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On codimension two and one splitting

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Abstract

According to Cappell and Shaneson, very often the secondary surgery obstruction O_2 to codimension two splitting is zero, but examples with $O_2 \neq 0$ exist. By considering homotopy equivalences with codomain the total space of a two disk bundle, we produce many examples with $O_2 \neq 0$ and other examples with $O_2 = 0$. Our approach is: "compare with codimension one splitting".

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1. The codimension two splitting problem

Here, CAT means the topological (TOP), the piecewise linear (PL) or the differentiable (DIFF) category. Details are to be found in [5]. If CAT is omitted, we mean any of the three categories.

Let $(X, \partial X)$ be a Poincaré pair with X connected. Let $(Y, \partial Y) \subset (X, \partial X)$, Y connected, be a Poincaré pair such that Y possesses a two-dimensional linear normal bundle in X. Let $f:(M, \partial M) \to (X, \partial X)$ be a homotopy equivalence split along the boundary, where M is an n-dimensional manifold with boundary. This means that f is transverse to ∂X and the restricted map $f|_Z : Z \to \partial Y$, $Z = f^{-1}(\partial Y)$, is a homotopy equivalence. The codimension two splitting problem (CTSP) is to find necessary and sufficient conditions for completion of the following diagram.

$$
\begin{array}{ccc}
M & \xrightarrow{f \text{ (homotopy equivalence of pairs)}} & X \\
\cup & & \cup \\
\begin{array}{c}\text{Locally flat}\\ \text{submanifold}\end{array} & \dashrightarrow_{\text{(homotopy equivalence of pairs)}} & Y
\end{array}
$$

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If the above diagram can be completed (after a homotopy of f relative the boundary if necessary) then, by definition, f is splittable along Y. It is possible to work with CAT = TOP, PL or DIFF. In addition, one can consider the simple splitting problem in which $(X, \partial X), (Y, \partial Y)$ are simple Poincaré pairs and the horizontal maps in the diagram above are required to be simple as well.

1.1. The surgery obstructions

For simplicity, unless explicitly required, orientation characters of surgery groups will be omitted [5,7].

As $f: M \to X$ is a homotopy equivalence, set $\nu = g^*\nu_M$. The bundle ν_M is the stable normal bundle of M and g is a homotopy inverse of f. Very basic bundle theory provides a bundle map $b: \nu_M \to \nu$ covering the homotopy equivalence $f: M \to X$. After a homotopy if necessary, we can assume that f is transverse to Y relative the boundary. Set $N = f^{-1}(Y)$. The abstract surgery obstruction is, by definition, the obstruction $\theta(f) = \theta(f|_N, b|_N) \in L^e_{n-2}(\pi_1 Y)$. The superscript e is s for the simple splitting problem. Otherwise, it is h. The vanishing of this obstruction is a necessary condition for the existence of a solution to the CTSP.

In higher codimensions (the hypothesis that Y has a linear normal bundle in X is artificial [1]), the vanishing of the abstract surgery obstruction enables one to use the normal cobordism extension property [1] (see also [6]) to produce a solution to the splitting problem [1]. In codimension two, one can use the normal cobordism extension property as well, provided the abstract surgery obstruction vanishes [5]. However, this may fail to produce a solution to the CTSP in even dimensions. The secondary obstruction measures this failure. More precisely, let n be even. Induction over the normal sphere bundle of $(Y, \partial Y) \subset (X, \partial X)$ induces a map $L^e_{n-1}(\pi_1 Y) \to L^e_{n+1}(\pi_*)$, where π denotes the projection of this bundle. Composing with natural maps, we obtain

$$\rho^e: L^e_{n+1}(\pi_1 Y) \to \Gamma^e_{n+1}(\phi),$$

ϕ is the homomorphism $(\mathcal{F}, \mathrm{id})$ from $\mathcal{F}$ to $\mathrm{id}_{\mathbb{Z}\pi_1 X}$, $\mathcal{F}: \mathbb{Z}\pi_1(X - Y) \to \mathbb{Z}\pi_1 X$ is the natural map. We have the following.

Theorem 1.1 [5, pp. 322, 323]. *Consider the CTSP specified by a homotopy equivalence $f: M \to X$, M an $(n \geqslant 7)$-dimensional manifold, $(X, \partial X)$ a Poincaré pair. Let $(Y, \partial Y) \subset (X, \partial X)$ be a Poincaré pair possessing a two-dimensional linear normal bundle. Assume that f is split along the boundary. If $\theta(f) = 0$, then f is splittable for n odd. If n is even, there is a well-defined obstruction $O_2(f) \in \Gamma^h_{n+1}(\phi)/\operatorname{Im} p^h$ that vanishes if and only if f is splittable. Similarly for the simple CTSP.*

Theorem 1.2 [5, p. 323]. *Let $f: M \to X$ be as in Theorem 1.1, n even. Assume $\theta(f) = 0$. If $\partial M = \partial X = \phi$ and $\pi_1(X - Y) \cong \mathbb{Z}$ or 0, then $O_2(f) = 0$. Similarly for the simple CTSP.*

In particular, very often $\theta(f) = 0$ implies that the CTSP has a solution. More precisely, $\theta(f) = 0$ implies that the CTSP has a solution in odd dimensions, and in even dimensions for the cases covered by Theorem 1.2.

The proof of Theorem 1.2 is a simple calculation in the case $\pi_1(X - Y) \cong 0$. In the case $\pi_1(X - Y) \cong \mathbb{Z}$, it is a generalization of the results of S. Lopez de Medrano [6].

1.2. Examples of the CTSP

Example 1.3.

$$\begin{array}{ccc} M^n & \longrightarrow & \mathbb{C}P^n \\ \cup & & \cup \\ ? & \dashrightarrow & \mathbb{C}P^{n-2} \end{array} \qquad \begin{array}{ccc} M^n & \longrightarrow & \mathbb{R}P^n \\ \cup & & \cup \\ ? & \dashrightarrow & \mathbb{R}P^{n-2} \end{array}$$

We have $\pi_1(\mathbb{C}P^n - \mathbb{C}P^{n-2}) \cong 0, \pi_1(\mathbb{R}P^n - \mathbb{R}P^{n-2}) \cong \mathbb{Z}$.

Therefore, for both CTSP above, the abstract surgery obstruction being zero will guarantee the existence of a solution. Of course, as $Wh(\pi) = 0$, $\pi = 1, \mathbb{Z}_2$, for the two examples above, splitting is the same as simple splitting. Here, $n \geqslant 7$.

Example 1.4. $\mathbb{C}P^n = E(\nu) \cup D_o$. D_o is a disk and ν is the canonical complex bundle over $\mathbb{C}P^{n-2}$. $E(\nu)$ is the total space of the associated disk bundle. Consider the CTSP (necessarily simple as $Wh(1) = 0$) below. CAT = TOP, PL, $n \geqslant 7$.

$$\begin{array}{ccc} M^n & \xrightarrow{h} & E(\nu) \\ \cup & & \cup \\ ? & \dashrightarrow & \mathbb{C}P^{n-2} \end{array}$$

Extend a CAT-homeomorphism $\partial M \to \partial D_o$ to a CAT-homeomorphism $\varphi : D \to D_o$, D a disk. By considering the associated CTSP

$$\begin{array}{ccc} M^n \cup D & \xrightarrow{h \cup \varphi} & \mathbb{C}P^n \\ \cup & & \cup \\ ? & \dashrightarrow & \mathbb{C}P^{n-2} \end{array},$$

we have the following: the CTSP of this example has a solution if and only if the abstract surgery obstruction vanishes. It is straightforward to check this by using the examples above.

Example 1.5. Y is a Poincaré complex, $X = Y \times D^2$, and M^n is an n-dimensional manifold, $n \geqslant 7$.

$$\begin{array}{ccc} M^n & \xrightarrow{f} & X \\ \cup & & \cup \\ ? & \dashrightarrow & Y \end{array}$$

The CTSP above has a solution if $\theta(f) = 0$. Similarly for the simple CTSP. The reason is the following: set $\pi = \pi_1 X$. We have $\pi_1(X - Y) = \pi \times \mathbb{Z}$. $\mathcal{F} : \mathbb{Z}[\pi \times \mathbb{Z}] \to \mathbb{Z}[\pi]$ is induced by the projection onto the first factor $\pi \times \mathbb{Z} \to \pi$. We have [5] a commutative diagram

$$\begin{tikzcd} L^e_{n+1}(\pi \times \mathbb{Z}) \arrow[r, "\varphi^*"] \arrow[dr, "\mathcal{F}_*"'] & \Gamma^e_{n+1}(\mathcal{F}) \arrow[d, "j_*"] \\ & L^e_{n+1}(\pi) \end{tikzcd}$$

As $\pi \times \mathbb{Z} \to \pi$ splits, $\mathcal{F}_*$ splits as well. Thus $\mathcal{F}_*$ is surjective. By [5, p. 320] we are done.

2. Comparing one- and two-dimensional splitting

The codimension one splitting problem (COSP) is defined the same way as the CTSP. The only difference is: Y above possesses a *one*-dimensional linear normal bundle in X above. The abstract surgery obstruction is similarly defined and is a necessary condition for solution.

Let η_0 be the canonical line bundle over $\mathbb{R}P^{n-2}$. Let $E(\eta_0)$ be the total space of the associated disk bundle over $\mathbb{R}P^{n-2}$. $E(\eta_0) = \mathbb{R}P^{n-1} - \operatorname{int} D_o$, where D_o is a disk. Take any homotopy projective space Q^{n-1} [7]. Let $Q_0^{n-1} = Q^{n-1} - \operatorname{int} D$, where D is a disk. Choose a homotopy equivalence of pairs $h : (Q_0^{n-1}, \partial Q_0^{n-1}) \to (E(\eta_0), S(\eta_0))$, where $S(\eta_0)$ is the total space of the sphere bundle associated to η_0. Consider the COSP, CTSP below, CAT = PL, TOP.

$$\begin{array}{ccc} Q_0^{n-1} & \xrightarrow{\quad h \quad} & E(\eta_0) \\ \cup & & \cup \\ ? & \dashrightarrow & \mathbb{R}P^{n-2} \end{array} \qquad \begin{array}{ccc} Q_0^{n-1} \times I & \xrightarrow{\quad h \times \mathrm{id}_I \quad} & E(\eta_0) \times I \\ \cup & & \cup \\ ? & \dashrightarrow & \mathbb{R}P^{n-2} \times \{\frac{1}{2}\} \end{array}$$

For simplicity we will identify $\mathbb{R}P^{n-2}$, with $\mathbb{R}P^{n-2} \times \{\frac{1}{2}\}$.

Theorem 2.1. *For $n \geqslant 7$, the two problems have, or have not, solutions jointly.*

Remark 2.2. Let $\rho : \mathbb{Z}_2 \times S^{n-1} \to S^{n-1}$ be a free involution such that $Q^{n-1} = S^{n-1}/\rho$. If the COSP above has a solution, we have

$$\begin{array}{ccc} Q_0^{n-1} & \xrightarrow{\quad h \quad} & E(\eta_0) \\ \cup & & \cup \\ Q^{n-2} & \xrightarrow[\quad h_0 \quad]{} & \mathbb{R}P^{n-2} \end{array} .$$

The space Q^{n-2} above is a homotopy projective space. Let η be the unique nontrivial line bundle over Q^{n-2}. Let T be a tubular neighborhood of Q^{n-2} in Q_0^{n-1}. The s-cobordism theorem applied to $Q_0^{n-1} - T(Q^{n-2})$ gives that $Q_0^{n-1} = E(\eta)$. Write $Q^{n-2} = S^{n-2}/\tau$, where $\tau\colon \mathbb{Z}_2 \times S^{n-2} \to S^{n-2}$ is a free involution. As $Q_0^{n-1} = E(\eta)$, ρ desuspends to a conjugate of τ [2]. Conversely, if ρ desuspends, the COSP above has a solution.

Proof of Theorem 2.1. If the COSP above has a solution, trivially the CTSP above also has a solution.

Assume that the CTSP above has a solution

$$\begin{array}{ccc} Q_0^{n-1} \times I & \xrightarrow{h \times \mathrm{id}_I} & E(\eta_0) \times I \\ \cup & & \cup \\ Q^{n-2} & \xrightarrow[\text{hom. eq.}]{} & \mathbb{R}P^{n-2} \end{array} .$$

By Lemma 2.1 [4, p. 40] the action $\rho\colon \mathbb{Z}_2 \times S^{n-1} \to S^{n-1}$ extends to a standard CAT = PL, TOP semifree $\mathbb{Z}_2$ action on S^n, also denoted by ρ. In addition, ρ admits a periodic knot $f\colon (S^{n-2}, \tau) \to (S^n, \rho)$ ($\tau\colon \mathbb{Z}_2 \times S^{n-2} \to S^{n-2}$ is a free CAT involution) [4]. Now, by Theorem 2.2 [4, p. 42], $\rho|_{S^{n-1}}$ desuspends to a conjugate of τ. Set $Q^{n-2} = S^{n-2}/\tau$. By the remark above, there is a solution

$$\begin{array}{ccc} Q_0^{n-1} & \xrightarrow{h} & E(\eta_0) \\ \cup & & \cup \\ Q^{n-2} & \xrightarrow[\text{hom. eq.}]{} & \mathbb{R}P^{n-2} \end{array}$$

for the COSP above. $\square$

Remark 2.3. Consider the situation of Theorem 2.1. S. Lopez de Medrano [6] produces homotopy projective spaces Q^{n-1}, CAT = PL, TOP with the following properties. Write $Q^{n-1} = S^{n-1}/\rho$, where $\rho\colon \mathbb{Z}_2 \times S^{n-1} \to S^{n-1}$ is a free involution. Then ρ is non-desuspendable. In addition, remove an open disk from Q^{n-1} obtaining Q_0^{n-1}, and choose a homotopy equivalence $h\colon Q_0^{n-1} \to E(\eta_0)$. It turns out that the COSP

$$\begin{array}{ccc} Q_0^{n-1} & \xrightarrow{h} & E(\eta_0) \\ \cup & & \cup \\ ? & \xrightarrow[\text{hom. eq.}]{} & \mathbb{R}P^{n-2} \end{array}$$

has $\theta(h) = 0$. For such examples, $O_2(h \times \mathrm{id}_I) \neq 0$.

Remark 2.4. It is possible in Theorem 2.1 to work with a homotopy projective space Q^{n-2} instead of $\mathbb{R}P^{n-2}$. Let η (respectively ε) be the unique nontrivial, (respectively trivial) line bundle over Q^{n-2}. Notice that $E(\eta \oplus \varepsilon) = E(\eta) \times I$ ($E(\,)$ means total space of the associated disk bundle). Let Q^{n-1} be a homotopy projective space. Take a disk D in Q^{n-1}. Set $Q_0^{n-1} = Q^{n-1} - \operatorname{int} D$. Choose a homotopy equivalence $h\colon Q_0^{n-1} \to E(\nu)$. The two splitting problems below have, or have not, solutions jointly.

$$\begin{array}{ccc} Q_0^{n-1} & \xrightarrow{h} & E(\eta) \\ \cup & & \cup \\ ? & \dashrightarrow & Q^{n-2} \end{array} \qquad \begin{array}{ccc} Q_0^{n-1} \times I & \xrightarrow{h \times \mathrm{id}_I} & E(\eta \oplus \varepsilon) \\ \cup & & \cup \\ ? & \dashrightarrow & Q^{n-2} \end{array}$$

Now, consider the (simple) splitting problem in which Y is a closed CAT = PL, TOP manifold, $|\pi_1(Y)|$ is finite and divisible by two. In addition, assume that the universal cover of Y is a sphere. Let η (respectively ε) be a nontrivial (respectively trivial) line bundle over Y. Assume $X = E(\eta \oplus \varepsilon) = E(\eta) \times I$, $M = N \times I$, $f = h \times \mathrm{id}_I$, $h: N \to E(\eta)$ is a (simple) homotopy equivalence. As the Browder–Livesay desuspension obstruction [2,6] will appear in our setting only for $n \equiv 0 \bmod 4$, we will work in this range of dimensions. Take a $(|\pi_1(Y)|/2)$-fold cover of N and glue a disk to obtain a homotopy projective space Q^{n-1}. Write $Q^{n-1} = S^{n-1}/\rho$, $\rho: \mathbb{Z}_2 \times S^{n-1} \to S^{n-1}$ is a free involution. We have the following.

Theorem 2.5. *Let* $n \geqslant 7, n = 0 \bmod 4$, *and assume that the abstract surgery obstruction of the CTSP*

$$\begin{array}{ccc} N \times I & \xrightarrow{h \times \mathrm{id}_I} & E(\eta \oplus \varepsilon) \\ \cup & & \cup \\ ? & \dashrightarrow & Y \end{array}$$

vanishes. Then, if ρ as above does not desuspend, $O_2(h \times \mathrm{id}_I) \neq 0$.

Proof. Take the $(|\pi_1(Y)|/2)$-fold cover of the CTSP above and use Remark 2.4. □

3. A particular CTSP

Consider the CTSP where Y is a closed manifold $X = E(\nu)$ and ν is an orientable two dimensional linear bundle over Y. In addition, assume Y simply connected. Let $q: E(\nu) \to Y$ be induced by bundle projection. Denote by $S(\nu)$ the associated sphere bundle. By the homotopy exact sequence of a fibration the map $\pi_1(S(\nu)) \to \pi_1(Y)$ induced by q, is a projection of a cyclic group to the trivial group. As a result [7], the map $L_{n+1}(\pi_1(X - Y)) \to L_{n+1}(\pi_1 X)$ induced by inclusion is an epimorphism. By [5, p. 320], if the abstract surgery obstruction vanishes then $O_2(f) = 0$ as well.

4. Concluding remarks

The phenomenon $O_2(f) \neq 0$ appeared in many of the examples presented above. In each of these cases, the normal bundle of Y in X is nonorientable. We leave the following problem for the reader: prove or disprove the following: assume ν orientable, then $O_2(f)$, if defined, must be zero. An interesting special case is given by the following.

Proposition 4.1. *Let X, Y be orientable manifolds, $\partial Y = \phi$, $\nu = \nu(Y \subset X)$ orientable. Assume $\pi_1(Y) \cong \pi_1(S(\nu)) \cong 0$, and $\pi_1(X - E(\nu))$ is 2-torsion free. Then, if $O_2(f)$ is defined, $O_2(f) = 0$.*

Proof. If $X = E(\nu)$, by the previous results, we are done. If not, use Cappell's splitting theorem [3] to split f along $S(\nu)$, reducing the proof to the case $X = E(\nu)$. □

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