^0(B) \to K^0(B)$ is an isomorphism, and we can compute in complex K-theory.

In complex K-theory we have the standard Bott class $\lambda_{L\otimes\eta} \in K^0_{\mathbb{Z}/2}(B; L\otimes\eta)$ given by the exterior algebra, and

$$\lambda_{L\otimes\eta}\cdot\gamma(L\otimes\eta) = \sum_{j=0}^{4}(-1)^j\lambda^j(L\otimes\eta). \tag{7.4}$$

This expression is easily seen to be $2+\lambda^2\eta-2t\eta$, since $\lambda^4\eta$ is trivial, and reduces, by application of the identity $2\lambda^2\eta = \eta^2-\psi^2\eta$ to $[\eta] = 4+mv$, to $(8-4mv)-(8+2mv)t$.

Since η is an $SU(4)$-bundle, the class $\lambda_{L\otimes\eta}$ lifts to a Bott class

$$\tilde{u} \in KO^0_{\mathbb{Z}/2}(B; L\otimes(\mathbb{R}\oplus\zeta)).$$

For this choice, we then obtain

$$(1-t)^4\tilde{u}\cdot\gamma(L\otimes(\mathbb{R}\oplus\zeta)) = (1-t)^4\lambda_{L\otimes\eta}\cdot\gamma(L\otimes\eta),$$

which gives the result.

The proof that $8 \mid m$ now follows easily from (6.1) with $q = 3$. We have 2-local integers a, b, c, d such that

$$(1+2(a+bv))(40-22mv)+80bv(81-54mv) = (c+dv)(8-mv).$$

At once we see that c is odd and that $-mc \equiv -(1+2a)\cdot 22m \pmod 8$. So 8 divides m. This completes the proof of Theorem 1.4.

8. *Proof of Theorems* 1.5 *and* 1.6

We use the notation introduced in the statements of the two theorems. The next proposition contains the main step in the proof.

PROPOSITION 8.1. *Suppose that* $r > 1$ *and write* $s = 2^{r-1}$. *Assume that either* (i) $\zeta = \mathbb{R} \oplus \lambda^s \oplus \lambda^s \oplus \lambda^s$ *or* (ii) $\zeta = \mathbb{R} \oplus \lambda^s \oplus \mathbb{C} \oplus \mathbb{C}$, *and that the sphere-bundle* ζ_B^+ *admits a fibrewise Hopf structure. Then* $m \leqslant r2^{r-1}$.

We use (6.1) to establish first the following lemma.

LEMMA 8.2. *Under the conditions of Proposition* 8.1, *if* $m \equiv 1 \pmod 4$, *then* $KO^1(B)$ *is torsion-free and the class* $2^r(1-[\lambda^s])$ *is zero in* $KO^0(B)$.

Computations are most easily performed using equivariant K-theory. Write $G = \mathbb{Z}/2^r$. Then we have $K_G^0(*) = \mathbb{Z}[z]/(z^{2^r}-1)$, where $z = [E]$. Complexification $KO_G^0(*) \to K_G^0(*)$ is injective and maps onto the subgroup generated by 1, $w := z^s$ and the elements $z^j + z^{-j}$ for $0 < j < s$.

Proof of Lemma 8.2. Writing mE for the m-fold direct sum, we have a Gysin sequence

$$\ldots \longrightarrow KO_G^0(*;mE) \xrightarrow{\gamma(mE)} KO_G^0(*)$$
$$\longrightarrow KO_G^0(S(mE)) = KO^0(B) \xrightarrow{\delta} KO_G^1(*;mE) \longrightarrow \ldots.$$

From this sequence one checks that $KO_G^0(*) \to KO^0(B)$ is surjective and $KO^1(B)$ is torsion-free (for $m \equiv 1 \pmod 4$).

We shall take $q = 2^r+1$. This means that ψ^q acts trivially on $K_G^0(*)$ and $KO^0(B)$, and simplifies computations considerably.

The calculations can all be done in the complex theory and come from the following standard result. Let F denote the standard 1-dimensional complex representation of the circle group $\mathbb{T}$. As before we write $\lambda_F \in K_{\mathbb{T}}^0(*;F)$ for the Bott generator. Then

$$\gamma(F)\cdot\lambda_F = 1-[F] \in K_{\mathbb{T}}^0(*) \tag{8.3}$$

and

$$\psi^q(\lambda_F) = (1+[F]+[F]^2+\ldots+[F]^{q-1})\cdot\lambda_F \in K_{\mathbb{T}}^0(*;F). \tag{8.4}$$

(See, for example, [3].)

In the notation of Section 6, we write $\tilde{u}\cdot\gamma(L\otimes(\mathbb{R}\oplus\zeta)) = \mathbf{e}(1-t)$. From (8.3) one sees that $\mathbf{e}$ is $\pm(1-t)(1-tw)^3$ in case (i), $\pm(1-t)^3(1-tw)$ in case (ii). In both cases these expressions simplify to give $\mathbf{e} = \pm 8(1+w)$. All that we shall use is

$$(1-w)\mathbf{e} = 0. \tag{8.5}$$

From (8.4) one obtains

$$\rho_+^q(\mathbb{R}\oplus\zeta)+\rho_-^q(\mathbb{R}\oplus\zeta)\,t = \begin{cases}(\tfrac{1}{2}(q+1)+\tfrac{1}{2}(q-1)\,t)(\tfrac{1}{2}(q+1)+\tfrac{1}{2}(q-1)\,tw)^3 & \text{in case (i),}\\ (\tfrac{1}{2}(q+1)+\tfrac{1}{2}(q-1)\,t)^3(\tfrac{1}{2}(q+1)+\tfrac{1}{2}(q-1)\,tw) & \text{in case (ii).}\end{cases}$$

Hence

$$(1-w)\rho^q_{-}(\mathbb{R}\oplus\zeta) = \begin{cases} -\frac{1}{2}q(q^2-1)\cdot(1-w) & \text{in case (i),} \\ \frac{1}{2}q(q^2-1)\cdot(1-w) & \text{in case (ii).} \end{cases} \tag{8.6}$$

Finally, multiply the criterion of (6.1) by $1-w$ to obtain in both cases the condition $\frac{1}{2}(q^2-1)(1-w) = 0$ in $KO^0(B)_{(2)}$. Hence $2^r(1-w) = 0$, since $r > 1$.

To prove (8.1) we now take $m = rs+1$ and show that $2^r(1-w)$ is non-zero in $KO^0(B)_{(2)}$. It is straightforward to check that $\gamma(mE)\cdot KO^0_G(*;mE)$ lies in the ideal $((z-2+z^{-1})^{(m+1)/2})$ of $KO^0_G(*)$. So it will suffice to show that

$$2^r(1-z^s) \text{ is non-zero in } \mathbb{Z}[z]/((z-1)^{m+1}, z^{2^r}-1). \tag{8.7}$$

This is a routine exercise, which runs as follows.

Put $X = z-1$. If $2^r(1-z^s)$ is zero, then we have

$$2^r(1-(1+X)^s) = PX^{m+1} + Q(1-(1+X)^{2s})$$

for some polynomials P,Q in $\mathbb{Z}[X]$. Hence the formal power series $2^r\cdot(1+(1+X)^s)^{-1}$ in $\mathbb{Q}[[X]]$ has integral coefficients up to degree $m-1$. Now $1+(1+X)^s$ can be written as $2(1+XR+\frac{1}{2}X^s)$ where R is a polynomial in X with integral coefficients. Expanding $2^{r-1}(1+XR+\frac{1}{2}X^s)^{-1}$ as $2^{r-1}\sum_{j\geq 0}(-1)^j(XR+\frac{1}{2}X^s)^j$, we see that the coefficient of X^{rs} is not integral.

This completes the proof of (8.1). The case $r = 1$ is covered by our final proposition.

PROPOSITION 8.8. *Let B be the real projective plane $P(\mathbb{R}^3)$, ζ the vector bundle $\mathbb{R}^{7-m}\oplus mH$, $0 \leqslant m \leqslant 7$. Then ζ^+_B admits a fibrewise Hopf structure if and only if m is 0 or 4.*

Proof. This follows from the spin condition, (1.3)(iii).

Proof of Theorem 1.5. Because the order of λ is 2^r, there is no loss of generality in assuming that a, b and c are prime to 15. The sufficiency of the condition is then established by (3.12). So suppose that either (i) $\nu_2(a) = \nu_2(b) = \nu_2(c)$ or (ii)$\nu_2(a) < \nu_2(b)$. If ζ^+_B admits a Hopf structure, so does its pull-back to $S(mE)/(\mathbb{Z}/2^{\alpha+1})$, where $\alpha = \nu_2(a)$. This contradicts (8.1) or (8.8).

Proof of Theorem 1.6. The case p odd and the sufficiency for $p = 2$ were dealt with in Section 3. To establish necessity for $p = 2$ we assume that V^+ admits a Hopf structure and form the associated bundle $\zeta = S(mE)\times_G V$ over the lens space B of (1.5), with $m > r2^{r-1}$. Since ζ is orientable, by (1.3), the representation V must be isomorphic to $\mathbb{R}\oplus E^a\oplus E^b\oplus E^c$ for some non-zero integers a, b, c with $\nu_2(a) \leqslant \nu_2(b) \leqslant \nu_2(c)$. The proof is completed by an appeal to (1.5).

REMARK 8.9. The case $G = \mathbb{Z}/2$ of (1.6) is included in the work of Iriye **[12]**. We note that the proof above depends upon nothing more than computations of Stiefel-Whitney classes.

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Mathematical Institute
24–29 St. Giles
Oxford OX1 3LB

Department of Mathematical Sciences
University of Aberdeen
Aberdeen AB9 2TY