BRANCHED CYCLIC COVERINGS

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The present paper outlines a solution to the following problem of Fox [4]: *When is the p-fold branched cyclic covering space of a manifold, with a manifold branching set, again a manifold?* The solution of this problem has many consequences for the study of cyclic group actions on manifolds. A few examples of applications are described below in Section 1 In Section 2 we study branched cyclic covers of S^3 and relate a result on these to the classical P. A. Smith conjecture and the above problem of Fox.

§1. *Solution of Fox's problem*

Let M^n and W^{n+2} be P.L. manifolds with M compact and $f: M \to W$ a P.L. embedding which is proper, i.e. $f(\partial M) = \partial W \cap f(M)$. A branched cyclic covering space of W along M is a simplicial complex Y equipped with a simplicial map $\pi: Y \to W$ so that Y is a branched cover of W along M [4] with $\pi^{-1}(M) \cong M$ a P.L. homeomorphism and $Y-M \to W-M$ a regular covering space with a finite cyclic group of covering translations. Note that we do not assume that $f(M)$ is a locally-flat submanifold of W. It is easy to see that in general W has a p-fold cyclic cover branched along M if and only if there is a class of order p in $H^1(W-M; Z_p)$ which under the composition of the natural maps $H^1(W-M; Z_p) \to H^1(\bar{\partial}E; Z_p) \to H^2(E, \bar{\partial}E; Z_p)$ goes to a mod p Thom class [3] of the regular neighborhood E of M in W, with $\bar{\partial}E = \partial E -$ interior $(\partial W \cap E)$. If $W = E$ is a regular neighborhood of M, this condition just means that the integral Thom class of E, defined by analogy with the Thom class of a bundle, is divisible by p.

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Fox observed that if M^n is a locally-flat submanifold of W^{n+2}, Y is certainly a manifold [4]. A precise but entirely local answer to Fox's problem for P.L. but not necessarily locally-flat submanifolds can be given as follows. Regard M as a subcomplex of a triangulation of W. For any point in the interior of an i-simplex Δ^i_α of M, the link pair in (M, W) is the i-th suspension of a P.L. locally flat knot pair (S^{n-1-i}, S^{n+1-i}) [11]. Let X_α be the manifold which is the p-fold cyclic cover of S^{n+1-i} along this locally-flat S^{n-1-i}. It is easy to see that Y is a P.L. manifold if and only if each such X_α is a sphere. While this reduces Fox's problem to questions about locally-flat P.L. submanifolds, it is too local to be very useful in applications.

We are thus led to a reformulation of Fox's problem. First note that outside of a regular neighborhood of the branching set M, Y is certainly a manifold. Thus, Fox's problem is solved by determining which branched cyclic covers of a manifold regular neighborhood E^{n+2} of M^n are again P.L. manifolds. Two manifold oriented regular neighborhoods E_0^{n+2} and E_1^{n+2} of M^n are said to be concordant if there is an oriented regular neighborhood V of $M \times 1$ which restricts to regular neighborhoods E_1 of $M \times 1$ and $-E_0$ of $M \times 0$. Recall the classifying space for oriented regular neighborhoods $BSRN_2$ constructed in [3] using results of [11] and analyzed using methods of [2]. See also [6], [1]. Concordance classes of manifold oriented codimension two regular neighborhoods of M are in 1 to 1 correspondence with elements of $[M, BSRN_2]$. Theorem 1 provides a global answer to the following formulation of Fox's problem. *Which* P.L. *oriented manifold regular neighborhoods of* M *are concordant to regular neighborhoods which have manifold* p-*fold cyclic covers branched along* M? In applying Theorem 1 it is useful to recall that if $f_0 : M \to W^{n+2}$ is a P.L. embedding, $n \geq 4$, with E_0 the regular neighborhood of $f_0(M)$ in W, and E_1 is a manifold regular neighborhood of M which is concordant to E_0, then there is an ambient concordance of f_0 in W to a P.L. embedding f_1 with E_1 P.L. homeomorphic to a regular neighborhood of $f_1(M)$ [3].

THEOREM 1. *There exists a classifying space* $BSRN_2^{Z_p}$ *equipped with a natural map* $\pi: BSRN_2^{Z_p} \to BSRN_2$ *such that an oriented manifold regular neighborhood* E_0^{n+2} *of the* P.L. *manifold* M^n *is concordant to an oriented manifold regular neighborhood* E_1 *of* M *with a manifold* p-*fold branched cyclic cover if and only if the map* $g: M \to BSRN_2$ *which classifies* E_0 *lifts to a map* $\bar{g}: M \to BSRN_2^{Z_p}$ *so that* $\pi\bar{g}$ *is homotopic to* g.

This classifying space for branched cyclic covers $BSRN_2^{Z_p}$ has non-finitely generated homotopy groups in even dimensions greater than 2. To see this, note that an element of $\pi_i(BSRN_2^{Z_p})$ is represented by a regular neighborhood E^{i+2} of S^i, which after being modified within its concordance class may be assumed to be locally-flat except possibly at one point P of S^i. The p-fold cyclic branched cover of the link pair of P in (S^i, E^{i+2}) is then, by the local criteria for branched covering spaces to be manifolds discussed above, a sphere equipped with a semi-free Z_p action with a knot as fixed points. This construction defines a map which is an isomorphism (except for $i = 2$, when it has kernel Z) of $\pi_i(BSRN_2^{Z_p})$ to the groups of concordance classes of "(i+2)-dimensional counterexamples to the P. A. Smith conjecture" defined and algebraically analyzed in [2, §11]. In particular, $\pi_{2i}(BSRN_2^{Z_p})$ is not finitely generated for $i > 1$, $\pi_{2i+1}(BSRN_2^{Z_p}) = 0$ for p odd, and $\pi_2(BSRN_2^{Z_p}) = Z$ if the classical P.A. Smith conjecture is true for Z_p actions on S^3. Thus, as a consequence of [13], $\pi_2(BSRN_2^{Z_2}) = Z$.

The detailed homotopy type of $BSRN_2^{Z_p}$ can be studied by combining the homology surgery method of studying codimension two embedding problems of [2], the global approach to non-locally flat embeddings developed in [3] and generalizations of the characteristic variety theorem developed by Sullivan [12] to study G/PL. That the characteristic variety theorem could be generalized to spaces other than G/PL was observed by J. Morgan and by L. Jones.

As an application of Theorem 1 we will consider the following problem: Which oriented closed manifolds M^n are the codimension two fixed points of semi-free actions on S^{n+2}? Theorem 3 which answers this problem will combine the following criteria for M to have a P.L. embedding in S^{n+2} with the condition that M be a Z_p-homology sphere which is imposed by P.A. Smith theory.

THEOREM 2 [3]. *Let* M^n *be a closed* P.L. *manifold with* $\pi_{n+1}(\Sigma M) \to H_{n+1}(\Sigma M)$ *onto. Then there is a* P.L. *embedding* $M \subset S^{n+2}$.

The relationship between the dimension k of the non-locally flat points of the embedding of M^n in S^{n+2} and the characteristic classes of M, developed in [3] shows that in many cases k must be at least n−4. Note that if M^n does have a P.L. embedding in S^{n+2}, then by the Thom-Pontrjagen construction, $\pi_{n+2}(\Sigma^2 M) \to H_{n+2}(\Sigma^2 M)$ is onto.

The following result is a kind of converse to P.A. Smith theory. Related results were obtained by L. Jones in high codimensions [7].

THEOREM 3. *Let* M^n *be a* Z_p *homology sphere with* $\pi_{n+1}(\Sigma M) \to H_{n+1}(\Sigma M)$ *surjective. Assume that* $H^2(M; Z_2) = 0$. *Then there exists a semi-free* P.L. *action of* Z_p *on* S^{n+2} *with* M *as fixed points.*

Many M which satisfy the hypothesis of this theorem do not have locally-flat embeddings in S^{n+2}. If the condition in Theorem 3 on the surjectivity of the Hurewicz map is dropped, we can still show, for n odd, that there is a Z_p homology sphere V^{n+2} with a semi-free Z_p action and with M as fixed points. The condition on $H^2(M; Z_2)$ arises from the 3-dimensional P.A. Smith conjecture in a manner which will be described below.

Another result which follows from Theorem 1, an analysis of $BSRN_2^{Z_p}$ and methods of [2], [3] is the following:

THEOREM 4. *Let* W^{n+2} *be an oriented compact* P.L. *manifold equipped with a semi-free* Z_p *action,* p *odd, with fixed points* $M^n \subset$ *interior* (W), *M an oriented closed P.L. manifold with* $H^2(M; Z_2) = 0$. *Then if* n *is odd or if* $\pi_1(W) = 0$, *for every closed P.L. manifold* M′ *homotopy equivalent to* M, *there exists a compact P.L. manifold* W′, *equipped with a semi-free P.L.* Z_p *action with* M′ *as fixed points, with* $(W', \partial W')$ *equivariantly homotopy equivalent to* $(W, \partial W)$.

The conditions on $H^2(M; Z_2)$ in the above results arise in the following way. In proving Theorems 3 and 4, we study a natural map of $BSRN_2^{Z_p}$ to G/PL and attempt to find a splitting of it. In particular, on the level of the second homotopy groups, we are trying to find a splitting of the map which assigns to a knot which is a counterexample to the classical P.A. Smith conjecture its Arf invariant. We thus propose the following weak form of the P.A. Smith conjecture, whose truth would imply the necessity of the conditions on $H^2(M; Z_2)$.

WEAK P.A. SMITH CONJECTURE. *Let* $K \subset S^3$, $K \cong S^1$ *be the fixed points of a P.L.* Z_p *action on* S^3, p *odd. Then is* $\Delta_K(-1) \equiv \pm 1$ *(modulo 8), where* $\Delta_K(t)$ *is the Alexander polynomial of the knot* $K \subset S^3$?

Fox [5] studied restrictions on $\Delta_K(t)$. However as his methods, which involve expressing homology in terms of $\Delta_K(t)$, apply in high dimensions, where for p odd the weak P.A. Smith conjecture is false [2], they alone will not suffice.

A result on $\pi_2(BSRN_2^{Z_p})$ is indicated at the end of Section 2 below.

§2. *Cyclic branched covering of* S^3

Let $\beta \subset S^3$ be a knot. Let V be a Seifert surface of β, with linking form L_V. Let L be a matrix for L_V with respect to some basis. Let L′ denote the transpose of L. If ξ is complex number of norm 1, let

$$K_\xi = L + L' - \xi L - \xi^{-1} L' .$$

Then K_ξ is a Hermitian form over the complex numbers; let $\sigma_\xi(\beta)$ denote its signature. Let $\Sigma(\beta, p)$ be the p-fold cyclic branched cover of S^3 along β, with the induced orientation.

THEOREM 5. *The* p*-fold cyclic cover* $\Sigma(\beta, p)$ *bounds a parallelizable manifold with signature*

$$\sum_{i=1}^{p-1} \sigma_{\xi^i}(\beta) ,$$

ξ *a primitive* p^{th} *root of unity.*

NOTES:

1. $\sigma_\xi(\beta) = \sigma_{\xi^{-1}}(\beta)$.
2. The function $\sigma_\xi(\beta)$ is actually a cobordism invariant of β.
3. $\sigma_\xi(\beta)$ is continuous in ξ, except possibly at the negatives of the roots of the Alexander polynomial of β.
4. The manifold constructed to bound $\Sigma = \Sigma(\beta, p)$ is simply connected and has even middle betti number.
5. Analogous results are true in high dimensions.

Theorem 5 has been obtained independently by L. Kauffman [14].

Proof of Theorem 5. Consider

$$P = \beta \times I \bigcup_{\beta \times 0} V \times 0 \subset S^3 \times I \bigcup_{S^3 \times 0} D^4 \cong D^4 .$$

Then the p-fold cyclic branched cover Q of D^4 along P is a 4-manifold with boundary $\Sigma(\beta, p)$. Clearly

$$Q = (\hat{\Sigma} \times I) \cup p(D^4) \bigcup_{P \times S^1} P \times D^2$$

where $\hat{\Sigma}$ is the part of Σ lying over the closure of the complement of a tubular neighborhood of β, and $p(D^4)$ is attached to $\hat{\Sigma} \times 1$ along the subset of its boundary pS^3 $S^3 \cup \cdots \cup S^3$ consisting of p copies of the closure of the complement of a tubular neighborhood of V in S^3. (If $\pi : \to S^3$ is the projection, $\pi|\pi^{-1}(S^3 - V)$ is the trivial p-fold covering space.)

Let $\hat{Q} = \hat{\Sigma} \times I \cup pD^4 \subset Q$. By excision $H_2(\hat{Q}) \cong H_2(\hat{Q}, pD^4) \cong H_2(p(V \times I, V \times \partial I)) \cong H_1(pV)$. Moreover, the mapping $H_2(\hat{Q}) \to H_2(Q)$ is surjective, as the composite

$$H_2(Q, \hat{Q}) \xrightarrow{\cong} H_2(P \times D^2, P \times S^1) \longrightarrow H_1(P \times S^1) \dashrightarrow H_1(\hat{Q}) \overset{\ell}{\dashrightarrow} Z$$

ℓ = linking number with P, is a monomorphism. Hence $\hat{Q}$ and Q have the same index. Since $\hat{Q}$ is an unbranched cover of a subset of S^3, it is parallelizable, i.e. for $x \in H_2(\hat{Q})$, $x \cdot x \equiv O(2)$. Hence Q is also parallelizable.

A basis of $H_2(Q)$ is obtained by pushing circles representing a basis of $H_1(V)$ in each component of $\pi^{-1}V$ in $\Sigma \times 0$ to each of the boundary components of a neighborhood of $\pi^{-1}V$ and making the results bound in the corresponding copy of D^4. With respect to the basis thus obtained from the basis of H_1V used to obtain L from L_V, it is easy to see that the intersection form on $H_2(\hat{Q})$ has the matrix

$$K = \begin{pmatrix} L+L' & -L & 0 & \cdots & & 0 & -L' \\ -L' & L+L' & 0 & \cdots\cdots & & 0 & 0 \\ 0 & -L' & L+L' & -L & \cdots & 0 \cdots & 0 \\ \vdots & & & & & & 0 \\ \vdots & & & & & & -L \\ -L & 0 & 0 & \cdots & & -L' & L+L' \end{pmatrix}.$$

Let H be the matrix

$$\begin{pmatrix} I & I & \xi^2 I & \cdots & \xi^{p-1} I \\ & \xi^2 I & \xi^4 I & \cdots & \xi^{2p-1} I \\ \vdots & & & & \\ I & \xi^{p-1} I & & \cdots & \xi^{(p-1)^2} I \end{pmatrix}$$

where ξ is a primitive p^{th} root of 1 and I is the identity matrix of the same size as L. Then $\bar{H}'KH$ is the matrix

$$p\begin{pmatrix} K_{\xi^0} & & & 0 \\ & K_{\xi^1} & & \\ & & \ddots & \\ 0 & & & K_{\xi^{p-1}} \end{pmatrix}$$

and $\bar{H}'H = pI$. The theorem follows.

NOTE. One can easily show that the intersection form on $H_2(Q)$ has the matrix

$$\begin{pmatrix} K_{\xi} & & & 0 \\ & K_{\xi^1} & & \\ & & \ddots & \\ 0 & & & K_{\xi^{p-1}} \end{pmatrix}.$$

From this and Poincaré Duality, we may recover all known results on $H_1(\Sigma)$.

EXAMPLE: β = trefoil knot, $p = 5$. Then

$$L = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix},$$

so that $\sigma_\xi(\beta) = -2 = \sigma_{-1}(\beta)$ for

$$\xi = e^{2\pi it}, \quad 1/6 < t < 5/6 ,$$

and

$$\sigma_\xi(\beta) = 0 \quad \text{if} \quad -1/6 < t < 1/6 .$$

Thus $\sum_{i=1}^{4} \sigma_\xi(\beta) = -8$.

In fact, it is well known [10] that the 5-fold branched cyclic cover of 3_1 is binary dodecahedral space ("Poincaré space").

As a consequence of Theorem 5 and Rohlin's Theorem [9], and results of [2] the natural periodicity map $\pi_2(BSRN_2^Z p) \to \pi_6(BSRN_2^Z p)$ is seen to be not surjective.

BIBLIOGRAPHY

[1] Cappell, S. E., and Shaneson, J. L., "Nonlocally flat embeddings, smoothings and group actions." Bull. Amer. Math. Soc. 79 (1973), 577-582.

[2] ________________, "The codimension two placement problem and homology equivalent manifolds." Ann. of Math. 99 (1974), 277-348.

[3] ________________. "P.L. embeddings and their singularities." To appear.

[4] Fox, R. H., "Covering spaces with singularities," in "Algebraic Geometry and Topology, A Symposium in honor of S. Lefschetz." Ed. R. H. Fox, et al., Princeton, 1957, 243-257.

[5] ________, "A quick trip through knot theory." Topology of 3-manifolds and related topics. Ed. M. K. Fort, Jr., Prentice Hall 1962, 95-99.

[6] Jones, L., "Three characteristic classes measuring the obstruction to P.L. local unknottedness." Bull.Amer. Math. Soc. 78 (1972), 979-980.

[7] ________, A converse to the fixed point theory of P. A. Smith. Ann. of Math. 94 (1971), 52-68.

[8] Neuwirth, L. P., "Knot groups." Ann. of Math. Studies 56, Princeton 1965.

[9] Rohlin, V. A., "A new result in the theory of 4-dimensional manifolds." Doklady 8 (1952), 221-224.

[10] Seifert, H., and Threlfall, W., "Lehrbach der Topologie." Akad. Verlagsgesesellschaft, B. G. Teubner, Leipzig, 1934.

[11] Stone, D., "Stratified Polyhedra." Springer Lecture Notes in Math. 252, New York and Berlin 1972.

[12] Sullivan, D., "Geometric Topology Seminar Notes." Princeton University 1967 (mimeographed notes).

[13] Waldhausen, F., "Uber Involutionen der 3-Sphare." Topology 8 (1969), 81-91.

[14] Kauffman, L., to appear.