The codimension two placement problem and homology equivalent manifolds

By SYLVAIN E. CAPPELL[1,2] and JULIUS L. SHANESON[1,2]

PREFACE

In this paper new methods of classifying smooth, piecewise linear (P.L.) or topological submanifolds are developed as consequences of a classification theory for manifolds that are homology equivalent, over various systems of coefficients. These methods are particularly suitable for the placement problem for submanifolds of codimension two. The role of knot theory in this larger problem is studied systematically by the introduction of the local knot group of an arbitrary manifold. Computations of this group are used to determine when sufficiently close embeddings in codimension two "differ" by a knot. A geometric periodicity is derived for the knot cobordism groups.

The methods of this paper can also be applied to get classification results on submanifolds invariant under group actions and on submanifolds fixed by group actions. In particular, algebraic calculations of the equivariant knot cobordism groups are given in this paper and some geometric consequences are derived. Other applications of the methods of this paper include a general solution of the codimension two surgery problem, given below, and corresponding results on smoothings of Poincaré embeddings in codimension two.

The proofs of many of the results use computations of new algebraic K-theory functors. In a future paper, the present methods will be applied to the study of P.L. embeddings and their singularities.

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Introduction

In order to get significant classification results, the study of embeddings in codimension two, i.e., embeddings of a manifold M^n in W^{n+2}, has usually been restricted to the case in which M and W are spheres: i.e., knot theory. The peculiar difficulties in the study of codimension two embeddings of M in W are due to the fact that the homomorphism $\pi_1(W - M) \to \pi_1(W)$, though always surjective, may not be an isomorphism. It is rather hard to conclude directly, from a knowledge of M, W, and the homotopy class of the embedding, enough information about $\pi_1(W - M)$ or, more generally, about the homotopy type of $W - M$, to give satisfactory geometric information about $W - M$. It is therefore natural to use instead the weaker information consisting of the homology groups of $W - M$ with coefficients in the local system of $\pi_1 W$. More precisely, we need a classification theory for manifolds that are only homotopy equivalent to $W - M$ in the weak sense defined by coefficients in $\pi_1 W$, i.e., manifolds *homology equivalent* to $W - M$ over the group ring $\mathbf{Z}[\pi_1 W]$. In knot theory ($M = S^n$, $W = S^{n+2}$), for example, while the homotopy type of the complement $S^{n+2} - S^n$ may be very complicated,[1] it is in any case always a homology circle. Moreover, this fact together with other elementary data serves to characterize knot complements. In this paper we develop a general theory for classifying manifolds that are homology equivalent over a local coefficient system and perform calculations in this theory to get results on codimension two embeddings.

Even in the case of knot theory, the systematic study of homology equivalences leads to new understanding and new results. For example, we

[1] The simple homotopy type of the knot complement, even relative to the boundary, does not, in any case, characterize a knot [C1].

prove a geometric periodicity theorem for high dimensional knot cobordism groups. Two embeddings of a manifold, $f_i: X \to Y$, $i = 0, 1$, are said to be *concordant* if there is an embedding $F: X \times I \to Y \times I$ with $F(x, i) = f_i(x)$, $i = 0, 1$, $I = [0, 1]$ the unit interval. (In the P.L. and topological case, only locally flat embeddings are under consideration in this paper. Non-locally flat embeddings will be studied in [16], [17].) Knots are said to be *cobordant* if they are ambient isotopic to concordant knots.[1] This equivalence relation was introduced by Fox and Milnor [23], in the classical case $X = S^1$, $Y = S^3$. The cobordism classes of knots form a group, with addition defined by connected sum. Kervaire [25] studied analogous groups for higher dimensions and proved that they vanish in even dimensions and are very large in odd dimensions. Using our methods, we give a conceptually simple proof (see Corollary 13.11, and also note 2 to Corollary 6.5) of the vanishing of the even dimensional knot cobordism groups; it is a consequence of the vanishing of the obstruction group to odd dimensional surgery to obtain an (integral) homology equivalence. This proof extends to show the vanishing of the even dimensional equivariant knot cobordism groups (see 10.5).

Levine [31], [32] computed the odd dimensional P.L. or smooth groups (except in the classical case) and deduced an algebraic periodicity for high dimensional knot cobordism groups. In [14], it was shown that for topological knots periodicity applies all the way down to the case of S^3 in S^5. In the present paper, we employ a new algebraic description (6.4, 13.10) of knot cobordism in terms of Hermitian or skew-Hermitian quadratic forms over $\mathbf{Z}[t, t^{-1}]$, the ring of finite Laurent series with integer coefficients, which become unimodular when one puts $t = 1$.

An especially simple formulation of geometric periodicity for knots is obtained by comparing, for a simply connected closed manifold M^k, the embeddings of $S^n \times M$ in $S^{n+2} \times M$ with the embeddings of spheres S^n in S^{n+2} and S^{n+k} in S^{n+k+2}. A cobordism class x represented by $f: S^n \to S^{n+2}$ determines the cobordism class $\delta(M, n)(x)$ of embeddings of $S^n \times M$ in $S^{n+2} \times M$ represented by $f \times \mathrm{id}_M$. A cobordism class y of embeddings of S^{n+k} in S^{n+k+2} determines, by connected sum with $f_0 \times \mathrm{id}_M$, $f_0: S^n \to S^{n+2}$ the usual inclusion, a cobordism class $\alpha(M, n)(y)$ of embeddings of $S^n \times M$ in $S^{n+2} \times M$. Let $G_n(M)$ be the cobordism classes of embeddings of $S^{n+2} \times M$ in $S^n \times M$ that are homotopic to $f_0 \times \mathrm{id}_M$; see § 13 for the precise definition of cobordism in this context. Let $G_n = G_n(pt)$, the knot cobordism group.

[1] For P.L. and topological knots, cobordism implies concordance.

THEOREM (13.1). *Assume* $n \geqq 2$ *and* M^k *is a simply-connected closed* P. L. *or topological manifold. Then* $\alpha(M, n)\colon G_{n+k} \to G_n(M^k)$ *is a one-to-one, onto map.*

In other words, up to cobordism every embedding of G_n in $G_n(M^k)$ can be pushed into the usual embedding except in the neighborhood of a point, and this can be done in a unique way.

THEOREM (13.2). *If* $k \equiv 0 \pmod 4$ *and* M *has index* ± 1, *and if* $n > 3$, *in the* P. L. *case, or* $n \geqq 3$, *in the topological case, then* $\delta(M, n)\colon G_n \to G_n(M^k)$ *is a one-to-one, onto map.*

In particular, let $M = CP^2$, the space of lines is complex three-space.

GEOMETRIC PERIODICITY THEOREM (13.3). *The map*

$$\alpha(CP^2, n)^{-1}\delta(CP^2, n)\colon G_n \to G_{n+4}$$

is an isomorphism for $n > 3$, *in the* P. L. *case, and for* $n \geqq 3$, *in the topological case.*

This geometric periodicity of knot theory leads to a homotopy-theoretic periodicity in the classifying spaces for P. L. singularities [17].

The methods of this paper also apply to the study of smooth (or locally flat P. L. or topological) knots invariant under a free action of a cyclic group or fixed under a semi-free action. S. Lopez de Medrano [35], [36] proved a number of important results on knots invariant under free $\mathbf{Z}_2$-actions. His work and the ideas of our earlier work [14], [47] suggested the role that homology equivalences could play in codimension two. Our analysis of invariant spheres in codimension two breaks up naturally into two problems; first the determination of which actions can be obtained by the restriction of a given action to an invariant sphere, and then the classification up to equivariant cobordism of the equivariant embeddings of a given free action on a sphere in another.

On even dimensional spheres, only the cyclic group $\mathbf{Z}_2$ can act freely, and Lopez de Medrano has determined which actions admit invariant spheres in codimension-two; his result is reproved and interpreted in our context in Appendix I. The following result asserts that the even-dimensional equivariant knot cobordism groups vanish:

THEOREM (10.5). *Let* Σ^{2k+2}, $k \geqq 3$, *be a (homotopy) sphere equipped with a free* $\mathbf{Z}_2$*-action. Then any two invariant spheres of* Σ^{2k+2} *are equivariantly cobordant.*

On odd dimensional spheres, the results are slightly easier to state if the group has odd order.

THEOREM (9.4). *Let* Σ^{2k+1} *and* Σ^{2k-1} *be spheres with free* P.L. *actions* ρ *and* τ, *respectively, of* $\mathbf{Z}_s$, *s odd. Then there is an equivariantly "nice"* (i.e., *smooth or equivariantly locally flat*) *embedding of* Σ^{2k-1} *in* Σ^{2k+1} *if and only if* Σ^{2k-1}/τ *is normally cobordant to a desuspension of the homotopy lens space* Σ^{2k+1}/ρ.

Note that normally cobordant implies homotopy equivalent. Note also that each normal cobordism class contains several inequivalent actions [8].

In a future paper (see the announcement [16]) it will be shown, as a consequence of a general theory of P.L. embeddings that there is an equivariant P. L. embedding, not necessarily equivariantly locally flat, of Σ^{2k-1} in Σ^{2k+1} if and only if Σ^{2k-1}/τ and Σ^{2k+2}/ρ are homotopy equivalent.

As a consequence of similar criteria for invariant spheres of high codimension, one can prove the following:

COROLLARY. *Let* τ *be a free action of* $\mathbf{Z}_s$ *on the* (*homotopy*) *sphere* Σ^{2j-1} *that is the restriction of the free action* ρ *on* Σ^{2k+1} *to an invariant sphere. Assume s is odd. Then* Σ^{2j+1} *is a characteristic sphere of* ρ (i.e., Σ^{2j+1}/τ *is a characteristic submanifold of* Σ^{2k+1}/τ) *if and only if there is a tower.*

$$\Sigma^{2j+1} \subset \Sigma^{2j+3} \subset \cdots \subset \Sigma^{2k-1} \subset \Sigma^{2k+1}$$

of invariant spheres.

Characteristic submanifolds are discussed in [35], [36], [6], and below. An obstruction theory computation shows that for $2j + 1 > (1/2)(2k + 1)$, the quotient space of an invariant sphere of dimension $(2j + 1)$ in an action on a $(2k + 1)$-sphere is a characteristic submanifold if and only if its normal bundle splits into a sum of plane bundles.

The determination, for an action ρ of $\mathbf{Z}_s$, $s = 2p$, on Σ^{2k+1}, of which actions appear as invariant spheres is somewhat more complicated. For $p = 1$, this was done by Medrano. The general situation, described in § 9 (see 9.2) below, is that corresponding to each "homotopy desuspension" L of Σ^{2k+1}/ρ there is, for k odd, precisely an entire normal cobordism class of homotopy lens spaces (or projective spaces) homotopy equivalent to L occurring as characteristic submanifolds of Σ^{2k+1}/ρ. For k even, there is an obstruction in $\mathbf{Z}_2$ to the existence of any characteristic submanifold homotopy equivalent to L, and if it vanishes, then an entire normal cobordism class occurs.

Our calculation of equivariant knot cobordism (see § 10) will be given using our new algebraic K-theoretic Γ-functors. A closely related computation of the knots fixed by semi-free actions yields the following:

THEOREM (11.2). *For any integer m and any knot $\kappa\colon S^n \to S^{n+2}$, $n \geqq 3$, $\kappa \# \cdots \# \kappa$ (connected sum m times) is cobordant to a knot fixed under a semi-free action of $\mathbf{Z}_m$. Moreover, every element of $8\mathbf{Z}$, for $n \equiv -1$ (mod 4), or $\mathbf{Z}_2$, for $n \equiv 1$ (4), occurs as the index, or invariant, of a knot fixed under a semi-free action of $\mathbf{Z}_m$.*

Of course, in the P.L. and topological cases only actions that are "nice" in the neighborhood of the fixed points are under consideration. For $n = 3$, in the P.L. and smooth case one can only realize $16\mathbf{Z}$. Theorem 11.2 below is only stated in the topological case.

In Chapter II and § 12 we study the question of when two sufficiently close embeddings f_0 and f_1 of M^n in W^{n+2} are concordant, at least up to taking connected sum with a knot. If f_1 is sufficiently close to f_0, it will lie in a bundle neighborhood. We therefore consider cobordism classes of embeddings of M, in the total space $E(\xi)$ of the disk bundle of a 2-plane bundle ξ over M, homotopic to the zero-section. Such an embedding is called a *local knot* of M in ξ, and the set of cobordism classes is denoted $C(M, \xi)$ or just $C(M)$ if ξ is trivial. (In the sequel, we write $C_O(M; \xi)$, $C_{\mathrm{PL}}(M; \xi)$, $C_{\mathrm{TOP}}(M; \xi)$ to distinguish the various categories; in the last two, local flatness is understood.) $C(M, \xi)$ is a monoid; the operation is called *composition* or *tunnel sum* and is defined as the composition $\hat{\iota}_1\iota_2$, where ι_1 and ι_2 are local knots of M in ξ and $\hat{\iota}_1$ is a thickening of ι_1 to an embedding of $E(\xi)$ in itself. We discuss only the case when M is closed; various relativizations exist.

THEOREM. (See 4.5, 5.3, 6.2, 6.3, and 6.5.) *For $n = \dim M \geqq 3$, $C(M, \xi)$ is a group under composition, and for $n \geqq 4$ it is abelian. For $n \neq 1$, $C(S^n)$ is isomorphic to the n-dimensional knot cobordism group. Connected sum with the zero-section defines a homomorphism $\alpha\colon C(S^n) \to C(M, \xi)$. In the* P.L. *or topological case, α is a monomorphism onto a direct summand, provided ξ is trivial, and $n \geqq 4$.*

We will compute $C(M, \xi)$ in terms of an exact sequence involving the Γ-groups to be described below. In particular, for $\dim M \equiv 1$ (mod 2), it is caught in an exact sequence of Wall surgery groups and hence tends to be fairly small. For example, one has

THEOREM (7.2). *For $n = \dim M \geqq 4$ even, there is an injection*

$$\bar{\rho}\colon C(M) \longrightarrow L^h_{n+1}(\pi_1 M)\ .$$

The map $\bar{\rho}$ has a geometric definition in terms of a surgery obstruction of a type of Seifert surface for local knots.

On the other hand, for M odd-dimensional, $C(M, \xi)$ is not, in general, finitely generated. For simply-connected M, the main result is the following:

THEOREM (6.5 and 6.6). *Let M be a simply-connected closed n-manifold, $n \geqq 4$, and let ξ be a 2-plane bundle over M. Then $\alpha\colon C(S^n) \to C(M, \xi)$ is onto, and is an isomorphism for ξ trivial, in the* P.L. *and topological cases. For n even, $C(M, \xi) = 0$.*

We draw some consequences for the study of close embeddings.

THEOREM. (See 12.1 and following discussion.) *Let $f_0\colon M^n \to W^{n+2}$ be an embedding (locally flat, of course) of the closed, simply-connected manifold M in the (not necessarily compact) manifold W. Assume $n \geqq 5$. Let f be another embedding, sufficiently close to f_0 in the C_0 topology. Then if the normal bundle ξ of f is trivial, or if n is even and the Euler class of ξ is not divisible by two, or if $n \equiv 2 \pmod 4$ and the Euler class of ξ is divisible only by two; then, after composition with a homeomorphism (or diffeomorphism or* P.L. *homeomorphism) of M homotopic to the identity, f is concordant to f_0, for n even and to the connected sum of f_0 with a knot, for n odd.*

The importance of simple connectivity of M is demonstrated by the next result.

THEOREM (7.3 and 14.5). *Let $T^n = S^1 \times \cdots \times S^1$, $n \geqq 4$ and even. Then, in the* P.L. *category,*

$$C(T^n) = [\Sigma(T^n - pt); G/\mathrm{PL}]\,,$$

and every element of $C(T^n)$ can be represented by an embedding arbitrarily close to the zero-section $T^n \subset T^n \times D^2$. $C(T^n)$ is generated by products with various $T^{n-i} \subset T^n$ of the connected sum of $T^i \subset T^i \times D^2$ with knots of dimension i.

This follows from 7.2, as quoted above, and from known results in ordinary surgery theory. Thus $C(T^n)$ is a direct sum of copies of $\mathbf{Z}$ and $\mathbf{Z}_2$ corresponding to the index or Arf invariant of knots sitting along various sub-tori of non-zero codimension, as described in the theorem. Thus knots with vanishing index or Arf invariant (a huge supply of them exists) disappear from sight when placed along a sub-torus. In another paper, we will show exactly how these knots reappear in the classification of *non-locally flat* cobordism classes of non-locally flat embeddings of T^n in $T^n \times D^2$.

Given a manifold Y^{n+2} and a submanifold X^k, the problem of making a homotopy equivalence $f: W \to Y$ transverse regular to X, with $f|f^{-1}X: f^{-1}X \to X$ a homotopy equivalence, is called the ambient surgery problem. There is always an abstract surgery obstruction, an element of $L_k(\pi_1 X)$, to solving this problem. We solve the codimension two surgery problem, i.e., the case $k = n$, using the methods of surgery to obtain homology equivalences. The odd dimensional result resembles the results of higher codimension [5]; *if the abstract surgery obstruction vanishes the problem can be solved and all the manifolds homotopy equivalent to X in one normal cobordism class can occur as $f^{-1}X$.* In even dimensions, there is an additional obstruction to this problem, defined in terms of the Γ-functors. This obstruction often can be interpreted in terms of knot cobordism groups. Note that in codimension two, even if $f|f^{-1}X: f^{-1}X \to X$ is a homotopy equivalence, $f|: (W - f^{-1}X) \to (Y - X)$ need only be a $\mathbf{Z}[\pi_1 Y]$-homology equivalence.

As an application of our codimension two surgery, one can study the problem of finding locally flat spines in codimension two. This problem has been studied by Kato-Matsumoto [29] and Matsumoto [39], using different methods of codimension two surgery in this special case.[1] In a forthcoming paper, we will apply our methods to the classification of non-locally flat spines.

Chapter I develops the theory of homology surgery. Let $\mathbf{Z}[\pi]$ be the integral group ring of the group π, with a usual involution determined by a homomorphism $w: \pi \to \{\pm 1\}$, and let $\mathcal{F}: \mathbf{Z}[\pi] \to \Lambda$ be a homomorphism of rings with unit and involution. It is convenient, though not always essential, to assume $\mathcal{F}$ is surjective.

THEOREM. *Let $(Y^n, \partial Y)$ be a manifold pair (or even just a Poincaré pair over Λ) with $(\pi_1 Y, w^1 Y) = (\pi, w)$, $n \geqq 5$. A normal map (f, b), $f: (X, \partial X) \to (Y, \partial Y)$, of degree one, inducing a homology equivalence over Λ of boundaries, determines an element $\sigma(f, b)$ of an algebraically defined abelian group $\Gamma_n^h(\mathcal{F})$. The element $\sigma(f, b)$ vanishes if and only if (f, b) is normally cobordant, relative the boundary, to a homology equivalence over Λ.*

This result, together with a realization theorem for elements of $\Gamma_*^h(\mathcal{F})$ (see 1.8 and 2.2) and a special study of homology surgery for manifolds $(Y, \partial Y)$ with $\pi_1 \partial Y = \pi_1 Y$ (see 3.1) leads, by a procedure analogous to

[1] They try to modify a submanifold to a spine by ambient surgery, as contrasted with our approach of modifying a transverse inverse image and its complement by abstract surgeries. In the even dimensional case, Matsumoto obtains an obstruction group of Seifert forms that corresponds, in the case of knots, to the expression $tV + V'$, V a Seifert matrix with transpose V'.

[58, § 9] to a general relative theory for homology surgery. In this theory a periodicity theorem which asserts that $\sigma((f, b) \times \mathrm{id}_{CP^2}) = \sigma(f, b)$ plays an important role. An analogous theory for simple homology equivalences, with absolute groups $\Gamma^s_n(\mathcal{F})$, is also developed in Chapter I.

If $\mathcal{F}$ is the identity of $\mathbf{Z}[\pi]$, then $\Gamma_n(\mathcal{F})$ is the Wall group $L_n(\pi)$, and $\sigma(f, b)$ is the usual surgery obstruction of Wall.

For $n = 2k$, $\Gamma_n(\mathcal{F})$ is defined as a Grothendieck group of $(-1)^k$-symmetric Hermitian forms over $\mathbf{Z}\pi$ that become non-singular forms on stably free modules when tensored with Λ. If $\mathcal{F}$ is onto, so is the natural map $\Gamma_{2k}(\mathcal{F}) \to L_{2k}(\Lambda)$. For $\mathcal{F}$ onto, we show directly that $\Gamma_{2k+1}(\mathcal{F})$ is a subgroup of $L_{2k+1}(\Lambda)$. Geometrically, this implies that in odd dimensions the vanishing of an obstruction in a Wall surgery group is enough to permit completion of homology surgery.

Further calculations of Γ-groups will appear in a future paper.

In terms of Γ-groups one has a calculation of $C(M, \xi)$.

THEOREM (5.2). *If* $n = \dim M \geqq 4$, *then there is an exact sequence*

$$0 \longrightarrow C_H(M, \xi) \xrightarrow{\Sigma} \Gamma_{n+3}(\phi) \xrightarrow{\rho} \operatorname{coker} s_H\ ,$$
$$H = O,\ \mathrm{PL},\ \mathrm{TOP}\ .$$

Here ϕ is the diagram

$$\begin{array}{ccc} \mathbf{Z}[\pi_1(\partial E)] & \xrightarrow{\mathrm{id}} & \mathbf{Z}[\pi_1(\partial E)] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle p_*} \\ \mathbf{Z}[\pi_1(\partial E)] & \xrightarrow{p^*} & \mathbf{Z}[\pi_1 M]\ , \end{array}$$

p the projection of ξ, and $s_H\colon [\Sigma E; G/H] \to L^s_{n+3}(p_*)$ is the usual map, $E = E(\xi)$. This Γ-group is calculated by an exact sequence involving absolute groups as the other terms (see 3.2). Our results on equivariant knot cobordism (§ 10) are also stated using Γ-groups.

Chapter I: Surgery with coefficients

1. The even dimensional absolute case

Let π be a finitely presented group and $w\colon \pi \to \mathbf{Z}_2$ a homomorphism of π to the group of two elements. Let the integral group ring $\mathbf{Z}\pi$ have the conjugation determined by the formula $\bar{g} = w(g)g^{-1}$ for $g \in \pi$. Throughout this section $\mathcal{F}\colon \mathbf{Z}\pi \to \Lambda$ will denote an epimorphism of rings (with 1 and) with involution. (Actually, the assumption that $\mathcal{F}$ is onto is not used until Lemma 1.2.) The most important case is the case of $\Lambda = \mathbf{Z}\pi'$ and $\mathcal{F}$ a map induced by an epimorphism of groups.

Let $\mathrm{Wh}(\mathcal{F}) = K_1(\Lambda)/\mathcal{F}(\pm\pi)$. Note that in case $\mathcal{F}$ is induced by an onto map of π to π', $\mathrm{Wh}(\mathcal{F}) = \mathrm{Wh}(\pi')$. Let $f: A \to B$ be a map of finite connected CW complexes, with $\pi = \pi_1 B$, and suppose that f induces isomorphisms on homology groups with local coefficients in Λ. Let C_f be the mapping cylinder of f. Let $\tilde{C}_f$ be its universal covering space, with $\tilde{A} \subset \tilde{C}_f$ the induced covering over A. Let C_* be the cellular chains of $(\tilde{C}_f, \tilde{A})$, a chain complex of free $\mathbf{Z}\pi$-modules with basis determined by choosing lifts of the cells of $(\tilde{C}_f, \tilde{A})$. Then $C_* \otimes_{\mathbf{Z}\pi} \Lambda$ is acyclic, and its torsion $\Delta_{\mathcal{F}}(f) \in \mathrm{Wh}(\mathcal{F})$ is a well-defined invariant of the homotopy class of f; see [40] for more details. If $\Delta_{\mathcal{F}}(f) = 0$ we say that f is a *simple homology equivalence over* Λ (or, if more precision is needed, *over* $\mathcal{F}$).

An isomorphism of *stably* based Λ-modules will be called $\mathcal{F}$-*simple*, or just *simple* if there is no danger of confusion, if and only if it represents the zero element of $\mathrm{Wh}(\mathcal{F})$. A stable basis of a Λ-module will be said to be in the ($\mathcal{F}$)-*preferred class*, with respect to a given stable basis, if and only if the stable automorphism given by change of basis is simple with respect to the given basis.

Let $\eta = \pm 1$. Let $I_\eta = \{\lambda - \eta\bar{\lambda} \mid \lambda \in \mathbf{Z}\pi\}$. By a *special* η-*form* over $\mathcal{F}$ is meant a triple (H, φ, μ), H a finitely-generated (right) $\mathbf{Z}\pi$-module, $\varphi: H \times H \to \mathbf{Z}\pi$ a $\mathbf{Z}$-bilinear map, $\mu: H \to \mathbf{Z}\pi/I_\eta$, satisfying the following properties:

(Q1) $\varphi(x, y\lambda) = \varphi(x, y)\lambda$, $\quad \forall x, y \in H$ and $\forall \lambda \in \mathbf{Z}\pi$;

(Q2) $\varphi(x, y) = \eta\overline{\varphi(y, x)}$, $\quad \forall x, y \in H$;

(Q3) $\varphi(x, x) = \mu(x) + \eta\overline{\mu(x)}$, $\quad \forall x \in H$;

(Q4) $\mu(x + y) - \mu(x) - \mu(y) \equiv \varphi(x, y) \bmod I_\eta$, $\quad \forall x, y \in H$;

(Q5) $\mu(x\lambda) = \bar{\lambda}\mu(x)\lambda$, $\quad \forall x \in H$;

(Q6) $H_\Lambda = H \otimes_{\mathbf{Z}\pi} \Lambda$ is stably based, and the map

$$A\varphi_\Lambda: H_\Lambda \to \mathrm{Hom}_\Lambda(H_\Lambda, \Lambda) \text{ given by } A\varphi_\Lambda(x)(y) = \varphi_\Lambda(x, y),\ \varphi_\Lambda$$

induced by φ, is a simple isomorphism with respect to a preferred class of stable bases and its dual.

Note that by (Q6), $(H_\Lambda, \varphi_\Lambda, \mu_\Lambda)$ is a special η-Hermitian form over Λ, in the sense of Wall [58].

The special η-forms form a semi-group under orthogonal direct sum, denoted $\perp$.

We say the η-form α is strongly equivalent to zero (write $\alpha \approx 0$) if $\exists$ a submodule $K \subset H$ with the following properties:

(PS1) $\varphi(x, y) = 0$ and $\mu(x) \equiv 0$ $\forall x, y \in K$; and

(PS2) The image of K_Λ in H_Λ is a *subkernel* in the sense of Wall [58, Lemma 5.3].
The submodule K will be called a *pre-subkernel.*

If $\alpha = (H, \varphi, \mu)$ is an η-form, we define $-\alpha = (H, -\varphi, -\mu)$.

LEMMA 1.1. *$\alpha \perp (-\alpha) \perp \kappa \approx 0$, κ a kernel over* $\mathbf{Z}\pi$.

Proof. Let $\alpha = (H, \varphi, \mu)$. Adding a *kernel* over $\mathbf{Z}\pi$ [58, p. 47], we may assume that H_Λ is free. Let $K \subset H \oplus H$ be the diagonal submodule; i.e., $K = \{(x, x) \mid x \in H\}$. Clearly $\varphi \perp (-\varphi)$ and $\mu \perp (-\mu)$ vanish on K. The same argument as in [58, Lemma 5.4] shows that K satisfies (PS2).

Now say $\alpha \sim \beta$ (α is equivalent to β) if and only if $\alpha \perp (-\beta) \approx 0$. Let $\Gamma_\eta(\mathcal{F})$ be the set of equivalences classes of η-forms under the equivalence relation generated by $\sim$; $\perp$ induces the structure of an abelian group on $\Gamma_\eta(\mathcal{F})$. We also write $\Gamma_\eta(\mathcal{F}) = \Gamma_{2k}(\mathcal{F})$ for $\eta = (-1)^k$. Note that the η-form α represents zero in $\Gamma_\eta(\mathcal{F})$ if and only if there exists an η-form β with $\beta \approx 0$ and $\alpha \perp \beta \approx 0$. Clearly $\Gamma_\eta(\mathcal{F})$ depends functorially on $\mathcal{F}$.

LEMMA 1.2. *Each η-form $\alpha = (H, \varphi, \mu)$ is equivalent to a form $\alpha_0 = (H_0, \varphi_0, \mu_0)$ with H_0 free, with a basis whose image in $H_0 \otimes \Lambda$ is in the preferred class.*

Proof. After adding a kernel over $\mathbf{Z}\pi$, if necessary, let $y_1, \cdots, y_m$ be a basis of H_Λ in the preferred class. Since $\mathcal{F}$ is an epimorphism, we may write $y_i = x_i \otimes 1$, $x_i \in \mathbf{Z}\pi$. Let H_0 be the free $\mathbf{Z}\pi$-module with basis $x_1, \cdots, x_m$ and let $p\colon H_0 \to H$ be the homomorphism determined by $p(x_i) = x_i$. Let $\varphi_0(x, y) = \varphi(px, py)$, $\mu_0(x) = \mu(px)$. The base $x_1, \cdots, x_m$ provides $(H_0)_\Lambda$ with a basis also, and it is clear that $\alpha_0 = (H_0, \varphi_0, \mu_0)$ is an η-form, with $x_1 \otimes 1, \cdots, x_m \otimes 1$ in the preferred class. Let $K = \{(px, x) \mid x \in H_0\}$; then (PS1) for K is clear and (PS2) follows by [58, 5.4]. Hence $K \subset H \otimes H_0$ is a pre-subkernel for $\alpha \perp (-\alpha_0)$; i.e., $\alpha \sim \alpha_0$.

LEMMA 1.3. *The η-form α represents zero in $\Gamma_\eta(\mathcal{F})$ if and only if there exists a kernel κ over $\mathbf{Z}\pi$ with $\alpha \perp \kappa \approx 0$.*

Proof. Suppose α represents zero. Then there exists $\beta \approx 0$ with $\alpha \perp \beta \approx 0$. Write $\beta = (H, \varphi, \mu)$. Then, by (PS1) and (PS2), there exist elements $x_1, \cdots, x_r$ in H such that φ and μ vanish on the submodule spanned by these elements and such that there exist $y_1, \cdots, y_r$ with $x_1 \otimes 1, \cdots, x_r \otimes 1$, $y_1 \otimes 1, \cdots, y_r \otimes 1$ a preferred basis of H_Λ, with φ_Λ and μ_Λ trivial on the submodule spanned by $y_1 \otimes 1, \cdots, y_j \otimes 1$, and with $\varphi_\Lambda(x_i \otimes 1, y_j \otimes 1) = \delta_{ij}$. The elements $y_i, \cdots, y_r$ are found by lifting to H a suitable basis of the dual subkernel to the image in H_Λ of a pre-subkernel; the $x_1, \cdots, x_r$ arise by lifting a suitable basis of this image to the pre-subkernel itself; again this uses the fact that $\mathcal{F}$ is onto.

Let H_0 be the free $\mathbf{Z}\pi$-module on $\{x_1, \cdots, x_r, y_1, \cdots, y_r\}$. Let $p: H_0 \to H$ be the homomorphism with $p(x_i) = x_i$ and $p(y_i) = y_i$. Define $\varphi_0(x, y) = \varphi(px, py)$ and $\mu_0(x, y) = \mu(px)$. Let $(H_0)_\Lambda$ have the stable basis $x_1 \otimes 1, \cdots, y_r \otimes 1$ (there is a certain abuse of notation here). It is easy to see that if K is a pre-subkernel for $\alpha \perp \beta$, then $((\mathrm{id}_\alpha) \perp p)^{-1}K$ is a pre-subkernel for $\alpha \perp \beta_0$, where id_α denotes the identity automorphism of α. In particular $\alpha \perp \beta_0 \approx 0$.

Now let κ be the kernel over $\mathbf{Z}\pi$ of dimension $2r$, with standard basis $e_1, \cdots, e_1, f_i, \cdots, f_r$; i.e., if $\kappa = (K, \rho, \nu)$, $\rho(e_i, f_j) = \delta_{ij}$, $\rho(e_i, e_j) = \rho(f_i, f_j) = \nu(e_i) = \nu(f_1) = 0$, and $e_1, \cdots, e_r, f_1, \cdots, f_r$ is in the preferred class. We define a homomorphism $h: \beta_0 \to \kappa$ by setting

$$h(x_i) = \textstyle\sum_{j=1}^{r} e_j(\eta\varphi_0(y_j, x_i)) \quad \text{and}$$
$$h(y_k) = f_k + \gamma_k e_k + \textstyle\sum_{j>k} e_j(\eta\varphi_0(y_j, y_k)) ,$$

where $\gamma_k \equiv \mu_0(y_k) \bmod I_\eta$. It is easy to verify that h preserves forms by checking it on basis elements. For example,

$$\rho(h(x_i), h(y_k)) = \eta\overline{\varphi_0(y_k, x_i)} = \eta^2\varphi_0(x_i, y_k) = \varphi_0(x_i, y_k) .$$

Further $(\mathrm{id}_\alpha \perp h) \otimes \mathrm{id}_\Lambda = \mathrm{id}_{\alpha_\Lambda} \perp (h \otimes \mathrm{id}_\Lambda)$ is an isomorphism of the special η-Hermitian forms $(\alpha \perp \beta_0)$ and $(\alpha \perp \kappa)_\Lambda$. It now follows that if L is a pre-subkernel for $\alpha \perp \beta_0$, then $(\mathrm{id}_\alpha \perp h)(N)$ is a pre-subkernel for $\alpha \perp \kappa$; i.e., $\alpha \perp \kappa \approx 0$.

If we omit the words "simple" and "preferred class of basis" from the above discussion, we obtain groups $\Gamma^h_\eta(\mathcal{F}) = \Gamma^h_{2k}(\mathcal{F})$, $\eta = (-1)^k$. The lemmas remain valid, by easier versions of the same proofs. When we wish to emphasize the distinction between the two types of groups, we write $\Gamma_\eta = \Gamma^s_\eta$. There are natural homomorphisms $\Gamma^s_\eta \to \Gamma^h_\eta$.

For $e = s, h$ there are natural homomorphisms

$$L^e_{2k}(\pi, w) \longrightarrow \Gamma^e_{2k}(\mathcal{F}) \text{ and } \Gamma^e_{2k}(\mathcal{F}) \longrightarrow L^e_{2k}(\Lambda, \pm\mathcal{F}\pi) ;$$

the second homomorphism will be generically called j_*, and is clearly an epimorphism. By $L^e_{2k}(\Lambda, \pm\mathcal{F}\pi)$ we mean $L^h_{2k}(\Lambda)$ if $e = h$ and, for $e = s$, we mean the Wall group of special η-Hermitian forms defined using vanishing in Wh $(\mathcal{F})$ as the criterion for "simplicity". If $\Lambda = \mathbf{Z}\pi'$ and $\mathcal{F}$ is induced by a homomorphism of groups,

$$L^e_{2k}(\Lambda, \pm\mathcal{F}\pi) = L^e_{2k}(\pi', w') .$$

For $\Gamma^h_\eta(\mathcal{F})$, one actually needs only that $\mathcal{F}$ be *locally epic*; i.e., $\forall\lambda_1, \cdots, \lambda_k \in \Lambda$, there exists a unit u of Λ with $\lambda_1 u, \cdots, \lambda_k u$ in $\mathcal{F}(\mathbf{Z}\pi)$.

Note that for $\Lambda = \mathbf{Z}\pi$ and $\mathcal{F}$ = identity, $\Gamma^e_{2k}(\mathcal{F}) = L^e_{2k}(\pi, w)$. As another essentially known example, let R be an extension of $\mathbf{Z}$ contained in the rationals. Let $\mathcal{F}$ be the inclusion of $\mathbf{Z}\pi$ in $R\pi$. Then $\mathcal{F}$ is locally epic. Using Lemmas 1.2 and 1.3, it is not hard to show that $\Gamma^h_{2k}(\mathcal{F}) = L^h_{2k}(R\pi)$.

Before applying this algebra to geometric problems, we will need the following result, essentially a consequence of Theorem 3.3 of [11]:

LEMMA 1.4. *Let C_* be a finite chain complex of free $\mathbf{Z}\pi$-modules. Suppose $H_i(C_*) = 0$ for $i < m$. Then the natural map $H_m(C_*) \otimes \Lambda \to H_m(C_* \otimes \Lambda)$ is an isomorphism.*

Proof. Let $C_N \xrightarrow{\partial_N} \cdots \to C_i \xrightarrow{\partial_i} C_{i-1} \to 0 \cdots \to 0$ be the chain complex C_*, with $i \leqq m$. Since $H_{i-1}(C_*) = 0$, $0 \to C'_i \to C_i \xrightarrow{\partial_i} C_{i-1} \to 0$ is exact, where $C'_i = \ker \partial_i$. Hence there exists a free module F with $C'_i \oplus F$ free. Consider the chain complex

$$C'_*\colon C_N \xrightarrow{\partial_N} \cdots \longrightarrow C_{i+1} \oplus F \xrightarrow{\partial_{i+1} \oplus 1} C'_i \oplus F \longrightarrow 0 \longrightarrow \cdots .$$

The obvious map $C'_* \to C_*$ (which is trivial on the two summands F) induces isomorphisms of homology groups and hence, by a theorem of [61], is a chain homotopy equivalence. So by induction we need only consider a complex that vanishes in degrees less than m; for such a complex the result follows from right exactness of tensor products.

By a *simple Poincaré* (or just *Poincaré*) *complex, pair, triad*, etc. *over* $\mathcal{F}$, we mean just the same thing as in Chapter 2 of [58] (*respectively*, [59, § 2]), except that cap product with a (possibly infinite) chain representing the fundamental class is only required to be a simple chain equivalence (*resp.* chain equivalence) after tensoring all relevant chain complexes with Λ. For the notion of *simple chain equivalence*, we require the vanishing in Wh $(\mathcal{F})$ of the torsion of the algebraic mapping cone of the chain equivalence under study; [40, p. 382]. A Poincaré complex over $\mathcal{F}$ has a (twisted) integral fundamental class; actually one could carry out the subsequent theory with a fundamental class over Λ, with some care.

Let (X, Y) be a simple Poincaré pair, of dimension $2k$, $k \geqq 3$, over $\mathcal{F}$, X connected, and suppose that $\pi_1 X = \pi$ and $w\colon \pi \to \mathbf{Z}_2$ is the orientation character of X. Let

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \xi \\ \big\downarrow & & \big\downarrow \\ (M, \partial M) & \xrightarrow{f} & (X, Y) \end{array}$$

be a *normal map* of degree one [3], [58] of the manifold pair $(M, \partial M)$ into (X, Y). M may be a smooth, PL, or topological manifold, and ν_M denotes its stable normal bundle. We will concentrate on the smooth case in the rest of this chapter; the transition to the PL case is standard, and for the topological case one appeals to the results of Lees [30], Kirby-Siebenmann [27], [28].

Let $\partial Y = Y_1 \cup \cdots \cup Y_t$ be the decomposition of ∂Y into components, and let $\rho_i\colon \mathbf{Z}\pi_1 Y_i \to \mathbf{Z}\pi$ be induced by inclusion. Let $M_i = (f \mid \partial M)^{-1} Y_i$. Then we assume the following:

> $f \mid M_i\colon M_i \longrightarrow Y_i$ *induces isomorphism of homology groups with local coefficients in* Λ, *and* $\Delta_{\mathcal{F}\circ\rho_i}(f \mid M_i) = 0$.

We will usually say, in the future, that f induces a simple homology equivalence of boundaries over Λ.

By $K_i(M, B)$, $K_i(M, \partial M; B)$, $K^i(M; B)$ etc., B a $\mathbf{Z}\pi$-module, we mean the same thing as in [58, § 2]. When $B = \mathbf{Z}\pi$, it will be omitted from the notation.

By surgery we may assume, after a normal cobordism relative the boundary, that f is k-connected. Let $\eta = (-1)^k$. Suppose first that (X, Y) is an actual Poincaré pair. Then intersection and self-intersection forms $\varphi\colon K_k(M) \times K_k(M) \to \mathbf{Z}\pi$ and $\mu\colon K_k(M) \to I_\eta$, respectively, are defined. This is done exactly as in [58, § 1] using the fact that the connecting homomorphism $H_{k+1}(f) \to K_k(M)$ is an isomorphism, a consequence of Poincaré duality, and the Hurewicz isomorphism $H_{k+1}(f) \cong \pi_{k+1}(f)$.

In the general case, these forms are defined on $H_{k+1}(f)$; note that by Poincaré duality over Λ, $H_{k+1}(f; \Lambda) = K_k(M; \Lambda)$, and by 1.4, $H_{k+1}(f; \Lambda) = H_{k+1}(f) \otimes \Lambda$; the Hurewicz isomorphism still holds. However, in order to keep the notation uncluttered and the exposition more in line with [58], and since we do not use this case in this paper, we abuse the notation and continue to write $K_{k+1}(M)$, φ, and μ for $H_{k+1}(f) \cong \pi_{k+1}(f)$ with its intersection and self-intersection forms.

It follows exactly as in [58, 5.2], that $\alpha(f, b) = \big(K_k(M), \varphi, \mu\big)$ satisfies (Q1)–(Q5). It follows exactly as in [58, 2.2], over the coefficients Λ, that $K_k(M; \Lambda) = H_{k+1}(f; \Lambda)$ is stably free. Given a stable basis, the torsion $\Lambda_{\mathcal{F}}(f)$ is defined, and there is a unique class of stable bases with respect to which this torsion vanishes [40, § 3, 4]. Similarly, $K^k(M; \Lambda)$ is stably based, and, recalling that $K_*(\partial M; \Lambda) = 0$, it is not hard to see (compare [58, 2.5 and 2.6], [40] and [41]) that Poincaré duality yields a simple isomorphism $K_k(M; \Lambda) \to K^k(M; \Lambda)$. Since $K_i(M; \Lambda) = 0$ for $i < k$, the analogue of 1.4 for the functor Hom, over Λ, implies that the natural map gives an isomorphism $K^k(M; \Lambda) \cong \mathrm{Hom}_\Lambda(K_k(M; \Lambda); \Lambda)$. This isomorphism carries the preferred class of stable bases to the class of dual bases to the preferred class of bases of $K_k(M; \Lambda)$; compare [58, § 2]. Under this isomorphism, and using 1.4, the duality map corresponds to the adjoint $A\varphi_\Lambda$. Thus $\alpha(f, b)_\Lambda = (K_k(M) \otimes \Lambda, \varphi_\Lambda, \mu_\Lambda)$ satisfies (Q6); i.e., $\alpha(f, b)_\Lambda$ is an η-form.

By $\sigma(f, b) \in \Gamma_{2k}(\mathcal{F})$ we denote the element represented by $\alpha(f, b)$.

Now suppose that (f_1, b_1) is another normal map, k-connected, inducing a simple homology equivalence of boundaries over Λ.

PROPOSITION 1.5. *If (f, b) and (f_1, b_1) are normally cobordant relative the boundary, then $\sigma(f, b) = \sigma(f_1, b_1)$.*

Remark. Actually one can show, by the same argument, a stronger result analogous to [46, Th. 1.2], a result due to Wall for Wall groups. However, this result also follows as a formal consequence of § 3 below.

Proof. Let (F, B), $F: W \to X \times I$ be the normal cobordism; i.e., $\partial W = M \cup (\partial M \times I) \cup M_1$, where $\partial M_1 = \partial M \times 1$, and $(F, B) \mid M = (f, b)$, $(F, B) \mid M_1 = (f_1, b_1)$, and $F(\partial M \times I) \subset Y \times I$. By surgery in the interior of W, we may assume that F is k-connected. By handle subtractions [58], we can also kill $K_k(W, M)$ and $K_k(W, M_1)$; as f and f_1 are k-connected, the effect of these subtractions on (f, b) and (f_1, b_1) will be to perform surgery on trivial $(k-1)$-spheres; i.e., to take connected sum with copies of $S^k \times S^k$. This will add kernels over $\mathbf{Z}\pi$ to $\alpha(f, b)$ and $\alpha(f_1, b_1)$, leaving their classes in $\Gamma_{2k}(\mathcal{F})$ unchanged. So we may assume $K_k(W, M) = K_k(W, M_1) = 0$.

Let φ_1 and μ_1 be intersection and self-intersection forms on the module $K_k(M_1)$, a stably based module after taking tensor product with Λ. Then $K_k(M \cup M_1) = K_k(M) \oplus K_k(M_1)$, and on this module intersection and self-intersection forms are given by $\varphi_1 \perp (-\varphi)$ and $\mu_1 \perp (-\mu)$; the change of sign is due to the orientation convention $\partial[X \times I] = [X \times 1] - [X \times 0]$. So we have to show $(K_k(M \cup M_1), \varphi_1 \perp (-\varphi), \mu_1 \perp (-\mu)) \sim 0$.

Note that $K_*(\partial W; \Lambda) = K_*(M_1 \cup M; \Lambda)$ and $K_*(W, \partial W; \Lambda) = K_*(W, M \cup M_1; \Lambda)$, since $K_*(\partial M; \Lambda) = 0$. By duality over Λ, $K_i(W, \partial W; \Lambda) = 0$ for $i \neq k+1$. So $K_{k+1}(W, \partial W; \Lambda)$ is a stably based Λ-module; we take for the preferred class of stable bases the unique class with respect to which the torsion in $\mathrm{Wh}(\mathcal{F})$ of the map $F: (W, \partial W) \to (X \times I, \partial(X \times I))$ of pairs vanishes. By adding trivial k-handles to W along M or M_1, we can increase the rank of $K_{k+1}(W, \partial W; \Lambda)$ until the stable basis can be realized by an actual basis; again this will only add kernels over $\mathbf{Z}\pi$ to $\alpha(f, b)$ and $\alpha(f_1, b_1)$.

Similarly, $K_k(W; \Lambda)$ is stably based and so, by adding trivial k-handles as above, may be taken to be based. Then it is evident from [40, 3.2] the following is a *based* exact sequence:

(1.5.1) $$0 \longrightarrow K_{k+1}(W, \partial W; \Lambda) \longrightarrow K_k(\partial W; \Lambda) \longrightarrow K_k(W; \Lambda) \longrightarrow 0 .$$

Further duality $K_{k+1}(W, \partial W; \Lambda) \to K^k(W; \Lambda) = \mathrm{Hom}_\Lambda(K_k(W; \Lambda); \Lambda)$ carries the basis to a basis in the preferred class of bases dual to the preferred bases of $K_k(W; \Lambda)$; this follows by considering torsions defined using cochains and cohomology instead of chains and homology and relating them by duality, as in [58, § 2] for the usual case. From all this and 1.4, it follows (compare the proof of 5.7 in [58]) that the image of $K_{k+1}(W, M_1 \cup M_2)$ in $K_k(M_1 \cup M_2)$ satisfies (PS2).

The proof that (PS1) is satisfied is almost identical with [58, 5.7] or [59, 7.3] and is left to the reader.

PROPOSITION 1.6. *Let (f, b) be a normal map of degree one inducing a simple homology equivalence over Λ on boundaries. Suppose f is itself a simple homology equivalence over Λ. Then $\sigma(f, b) = 0$.*

Proof. This is proved using a slight modification of the preceding proof. Let (F, B), $F: W \to X \times I$, be a normal cobordism, relative the boundary of $f: M \to X$ to $f': M' \to X$, f' k-connected. Again, by handle subtraction, it may be assumed that $K_k(W, M') = 0$. As in 1.5, if $\alpha(f', B \mid M') = (K_k(M'), \varphi', u')$, φ' and u' vanish on the image of $\partial: K_{k+1}(W, M') \to K_k(M')$. Since $K_i(M, \Lambda) = 0$, the based sequence (1.5.1) will still hold here, after further subtractions. Hence the image of ∂ will be a pre-subkernel for $\alpha(f', B \mid M)$.

Given any normal map (f, b), $f: (M^{2k}, \partial M) \to (X, Y)$ of degree one, inducing a simple homology equivalence of boundaries, over Λ, we define $\sigma(f, b) = \sigma(f_0, b_0)$, where (f_0, b_0) is a k-connected normal map that is normally cobordant, relative to boundary, to (f, b).

THEOREM 1.7. *The invariant $\sigma(f, b)$ depends only upon the normal cobordism class of (f, b) and vanishes if and only if (f, b) is normally cobordant to a simple homology equivalence over Λ.*

Proof. By 1.5 and 1.6, it remains only to show that if $\sigma(f, b) = 0$, then (f, b) is normally cobordant to a simple homology equivalence over Λ. By surgery, we may assume that f is k-connected, so that $\alpha(f, b)$ is defined. By Lemma 1.3, there exists a kernel κ over $\mathbf{Z}\pi$ with $\alpha(f, b) \perp \kappa \approx 0$. We may realize the orthogonal sum geometrically by taking connected sum with copies of $S^k \times S^k$; so we may as well assume that $\alpha(f, b) \approx 0$.

Hence there exists a subkernel $H \subset K_k(M)$. Since $\mathcal{F}$ is (locally) epic, we can find elements $x_1, \cdots, x_r \in H$ whose images in $K_k(M; \Lambda) = K(M) \otimes \Lambda$ form a preferred basis for a subkernel of $\alpha(f, b)_\Lambda$. Since the intersection and self-intersection forms vanish on the subspace generated by $x_1, \cdots, x_r$, these elements can be represented by disjointly embedded framed spheres. Surgery can be performed on these spheres; the same argument as in [59, 5.6] shows that the result is a simple homology equivalence Λ.

ADDENDUM TO 1.7. *If $\sigma(f, b) = 0$, f is normally cobordant to a $(k-1)$-connected simple homology equivalence over* Λ.

Now we consider the special case $X^{2k} = Y \times I$, $k \geqq 3$. Let (h, c), $h: (P, \partial P) \to (Y, \partial Y)$, be a simple homology equivalence of pairs over Λ, P a manifold.

THEOREM 1.8. *Let $\gamma \in \Gamma_{2k}(\mathcal{F})$. Then there exists a normal cobordism (f, b) of (h, c), relative the boundary, to a simple homology equivalence over Λ, with $\sigma(f, b) = \gamma$.*

Proof. Let γ be represented by (H, φ, μ), with H a free module with basis $x_1, \cdots, x_m$ so that $x_1 \otimes 1, \cdots, x_m \otimes 1$ is in the preferred class of bases of $H \otimes \Lambda = H_\Lambda$. Let $f_i^0: S^{k-1} \times D^k \to \operatorname{Int} P$, $1 \leqq i \leqq m$, be disjoint, unlinked, unknotted embeddings. According to [59, p. 53], we can subject the f_i^0 to regular homotopies $\eta_i: S^{k-1} \times D^k \times I \to (\operatorname{Int} P) \times I$, with intersection numbers $\eta_i \cdot \eta_j = \varphi(x_i, x_j)$ and with the self-intersection number of η_i equal to $\mu(x_i)$. We will also assume that η_i is constant on $S^{k-1} \times D^k \times [0, 1/4]$ and that $S^{k-1} \times D^k \times [0, 1/4] = \eta_i^{-1}(P \times [0, 1/4])$. Let f_i^1 be the final stage of η_i, and let (f, b), $f: M \to Y \times I$, be obtained from $h \times 1: P \times I \to Y \times I$ by attaching handles to $P \times I$ with attaching maps f_i^1. Then $K_k(M, P)$ is isomorphic to H, and under the obvious isomorphism, intersection and self-intersection forms correspond to φ and μ, respectively.

To analyze (f, b), we must make it k-connected. To do this, we perform surgery on $(f, b) \mid P \times [0, 1/4]$, relative the boundary, to obtain (f', b'), $f': U \to Y \times [0, 1/4]$ a k-connected map. Let $W = (M - P \times [0, 1/4]) \cup U$, and define (f_0, b_0) on W to be the union of (f, b) and (f', b'). By Van Kampen's theorem, f_0 is an isomorphism on fundamental groups, and it is not hard to check, using homology with local coefficients in $\mathbf{Z}\pi$, that f_0 is k-connected.

By excision, $K_*(W, U) = K_*(M, P \times [0, 1/4]) = K_*(M, P)$, and this identification respects intersection and self-intersection numbers. Hence we have the following exact sequence

$$0 \longrightarrow K_k(U) \longrightarrow K_k(W) \longrightarrow K_k(M, P) \longrightarrow 0\ ,$$

with the maps preserving intersection and self-intersection forms. Since the f_i^0, and hence the $f_i^{1/4}: S^{k-1} \times D^k \times 1/4 \to P \times 1/4$, are unlinked and unknotted, there is a splitting of the sequence that exhibits $K_k(W)$, with its intersection and self-intersection forms, as the orthogonal direct sum of $K_k(U)$ and $K_k(M, P)$ with their forms. Hence the form on $K_k(W)$ is a special $(-1)^k$ form over $\mathcal{F}$; by [Q6] and the same argument (over Λ) as in [58, 5.8], it follows that f_0 induces a simple homology equivalence of boundaries over Λ. The split exact sequence above now implies

$$\alpha(f_0, b_0) = \alpha(f', b') \perp (H, \varphi, \mu) .$$

But $\sigma(f', b') = 0$, by Theorem 1.7. So

$$\sigma(f_0, b_0) = \rho(f, b) = \gamma ,$$

which proves the theorem.

ADDENDUM TO 1.8. *If h was i-connected, $i < k$, then so is the normal map into $Y \times 1$ obtained from h by the surgeries described above.*

Proof. To obtain this normal map, only surgery on $(k-1)$-spheres was performed.

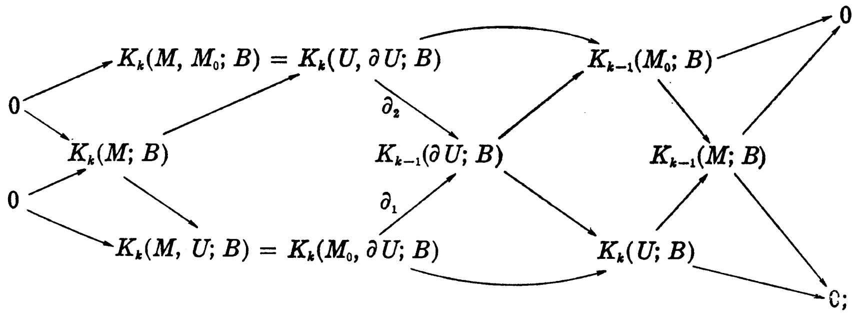
If we omit the word "simple" throughout the entire discussion, we obtain the theory for surgery to get just homology equivalence over Λ. The obstructions $\sigma(f, b)$ will lie in groups $\Gamma_{2k}^h(\mathcal{F})$ mentioned above. The analogues of 1.5–1.8 remain valid and are easier to prove. We will refer to these numbers for these propositions and their non-simple analogues throughout the rest of this paper.

2. The odd dimensional absolute case

Let $\mathcal{F}: \mathbf{Z}\pi \to \Lambda$ be an epimorphism of rings, as in § 1. Let (X^{2k-1}, Y) be a Poincaré pair over $\mathcal{F}$, $k \geqq 3$. As in [60], we may write $X = X_0 \cup D^{2k-1}$ with $(X_0; Y, S^{2k-1})$ a Poincaré triad, over $\mathcal{F}$. In this section we discuss the non-simple case; the simple case is actually the easier of the two in these dimensions.

Let (f, b), $f: (M, \partial M) \to (X, Y)$, be a degree one normal map inducing a homology equivalence of boundaries, over Λ. By surgery, one can make f $(k-1)$-connected. Let $f_i: S^{k-1} \times D^k \to \operatorname{Int} M$ be disjoint framed embeddings representing a set of generators of $K_{k-1}(M; \Lambda)$. Let U be the union of the images of the f_i; we may assume $f(U) \subset D^{2k-1}$, $f(M_0) \subset X_0$, $M_0 = \operatorname{cl}(M - U)$.

For any $\mathbf{Z}\pi$-module B we have the diagram $*(B)$:



and for $B = \Lambda$, $*(B)$ has exact arcs (compare [58, p. 56]). (Again, if (X, Y) is not a Poincaré pair, we should really use homology groups of suitable maps, over $\mathbf{Z}\pi$ as in § 1.)

Applying essentially the arguments of Theorem 1.5, one may show that after a stabilization the image of ∂_1 is a pre-subkernel for $K_{k-1}(\partial U)$, which we identify, as in [59], with a standard kernel over $\mathbf{Z}\pi$. The stabilization is achieved by adding some trivial elements to our set of generators of $K_{k-1}(M; \Lambda)$.

The image of ∂_2 is the standard subkernel of the kernel $K_{k-1}(\partial U)$. So by 5.3.1 of [58], any isomorphism of $K_k(U, \partial U; \Lambda)$ with $K_k(M, U; \Lambda)$ will extend to an automorphism α, say, of the standard kernel $K_{k-1}(\partial U; \Lambda)$. Then α represents an element of the Wall [58, 17D] group $L^h_{2k-1}(\Lambda)$, and the arguments of [58, § 6], essentially unmodified, show that this element, denoted $\sigma(f, b)$, is a well-defined invariant of the normal cobordism class of (f, b) relative the boundary and vanishes for f a homology equivalence over Λ.

Conversely, suppose that $\sigma(f, b) = 0$; i.e., α represents zero in $L^h_{2k-1}(\Lambda)$. Then after stabilizing we may write

$$\alpha = \Sigma_r \alpha_1 \cdots \alpha_n$$

where each α_i stably represents an element of $RU(\Lambda)$ or is of the form $H(A)$, A a non-singular matrix over Λ. The notation here is exactly as in [59, § 6], except that to measure Whitehead torsion we replace Wh (π) by Wh $(\mathcal{F})$. The stabilization can be achieved by adding some trivial classes in $K_{k-1}(M)$; this changes α to $\alpha \oplus \sigma$ and replaces Σ_r by Σ_{r+1}. Here we are using the description of $L^h_{2k+1}(\Lambda)$ as the quotient of $U(\Lambda)$, (analogous to $SU(\Lambda)$ in [59] except that we forget about Whitehead torsion considerations) by the subgroup generated by $RU(\Lambda)$ and $\{H(A) \mid A \text{ non-singular}\}$. This is clearly equivalent to the description mentioned, for example, in [46].

However, $[U(\Lambda), U(\Lambda)] = [SU(\Lambda), SU(\Lambda)] \subset RU(\Lambda)$; the inclusion is proved by Wall [58] and the equality follows, for example, from an identity of Vaserstein [57]. (See also [50], where the analogue for unitary Steinberg groups is given.) So, after further stabilizations, we may assume that

$$\alpha = \Sigma_r H(A)\beta ,$$

A non-singular and $\beta \in RU(\Lambda)$. We may change α to $\Sigma_r H(A)$ by doing surgeries and altering some arbitrary choices, exactly as in [59, § 6]; this uses the fact that $\mathcal{F}$ is onto. It is quite easy to see from $*(\Lambda)$ that for such an α, $K_{k-1}(M; \Lambda) = 0$, which makes f a homology equivalence over Λ.

To define $\sigma(f, b)$ for an arbitrary normal map, put $\sigma(f, b) = \alpha(f', b')$, where (f', b') is a $(k-1)$-connected normal map normally cobordant, relative the boundary, to (f, b).

Then we have shown the following:

PROPOSITION 2.1. *The invariant $\sigma(f, b) \in L^h_{2k-1}(\Lambda)$ depends only upon the normal cobordism class of (f, b) and vanishes if and only if (f, b) is normally cobordant relative the boundary to a homology equivalence with coefficients in Λ. Further, if $\sigma(f, b) = 0$, (f, b) is actually normally cobordant relative the boundary to a $(k-1)$-connected simple homology equivalence over Λ.*

However, recall that in $*(\mathbf{Z}\pi)$ the image of ∂_1 was actually a pre-subkernel. So define $\Gamma^h_{2k-1}(\mathcal{F})$ to be the subgroup of elements of $L^h_{2k-1}(\Lambda)$ that have representatives $\alpha \in U(\Lambda)$ with the property that there exists a pre-subkernel of the standard kernel over $\mathbf{Z}\pi$ whose image in the standard kernel over Λ, after tensoring with Λ, is precisely the image under α of the standard subkernel (i.e., image of ∂_1 in $*(\Lambda)$). One sees that this is a subgroup by using the orthogonal sum representation of addition in $L^h_{2k}(\Lambda)$. The invariant $\sigma(f, b)$ defined above always lies in $\Gamma^h_{2k-1}(\mathcal{F})$.

We usually write $j_*: L^h_{2k-1}(\mathcal{F}) \to L^h_{2k-1}(\Lambda)$ for the inclusion.

Next, let $h: (P^{2k-2}, \partial P) \to (X, \partial X)$, P a manifold, be a homology equivalence over Λ, and let $c: \nu_P \to \xi$ be a bundle map over h.

THEOREM 2.2. *Let $\gamma \in \Gamma^h_{2k-1}(\mathcal{F})$, $k \geqq 4$. Then there is a normal cobordism, (f, b), relative the boundary, of (h, c) to a homology equivalence over Λ, with $\sigma(f, b) = \gamma$.*

Proof. First, let (F_0, B_0), F_0: $W_0 \to X \times [0, 1/2]$, be a normal cobordism of (h, c) to (h_0, c_0), where

$$h_0: T = P \# r(S^{k-1} \times S^{k-1}) \longrightarrow X \times 1/2$$

is obtained by boundary connected sum along $(\operatorname{Int} P) \times 1/2$ of $P \times [0, 1/2]$

and r copies $S^{k-1} \times D^k$. Choose r so that γ has a representative $\alpha \in U_r(\Lambda)$ exhibiting γ as an element of $\Gamma^h_{2k-1}(\mathcal{F})$, as described above. Hence if $e_1, \cdots, e_r$ is the standard base of the subkernel of $K_{k-1}(T; \Lambda)$; i.e., e_i is carried by the i^{th} copy of $S^{k-1} \times pt$; then we may choose $x_1, \cdots, x_r \in K_{k-1}(T)$ whose images in $K_{k-1}(T; \Lambda)$ are $\alpha(e_1), \cdots, \alpha(e_r)$, respectively, and such that intersection and self-intersection forms vanish on the submodule they generate. Represent $x_1, \cdots, x_r$ by framed embedded spheres of dimension $(k-1)$ and perform surgery to obtain a normal cobordism (F_1, B_1) of (h_0, c_0), $F: W_1 \to X \times [1/2, 1]$. Let $(f, b) = (F_0, B_0) \cup (F_1, B_1)$.

If f were $(k-1)$-connected, we could conclude exactly as in [58, § 6] that $\sigma(f, b) = \gamma$. One can deal with this point exactly as we dealt with a similar problem in 1.8; perform surgery near $P \times 0 \subset \partial(W_0 \cup W_1)$ to make f $(k-1)$-connected and observe that the surgery obstruction of the resulting normal map is represented by $\alpha \perp \beta$, where, using 2.1, β represents zero in $L^h_{2k-1}(\Lambda)$. Hence $\sigma(f, b) = \gamma$. We leave the details to the reader.

ADDENDUM TO 2.2. *If h is i-connected, $i < (k-1)$, so is the normal map into $X \times 1$ obtained by the surgeries described above.*

In the simple case we get obstructions $\sigma(f, b)$ lying in $\Gamma_{2k-1}(\mathcal{F}) = \Gamma^s_{2k-1}(\mathcal{F}) \subset L_{2k-1}(\Lambda, \pm\mathcal{F}\pi)$; the last term denotes a Wall group $SU(\Lambda)/RU(\Lambda)$; $SU(\Lambda)$ consists of stabilizations of elements that have vanishing torsion in Wh$(\mathcal{F})$ with respect to the usual basis of the standard kernel; $RU(\Lambda)$ is generated by $TU(\Lambda)$ and σ; $TU(\Lambda)$ consists of stabilizations of elements of $U(\Lambda)$ that preserve the standard subkernel and induce on it an automorphism with vanishing torsion in Wh$(\mathcal{F})$. Note that for the case $\Lambda = \mathbf{Z}\pi'$ and $\mathcal{F}$ induced by a group homomorphism, $L_{2k-1}(\Lambda, \pm\mathcal{F}\pi) = L^s_{2k-1}(\pi')$. There are natural maps $\Gamma^s_{2k-1} \to \Gamma^h_{2k-1}$.

3. The general case

The purpose of this section is to define relative groups for the surgery problem with coefficients. This could be done using algebraically defined obstruction groups as in § 1, 2, essentially along the lines of the thesis of Sharpe [49]. Instead, we proceed as in § 9 of [59] to construct such groups abstractly and fit them into appropriate exact sequences. This will provide sufficient information for the applications to follow. However, it should be pointed out that the algebraic descriptions given above for the absolute groups, especially the realization theorems 1.8 and 2.2, are essential to prove an analogous realization theorem for relative groups. This result is in turn essential in applications.

The main point in defining relative groups in surgery theory is to show geometrically that such groups must vanish for the identity map. More precisely, let $(X^n; Y_-, Y_+)$ be a simple Poincaré triad over $\mathcal{F}: Z\pi \to \Lambda$, $\pi = \pi_1 X$, X connected. Assume that Y_+ is connected and the natural map $\pi_1 Y_+ \to \pi_1 X$ is an isomorphism; write $\pi_1 Y_+ = \pi$. Let $(f, b), f:(M^n; \partial_- M, \partial_+ M) \to (X; Y_-, Y_+)$ be a degree one normal map with $f \mid \partial_- M: (\partial_- M, \partial(\partial_- M)) \to (Y_-, \partial Y_-)$ a simple homology equivalence over Λ.

THEOREM 3.1. *The normal map (f, b) is normally cobordant, relative $\partial_- M$, to a simple homology equivalence of triads, over Λ, $n \geqq 6$.*

Remarks. 1. This generalizes Theorem 3.3 [58]. However, the present proof differs somewhat from that of [58], even for the case of Wall groups (i.e., $\mathcal{F}$ = identity).

2. A similar result is valid in the non-simple case.

Proof of 3.1. Let $(f_\pm, b_\pm)$ denote the restriction of (f, b) to $\partial_+ M$ and $\partial_- M$, respectively.

Case I. $n = 2k + 1$. Then the argument of Theorem 1.5 applies to show that $\sigma(f_+, b_+) \in \Gamma^s_{2k}(\mathcal{F})$ vanishes. (Compare [58, 5.6 and 5.7].) So let (F, B), $F: W \to Y_+ \times [0, 1/2]$ be a normal cobordism, relative the boundary, of (f_+, b_+) to a simple homology equivalence over Λ, (h, c) say. Then $(f, b) \cup (F, B)$ is a normal map from $M \cup_{\partial_+ M} W$ to $X \cup_{Y_+} (Y_+ \times [0, 1/2])$ that induces a simple homology equivalence of boundaries, over Λ. Let $\gamma = \sigma(f \cup F, b \cup B) \in \Gamma_{2k+1}(\mathcal{F})$ be its surgery obstruction.

By Theorem 2.2. there exists a normal cobordism (G, C), $G: V \to Y_+ \times [1/2, 1]$, relative the boundary, of (h, c) to a simple homology equivalence over Λ, with $\sigma(G, C) = -\gamma$.

By additivity,

$$\sigma(f \cup F \cup G, b \cup B \cup C) = \sigma(F \cup f, B \cup b) + \sigma(G, C) = 0 .$$

In these dimensions these obstructions lie in a subgroup of a Wall group, so that the required additivity over unions follows exactly the same way as for Wall obstructions; we omit the details. So there is a normal cobordism, relative the boundary, of $(f \cup F \cup G, b \cup B \cup C)$ to a simple Λ-homology equivalence. Clearly this normal cobordism may be viewed as a cobordism of (f, b), relative $\partial_- M$; it is the desired cobordism.

Case II. $n = 2k$. By surgery we may assume that f_+ is $(k-1)$-connected and f is k-connected. Hence $K_i(M, \partial_+ M; \Lambda) = 0$ for $i \neq k$, by Poincaré duality over Λ. Hence, as in the proof of 2.3 of [58], $K_k(M, \partial_+ M; \Lambda)$ is stably based. After handle subtractions if necessary, let $x_1, \cdots, x_r$ be a basis of $K_k(M, \partial_+M; \Lambda)$ that is in the preferred class of bases.

By the Hurewicz theorem and the connectivity assumptions, the classes x_i can be represented by framed k-disks in (M, ∂_+M). By transversality we may assume the immersions are regular and mutually transverse; in particular the boundary spheres will be disjointly embedded in ∂_+M. Using the 1-connectedness of (M, ∂_+M), one may eliminate all these intersections and self-intersections by piping across the boundary (see [42, p. 73–84]), obtaining regularly homotopic disjoint framed embeddings. Perform handle subtractions using those embeddings. Then the same argument as in § 4 of [59], over Λ, shows that the result is a simple homology equivalence over Λ.

With the aid of Theorem 3.1, the groups $\Gamma^e_*(\mathcal{F})$, $e = s$ or h, can be defined geometrically as in § 9 of [59], π finitely presented. The "objects" and "restricted objects" are the same as in [59], except for two changes: the targets can be Poincaré triads or simple Poincaré triads over $\mathcal{F}$, and homotopy or simple homotopy equivalences in [59] are only required to be homology or simple homology equivalences over Λ. It follows from 3.1 that the group of objects and the set of restricted objects (with $K = K(\pi, 1)$) are naturally isomorphic to each other and to $\Gamma^e_n(\mathcal{F})$, in dimension $n \geqq 6$. (Note: If $\mathcal{F} = \mathrm{id}$, we actually obtain a slight variation from [59, § 9] in terms of what is required of the restriction of a normal map to the boundary. The present approach seems more natural, even for Wall groups.)

Let $\phi = (\alpha, \alpha')$ be a homomorphism from $\mathcal{F}$ to $\mathcal{G}\colon \mathbf{Z}\pi' \to \Lambda'$; i.e., the following square commutes:

$$\begin{array}{ccc} \mathbf{Z}\pi & \xrightarrow{\mathcal{F}} & \Lambda \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ \mathbf{Z}\pi' & \xrightarrow{\mathcal{G}} & \Lambda' \end{array} .$$

Then we may define, similarly, $\Gamma^e_n(\phi)$, $n \geqq 6$, functorial in ϕ. There are natural maps $\Gamma^s_n(\phi) \to \Gamma^h_n(\phi)$.

PROPOSITION 3.2. *For $n \geqq 6$, there is a natural exact sequence*

$$\Gamma^e_{n+1}(\phi) \xrightarrow{\partial} \Gamma^e_n(\mathcal{F}) \xrightarrow{\phi_*} \Gamma^e_n(\mathcal{G}) \longrightarrow \Gamma^e_n(\phi) .$$

The proof is immediate from the definitions in terms of unrestricted objects. The map ∂ corresponds to taking a suitable part of the boundary of a normal map. Geometrically ϕ_* may be described as "weakening the coefficients"; in terms of the descriptions in § 1 and § 2, ϕ_* is just the map induced functorially by ϕ. If one thinks of $\Gamma^e_n(\mathcal{G})$ as described in this section, the unlabeled map can be described geometrically by introducing a trivial boundary component to the normal map (e.g., use the empty set or add the identity map on a disjoint cell), which becomes the part of the boundary corresponding to $\mathbf{Z}[\pi] \to \Lambda$. One may also "free" a portion of the boundary on which one already has a (simple) homology equivalence over Λ.

If $\Lambda = \mathbf{Z}\Pi$, $\Lambda' = \mathbf{Z}\Pi'$ are group rings and α, α', $\mathcal{F}$ and $\mathcal{G}$ are induced by group homomorphisms, we may think of ϕ as induced from a diagram

$$\begin{array}{ccc} K(\pi, 1) & \longrightarrow & K(\Pi, 1) \\ \downarrow & & \downarrow \\ K(\pi', 1) & \longrightarrow & K(\Pi', 1) \end{array}$$

of Eilenberg-MacLane spaces. Hence, from the description of Wall groups in § 9 of [58] and the preceding, it is easy to see that there is a natural map $j_*\colon \Gamma^e_n(\phi) \to L^e_n(\alpha')$, so that the following diagram commutes:

$$\begin{array}{ccccccc} \Gamma^e_{n+1}(\phi) & \xrightarrow{\partial} & \Gamma^e_n(\mathcal{F}) & \xrightarrow{\phi_*} & \Gamma^e_n(\mathcal{G}) & \longrightarrow & \Gamma^e_n(\phi) \\ \downarrow j_* & & \downarrow j_* & & \downarrow j_* & & \downarrow j_* \\ L^e_{n+1}(\alpha') & \xrightarrow{\partial} & L^e_n(\Pi) & \xrightarrow{\alpha'_*} & L^e_n(\Pi') & \longrightarrow & L^e_n(\alpha') \end{array} \tag{3.2.1}$$

the other maps j_* have been already defined.

The relative groups solve a relative surgery problem. That is, let (f, b), $f\colon (M^n; \partial_- M, \partial_+ M) \longrightarrow (X; Y_-, Y_+)$ be a normal map into the simple Poincaré triad $(X; Y_-, Y_+)$ over $\mathcal{G}$, of degree one, with $\pi_1 Y_+ = \pi$, $\pi_1 X = \pi$, and α the natural map induced by inclusion. (X and Y_+ are assumed connected.) We assume $(Y_+, \partial Y_+)$ a simple Poincaré pair over $\mathcal{F}$. Assume $f_-\colon \partial_- M \to Y_-$ is a simple homology equivalence over Λ' and $f_- \mid \partial(\partial_- M)\colon \partial(\partial_- M) \to \partial Y_-$ is a simple homology equivalence over Λ.

THEOREM 3.3. *Let $n \geqq 6$. The element $\sigma(f, b) \in \Gamma^s_n(\phi)$ represented by (f, b) vanishes if and only if (f, b) is normally cobordant relative $\partial_- M$ to (g, c), $g\colon (N; \partial_- N, \partial_+ N) \to (X; Y_-, Y_+)$ (so $\partial_- N = \partial_- M$) with $g \mid \partial_+ N\colon \partial_+ N \to Y_+$ a simple homology equivalence over Λ and g a simple homology equivalence over Λ'.*

A similar result holds in the non-simple case. Whenever we write Γ_n, we always mean Γ^s_n. The theorem may be derived formally from 3.2, roughly as follows: since $\partial\sigma(f, b) = 0$, we first do surgery on the boundary to get a simple homology equivalence over Λ. The resulting normal map has a surgery obstruction in $\Gamma_n(\mathcal{F})$ which, by hypothesis, has vanishing image in $\Gamma_n(\phi)$. Hence it is in the image of ϕ_*. Therefore, it may be killed by a further normal cobordism of the boundary. Surgery can then be performed relative the boundary to obtain the normal map (g, c).

ADDENDUM TO 3.3. *In Theorem* 3.3, *g can be taken to induce isomorphisms of $\pi_1 N$ to $\pi_1 X$ and of $\pi_1 \partial_+ N$ to $\pi_1 Y$.*

This is easy to see from the preceding proof and the same result in the absolute case; the absolute case is in turn easy to check from § 1, 2 (see 2.1, 1.7).

Using 3.3, 1.8, 2.2, and addenda, one can imitate the proof of 10.4 of [59] to derive a realization theorem, namely: suppose $X^n = Z \times I$, $(Z; Z_-, Z_+)$ a simple Poincaré triad over $\mathcal{F}$, with $(Z_+, \partial Z_+)$ a simple Poincaré pair over $\mathcal{F}$, where $\pi_1 Z_+ = \pi$, $\pi_1 Z = \pi'$, and α is the natural map.

THEOREM 3.4. *Let (h, c) be a normal map, $h: (M^n, \partial_\pm M) \to (Z; Z_\pm)$ a simple homology equivalence of triads over Λ', with $h \mid \partial_+ M: \partial_+ M \to Z_+$ a simple homology equivalence over Λ. Suppose $h: M \to Z$ is i-connected, $2i < n$, or $h \mid \partial_+ M: \partial_+ M \to Z_+$ is j-connected, $2j < n - 1$. Let $n \geqq 7$, and $\gamma \in \Gamma_n(\phi)$. Then there exists a normal cobordism, relative $\partial_- M$, (F, B), to a map satisfying the same conditions as h, with $\sigma(F, B) = \gamma$.*

A similar result holds in the non-simple case.

It follows from this section that surgery obstructions for homology equivalence satisfy natural properties and additive formulae similar to the corresponding properties and formulae for Wall groups. For the absolute case, such properties can also be derived directly from the definitions in § 1, 2, using the special forms given in 1.8 and 2.2. The general results can then be derived formally from 3.3 and the absolute case.

Let CP^2 denote the complex projective plane.

THEOREM 3.5. *Let (f, b), $f: (M^n, \partial M) \to (X, Y)$, $n \geqq 5$ be a normal map, (X, Y) a simple Poincaré pair over Λ, $\mathcal{F}: \mathbf{Z}\pi \to \Lambda$ a homomorphism as above where $\pi = \pi_1 X$. Assume f induces a simple homology equivalence over Λ. Then, in $\Gamma^e_n(\mathcal{F}) = \Gamma^e_{n+4}(\mathcal{F})$,*

$$\sigma(f, b) = \sigma\big((f, b) \times CP^2\big) .$$

The proof is the same as for [59, 9.9]. Applying the five lemma yields the following, ϕ as above:

COROLLARY 3.6. *Taking products with CP^2 induces isomorphisms*

$$\Gamma^e_n(\phi) \longrightarrow \Gamma^e_{n+4}(\phi) , \qquad n \geqq 7 .$$

Remark 3.7. Let (f, b) $f: W^m \to X \times I$, X closed, be a normal map that induces homology equivalences on boundary components over Λ,

$\mathcal{F}: \mathbf{Z}\pi_1 X \to \Lambda$ an epimorphism. Suppose that $\Delta_{\mathcal{F}}(f \mid \partial_- W) = \Delta_{\mathcal{F}}(f \mid \partial_+ W)$. Then one can actually define $\sigma(f, b) \in \Gamma^s_m(\mathcal{F})$, which vanishes if and only if f is normally cobordant to a homology equivalence over Λ with torsion $\Delta_{\mathcal{F}}(f \mid \partial_- W)$. All obstructions can be realized beginning with a given homology equivalence over Λ. One can sum this up, after changing a sign in one of the boundary components to account for orientations, by saying that obstructions in Γ^s_* are still defined under the hypotheses that the *total* Whitehead torsion, i.e., the sum of the torsions of the components of the boundary, vanishes. A similar remark applies to the relative groups defined in this section. We leave the details to the reader.

Chapter II: The local codimension two problem

4. The group of local knots of a manifold

Let M^n be a closed, connected, smooth (piecewise linear, topological) manifold. Let ξ be a 2-plane bundle over M, with disk bundle $E = E(\xi)$ and boundary sphere bundle $S = \partial E$. Let $\pi' = \pi_1 M = \pi_1 E$, $\pi = \pi_1 S$, $\mathcal{F}: \mathbf{Z}\pi \to \mathbf{Z}\pi' = \Lambda$ the map induced by $S \subset E$ or, equivalently, by projection p of $E(\xi)$ onto M. The involution on $\mathbf{Z}\pi$ is given by $\bar{g} = w(g)g^{-1}$, $g \in \pi$, where $w: \pi \to \{\pm 1\}$ is the orientation character of S. This notation will remain constant throughout this section.

By a smooth (*resp.* P.L., *topological*) *local knot of M* (in ξ) is meant a smooth (*resp.* P.L. locally flat, topologically locally flat) embedding $\iota: M \to E - S$ homotopic to the zero section $M \subset E$.

Two local knots ι and ι_1 are said to be *equivalent* if there exists a diffeomorphism (*resp.* P.L. homeomorphism, homeomorphism) $\varphi: (E, S) \to (E, S)$, homotopic to the identity as a map of pairs, with $\varphi\iota = \iota_1$.

The local knots ι and ι_1 are said to be *concordant* if there is a smooth (*resp.* P.L. locally flat, topologically locally flat) embedding of $M \times I$ in $E \times I$ that restricts to $(\iota, 0)$ and $(\iota_1, 1)$ on $M \times 0$ and $M \times 1$, respectively.

Two local knots are said to be *cobordant* if they are equivalent to concordant knots. Cobordism is easily seen to be an equivalence relation. Let $C_O(M, \xi)$ (*resp.* $C_{\mathrm{PL}}(M, \xi)$, $C_{\mathrm{TOP}}(M, \xi)$) denote the cobordism classes of smooth (*resp.* P.L., topological) local knots of M in ξ.

Assertions about "local knots" with no additional adjective are meant to cover all three categories. Proofs of such assertions will usually apply as written at least in the smooth and P.L. category. In the topological category, the arguments can be modified using [30], [27, 2].

LEMMA 4.1. *Let $\iota: M \to E(\xi)$ be a local knot. Then the inclusion*

$S \subset E - \iota(M)$ is a simple homology equivalence over $\mathbf{Z}\pi' = \Lambda$.

Proof. By Poincaré duality over Λ and the fact that ι is a simple homotopy equivalence.

LEMMA 4.2. *Let $\iota: M \to E$ be a local knot with tubular neighborhood $T \subset E - S$. Let $W = \mathrm{cl}(E - T)$. Then there exists a map $f: W \to S$, with $f \mid S$ the identity and $f \mid \partial T: \partial T \to S$ a bundle map, so that pf is homotopic to $p \mid W$. Further, f is unique up to a homotopy, relative S, that is an isotopy of bundle maps on ∂T. In particular, ι has normal bundle equivalent to ξ.*

Note. A similar result, in a relative form, holds for a concordance $M \times I \to E \times I$, and is proved the same way.

Proof. Consider the lifting problem

$$\begin{array}{ccc} S & \xrightarrow{\mathrm{id}} & S \\ \cap & \nearrow & \downarrow{\scriptstyle p|S} \\ W & \xrightarrow{p|W} & M\,. \end{array}$$

The obstructions to lifting $p|W$ to S, *rel* S, all lie in cohomology groups of (W, S) with local coefficients in $\pi_i(S^1)$, module over $\mathbf{Z}\pi'$ by the action determined by the circle fibration $S(\xi)$. By Lemma 4.1, such groups must vanish. So there exists $\bar{f}: W \to S$, solving the lifting problem. Let $q: \partial T \to M$ be the projection of the normal bundle. Since q and $p \mid \partial T$ are homotopic, as their obvious extensions to T are, we may suppose by the covering homotopy and homotopy extension properties that $p(\bar{f} \mid \partial T) = q$, after a homotopy relative S.

Clearly $\bar{f} \mid \partial T$ has degree one, and so induces an epimorphism of fundamental groups. It follows by considering the exact sequence of homotopy groups that $\bar{f} \mid \partial T$ is a (homotopy) equivalence of spherical fiber spaces. Since G_2/O_2 is contractible, $\bar{f} \mid \partial T$ is homotopic, through maps of spherical fiber spaces, to a bundle map. The homotopy extension property then yields the desired map f.

The uniqueness assertions follow from similar considerations for $\iota \times 1_I: M \times I \to E \times I$.

We say $\iota: M \to E$, a local knot, is 1-*simple* if the inclusion $S \subset E - \iota(M)$ induces an isomorphism of fundamental groups. Using 4.2, it is easy to see that ι is 1-simple if and only if $\partial T \subset W$ induces an isomorphism of fundamental groups, T and W as in 4.2.

LEMMA 4.3. *Every local knot of M^n, $n \geqq 3$, is concordant to a 1-simple local knot.*

The reader will observe that the argument we are about to give actually makes the inclusion of S as highly connected as possible without being a homotopy equivalence. We will never need so strong a result; actually it seems probable that even Lemma 4.3 can be omitted, at the expense of complicating later portions of the exposition.

Proof of 4.3. Let $\iota: M \to E$ be a local knot. Let T, $f: W \to S$ be as in Lemma 4.2. Let $F = (f, \gamma)$, $F: W \to S \times I$, γ a Morse function with $\gamma^{-1}(0) = \partial T$, $\gamma^{-1}(1) = S$. It is easy to find a map B of stable normal bundles, covering F. (Compare § 5, second paragraph.) By surgery, we may find a normal cobordism, relative the boundary, of (F, B) to (F', B') with F' $[(n + 2)/2]$-connected. Since F is a simple homology equivalence over Λ, by 4.1, $\sigma(F', B') \in \Gamma^s_{n+2}(\mathcal{F})$ vanishes. Hence, by performing surgery on $[(n + 2)/2]$-spheres, we can find (F'', B''), normally cobordant relative boundary to (F', B'), with F'' a simple homology equivalence over Λ. Clearly F'' will be $[(n + 1)/2]$-connected.

The normal cobordism of (F, B) to (F'', B'') may have a non-vanishing obstruction in $\Gamma_{n+3}(\mathcal{F})$; since surgery obstructions add over unions, this may be killed by adding a further normal cobordism of (F'', B''). By the addenda to 1.8 and 2.2, we may do this without lowering the connectivity of F''. So, changing notation, we may assume that if (G, C) is the normal cobordism of (F, B) to (F'', B''), relative boundary, $G: Z \to S \times I \times I$ is a simple homology equivalence over $\mathbf{Z}\pi'$. By the argument of the preceding paragraph, we can suppose G induces isomorphisms of fundamental groups.

Let $X = (T \times I) \cup Z$; note that $(T \times I) \cap Z = \partial T \times I$. Clearly X is a homology s-cobordism over $\mathbf{Z}\pi'$. Let V be the domain of F''. Since $\pi' = \pi_1 X = \pi_1 E = \pi_1(T \times 1 \cup V)$, X is actually an s-cobordism, relative boundary. Apply the s-cobordism theorem; let $\varphi: X \to E \times I$ a diffeomorphism (*resp.* P.L. homeomorphism, homeomorphism) with $\varphi \mid E: E \to E \times 0$ the obvious map. Then the desired concordance is given by the composition

$$M \times I \xrightarrow{\iota \times 1} T \times I \subset X \xrightarrow{\varphi} E \times I .$$

LEMMA 4.4. *Let $\iota: M \to E$ be a smooth* (P.L., *topological*) *local knot. Let W be the closed complement of a tubular neighborhood T of $\iota(M)$. Let $\varphi: \partial T \longrightarrow \partial T$ be a diffeomorphism* (*resp.* P.L. *homeomorphism, homeomorphism*) *and Φ a homotopy of φ to the identity. Then Φ extends to a homotopy of the identity of $(W, \partial W)$ to a diffeomorphism* (*resp.* P.L. *homeomorphism, homeomorphism*) *extending φ, assuming $n \geqq 3$.*

Proof. Consider $\mathcal{S}_H(\partial T \times I, \partial T \times \partial I)$, as in § 10 of [58], for example,

$H = O$, PL, TOP. (In [58], $\mathcal{S}_O$ is called $\mathcal{S}_{\text{Diff}}$.) Then Φ represents an element of this set, and it suffices (using the relative s-cobordism theorem) to show that the natural boundary homomorphism

$$\partial\colon \mathcal{S}_H(W \times I,\ W \times \partial I) \longrightarrow \mathcal{S}_H(\partial T \times I,\ \partial T \times \partial I)$$

is an epimorphism.

The following diagram has exact rows and commutes; the exactness of the top row uses the 1-simplicity of ι:

$$\begin{array}{ccccccccc}
[\Sigma^2 W_+; G/H] & \longrightarrow & L_{n+4}(\pi \cup \pi \longrightarrow \pi) & \longrightarrow & \mathcal{S}_H(W \times I,\ W \times \partial I) & \longrightarrow & [\Sigma W_+; G/H] & \longrightarrow & L_{n+3}(\pi \cup \pi \longrightarrow \pi) \\
\downarrow & & \downarrow \partial & & \downarrow \partial & & \downarrow & & \downarrow \partial \\
[\Sigma \partial T_+; G/H] & \longrightarrow & L_{n+3}(\pi) & \longrightarrow & \mathcal{S}_H(\partial T \times I,\ \partial T \times \partial I) & \longrightarrow & [\Sigma \partial T_+; G/H] & \longrightarrow & L_{n+2}(\pi)\,.
\end{array}$$

The unlabeled maps are isomorphisms because $\partial T \subset W$ is, in particular by Lemma 4.1, a homology equivalence over **Z**. The maps on surgery groups are essentially well known to be isomorphisms; see [58]. Hence the central map is an isomorphism, proving the lemma.

Now we can define the group structure on $C_H(M^n; \xi)$, $n \geqq 3$. Let $\iota_1, \iota_2\colon M \to E$ be two local knots. We define the *composition*, or *tunnel sum* of ι_1 and ι_2, written $\iota_1 \circ \iota_2$, as follows: let T be a tubular neighborhood of $\iota_1(M)$ and let $g\colon T \to E(\xi)$ be the canonical extension (to a bundle map) of $f \mid \partial T$, f as in Lemma 4.2. Then we define $\iota_1 \circ \iota_2$ to be the composition

$$M \xrightarrow{\ \iota_2\ } E \xrightarrow{\ g^{-1}\ } T \subset E\,.$$

Clearly $\iota_1 \circ \iota_2$ is well-defined, up to isotopy, in terms of ι_1 and ι_2.

Alternatively, let W be the closed complement of T. Then $\iota_1 \circ \iota_2$ can be defined as the composition

$$M \xrightarrow{\ \iota_2\ } E \subset E \underset{(f \mid \partial T)}{\cup} W \xrightarrow{\ g^{-1} \cup \mathrm{id}_W\ } T \cup W = E\,,$$

where the identification space is obtained by identifying $x \in \partial T$ with $f(x) \in S$. It is clear that these two definitions are the same.

PROPOSITION 4.5. *The cobordism of $\iota_1 \circ \iota_2$ depends only upon the cobordism classes of ι_1 and ι_2. The operation of tunnel sum induces the structure of a group on $C_H(M, \xi)$, $H = O$, PL, TOP.*

Proof. The equivalence class of $\iota_1 \circ \iota_2$ is clearly not affected by the composition of ι_1 with a diffeomorphism (P.L. homeomorphism, homeomorphism) of (E, S) homotopic to the identity.

A concordance $\alpha\colon M \times I \to E \times I$ of ι_2 to ι_2' leads to a concordance

$$M \times I \xrightarrow{\alpha} E \times I \subset (E \underset{f|\partial T}{\cup} W) \times I \xrightarrow{(g^{-1} \cup \mathrm{id}_W) \times 1} E \times I$$

of $\iota_1 \circ \iota_2'$.

Suppose $\beta\colon M \times I \to E \times I$ is a concordance of ι_1 with ι_1'. Let U be a tubular neighborhood of $\beta(M \times I)$, respecting the boundary, with closed complement Z. By the note following Lemma 4.2, there is a canonical bundle map $G\colon U \to E \times I$, extending the bundle maps on the ends provided by 4.2. The following composition is a concordance between $\iota_1 \circ \iota_2$ and $\iota_1' \circ \iota_2$:

$$M \times I \xrightarrow{\iota_1 \times 1} E \times I \subset (E \times I) \underset{G|\partial U}{\cup} Z \xrightarrow{G^{-1} \cup \mathrm{id}_Z} U \cup Z = E \times I\,.$$

To complete the proof that tunnel sum induces a well-defined operation on $C_H(M; \xi)$, it suffices to show that the cobordism class of $\iota_1 \circ \iota_2$ is unchanged under composition of ι_2 with a diffeomorphism (P.L. homeomorphism, homeomorphism). By Lemma 4.3 and what has been shown already, ι_2 may be assumed 1-simple without altering the cobordism class of $\iota_1 \circ \iota_2$. In this case the result follows from Lemma 4.4.

It is obvious that tunnel sum on cobordism classes is associative and that the zero-section $M \subset E$ is a left and right identity.

To show that inverses exist, let $\iota\colon M \to E$ be a 1-simple local knot. Let T be a tubular neighborhood of $\iota(M)$ with closed complement W, let $g\colon T \to E$ be the canonical extension to T of $f \mid \partial T$, f as in Lemma 4.2, and let $V = W \times 0 \cup S \times I \cup E \times I \subset E \times I$. Then it follows from 1-simplicity and Lemma 4.1 that $(E \times I; T, V)$ is an s-cobordism. Hence there exists a diffeomorphism (a P.L. homeomorphism or homeomorphism) $\alpha\colon E \times I \to E \times I$ with $\alpha \mid T = g$. Let $\bar{\iota}$ be the local knot $M \subset E \times 1 \subset V \xrightarrow{\alpha|V} (E \times 1) \cup (S \times I) \cong E$, where the first inclusion is the 0-section. Then we may view the composite

$$M \times I \xrightarrow{\iota \times 1} E \times 1 \xrightarrow{\alpha} E \times I$$

as a concordance of the 0-section to $\bar{\iota} \circ \iota$. So $\bar{\iota}$ represents the inverse of ι in $C_H(M; \xi)$. This completes the proof of Proposition 4.5.

The final result of this section will be useful later.

LEMMA 4.6. *Let* $\iota\colon M \to E(\xi)$ *be a smooth* (P.L., *topological*) *local knot. Let* $T, W, f\colon W \to S$ *be as in Lemma* 4.2, *and let* $F = (f, \gamma)$, $\gamma\colon (W; \partial T, S) \to (I; 0, 1)$, *be a Morse function. Let* $g\colon T \to E$ *be the canonical extension* (i.e., *by a bundle map*) *to* $g\colon T \to E$. *Then* $h = g \cup F\colon (T \cup W, S) \to (E \cup S \times I,\ S \times 1) \cong (E, S)$ *is homotopic* (*as a map of pairs*) *to a diffeomorphism* (*resp.* P.L. *homeomorphism, homeomorphism*).

Proof. We will show that h is homotopic to a bundle map. Since ι is homotopic to the 0-section, $M \subset E$, ph is homotopic to p; let $H: E \times I \to M$, $H(x, 0) = ph(x)$, $H(x, 1) = p(x)$, be a homotopy. By the covering homotopy extension property for the circle fibration $p \mid S$, we may lift $H \mid S \times I$ to $\tilde{H}: S \times I \to S$ with $p\tilde{H} = H$ and $\tilde{H} \mid S \times 0 = h \mid S =$ identity of S. By the covering homotopy extension property of the disk fibration p, $\tilde{H}$ may be extended to $\bar{H}: E \times I \to E$ with $p\bar{H} = H$, with $\bar{H}(x, 0) = h(x)$, $x \in E$. Write $k(x) = H(x, 1)$. Then k is a map of fiber spaces with fiber (D^2, S^1). Since G_2/O_2 is contractible, it follows that k is homotopic through fiber space maps to a bundle map.

5. Local knots and surgery with coefficients

Let $\iota: M^n \to E(\xi)$ be a local knot of M in ξ. Let T be a tubular neighborhood, with closed complement W. Let f: $W \to S$ be the canonical map, as in Lemma 4.2. Choose a Morse function γ on W, and let

$$F = (F, \gamma): (W; \partial T, S) \longrightarrow (S \times I; S \times 0, S \times 1) .$$

By Lemma 4.1 and the existence of the augmentation $\mathbf{Z}\pi' = \Lambda \to \mathbf{Z}$, $S \subset W$ is an integral homology equivalence. Hence $[W; BO] \to [S; BO]$ and $[W; O] \to [S; O]$ are isomorphisms[1]. So there exists a map B of stable normal bundles, unique up to isotopy of bundle maps, with $B \mid S$ the identity. By Lemma 4.1, F is a simple homology equivalence over Λ.

Now, (F, B) represents an element in the normal cobordism class, relative $S \times 0$, of normal maps into $S \times I$. By a theorem of Sullivan [51], these normal cobordism classes are in 1-1 correspondence with $[S \times I/S \times 0; G/H] = 0$, $H = O$, PL, TOP. Hence (F, B) is normally cobordant, relative ∂T, to a diffeomorphism (P.L. homeomorphism, homeomorphism); for example, to $(f \mid \partial T) \times \mathrm{id}: \partial T \times I \to S \times I$. Denote the normal cobordism by (G, C). By similar considerations, (G, C) is unique up to normal cobordism relative the part of the boundary corresponding under G to $S \times I \times 0 \cup S \times 0 \times I \cup S \times I \times 1$. (Recall the target of G is $S \times I \times I$.)

Let ϕ be the diagram

$$\begin{array}{ccc} \mathbf{Z}\pi & \xrightarrow{\mathrm{id}} & \mathbf{Z}\pi \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathcal{F}} \\ \mathbf{Z}\pi & \xrightarrow{\mathcal{F}} & \mathbf{Z}\pi' = \Lambda , \end{array}$$

considered as a map from $\mathrm{id}_{\mathbf{Z}\pi}$ to $\mathcal{F}$. Then what has just been said (together with the tubular neighborhood theorem, relative S) implies that

[1] $[X, Y] =$ homotopy classes of maps $X \to Y$.

$$\Sigma_0(\iota) = \sigma(G, C) \in \Gamma_{n+3}(\phi)$$

is a well-defined invariant of ι, $n \geqq 3$.

PROPOSITION 5.1. *For* $n \geqq 3$, $\Sigma_0(\iota)$ *depends only upon the cobordism class of* ι *and induces a homomorphism*

$$\Sigma\colon C_H(M, \xi) \longrightarrow \Gamma_{n+3}(\phi) ,$$

$H = O$, PL, TOP.

Proof. It suffices to show that $\Sigma_0(\iota_1 \circ \iota_2) = \Sigma_0(\iota_1) + \Sigma_0(\iota_2)$ and $\Sigma_0(\iota) = 0$ if ι is cobordant to the zero-section.

To prove the additivity, let W_i, W, $i = 1, 2$, be the closed complements of tubular neighborhoods T_i, T of $\iota_i(M)$ and $\iota(M)$, respectively, where $\iota = \iota_1 \circ \iota_2$. Let (F_i, B_i) be as in the definition of $\Sigma_0(\iota_i)$, $\iota = 1, 2$, respectively. Write $F_1(x) = (g(x), 0)$, $x \in \partial T$. Then $W = W_2 \cup_{g^{-1}} W_1$. Since W_1 and W_2 are homology products, we may assume g carries B_1 to $B_2 \mid S = \mathrm{id}$, following an isotopy of bundle maps. Then for (F, B) in the definition of $\Sigma_0(\iota)$ we may take $(F, B) = (F_2, B_2) \cup (F_1', B_1)$, $F\colon W \to S \times [0, 2]$, where $F_2'(x) = (y, t + 1)$ if $F_1(x) = (y, t)$.

Now let (G_i, C_i), $G_i\colon Z_i \to S \times [2 - i, 3 - i] \times [0, 1]$, $i = 1, 2$ be normal cobordisms of (F_1', B_1) and (F_2, B_2), relative ∂T_i, to $(F_i \mid \partial T_i) \times \mathrm{id}$, respectively. Let

$$Z = Z_2 \cup_{W_2 \times 1} W_2 \times [0, 1] \cup_{g^{-1} \times \mathrm{id}_I} Z_1$$

and let (G, C),

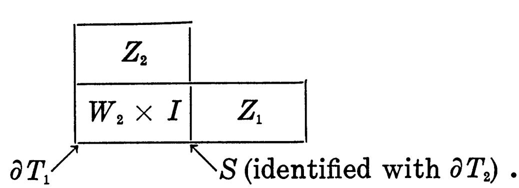
$$G\colon Z \longrightarrow S \times [0, 1] \times [1, 2] \cup S \times [0, 1] \times [0, 1] \cup S \times [1, 2] \times [0, 1]$$

be the union

$$(G_2', C_2) \cup (F_2 \times \mathrm{id}, B_2 \times \mathrm{id}) \cup (G_1, C_1) ,$$

where $G_2'(x) = (x, s, t + 1)$ if $G_2(x) = (x, s, t) \in S \times I \times I$.

Here is a schematic picture of Z:



The range of G is obviously diffeomorphic (P.L. homeomorphic, homeomorphic) to $S \times [0, 2] \times [0, 2]$, and it is clear that

$$\Sigma_0(\iota) = \sigma(G, C) .$$

On the other hand, surgery obstructions add over unions along homology equivalence over Λ (see § 3). Further, by 4.1, F_2 is a homology equivalence over Λ.

Hence

$$\sigma(G, C) = \sigma(G_1, C_1) + \sigma(G_2, C_2) = \Sigma_0(\iota_1) + \Sigma_0(\iota_2) .$$

So this proves the additivity.

Next, suppose ι represents zero in $C_H(M; \xi)$. It is not hard to show that $\Sigma_0(\iota)$ is an invariant of equivalence class. Further, it is clear that if ι is cobordant to $M \subset E$, the 0-section, then an equivalent local knot is concordant to the 0-section. Therefore, consider a concordance $\bar{\iota}: M \times I \to E \times I$ of ι with the 0-section; we have to show that $\Sigma_0(\iota) = 0$. But this follows easily by using the relative form of Lemma 4.2 to construct a normal cobordism (G, C) as in the definition of $\Sigma_0(\iota)$ with G a simple homology equivalence over Λ and a simple homotopy equivalence on the part of boundary mapping to $S \times 0 \times I$; i.e., $\Sigma_0(\iota) = \sigma(G, C) = 0$. So this concludes 5.1.

Let $j_*: \Gamma_{n+3}(\phi) \to L_{n+3}(\mathcal{F})$ be the natural map, defined in § 3. Let[1]

$$s_H: [\Sigma E(\xi); G/H] \longrightarrow L^s_{n+3}(\mathcal{F}) , \qquad H = O, \text{PL}, \text{TOP},$$

be the map defined by taking surgery obstructions of normal cobordism classes [3], [58]. Let ρ be the composition of j_* with the quotient map to the cokernel of s_H.

THEOREM 5.2. *Let $n = \dim M$ be at least four. Then the following sequence is exact:*

$$0 \longrightarrow C_H(M; \xi) \xrightarrow{\Sigma} \Gamma_{n+3}(\phi) \xrightarrow{\rho} \operatorname{coker} s_H .$$

COROLLARY 5.3. *The group $C_H(M^n, \xi)$ is abelian, $n \geqq 4$.*

Notes. 1. For the case $n = 3$, one can show $\operatorname{Im} \Sigma = \ker \rho$. All of Theorem 5.2 still holds provided that, in the definition of $C_H(M; \xi)$, one replaces "concordance" by "s-cobordism". For the case M a simply-connected 3-manifold, this change makes no difference, by [48] or [53, 6.1]. In particular, 5.2 holds precisely for $M = S^3$.

2. It seems that 5.3 should have a more direct geometric proof, perhaps using the s-cobordism theorem only. Some such theorem probably must be used, in view of the difficulties that arise in trying to define and study a group $C(S^1, \xi)$, ξ trivial.

COROLLARY 5.4. *If $M = S^3$ or $\dim M \geqq 4$ and M is in the appropriate*

[1] Strictly speaking we should use ΣE_+, $E_+ =$ the union of E with a disjoint point. As G/H is simply-connected, we shall ignore this point.

category, the natural maps $C_O(M, \xi) \to C_{\mathrm{PL}}(M; \xi)$ *and* $C_{\mathrm{PL}}(M; \xi) \to C_{\mathrm{TOP}}(M, \xi)$, *are monomorphisms.*

This is actually not difficult to see directly, as in [14].

Proof of 5.2. To show Σ is 1-1, let $\iota: M \to E$ be a local knot with $\Sigma_0(\iota) = 0$. Let T be a tubular neighborhood of $\iota(M)$ with closed complement W. Let (F, B), (G, C) be as in the definition of Σ_0, so that by assumption $\sigma(G, C) = 0$. Write

$$G: (Z; \partial_- Z, \partial_+ Z) \longrightarrow (S \times I \times I; Y, S \times 1 \times I)\,,$$

where $Y = S \times I \times 0 \cup S \times 0 \times I \cup S \times I \times 1$ and $\partial_- Z = W \cup_{\partial T} \partial T \times [0, 2]$ with a "corner" at $\partial T \times 1$. By Theorem 3.3, we may assume following a normal cobordism relative $\partial_- Z$, that G is a simple homology equivalence over $\mathbf{Z}\pi' = \Lambda$ and $G \mid \partial_+ Z: \partial_+ Z \to S \times 1 \times I$ is a simple homology equivalence over $\mathbf{Z}\pi$. By the addendum to 3.3, we may assume that G and its restriction $\partial_+ Z$ induce isomorphisms of fundamental groups; note that this is already the case for $G \mid \partial_- Z$. In particular, $G \mid \partial_+ Z$ will be a simple homotopy equivalence; i.e., $\partial_+ Z$ is an s-cobordism.

Let $D = T \times I \cup_{\partial T \times I} Z$. Then view $(D, \partial_+ Z)$ as a cobordism from $(E, S) = (T \cup W, S)$ to $(T \cup_{\partial T} \partial T \times [1, 2], \partial T \times 2)$. By Van-Kampen's theorem, and the preceding paragraph, $\pi_1 D = \pi'$, and the inclusions of E and $T \cup \partial T \times I$ induce isomorphisms of fundamental groups. Again by the preceding paragraph, it is easy to see that D is a homology s-cobordism over $\Lambda = \mathbf{Z}\pi'$. Hence $(D, \partial_+ Z)$ is an s-cobordism. By the s-cobordism theorem, let $\alpha: (D, \partial_+ Z) \to (E \times I, S \times I)$ be a diffeomorphism (P.L. homeomorphism, homeomorphism) with $\alpha(x) = (x, 0)$ for $x \in E$.

Let $g: T \to E$ be the canonical map extending $F \mid \partial T: \partial T \to S \times 0 = S$. Consider the concordance

$$M \times I \subset E \times I \xrightarrow{g^{-1} \times 1} T \times I \subset D \xrightarrow{\alpha} E \times I\,;$$

the first inclusion denotes the zero-section. On $M \times 0$, this concordance carries $(x, 0)$ to $(\iota x, 0)$. So it suffices to show that

$$M \subset E \xrightarrow{g^{-1}} T \cup_{\partial T} \partial T \times [1, 2] \xrightarrow{\alpha} E$$

is equivalent to the 0-section.

Let $\varphi: E \to E$ be any diffeomorphism (P.L. homeomorphism, homeomorphism) with $\varphi \mid M$ homotopic to the inclusion $M \subset E$ as the zero-section. By the argument of 4.6, φ is homotopic, as a map of the pair (E, S) to itself, to a bundle map, μ say. By definition, $\varphi\mu^{-1} \mid M: M \to E$ is equivalent to the zero-section. On the other hand, $\varphi\mu^{-1} \mid M = \varphi \mid M$. So $\varphi \mid M$ is equivalent to the zero-section. (*Note*: Let $\iota: M \to E$ be a local knot, and let $\varphi: E \to E$ be a diffeomorphism (P.L. homeomorphism, homeomorphism) with $\varphi\iota$ homotopic to ι. Let $\varphi_*: \Gamma_{n+3}(\phi) \to \Gamma_{n+3}(\phi)$ denote the map induced by the map induced by φ on fundamental groups. Then one can show easily, by naturality of surgery obstructions,

$$\Sigma_0(\varphi\iota) = \varphi_*\Sigma_0(\iota) \; . \tag{5.2.1}$$

It is not hard to show that composition with φ induces a map $\varphi^!: C_H(M, \xi) \to C_H(M, \xi)$, and we have,

$$\Sigma\varphi^! = \varphi_*\Sigma \;.)$$

To prove that $\rho\Sigma = 0$, let (G, C) be as in the definition of $\Sigma_0(\iota)$, $\iota: M \to E$ a local knot, T and W as above. Then, by the naturality of surgery obstructions (see § 3 and [59, § 9]), the following holds in $L_{n+3}(\mathcal{F})$:

$$\sigma\big((G, C) \cup \mathrm{id}_{T\times I}\big) = j_*\big(\sigma(G, C)\big) \; .$$

Hence, by Lemma 4.6, $j_*\big(\sigma(G, C)\big)$ acts trivially on the class in $\mathcal{S}_H(E)$ represented by the identity of E. The surgery exact sequence [58], [3]

$$[\Sigma E; G/H] \xrightarrow{s_H} L_{n+3}(\mathcal{F}) \longrightarrow \mathcal{S}_H(E)$$

implies that $j_*\big(\sigma(G, C)\big) \in \mathrm{Im}\, s_H$; i.e., $\rho\big(\Sigma(\iota)\big) = \rho\Big(j_*\big(\sigma(G, C)\big)\Big) = 0$.

Conversely, given $\gamma \in \Gamma_{n+3}(\phi)$, we may find a normal map (G, C), with $\gamma = \sigma(G, C)$, which is a normal cobordism, relative $S \times 0$, of the identity of $S \times I$ to a simple homology equivalence $f: (V, \partial V) \to (S \times I,\; S \times \partial I)$ over Λ that induces a simple homology equivalence over $\mathbf{Z}\pi$ of boundaries; also we can suppose f induces an isomorphism of $\pi_1 V$ with $\pi_1 S$ and of the fundamental groups of the boundary components. Consider

$$\mathrm{id}_E \cup f: E \cup_{S\times 0} V \longrightarrow E \cup S \times I \cong E \; .$$

By Van-Kampen's theorem and the properties of f, $\mathrm{id}_E \cup f$ is a simple homotopy equivalence of manifold pairs. Suppose $\rho\gamma = 0$. Then by the naturality and the surgery sequence mentioned in the preceding paragraph, $\mathrm{id}_E \cup f$ is homotopic, as a map of manifold pairs, to a diffeomorphism (or P.L. homeomorphism, homeomorphism) φ; i.e., $\mathrm{id}_E \cup f$ is trivial in $\mathcal{S}_H(E)$.

Let ι be the local knot

$$M \subset E \subset E \cup V \xrightarrow{\varphi} E \; .$$

Then it is easy to see that $\Sigma(\iota) = -\gamma$. So $-\gamma$ and, hence, γ are in the image of Σ.

To conclude this section, we observe that in the present case the sequence of 3.2 becomes the following:

$$\cdots \longrightarrow L_{n+3}(\pi) \longrightarrow \Gamma_{n+3}(\mathcal{F}) \longrightarrow \Gamma_{n+3}(\phi) \longrightarrow L_{n+2}(\pi) \longrightarrow \Gamma_{n+2}(\mathcal{F}) \longrightarrow \cdots \; .$$

6. Relation with knot cobordism

Let $\iota: M^n \to E(\xi)$ be a local knot, and let $B^n \subset M^n$ be a closed ball. Then ι is said to be in *normal form* (with respect to B) if $\iota \mid B$ is the restriction of the zero-section to B and if $\iota(M) \cap E(\xi \mid B) = \iota(B)$. If a local knot is in normal form with respect to B, it can be placed in normal form with respect to any given ball by an isotopy.

LEMMA 6.1. *Every 1-simple local knot $\iota: M^n \to E$ is equivalent (even isotopic) to a local knot in normal form (with respect to an arbitrary ball), provided $n \geqq 3$.*

Proof. Let $B^n \subset M$ be a closed ball. A standard argument shows that after an isotopy, it may be supposed that $\iota \mid B$ is the restriction of the zero-section. Let $x \in \operatorname{Int} B$, and let E_x be the fiber of $E(\xi)$ over x. Suppose $\iota(M) \cap E_x = \{x\}$. Then $\iota(B') \cap E(\xi \mid B') = \iota(B')$, by compactness, for a smaller ball B', and so ι is in normal form with respect to B'.

In any case, we may suppose $\iota(M)$ meets E_x, transversely, in points. Suitably orienting E_x, its integral intersection number with $\iota(M)$ will be $+1$. (In case M is not orientable, we use twisted integer coefficients; see [58, 2], for example.) Let $z, w \in (\iota(M) - x) \cap E_x$ be two points with opposite intersection numbers, and choose paths l from z to w in $\iota(M) - \iota(M) \cap E_x$ and l' from w to z in $E_x - E_x \cap \iota(M)$. Push $l \cup l'$ slightly to get it off $E_x \cup \iota(M)$; let $\mathcal{O}$ be the resulting loop. If $\mathcal{O}$ is null-homotopic in $E - E_x \cup \iota(M)$, then we could span it by an embedded 2-disk and carry out the Whitney process to remove the two points z and w from the intersection (see [42, p. 78–82]). By general position, it suffices to have $\mathcal{O}$ null-homotopic in $E - \iota(M)$.

Since $\pi_1 E = \pi_1 M$, a suitable choice of the path l in $\iota(M)$ will assure that $\mathcal{O}$ is contractible in E. From the 1-simplicity and Van Kampen's theorem, $\pi_1(E - \iota(M)) \to \pi_1(E)$ has kernel generated by the homotopy class $[C]$ of a meridian C, which may be taken near w. (A *meridian* is the boundary of a fiber of a tubular neighborhood.) $\mathcal{O}$ may be (pre-) multiplied by $[C]^k$ by twisting l k times around M when pushing it off M to create $\mathcal{O}$. So we may take $\mathcal{O}$ to be contractible in $E - \iota(M)$ and remove the points z and w. Proceeding inductively, the result follows.

Let C_n^H denote the group of cobordism classes of smooth ($H = O$), piecewise linear locally flat[1] ($H = \mathrm{PL}$), or topologically locally flat ($H = \mathrm{TOP}$) embeddings of S^n in S^{n+2}. Then *connected sum* gives a well-defined map,

[1] By smoothing theory [37], C_n^{PL} are the knot groups studied in [30].

written $(x, y) \to x \# y$, from $C_n^H \times C_H(M, \xi)$ to $C_H(M, \xi)$. (We assume orientations have been fixed once and for all.) Define

$$\alpha = \alpha_H(M, \xi): C_n^H \to C_H(M, \xi)$$

by $\alpha(x) = x \# \iota_0$, ι_0, the zero-section. Since the group structure in C_n^H is defined by connected sum, it follows easily that

$$(x + y) \# z = x \# (y \# z) .$$

On the other hand, by considering two local knots of M in E that are in normal form with respect to the same ball (their composition will also have this property), it is easy to see that

$$x \# (z \circ u) = (x \# z) \circ u = z \circ (x \# u) .$$

In particular, if z is the neutral element of $C_H(M, \xi)$,

$$\begin{aligned}(x + y) \# z = (x + y) \# (z \circ z) &= x \# \big(y \# (z \circ z)\big) \\ = x \# \big((y \# z) \circ z\big) &= (y \# z) \circ (x \# z) .\end{aligned}$$

Thus we have

PROPOSITION 6.2. *The map $\alpha(M, \xi)$ is a homomorphism.*

Note. This proposition and the usual description of inverses in knot cobordism suggests that if $\iota: M \to E(\xi)$ is a local knot, the inverse of its class in $C_H(M, \xi)$ should be represented by βi, where β is a bundle map over the identity that reverses orientations of the fibers.

PROPOSITION 6.3. *Let $n \geqq 3$. Then $\alpha_H(S^n, \xi)$ is an isomorphism, $H = O$,* PL, TOP.

Note that ξ must be trivial. For $n = 2$, one can show, using some of the methods in this paper, that $C_H(S^2, \xi) = 0$, so that α will be an isomorphism for $n = 2$ also.

Proof. Let $\iota: S^n \to S^n \times D^2$ be in normal form. Let h be the composite of ι with the inclusion $S^n \times D^2 \subset S^{n+2}$ of a tubular neighborhood of the unknotted sphere. Then it is easy to see that $[\iota] = \alpha[h]$. Hence, by 6.1 and 4.3, $\alpha_H(S^n, \xi)$ is onto.

Given $h: S^n \to S^{n+2}$, it is well known that h can be factored, after an isotopy, as

$$S^n \xrightarrow{\iota} S^n \times D^2 \subset S^{n+2} ,$$

as above, ι a local knot. Using the unknotting theorems of [33], [48], [52] (see [14] for the topological case) it follows that if ι is equivalent to the zero-section, then h will be unknotted. Hence a concordance of ι to a local knot equivalent to the zero-section leads, by gluing in $D^{n+1} \times S \times I$ to get $S^{n+2} \times I$ as target, to a cobordism of h to an unknotted embedding. Hence $\alpha_H(S^n, \xi)$ is also a monomorphism.

Let ϕ_0 be the diagram

$$\begin{array}{ccc} \mathbf{Z}[\mathbf{Z}] & \xrightarrow{\text{id}} & \mathbf{Z}[\mathbf{Z}] \\ \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \mathcal{F}_0} \\ \mathbf{Z}[\mathbf{Z}] & \xrightarrow{\mathcal{F}_0} & \mathbf{Z}\ , \end{array}$$

where $\mathcal{F}_0$ is the augmentation. Let $t \in \pi = \pi_1(S(\xi))$ be represented by a fiber; we suppose, the orientations having been fixed, that t represents a positively oriented generator. Identifying $\mathbf{Z}[\mathbf{Z}] = \mathbf{Z}[\pi_1 S^1]$ determines an obvious homomorphism

$$\beta: \phi_0 \longrightarrow \phi\ .$$

Notice that if ξ is the trivial bundle, so that $\pi' = \mathbf{Z} \times \pi$, then β has a left inverse.

By the functoriality of Γ-groups (§ 3), β induces a homomorphism $\beta_*: \Gamma_i(\phi_0) \to \Gamma_i(\phi)$.

THEOREM 6.4. *Suppose $n \geqq 3$. Then the following diagram commutes:*

$$\begin{array}{ccc} C_n^H & \xrightarrow{\Sigma\alpha} & \Gamma_{n+3}(\phi_0) \\ \downarrow{\scriptstyle \alpha(M,\xi)} & & \downarrow{\scriptstyle \beta_*} \\ C_H(M, \xi) & \xrightarrow{\Sigma} & \Gamma_{n+3}(\phi)\ . \end{array}$$

For $H = $ PL, TOP, $\Sigma\alpha: C_n^H \to \Gamma_{n+3}(\phi_0)$ is an isomorphism, for $H = O$ a monomorphism.

COROLLARY 6.5. *Let ξ be trivial. Assume $n \geqq 4$. Then $\alpha_H(M, \xi)$ is a monomorphism. If H is* PL *or* TOP, *its image is a direct summand.*

Notes. 1. For $n = 3$, one can show that $\Sigma\alpha$ is still an isomorphism for $H = $ TOP, and a monomorphism onto a subgroup of index two for $H = O$, PL; see [14]. One can further show that the corollary remains true for $n = 3$, at least if one replaces $C_H(M, \xi)$ by s-cobordism classes of local knots. For $n \geqq 4$, 6.5 is immediate from 6.3, 6.4, and the existence of a left inverse for β when ξ is trivial

2. From the exact sequence

$$\Gamma_{n+3}(\mathcal{F}_0) \longrightarrow \Gamma_{n+3}(\phi_0) \longrightarrow L_{n+2}(\mathbf{Z}) \longrightarrow \Gamma_{n+2}(\mathcal{F}_0)\ ,$$

we see that $\Gamma_{n+3}(\phi_0) = 0$ if n is even, namely: $\Gamma_{n+3}(\mathcal{F}_0) \subset L_{n+3}(e) = 0$ for n even and $L_{n+2}(\mathbf{Z}) \to \Gamma_{n+2}(\mathcal{F}_0)$ is monic because the composition with the natural map gives the natural map $L_{n+2}(\mathbf{Z}) \to L_{n+2}(e)$, an isomorphism [46]. Thus we recover the theorem of Kervaire [25] on the vanishing of even dimensional knot groups. For odd dimensions, we may view 6.3 and 6.4 as a new calculation of knot groups.

THEOREM 6.6. *Let M^n, $n \geqq 4$, be simply-connected.* Let $H =$ PL, TOP. *Then $\alpha_H(M, \xi)$ is onto. In particular, $C_H(M^n, \xi) = 0$ if n is even. Moreover, for n even $C_O(M, \xi) = 0$ also.*

This result can also be proven using methods of [29], [39].

This theorem asserts that every PL or topological knot of M is the connected sum of a knot with the zero-section, up to cobordism. By 6.5, the cobordism class of the knot is uniquely determined, for ξ trivial.

Notes. 1. For $n = 3$, Theorem 6.6 remains valid, $H = O$, PL, or TOP. The proof of this uses the note following 5.2.

2. The reader may observe that a result for $H = O$ and n odd can be stated in terms of the inertial group of M.

Proof of 6.6 (assuming 6.4). Suppose first that n is even. Then the commutative square

$$\begin{array}{ccc} [\Sigma E; G/H] & \xrightarrow{s_H} & L_{n+3}(\mathcal{F}) \\ \cong \uparrow p^* & & \uparrow p^* \\ [\Sigma M; G/H] & \xrightarrow{s_H} & L_{n+1}(e) = 0 , \end{array}$$

p the projection of ξ, shows that the upper map is trivial. (The right map p^* is defined geometrically [58, § 9] by inducing bundles over normal maps.) So by Theorem 5.2 and the results of § 3, we have the following commutative diagram with exact rows and column:

$$\begin{array}{ccccccc} & & & & \Gamma_{n+3}(\mathcal{F}) & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & C_H(M, \xi) & \xrightarrow{\Sigma} & \Gamma_{n+3}(\phi) & \xrightarrow{j_*} & L_{n+3}(\mathcal{F}) \\ & & & & \downarrow \partial & & \downarrow \partial' \\ & & & & \Gamma_{n+2}(\pi) & = & L_{n+2}(\pi) . \end{array}$$

So $\partial\Sigma = 0$. But $\Gamma_{n+3}(\mathcal{F}) \subset L_{n+3}(e) = 0$, by § 2. Hence Σ is trivial. So, by Theorem 5.2, $C_H(M, \xi) = 0$.

Suppose that n is odd. The theorem then follows from 6.4 and 5.2 if we can show that the image of $\beta_*: \Gamma_{n+3}(\phi_0) \to \Gamma_{n+3}(\phi)$ contains the image of $\Sigma: C_H(M, \xi) \to \Gamma_{n+3}(\phi)$. The following commutative diagram has exact rows:

$$\begin{array}{ccccccc}
\Gamma_{n+3}(\mathcal{F}_0) & \longrightarrow & \Gamma_{n+3}(\phi_0) & \longrightarrow & L_{n+2}(\mathbf{Z}) & \longrightarrow & 0 = \Gamma_{n+2}(\mathcal{F}_0) \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\beta_*} & & \downarrow{\scriptstyle\tau} & & \\
\Gamma_{n+3}(\mathcal{F}) & \longrightarrow & \Gamma_{n+3}(\phi) & \longrightarrow & L_{n+2}(\pi) & \longrightarrow & 0 = \Gamma_{n+2}(\mathcal{F})\ .
\end{array}$$

The maps δ and τ are induced by the restriction β to suitable portions of the square ϕ_0. In particular, δ is induced by an epimorphism $\mathbf{Z}[\mathbf{Z}] \to \mathbf{Z}[\pi]$ and so is easily seen to be onto. Hence, by a simple diagram chase, it suffices to prove that the image of $\partial\Sigma: C_H(M, \xi) \to L_{n+2}(\pi)$ is contained in the image of τ.

Let $\lambda: L_{n+1}(e) \to L_{n+2}(\pi)$ be the composite of p^* and the natural map ∂'. The square at the start of this proof, the diagram

$$\begin{array}{ccc}
[\Sigma S(\xi); G/H] & \xrightarrow{s_H} & L_{n+2}(\pi) \\
\uparrow & & \uparrow{\scriptstyle\partial'} \\
[\Sigma E; G/H] & \xrightarrow{s_H} & L_{n+3}(\mathcal{F})\ ,
\end{array}$$

and Theorem 5.2 imply that the image of $\partial\Sigma$ is contained in the image of λ. But the diagram

$$\begin{array}{ccc}
L_{n+1}(e) & \xrightarrow{\bar{\lambda}} & L_{n+2}(\mathbf{Z}) \\
& \searrow{\scriptstyle\lambda} & \downarrow{\scriptstyle\tau} \\
& & L_{n+2}(\pi)
\end{array}$$

also commutes, where $\bar{\lambda}$ is a map defined geometrically by crossing with a circle. This is seen using the facts that λ can be described geometrically as inducing over a circle bundle, that an arbitrary element of $L_{n+1}(e)$ can be realized by a connected sum with a normal map into a disk ("local character" of surgery obstructions [3], [58]), and by the "naturality property" of surgery obstructions [58]. We leave the details as an exercise in surgery theory. This diagram implies Image $\lambda \subset$ image τ, completing the proof.

Proof of 6.4. By 6.3, to prove the second statement, it suffices to show that $\Sigma: C_H(S^n, \xi) \to \Gamma_{n+3}(\phi_0)$ is an isomorphism for $H =$ PL, TOP, and monic for $H = O$. By 5.2, Σ is always monic.

For $H=$PL, TOP, it is well-known, [3], [51], [58] that $s_H: [\Sigma S^n; G/H] \to L_{n+1}(e)$ is an isomorphism. The following diagram commutes, where the right vertical map is given geometrically by crossing normal maps with D^2 and the left one is the obvious map:

$$\begin{array}{ccc} [\Sigma(S^n \times D^2); G/H] & \xrightarrow{s_H} & L_{n+3}(\mathcal{F}) \\ \uparrow\cong & & \uparrow\omega \\ [\Sigma S^n; G/H] & \xrightarrow{s_H} & L_{n+1}(e) . \end{array}$$

So we have to show that ω is an epimorphism.

The sequence

$$L_{n+3}(\mathbf{Z}) \longrightarrow L_{n+3}(e) \longrightarrow L_{n+3}(\mathcal{F}) \xrightarrow{\partial''} L_{n+2}(\mathbf{Z}) \longrightarrow L_{n+2}(e)$$

is exact [58, §9]. By [46, 5.1] it follows that $L_{n+3}(\mathcal{F}) = 0$ for n even, and that ∂'' is an isomorphism for n odd. Clearly, $\partial''\omega$ is precisely $\bar{\lambda}: L_{n+1}(e) \rightarrow L_{n+2}(\mathbf{Z})$, an isomorphism by [46].

To prove commutativity of the square in 6.4, let $x \in C_n^H$ and choose a representative $h: S^n \rightarrow S^{n+2}$. Then it is easy to see that after an isotopy h may be assumed to be a composite

$$S^n \xrightarrow{\iota} S^n \times D^2 \subset S^{n+2} ,$$

where ι is a local knot and the inclusion is unknotted. We may assume that ι is normal with respect to a disk $B_0 \subset S^n$, the closure of whose complement we identify with a closed disk $B \subset M$. Identify $E(\xi \mid B)$ with $B \times D^2$. Let $\iota_0: M \rightarrow E$ be the zero-section. Then it is clear that $\alpha_H(M, \xi)(x)$ is represented by ι', where $\iota' \mid B = \iota \mid B$ and $\iota' = \iota_0$ outside B. We have already seen that ι represents $\alpha(S^n, \xi)(x)$. So we want to show that $\Sigma_0(\iota') = \beta_*\Sigma_0(\iota)$.

Choose a tubular neighborhood T of $\iota(S_n)$ so that, over B_0, T is a standard tube, $B_0 \times (1/2)D^2$, say, and so that $T \cap (B_0 \times D^2) = B_0 \times (1/2)D^2$; $(1/2)D^2$ denotes the disk of radius 1/2. Let W be the closure of the complement of T, and let W_0 be the closure of $B \times D^2 - T \cap (B \times D^2)$, so that

$$W = W_0 \cup \left(B_0 \times S^1 \times \left[\frac{1}{2}, 1\right]\right) .$$

Let $f: W \longrightarrow S^n \times S^1$ be as in Lemma 4.2; it is not hard to see that we may suppose $f \mid B_0 \times S^1 \times [1/2, 1]$ is just the natural projection on $B_0 \times S^1$. Let γ be a Morse function, $\gamma: W \rightarrow [1/2, 1]$ that restricts to the usual projection on $B_0 \times S^1 \times [1/2, 1]$. Let $F = (f, \gamma)$; we may find (see § 4, 5) a covering bundle map b of F with $b \mid B_0 \times S^1 \times [1/2, 1]$ the identity.

Note that $[B \times S^1 \times [1/2, 1]/\partial B \times S^1 \times [1/2, 1] \cup B \times S^1 \times 1/2; G/H] = 0$. So $(F \mid W_0, b \mid W_0)$ is normally cobordant relative $\partial B \times S^1 \times [1/2, 1] \cup \partial T \cup (B \times D^2)$ to a diffeomorphism (P.L. homeomorphism, homeomorphism). Let (G, C) be the normal cobordism. Clearly, if e is the identity normal cobordism of the identity of $B_0 \times S^1 \times [1/2, 1]$ to itself, then

$$\sigma(G, C) = \sigma\big((G, C) \cup e\big) = \Sigma\big([\iota]\big) = \Sigma_0(\iota) ,$$

by definition and naturality of surgery obstructions.

Let e_M be the identity normal cobordism of the identity of $S(\xi \mid M_0) \times [\xi, 1]$, $M_0 = \mathrm{cl}(M - B)$. Then it is clear that

$$\sigma\big((G, C) \cup e_M\big) = \Sigma_0(\iota') .$$

But, by naturality of surgery obstructions (§ 3), $\sigma\big((G, C) \cup e_M\big) = \beta_* \sigma(G, C)$. This completes the proof.

To conclude this section, we state an easy consequence of 5.2 and 3.1 (or 3.2).

THEOREM 6.7. *Let ξ be a 2-plane bundle over the simply connected manifold M^n, with Euler class a primitive element of $H^2(M)$ (so that $\pi_1 S(\xi) = 0$). Then, for $n \geqq 4$, $C_H(M, \xi) = 0$, $H = O$,* PL, TOP.

Thus the connnected sum of a knot with the zero-section in a bundle as in 6.7 is cobordant to the zero-section.

7. The case of the trivial bundle

In this section we outline some simplifications that occur when ξ is trivial; in this case we write $C_H(M)$ for $C_H(M, \xi)$. We have $\pi = \mathbf{Z} \times \pi'$ and $\mathcal{F}: \mathbf{Z}\pi \to \mathbf{Z}\pi'$, the natural map.

The exact sequence

$$L_{n+3}(\pi' \times \mathbf{Z}) \longrightarrow L_{n+3}(\pi') \longrightarrow L_{n+3}(\mathcal{F}) \longrightarrow L_{n+2}(\pi' \times \mathbf{Z}) \longrightarrow L_{n+2}(\pi')$$

shows that $L_{n+3}(\mathcal{F})$ is the kernel of $L_{n+2}(\pi' \times \mathbf{Z}) \to L_{n+2}(\pi')$; by [46] this is just $L^h_{n+1}(\pi')$, viewed as a subgroup of $L_{n+2}(\pi' \times \mathbf{Z})$ by crossing with a circle. This induces an isomorphism of $L_{n+3}(\mathcal{F})/s_H[\Sigma(M \times D^2); G/H]$ with $L^h_{n+1}(\pi')/s_H[\Sigma M; G/H]$. Let $\bar{\rho}$ be the composition of $\Sigma: C_H(M) \to \Gamma_{n+3}(\phi)$ with the map

$$\Gamma_{n+3}(\phi) \xrightarrow{j_*} L_{n+3}(\mathcal{F}) = L^h_{n+1}(\pi') .$$

Let $\psi: \Gamma_{n+3}(\mathcal{F}) \to \Gamma_{n+3}(\phi)$ be the appropriate map in 3.2. Note that $\phi_*: L_{n+2}(\pi) \to \Gamma_{n+2}(\mathcal{F})$ carries $L^h_{n+1}(\pi')$ trivially if n is odd, by [46, 5.1] and the fact from § 2 that the natural map $\Gamma_{n+2}(\mathcal{F}) \to L_{n+2}(\pi')$ is monic. Using this fact, Theorems 5.2 and 3.2, it is not hard to see the following:

THEOREM 7.1. *If $n = \dim M \geqq 4$, the following sequence is exact:*

$$L_{n+3}(\pi) \xrightarrow{\phi_*} \Gamma_{n+3}(\mathcal{F}) \xrightarrow{\Sigma^{-1}\psi} C_H(M) \xrightarrow{\bar{\rho}} L^h_{n+1}(\pi') .$$

If n is odd, the image of $\bar{\rho}$ is the image of $s_H: [\Sigma M; G/H] \to L^h_{n+1}(\pi')$. If n is even, then the image of ρ is the intersection with image s_H of the kernel of the restriction of $\phi_*: L_{n+2}(\pi) \to \Gamma_{n+2}(\mathcal{F})$ to $L^h_{n+1}(\pi') \subset L_{n+2}(\pi)$.

For $n = 3$, one has to modify the definition of $C_H(M)$ slightly to obtain this result, as usual.

The homomorphism $\bar{\rho}$ can be defined intrinsically as follows: Given $\iota\colon M \to M \times D^2 = E(\xi)$, ξ trivial, let W be the closure of the complement of a tubular neighborhood T of $\iota(M)$. Let $f\colon W \to M \times S^1$ be as in Lemma 4.2, and let $\pi_2\colon M \times S^1 \to S^1$ be the natural projection. Let $p\colon M \times D^2 \to M$ be the natural projection on M. Let $y \in S^1$ be a regular value of $\pi_2 \circ f$, and let $V = (\pi_2 f)^{-1}(y)$. It is not hard to find a bundle map b of normal bundles covering $p \mid V\colon V \to M$. Let $\gamma\colon V \to I$ be a Morse function. Let $g = (\pi \mid V, \gamma)$. Clearly g induces a homotopy equivalence, in fact a diffeomorphism (homeomorphism, P.L. homeomorphism), of boundaries and one may show:

ADDENDUM TO 7.1. $\bar{\rho}[i] = \sigma(g, b) \in L^h_{n+1}(\pi_1 M)$.

If n is even, then we get a strikingly small answer. For $\Gamma_{n+3}(\mathcal{F}) \xrightarrow{j_*} L_{n+3}(\pi')$ is a monomorphism, by § 2, and the natural map $L_{n+3}(\pi) \to L_{n+3}(\pi')$ is onto. Hence

COROLLARY 7.2. *If $n = \dim M$ is even, $\bar{\rho}\colon C_H(M) \to L^h_{n+1}(\pi_1 M)$ is a monomorphism.*

For example, $L_{\text{odd}}(\mathbf{Z}_p) = 0$, p odd, [34], [6]. Hence if M is orientable with $\pi_1 M = \mathbf{Z}_p$, then $C_H(M) = 0$.

To conclude, we compute local knottings of $T^n = S^1 \times \cdots \times S^1$, n even. If $H =$ PL, TOP, then $s_H\colon [\Sigma T^n; G/H] \to L_{n+1}(\mathbf{Z}^n)$ is a monomorphism, [24], [58], where $H =$ PL or TOP.

THEOREM 7.3. *For n even, $s_H^{-1}\bar{\rho}\colon C_H(T^n) \to [\Sigma T^n; G/H]$ is an isomorphism, $H =$ PL, TOP, $n \geqq 4$. A minimal set of generators for $C_{\text{TOP}}(T^n)$ can be obtained as follows: for each odd integer $k < n$, let α_k be a topologically locally flat knot of S^k in S^{k+2} with index equal to* 8 *($k \equiv -1 \bmod 4$) or* Arf *invariant ($k \equiv 1 \bmod 4$) equal to one. (For $k = 3$, see* [14].*) For each odd k, let $H^k_j \subset T^n$, $j = 1, \cdots, \binom{n}{k}$, be the standard subtori (given by holding certain factors in $S^1 \times \cdots \times S^1$ fixed), with complementary standard subtorus $\bar{H}^k_j$. Then a minimal set of generators of $C_{\text{TOP}}(T^n)$ is represented by the local knots*

$$T^n \xrightarrow{\beta} \bar{H}^k_j \times H^k_j \xrightarrow{\iota_0 \times \alpha_k} \bar{H}^k_j \times H^k_j \times D^2 \xrightarrow{\beta^{-1} \times 1} T^n \times D^2 ,$$

ι_0 the zero-section and β a suitable shuffling of co-ordinates.

A similar construction gives generators of $C_{\text{PL}}(T^n)$; for $k = 3$, α_k is taken to have index sixteen.

The proof follows easily from the addendum to 4.1, Corollary 7.2, and the well-known calculation of $L_{n+2}(Z^n)$, [46], [58] and of the map $s_H: [\Sigma T^n; G/H] \to L_{n+1}(Z^n)$, [24], [58].

In Chapter V, we will complete this result by calculating $C_H(T^n)$ for n odd; see 14.5.

Chapter III: Codimension two splitting and group actions

8. Codimension two splitting

Let $(X^n, \partial X)$ be a Poincaré pair (e.g., a compact manifold), X connected. Let $(Y, \partial Y) \subset (X, \partial X)$, Y connected, be a Poincaré (simple Poincaré) pair[1] having a linear normal bundle in $(X, \partial X)$. The case $\partial X \neq \emptyset$ is not excluded. Let $(M, \partial M)$ be a smooth, piecewise linear, or topological manifold pair and let $f: (M, \partial M) \to (X, \partial X)$ be a homotopy equivalence. Assume f is split (*resp.* simply split) along the boundary; i.e., $f \mid \partial M$ is transverse to ∂Y and f restricts to a homotopy (*resp.* simple homotopy equivalence) of $(f \mid \partial X)^{-1}(\partial Y)$ with ∂Y.

The (*simple*) splitting problem is the following: when is a homotopy equivalence f, (simply) split along the boundary, homotopic relative the boundary, to g transverse to Y with $g \mid g^{-1}Y: g^{-1}Y \to Y$ a (simple) homotopy equivalence? Such a g will be called (*simply*) *split*, and if it exists f will be said to be (*simply*) splittable along Y.

LEMMA 8.1a. *Suppose $f: (M^n, \partial M) \to (X, \partial X)$, as above, is normally cobordant, relative the boundary to a split (simply split) homotopy equivalence. Assume $n \geqq 5$. If n is odd, then f is (simply) splittable.*

b. *If instead n is even, then f is (simply) splittable if and only if f is normally cobordant to a (simply) split map via a normal cobordism with surgery obstruction in the image of the natural map $j_*: \Gamma^h_{n+1}(\mathcal{F}) \to L^h_{n+1}(\pi_1 X)$, $\mathcal{F}$ induced by the inclusion of $X - Y$ in X.*

Note. Of course, j_* is onto if the natural map from $L^h_{n+1}(\pi_1(X - Y))$ to $L^h_{n+1}(\pi_1 X)$ is onto. In particular, this is always the case if Y has a trivial normal bundle in X.

Proof. We ignore the boundary, since it plays no essential role in the proof. As a first step, we show that, under the hypothesis of 8.1, f is h-cobordant, relative boundary, to a split map.

Suppose n is odd. The normal cobordism f to a (simply) split map will

[1] In general the closure of the complement of a neighborhood will be a (simple) Poincaré pair over $Z[\pi_1 X]$, with boundary a simple Poincaré pair. A Poincaré embedding of Y in X can by definition be taken as an inclusion, as above, after changing the finite complex X to a homotopy equivalent one.

have a surgery obstruction $\gamma \in L^h_{n+1}(\pi_1 X)$. We will construct, given a (simply) split map $g: N \to X$, a normal cobordism of g to a simply split map, with surgery obstruction $-\gamma$. Having done this, we can paste the given normal cobordism to the constructed one along g to obtain a normal cobordism with vanishing obstruction in $L^h_{n+1}(\pi_1 X)$. Performing surgery yields the desired h-cobordism.

To construct the desired normal cobordism, let C be the complement of the interior of a closed disk bundle neighborhood of Y. Let $\mathcal{F}: \mathbf{Z}[\pi_1 C] \to \mathbf{Z}[\pi_1 X]$ be induced by the natural map, a surjection $\pi_1 C \to \pi_1 X$ by Van Kampen's theorem (or general position). By lifting matrices, it is obvious that the natural map $j_*: \Gamma^h_{n+1}(\mathcal{F}) \to L^h_{n+1}(\pi_1 X)$ is a surjection. Let $\delta \in \Gamma^h_{n+1}(\mathcal{F})$ with $j_*\delta = -\gamma$.

We may suppose that g is a bundle map on a tubular neighborhood T of $g^{-1}Y$, with $U = g^{-1}(C) =$ closure of $M - T$. Then it is not hard to see that $g \mid U: U \to C$ is a simple homology equivalence over $\mathbf{Z}[\pi_1 X]$. By Theorem 1.8, let (F, B) be a normal cobordism of $g \mid U$, relative boundary, to a simple homology equivalence h over $\mathbf{Z}\pi_1 X$, with $\sigma(F, B) = \delta$. By the same argument as in Lemma 4.3, we may assume, after a further homology s-cobordism over $\mathbf{Z}[\pi_1 X]$, if necessary, that h induces an isomorphism of fundamental groups; this will not alter $\sigma(F, B)$, by additivity.

Let W be the domain of F, so that we have

$$F: \big(W;\ U \cup (\partial U \times I),\ \partial_+ W\big) \longrightarrow (C \times I,\ C \times 0 \cup \partial C \times I,\ C \times 1)\ .$$

Let $V = T \times I \cup_{\partial U \times I} W$. Let $(G, C) = (F, B) \cup (g \mid T, b)$, b a suitable bundle map. By Van Kampen's theorem and a Meyer-Vietoris argument, $G \mid T \times 1 \cup \partial_+ W$ is a homology equivalence over $\mathbf{Z}[\pi_1 X]$ and induces an isomorphism of fundamental groups. Hence it is a homotopy equivalence. It is evidently (simply) split. By naturality (§ 3) (or by comparing the realization Theorem 1.8 with [59, 5.8]), it follows that $\sigma(G, C) = j_*\sigma(F, B) = j_*\delta = -\gamma$.

If n is even, the hypothesis of 8.1b allows us to apply the same argument.

Thus f is h-cobordant to a (simply) split map $g: N \to X$. An h-cobordism R will have torsion $\tau(R, N) = \tau \in \mathrm{Wh}\,(\pi_1 X)$. To show f is (simply) splittable, it suffices to show that g is h-cobordant to a (simply) split map by an h-cobordism with torsion $-\tau$. For then, pasting the given h-cobordism R to the constructed one along N yields an s-cobordism of f with a (simply) split homotopy equivalence, and the s-cobordism theorem implies the result. However, it is easy to see, from the construction of non-trivial h-cobordisms outlined in [40, § 11] for example, that they may be constructed by attaching handles entirely in the complement of any submanifold with complement having a fundamental group that maps onto $\pi_1 X$ by the natural map. In particular, a submanifold of codimension two can always be avoided. Hence the required h-cobordism can always be constructed.

Now consider $f\colon (M, \partial M) \to (X, \partial X)$, (simply) split along the boundary, as in the first paragraph of this section. Suppose f is transverse to Y. There is a natural bundle map b, as in [5], [58, § 11] for higher codimension, with domain the stable normal bundle of $f^{-1}Y$, covering $f \mid f^{-1}Y$. Let $\Sigma_e(f) \in L^e_{n-2}(\pi_1 Y)$, $e = s$ or h as appropriate, be the surgery obstruction of $(f \mid f^{-1}Y, b)$. This is easily seen to be an invariant of homotopy class of f, relative the boundary. We call $\Sigma_e(f)$ the *abstract surgery obstruction of* f.

THEOREM 8.2. *Let $n \geqq 7$ be odd. Let f be split (simply-split) along the boundary. Then $\Sigma_h(f)$ (resp. $\Sigma_s(f)$) vanishes if and only if f is splittable (resp. simply splittable).*

For higher codimensions, the analogous theorem is due to Browder [5].

Proof of 8.2. We consider only the non-simple case; e.g., $e = h$. The proof for the simple case is the same. It suffices to show that f is normally cobordant relative boundary to a split homotopy equivalence, by 8.1, assuming $\Sigma_h(f) = 0$. By surgery on $f \mid f^{-1}Y$ and the cobordism extension theorem, f is normally cobordant to a normal map (g, b), $g\colon N \to X$, with $g \mid g^{-1}Y\colon g^{-1}Y \to Y$ a homotopy equivalence. We may suppose g is actually a bundle map on a neighborhood T of $g^{-1}Y$. Let $U = g^{-1}(C)$, C the complement of the interior of a tubular neighborhood of Y. Then $\sigma(g \mid U, b \mid U) \in \Gamma^h_n(\mathcal{F})$ is defined, $\mathcal{F}\colon \mathbf{Z}[\pi_1 C] \to \mathbf{Z}[\pi_1 X]$ the natural map, as in the proof of 8.1. By naturality, $j_*\sigma(g \mid U, b \mid U) = \sigma(g, b) \in L^e_n(\pi_1 X)$. But $\sigma(g, b) = 0$, as (g, b) is normally cobordant to f. By § 2 and since n is odd, this implies that $\sigma(g \mid U, b \mid U) = 0$ also.

Hence, by 2.1, $(g \mid U, b \mid U)$ is normally cobordant, relative the boundary, to a homology equivalence over $\mathbf{Z}[\pi_1 X]$ that induces an isomorphism of fundamental groups. Gluing T back in along the appropriate part of the boundary and using $g \mid T$ to extend the homology equivalence, we obtain a split homotopy equivalence normally cobordant to f.

In the even dimensional case, it is still necessary for splittability that the obstruction $\Sigma_e(f)$ must vanish. In this case a secondary obstruction in $\Gamma^e_n(\mathcal{F})$ arises to completing the surgery to get a normally cobordant split map. In general, we know only that the image of this obstruction in $L^e_n(\pi_1 X)$ must vanish. However, if $\partial X = \partial M = \varnothing$, then one sees that the secondary obstruction actually lies in the image of $L^e_n(\pi_1 C)$ under the natural map of this group of $\Gamma_n^e(\mathcal{F})$. Hence one has the following result:

THEOREM 8.3. *Let $n \geqq 7$ be even. Let $f: M \to X$ be a homotopy equivalence, as in* 8.1, *but with $\partial M = \partial X = \varnothing$. Assume that the kernel of the natural map $i_*: L_n^h(\pi_1(X - Y)) \to L_n^h(\pi_1 X)$ vanishes under the natural map of $L_n^h(\pi_1(X - Y))$ into $\Gamma_n^h(\mathcal{F})$, $\mathcal{F}: Z\pi_1(X - Y) \to Z\pi_1 X$ the natural map. Assume also that $j_*: \Gamma_{n+1}^h(\mathcal{F}) \to L_{n+1}^h(\pi_1 X)$ is surjective.* (*See note following* 8.1.) *Then f is splittable if and only if its abstract surgery obstruction vanishes.*

For the simple splitting problem, the analogous result holds.

COROLLARY 8.4. *If $f: M \to X$ is as in* 8.3, *and if $\pi_1(X - Y) = Z$* or 0, *then f is splittable* (*or simply splittable*) *if and only if its abstract surgery obstruction $\Sigma_h(f)$ (or $\Sigma_s(f)$), vanishes.*

This extends the results of Lopez de Medrano [36].

Proof of 8.4. If $\pi_1(X - Y) = 0$, then $\pi_1 X = 0$ also and i_* is the identity. If $\pi_1(X - Y) = Z$ and $X - Y$ is orientable, then so is X and the composite

$$L_n(Z) \longrightarrow L_n^e(\pi_1 X) \longrightarrow L_n(e)$$

is an isomorphism [46]. In the non-orientable case, let $w: \pi_1 X \to Z_2$, be the orientation character. For $n \equiv 2 \pmod 4$,

$$L_n(Z_1, -) \longrightarrow L_n^e(\pi_1 X, w) \longrightarrow L_n(Z_2, -)$$

is an isomorphism (see [58], for example). For $n \equiv 0$ (4), this composite is trivial and we apply the result A. 1 of Appendix II. Also, $L_{n+1}(\pi_1 X) = O$ or Z_2 [34], [7], [62], [63], and in case it is Z_2, the natural map from $L_n(Z)$ is surjective [62], [63]; hence so is j_*.

For the general splitting problem in even dimensions, the secondary obstruction can be made a little more precise as follows: given $(Y, \partial Y) \subset (X, \partial X)$ as in the beginning of this section, induction over the normal sphere bundle induces a map $L_{n-1}^e(\pi_1 Y) \to L_{n+1}^e(\pi_*)$, π the projection of this bundle. Composing with natural maps, we obtain $\rho^e: L_{n+1}^e(\pi_1 Y) \to \Gamma_{n+1}^e(\phi)$, ϕ the homomorphism $(\mathcal{F}, \text{id})$ from $\mathcal{F}$ to $\text{id}_{Z\pi_1 X}$, $\mathcal{F}: Z\pi_1(X - Y) \to Z\pi_1 X$ the natural map.

THEOREM 8.5. *Let $f: (M^n, \partial M) \to (X, \partial X)$ be as in* 8.1, *but with $n \geqq 6$ even. Then if the surgery obstruction $\Sigma_h(f)$ (resp. $\Sigma_s(f)$) vanishes, there is a well-defined obstruction in $\Gamma_{n+1}^h(\phi)/\text{Im}\,\rho^h$ (resp. $\Gamma_{n+1}^s(\phi)/\text{Im}\,\rho^s$) that vanishes if and only if f is splittable (resp. simply splittable).*

The proof is omitted, but we point out that in general all these obstructions may arise.

The results of this section imply results of [29] and [39] on the existence of locally flat spines. For example, suppose the P.L. manifold W^{n+2} has the homotopy type of the closed Poincaré complex Y^n, $n \geqq 4$. If $n = 4$, assume Y is a manifold. Then it is not hard to exhibit a natural Poincaré embedding $Y \subset X$, with linear normal bundle, and a homotopy equivalence $h: W \to X$. So for n odd, Theorem 8.2 applies to yield a locally flat spine for W provided that a well-defined surgery invariant vanishes. Hence if $\pi_1 W = 0$, a locally flat spine exists, for n odd. For n even, the surgery obstruction and the secondary obstructions mentioned above can be killed by introducing a singular point whose link pair corresponds to the appropriate secondary obstruction under the identification of § 6 of knot cobordism groups with Γ-groups. Thus, if $\pi_1 W = 0$, a spine with only one non-locally flat point exists.

A general theory of non-locally flat P.L. submanifolds is developed in [16], [17]. The phenomenon of "total spinelessness" will be studied in [64].

9. Invariant spheres and characteristic submanifolds for free actions of cyclic groups on odd dimensional spheres

Let $\mathbf{Z}_s$ be the cyclic group of order s. We study free P.L.[1] (orientation preserving) actions of $\mathbf{Z}_s$ on odd dimensional spheres. The quotient space of such an action is an oriented PL manifold with an identification of its fundamental group with $\mathbf{Z}_s$, a so-called "fake lens space". We will say two actions are *equivalent* (*homotopy equivalent*) if their quotient spaces are P.L. homeomorphic (*resp.* homotopy equivalent) by a map of degree 1 that preserves the identifications of fundamental groups with $\mathbf{Z}_s$. Many of the results of this section are also valid for the other categories.

A *suspension* of a free action is defined as any free action obtained from the given one by taking the join with a free action on S^1. The quotient manifold of a suspension will also be called a suspension of the quotient manifold. There is one suspension for each primitive s^{th} root of unity. Let M be a fake lens space. Then, by considering a homotopy equivalent classical lens space and using the classical criterion for them to be homotopy equivalent [44], [40], it is easy to see that the different suspensions of M are not homotopy equivalent. (Compare [58, 14 E. 9].) A suspension N of the fake lens space L is easily seen to be determined by the normal bundle, of L in M, a two-disk bundle with Chern class a generator of $H^2(L; \mathbf{Z}) = \mathbf{Z}_s$.

Let ρ be a free action of $\mathbf{Z}_s$ on S^{2k+1}, with quotient space $M = S^{2k+1}/\rho$. An (locally flat) invariant sphere (of codimension two) is a (always assumed

[1] This means the quotient space is given a P.L. manifold structure.

locally flat) submanifold $K^{2k-1} \subset S^{2k+1}$, P.L. homeomorphic to S^{2k-1}, invariant under the action ρ. It follows that there is a bundle neighborhood on which ρ carries fibers linearly to fibers. A (simple) characteristic submanifold for a homotopy equivalence $f: M \to N$, N a suspension of L^{2k-1}, is a submanifold $g^{-1}L$, where $g: M \to N$ is homotopic to f, transverse to L, and $g \mid g^{-1}L: g^{-1}L \to L$ is a (simple) homotopy equivalence. If $h: L \to L'$ is a homotopy equivalence, there exists a canonical extension $\bar{h}: N \to N'$, for a unique suspension N' of L', so that characteristic submanifolds for f and $\bar{h}f$ coincide.

A characteristic submanifold is clearly the quotient space for the restriction of ρ to an invariant sphere. On the other hand, let L^{2k-1} be the quotient space of the restriction of ρ to an invariant sphere. Then a little obstruction theory shows that, for the suspension N of L determined by the normal bundle of L in M, the identity of L extends to a homotopy equivalence $f: M \to N$ for which L is a characteristic submanifold. So we have

PROPOSITION 9.1. *The quotient spaces of invariant spheres of ρ are exactly the characteristic submanifolds of $M = S^{2k+1}/\rho$.*

Let N be a suspension of L^{2k-1}, L a fake lens space. Then we define $t_N^L: [N; G/\mathrm{PL}] \to \mathbf{Z}_2$ as follows:

$$t_N^L(x) = \omega_* \sigma(x \mid L) \text{ if } s \text{ and } k \text{ are both even;}$$
$$t_N^L(x) = 0 \qquad \text{otherwise;}$$

where $\sigma: [L; G/\mathrm{PL}] \to L_{2k-1}(\mathbf{Z}_s)$ is the surgery obstruction map and $\omega_*: L_{2k-1}(\mathbf{Z}_s) \to L_{2k-1}(\mathbf{Z}_2) = \mathbf{Z}_2$ is the natural map, k and s both even. If $f: M \to N$ is a homotopy equivalence, we may write $t_N^L(f)$ for $t_N^L(x)$, where x is the normal invariant of f.

THEOREM 9.2. *Let ρ be a free action of $\mathbf{Z}_s$ on S^{2k+1}, $k \geqq 2$. Let $M = S^{2k+1}/\rho$. Let τ be a free action of $\mathbf{Z}_s$ on S^{2k-1}. Then τ is homotopy equivalent to the restriction of ρ to an invariant sphere if and only if $L = S^{2k-1}/\tau$ has a suspension N homotopy equivalent to M by a homotopy equivalence $f: M \to N$ with $t_N^L(f) = 0$.*

COROLLARY 9.3. *For s and $k \geqq 2$ not both even, every free action of $\mathbf{Z}_s$ on S^{2k+1} has an invariant sphere of codimension two.*

Notes. 1. Actually, the results hold in the topological and smooth categories as well and are proved the same way. In the smooth category the statement about L in 9.2 will refer to the underlying P.L. homotopy lens spaces.

2. For $k = 1$, the analogous result to 9.2 is trivial.

3. The corollary follows from 9.2 and the fact that every homotopy lens space has the homotopy type of classical lens space. For s odd, 9.3 also follows from the unique desuspension theorem of [8]. For $s = 2$, it is a result of Lopez de Medrano [35]. The existence of characteristic submanifolds in higher codimension is due to W. Browder [6].

Proof of 9.2. The necessity of the condition that τ have the homotopy type of the restriction of ρ follows easily from 9.1. Conversely, suppose $f: M \to N$, N a suspension of $L = S^{2k-1}/\tau$. Then, by [58, 14. E. 4], the abstract surgery obstruction of f, with respect to $L \subset N$, vanishes if and only if $t_N^L(f) = 0$. (For s odd, this is due to Browder [6].) Hence, for $k \geqq 3$, 9.2 follows immediately from 8.2.

For $k = 2$, the composition of the surgery obstruction with ω_* induces an isomorphism $[L^3; G/\mathrm{PL}] \to \mathbf{Z}_2$. Hence the vanishing of $t_N^L(f)$ in this case implies that the restriction of f to the transverse inverse image $f^{-1}L$ is normally cobordant to a homotopy equivalence; in fact it is normally cobordant to the identity. Using this, the proof of 8.2 will carry through to show that, in fact, τ itself is the restriction of ρ to an invariant sphere.

If τ_1, τ_2 are free actions of $\mathbf{Z}_s$ on a sphere with the same homotopy type, let $f: S^i/\tau_1 \to S^i/\tau_2$ be a homotopy equivalence, of degree one, preserving identifications of fundamental groups with $\mathbf{Z}_s$. Then we say τ_1 is *normally cobordant* to τ_2 if and only if the normal invariant $\eta(f)$ of f in $[S^i/\tau_2, G/\mathrm{PL}]$ vanishes. Normal cobordism is an equivalence relation, in view of the formula $\eta(g \circ f) = \eta(g) + (g^{-1})^*\eta(f)$.

THEOREM 9.4. *Let τ_1 be equivalent to the restriction of a free action ρ on S^{2k+1} to an invariant sphere of codimension two. Let τ_2 be homotopy equivalent to τ_1. Assume $k \geqq 2$. Then τ_2 is the restriction of ρ to an invariant sphere if and only if τ_1 is normally cobordant to τ_2.*

Note. Smooth and topological analogues are valid also, and proved in the same way. For $k = 1$, Theorem 9.4 is trivial.

Proof. That τ_1 and τ_2 must be normally cobordant if they are restrictions of ρ to invariant spheres follows easily from 9.1 and transversality, using the natural homotopy equivalence of appropriate suspensions of τ_1 and τ_2.

To prove the converse, let N be a suspension of $L_1 = S^{2k-1}/\tau_1$ so that there is a homotopy equivalence $f: M \to N$, $M = S^{2k+1}/\rho$, with L_1 characteristic for f and $f \mid L_1$ the identity. (See the discussion preceding 9.1. Here we have $L_1 \subset M$ by identifying τ_1 with the equivalent restriction of ρ.) Let $g: W \to L \times I$ be a normal cobordism of the identity of L_1 to a homotopy equivalence $h: L_2 \to L_1$, $L_2 = S^{2k-1}/\tau_2$ of degree one and preserving identifications of fundamental groups. (Bundle maps are omitted from the notation.) By the cobordism extension theorem, we may extend g to a normal cobordism $G: V \to N \times I$ of f to a map $H: M' \to N$ with $W \subset V$, $G^{-1}(L \times I) = W$, $G \mid W = g$, and G a bundle map on a neighborhood of W.

Let R be a bundle neighborhood of L_1 in N, with interior $\mathring{R}$. Then the surgery obstruction $\sigma(H \mid H^{-1}(N - \mathring{R})) \in \Gamma^h_{2k+1}(\mathcal{F})$ is defined, $\mathcal{F}: \mathbf{Z}[\mathbf{Z}] \to \mathbf{Z}[\mathbf{Z}_s]$ a natural map induced by inclusion of $N - \mathring{R}$ in N. Its image under j_*, the natural map to $L^h_{2k+1}(\mathbf{Z}_p)$, is, by naturality, just $\sigma(H) = \sigma(f) = 0$. Hence $\sigma(H \mid H^{-1}(N - \mathring{R})) = 0$. So, after some surgeries avoiding L_2 if necessary, $H \mid H^{-1}(N - \mathring{R})$ can be assumed to be a homology equivalence over $\mathbf{Z}[\mathbf{Z}_s]$, as well as an isomorphism of fundamental groups, by 2.1. Hence H will be a homotopy equivalence.

So L_2 is a characteristic submanifold of a homotopy lens space normally cobordant to M. The proof of 8.1a shows that this implies that L_2 is a characteristic submanifold of M also, which implies the result. Observe that in the proof of 8.1, the codimension two submanifolds were never disturbed, as all surgeries, normal cobordisms, etc. took place in the complement.

10. Equivariant cobordism of invariant spheres in codimension two

Let ρ be a free action of $\mathbf{Z}_s$ on S^{2k+1}, in the P.L. category; i.e., S^{2k+1}/ρ is a P.L. manifold. Many of the results can be formulated for the smooth and topological categories also. Let R_0, $R_1 \subset S^{2k+1}$ be invariant (locally flat) spheres of codimension two. Let $S^{2k+1} \times I$ have the action given by ρ in the first factor and be trivial on the second. Then R_0 and R_1 are *equivariantly cobordant* if and only if there is a submanifold W of $S^{2k+1} \times I$, invariant under the action of $\mathbf{Z}_s$, P.L. equivariantly homomorphic to $R_0 \times I$, and meeting the boundary transversely in $\partial W = (R_0 \times 0) \cup (R_1 \times 1)$. In particular, $\rho \mid R_0$ and $\rho \mid R_1$ are equivalent. (More precisely, the above should be called equivariant *s-cobordism*. The methods of this section could also be used to study equivariant *h-cobordism*.)

Let $M = S^{2k+1}/\rho$, and let $L^{2k-1} = S^{2k-1}/\tau$ be a homotopy lens space. Suppose ρ has an invariant sphere equivalent to τ; in § 9 we determined when this happens. Then, by 9.1 (more precisely, by the discussion preceding 9.1), there is a unique suspension N of L and a homotopy equivalence $h: M \to N$, preserving orientation and identification of fundamental groups with $\mathbf{Z}_s$, for which S^{2k-1}/τ is a characteristic submanifold. A similar argument shows that equivariant cobordism classes of invariant spheres for which the restriction of ρ is equivalent to τ, denoted $C(\rho, \tau)$, correspond exactly to *cobordism* classes of (simple) characteristic submanifolds for h that are P.L. homeomorphic to L via a homeomorphism preserving orientation and identification of fundamental groups with $\mathbf{Z}_s$. Two (simple) characteristic submanifolds for h, $L_i = g_i^{-1}L$, $i = 0, 1$, are *cobordant* if and only if there exists a homotopy G of g_0 with g_1, transverse regular to $L \times I$, with $G \mid G^{-1}(L \times I)$ a simple homotopy equivalence onto $L \times I$.

Let $C(\rho)$ be the union of the sets $C(\rho, \tau)$, for all τ equivalent to the restriction of ρ to an invariant sphere.

Let $K \subset M$ be a characteristic submanifold for $h: M \to N$, P.L. homeomorphic to L by a homeomorphism that preserves *polarization* (i.e., orientation and identification of fundamental groups; see [58, 14. E. 3]). Then $K = g^{-1}L$, where $g: M \to N$ is homotopic to h, transverse to L, and $g \mid K: K \to L$ is homotopic to a P.L. homeomorphism. We may suppose $g \mid R(K): R(K) \to R(L)$ is a bundle map, where $R(K)$, for example, denotes a bundle neighborhood of K. Let $f: W \to D^{2k} \times S^1$ be the restriction of g to a map from $W = \mathrm{cl}\,(M - R(K))$ to $D^{2k} \times S^1 = \mathrm{cl}\,(N - R(L))$. Let $\mathcal{F}: \mathbf{Z}[\mathbf{Z}] \to \mathbf{Z}[\mathbf{Z}_s]$ be induced by the map on fundamental groups induced by inclusion $D^{2k} \times S^1 \subset N$. Then f is a homology equivalence over $\mathbf{Z}[\mathbf{Z}_s]$ and $f \mid \partial W$ is a homotopy equivalence of ∂W with $S^{2k-1} \times S^1$, homotopic to a P.L. homeomorphism. Furthermore, the torsion $\Delta_{\mathcal{F}}(f) \in \mathrm{Wh}\,(\mathcal{F}) = \mathrm{Wh}\,(\mathbf{Z}_s)$ is easily see to be precisely the torsion $\delta \in \mathrm{Wh}\,(\mathbf{Z}_s)$ of the map h.

Let $\mathcal{S} = \mathcal{S}^{\delta}_{\mathcal{F}}(D^{2k} \times S^1)$ denote homology s-cobordism classes, over $\mathbf{Z}[\mathbf{Z}_s]$, of homology equivalences $f: (W, \partial W) \to (D^{2k} \times S^1, S^{2k-1} \times S^1)$ with torsion δ that restrict to homotopy equivalences on the boundary. More precisely f and f' will represent the same element in $\mathcal{S}$ if there exists a relative cobordism (U, V) of $(W, \partial W)$ with $(W', \partial W')$, with V an s-cobordism and U a homology s-cobordism over $\mathbf{Z}[\mathbf{Z}_s]$, and an extension $F: (U, V) \to (D^{2k} \times S^1, S^{2k-1} \times S^1)$ of $f \cup f'$. The definition of f in the preceding paragraph defines an invariant $\theta(K) \in \mathcal{S}$ of the characteristic submanifold K, easily seen to depend only on the cobordism class of K. Using the fact [4], [38] that P.L. homeomorphisms of $S^1 \times S^m$, homotopic to the identity are pseudo-isotopic to the identity; the s-cobordism theorem (including Theorem 2.1 of [48]); and the fact (compare Lemma 4.3) that given a cobordism (U, V) as above, one can always find U', homology s-cobordant to U over $\mathbf{Z}[\mathbf{Z}_s]$, relative the boundary with $\pi_1 U' = \mathbf{Z}$, one can easily show the following:

PROPOSITION 10.1. *Let K and K' be characteristic submanifolds for*

$h\colon M^{2k+1} \to N$ that are P.L. *homeomorphic to L via a homeomorphism preserving polarizations. Then $\theta(K) = \theta(K')$ if and only if K and K' are cobordant, $k \geqq 2$.*

To calculate $\mathcal{S}$, let ϕ be the diagram

$$\begin{array}{ccc} \mathbf{Z}[\mathbf{Z}] & \xrightarrow{\text{id}} & \mathbf{Z}[\mathbf{Z}] \\ \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \mathcal{F}} \\ \mathbf{Z}[\mathbf{Z}] & \xrightarrow{\mathcal{F}} & \mathbf{Z}[\mathbf{Z}_s] \end{array},$$

a morphism from the identity of $\mathbf{Z}[\mathbf{Z}]$ to $\mathbf{Z}[\mathbf{Z}_s]$. The group $\Gamma^s_{2k+2}(\phi)$ acts on $\mathcal{S}^\delta_{\mathcal{F}}(D^{2k} \times S^1)$; when $\delta = 0$ this action is given by the realization Theorem 3.4 and is analogous to the usual action of Wall groups on $\mathcal{S}_{\text{PL}}(P)$, P a manifold; when $\delta \neq 0$ one appeals to 3.7. Since $\pi_1(G/\text{PL}) = 0$, the elements of $\mathcal{S}$ are all normally cobordant. Hence the action of $\Gamma^s_{2k+1}(\phi)$ is transitive.

Every map $f\colon W \to D^{2k} \times S^1$ representing an element of $\mathcal{S}$ is normally cobordant to the identity; this normal cobordism will have an obstruction in $\Gamma^h_{2k+2}(\phi)$. Further, if an element of $\Gamma^s_{2k+2}(\phi)$ acts trivially on the identity element, then it is represented by a normal map into $S^1 \times D^{2k} \times I$ that is a homotopy equivalence on the parts of the boundary corresponding to $S^1 \times D^{2k} \times \{0, 1\}$; hence it is in the image of $\Gamma^s_{2k+2}(\mathbf{Z}, \mathbf{Z}) = 0$ under the natural map, and so vanishes. These two facts, together with additivity of surgery obstructions over unions, imply that if $\xi \in \Gamma^s_{2k+2}(\phi)$ acts trivially on any element of $\mathcal{S}$, then the image of ξ in $\Gamma^h_{2k+2}(\phi)$ under the natural map vanishes. In Appendix I we will show that the natural map $\Gamma^s_{2k+2}(\phi) \to \Gamma^h_{2k+2}(\phi)$ is a monomorphism. So we have:

PROPOSITION 10.2. *The action of $\Gamma^s_{2k+1}(\phi)$ on $\mathcal{S}^\delta_{\mathcal{F}}(D^{2k} \times S^1)$ is transitive and free, $k \geqq 3$.*

If we fix a characteristic submanifold $K_0^{2k-1} \subset M$ for $h\colon M \to N$, P.L. homeomorphic to L via a map preserving polarization, then the action of $\Gamma^s_{2k+2}(\phi)$ on $\theta(K_0)$ gives a bijection $\Delta_{K_0} = \Delta\colon \Gamma^s_{2k+2}(\phi) \to \mathcal{S}$, and so we get an invariant $\Delta^{-1}\theta(K)$ that determines the cobordism class of K.

To determine the image of $\Delta^{-1}\theta$, let K be a characteristic submanifold for h, P.L. homeomorphic to L via a polarization preserving map. Then there is a map $F\colon M \times I \to N \times I$, homotopic to $h \times 1$, transverse to $L \times I$, so that if $W = F^{-1}(L \times I)$, then $\partial W = (K_0 \times 0) \cup (K \times 1)$. We may suppose

$$F_W = F \mid R(W)\colon R(W) \longrightarrow R(L) \times I$$

is linear on a bundle neighborhood $R(W)$ of W. Let G_W be the restriction of F to a map of the closure of the complement of $R(W)$ to the closure of the complement of $R(L) \times I$. Then it is clear that

$$\Delta^{-1}\theta(K) = \sigma(G_W) .$$

(Bundle maps will be omitted from the discussion, for convenience.)

Recall the natural map $j_*: \Gamma^s_{2k+2}(\phi) \to L^s_{2k+2}(\mathcal{F})$. By additivity over unions and naturality of surgery obstructions, $j_*\sigma(G_W) - \sigma(F_W)$ is just the image of $\sigma(F) \in L_{2k+2}(\mathbf{Z}_s)$ in $L_{2k+2}(\mathcal{F})$ under the natural map; this holds for any normal map into $N \times I$ with torsion δ on both ends (see 3.7 for the case $\delta \neq 0$). The sign is due to orientation considerations. Since F is a homotopy equivalence with torsion δ, $\sigma(F) = 0$. Thus

$$j_*\sigma(G_W) = \sigma(F_W) .$$

Let $p: L_{2k}(\mathbf{Z}_s) \to L_{2k+2}(\mathcal{F})$ be the map defined geometrically by inducing normal maps into $L \times I$ over the normal bundle of $L \times I$ in $N \times I$. Since F_W is a bundle map, $\sigma(F_W) = p\sigma(F_1)$, where F_1 is the restriction of F to a normal map of W to $L \times I$. By hypothesis, $F \mid K_0 \times 0$ and $F \mid K \times 1$ are homotopic, as maps into $L \times 0$ and $L \times 1$ respectively, to P.L. homomorphisms preserving polarization. Hence $\sigma(F_1)$ is in the image of the surgery map $s: [\Sigma L; G/\mathrm{PL}] \to L_{2k}(\mathbf{Z}_s)$. Thus $j_*\Delta^{-1}\theta(K)$ is in the image of $p \circ s$. It is straightforward to reverse the above argument to show that every element $\xi \in \Gamma_{2k+2}(\phi)$ with $j_*\xi \in \operatorname{Im}(p \circ s)$ is of the form $\Delta^{-1}\theta(K)$, K a characteristic submanifold P.L. homeomorphic to L by a polarization preserving map. Thus *cobordism classes of such charcteristic submanifolds are in one-one correspondence with* j_*^{-1} $(\operatorname{Im} p \circ s)$. In terms of invariant spheres, we have:

THEOREM 10.3. *Let ρ be a free action of $\mathbf{Z}_s$ on S^{2k+1}, $k \geqq 2$, in the* P.L. *category. Let τ be a free action on S^{2k-1}, equivalent to the restriction of ρ to a locally flat invariant sphere R. Let $L = S^{2k-1}/\tau$. Let K_0 be the quotient characteristic submanifold corresponding to R. Then the invariant $\Delta_{K_0}^{-1}\theta$, on the quotient characteristic submanifolds, gives a bijection of equivariant cobordism classes of invariant spheres R', with $\rho \mid R'$ equivalent to τ* (i.e., *of $C(\rho, \tau)$*) *to elements $\xi \in \Gamma_{2k+2}(\phi)$ so that $j_*\xi$ is in the image of*

$$[\Sigma L; G/\mathrm{PL}] \xrightarrow{s} L_{2k}(\mathbf{Z}_s) \xrightarrow{p} L_{2k+2}(\mathcal{F}) .$$

Actually the case $k = 2$ has not been adequately dealt with here. This case requires further special arguments which would clutter up the present exposition.

To interpret 10.3, consider the following diagrams, which commute:

$$\text{(a)}\quad \begin{array}{ccccccccc} L_{2k+2}(\mathbf{Z}) & \xrightarrow{\beta_1} & \Gamma_{2k+2}(\mathcal{F}) & \longrightarrow & \Gamma_{2k+2}(\phi) & \xrightarrow{\partial_1} & L_{2k+1}(\mathbf{Z}) & \longrightarrow & \Gamma_{2k+1}(\mathcal{F}) \\ \downarrow{=} & & \downarrow{j_*} & & \downarrow{j_*} & & \downarrow{=} & & \downarrow \\ L_{2k+2}(\mathbf{Z}) & \xrightarrow{\beta_2} & L_{2k+2}(\mathbf{Z}_s) & \longrightarrow & L_{2k+2}(\mathcal{F}) & \xrightarrow{\partial_2} & L_{2k+1}(\mathbf{Z}) & \longrightarrow & L_{2k+1}(\mathbf{Z}_s)\ ; \end{array}$$

$$\text{(b)}\quad \begin{array}{ccc} L_{2k}(\mathbf{Z}_s) & \xrightarrow{p} & L_{2k+2}(\mathcal{F}) \\ \downarrow{\tau_0} & & \downarrow{\partial_2} \\ L_{2k}(e) & \xleftarrow{\alpha} & L_{2k+1}(\mathbf{Z})\ . \end{array}$$

The diagram (a) is just the transcription of (3.2.1) to the present case. In (b) τ_0 is the transfer homomorphism, defined geometrically by passing to universal covering spaces; ∂_2 is the natural map in the exact sequence [58, § 3] for the Wall group of a pair; and α is defined in [46], and is an isomorphism [46, 5.1]. By $\widetilde{\Gamma}_{2k+2}(\mathcal{F})$ and $\widetilde{L}_{2k+2}(\mathbf{Z}_s)$ we denote the cokernels of β_1 and β_2, respectively.

If s is even and k is odd, then [58, B.A. 9] implies that ∂_1 and ∂_2 are trivial. So in this case p induces

$$\bar{p}\colon L_{2k}(\mathbf{Z}_s) \longrightarrow \widetilde{L}_{2k+2}(\mathbf{Z}_s)\ .$$

Let $A_{s,k} \subset \widetilde{\Gamma}_{2k+1}(\mathcal{F})$ be those elements ξ with $\tilde{j}_*(\xi)$ in the image of $\bar{p} \circ s$, $\tilde{j}_*$ induced by the natural map, for s even and k odd.

If s is odd *or* k is even, the last maps on the right in (a) are trivial, by [58, 14. E. 5] and 2.1. Further, recalling that the natural map $i_*\colon L_{2k}(e) \to L_{2k}(\mathbf{Z})$ is an isomorphism, it follows using (b) that the composite

$$L_{2k}(\mathbf{Z}) \xrightarrow{\beta_2} L_{2k}(\mathbf{Z}_s) \xrightarrow{p} L_{2k+2}(\mathcal{F}) \xrightarrow{\partial_2} L_{2k+1}(\mathbf{Z})$$

is a monomorphism; this also uses the fact that $\tau_0\beta_2 i_*(x) = sx$. So, in fact, the image of $\partial_2 p \beta_2$ is generated by sg, g a generator of $L_{2k+1}(\mathbf{Z})$. In particular, p induces a map

$$\tilde{p}\colon \widetilde{L}_{2k}(\mathbf{Z}_s) \longrightarrow \widetilde{L}_{2k+2}(\mathbf{Z}_s)\ .$$

In this case, let $A_{s,k}$ denote the inverse image under $\tilde{j}_*\colon \widetilde{\Gamma}_{2k+2}(\mathcal{F}) \to \widetilde{L}_{2k+2}(\mathbf{Z}_s)$ of the image of the composition of $\tilde{p}$ with the composite of $s\colon [\Sigma L; G/\mathrm{PL}] \to L_{2k}(\mathbf{Z}_s)$ and the quotient map of $L_{2k}(\mathbf{Z}_s)$ to $\widetilde{L}_{2k}(\mathbf{Z}_s)$.

Clearly the natural map induces a monomorphism of $A_{s,k}$ into $\Gamma_{2k+2}(\phi)$, which acted freely on $\mathcal{S}$. The reader will see that this leads to a free action of $A_{s,k}$ on $C(\rho, \tau)$, which can easily be given a direct description in terms of the action of elements of $\Gamma_{2k+2}(\mathcal{F})$ on the complements of characteristic submanifolds to produce new characteristic submanifolds.

If s is even and k odd, let $\rho(x) = 0$ for all $x \in C(\rho, \tau)$; otherwise let

$$\rho(x) = \frac{1}{s}\alpha\partial_1\Delta_{K_0}^{-1}\theta(x) \in L_{2k}(e) = \begin{cases}\mathbf{Z}, & k \text{ even} \\ \mathbf{Z}_2, & k \text{ odd}\,,\end{cases}$$

K as in 10.3. Then we may reformulate 10.3 as follows:

COROLLARY 10.4. *The group $A_{s,k}$ acts freely on $C(\rho, \tau)$ and ρ induces a bijection of the quotients space with* $\mathbf{Z}$, *k even, or* $\mathbf{Z}_2$, *k odd.*

Notes 1. If s is odd, then it follows from [45] (see also [8], [58], [6]) that the image of s lies entirely in $L_{2k}(e) \subset L_{2k}(\mathbf{Z}_s)$. Hence $A_{s,k}$ is just the kernel of $\tilde{j}_*: \tilde{\Gamma}_{2k+2}(\mathcal{F}) \to \tilde{L}_{2k+2}(\mathbf{Z}_s)$ in this case. It is easy to check, recalling the isomorphism $L_{2k+2}(e) \cong L_{2k+2}(\mathbf{Z})$ [46, 5.1], that $A_{s,k}$ is naturally isomorphic to the kernel of j_* itself.

2. Let $s^h: [\Sigma L; G/\mathrm{PL}] \to \tilde{L}^h_{2k}(\mathbf{Z}_s)$ be the surgery map. Then using 10.4 and 9.3, one can produce a one-one correspondence of $C(\rho)$ with $A_{s,k} \oplus L_{2k}(e) \oplus \tilde{L}_{2k}(\mathbf{Z}_s)/\mathrm{Im}\, s^h$, s and $(k+1)$ not both even, and with $A_{s,k} \oplus \tilde{L}_{2k}(\mathbf{Z}_s)/\mathrm{Im}\, s^h$ if s and $(k+1)$ are both even. Again, if s is odd s^h vanishes.

3. If s is odd and ρ is a suspension of τ, then the unique desuspension theorem [8, Cor. 1] provides a canonical choice for K_0.

To conclude this section, we consider a free action ρ of $\mathbf{Z}_2$ on a $2k$-dimensional (homotopy) sphere, in the smooth, P.L., or topological category. In this case, the quotient $M = S^{2k}/\rho$ is always homotopy equivalent to real projective space P^{2k}, and any (locally flat) invariant (homotopy) sphere is a characteristic submanifold for the (unique up to homotopy) homotopy equivalence $f: M \to P^{2k}$, with respect to the standard codimension two projective space $P^{2k-2} \subset P^{2k}$.

THEOREM 10.5. *There exists at most one cobordism class of characteristic submanifolds for f, $k \geqq 4$. Thus a smooth,* P.L. *(locally flat), or topological (locally flat) invariant sphere R (homotopy sphere in the smooth case) is unique up to equivariant cobordism, assuming $k \geqq 4$.*

For $k = 3$, a result can be stated in terms of a definition of cobordism sufficiently weakened to compensate for the lack of an s-cobordism theorem in dimension five.

Proof of 10.5. Let $F: M \times I \to P^{2k} \times I$ be a (simple, as $\mathrm{Wh}(\mathbf{Z}_2) = 0$) homotopy equivalence, transverse to $P^{2k-2} \times I$. Let $V = F^{-1}(P^{2k-2} \times I)$, and suppose $F \mid \partial V: \partial V \to P^{2k} \times \{0, 1\}$ is a homotopy equivalence. To prove the theorem we have to show that the characteristic submanifolds $(\partial V) \cap (M \times 0)$ and $(\partial V) \cap (M \times 1)$ are cobordant; we are taking for granted a relative analogue of 9.1.

By 13. A. 1 of [58], $L_{2k-1}(\mathbf{Z}_2^-) = 0$. Hence, by the cobordism extension theorem $F \mid V$ is normally cobordant to a homotopy equivalence. By the cobordism extension theorem, it follows that F is normally cobordant, relative the boundary, to $G\colon W \to P^{2k} \times I$ with $G \mid U\colon U \to P^{2k-2} \times I$ a homotopy equivalence, G transverse to $P^{2k-2} \times I$ and $U = G^{-1}(P^{2k-2} \times I)$.

We may suppose that G is a bundle map on a neighborhood $R(U)$ to a neighborhood $R(P^{2k-2} \times I)$. Let $H\colon Q \to B = S^1 \times D^{2k-1} \times I$ be the restriction of G to a normal map of the closure of the complements of these neighborhoods. Then, by Meyer-Vietoris sequences, H induces a (simple) homology equivalence over $\mathbf{Z}[\mathbf{Z}_2]$, so that $\sigma(H) \in \Gamma_{2k+1}(\mathcal{F})$ is defined, $\mathcal{F}\colon \mathbf{Z}[\mathbf{Z}] \to \mathbf{Z}[\mathbf{Z}_2]$ the natural map. By § 2 and 13. A. 1 of [59], this group is zero. So, by 2.1 H is normally cobordant, relative the boundary, to a homology equivalence over $\mathbf{Z}[\mathbf{Z}_2]$, $H'\colon Q' \to B$, say, that induces an isomorphism of fundamental groups. Clearly $R(U) \cup Q'$ will be an s-cobordism (recall $\mathrm{Wh}(\mathbf{Z}_2) = 0$), containing the s-cobordism U. The s-cobordism theorem now provides the required cobordism.

11. Knots as fixed points

Let C^{TOP}_{2k-1} denote the group of locally flat cobordism classes of locally flat topological knots of S^{2k-1} in S^{2k+1}. In this section we state without proof some results on the following question: which elements in C^{TOP}_{2k-1} can be represented by a knot that forms the fixed point set of a semi-free action of a cyclic group $\mathbf{Z}_s$ that is linear on the fibers of a bundle neighborhood of the fixed points? Such an action will be called *locally linear*. Similar results can be stated for C^{H}_{2k-1}, $H = O$ or PL; the statements are simpler in the present case because the groups C^{TOP}_{2k-1}, $k \geqq 2$, satisfy periodicity [14].

Let $\rho_k\colon C^{\mathrm{TOP}}_{2k-1} \to L_{2k}(e) = \begin{cases} \mathbf{Z} & \text{if } k \text{ is even} \\ \mathbf{Z}_2 & \text{if } k \text{ is odd} \end{cases}$ denote the homomorphism that measures the Arf invariant or one-eighth the index of a Seifert surface; for $k = 2$ we use the suspended Seifert surface of [14]. One can also describe ρ_k as the composition

$$C^{\mathrm{TOP}}_{2k-1} \xrightarrow{\alpha} C_{\mathrm{TOP}}(S^{2k-1}) \xrightarrow{\Sigma} \Gamma_{2k+2}(\phi_0) \xrightarrow{\partial} L_{2k+1}(\mathcal{F}_0) \xrightarrow{\partial} L_{2k}(\mathbf{Z}) = L_{2k}(e) ;$$

see § 6. From Theorem 6.4 it follows that the kernel ρ is isomorphic to $\tilde{\Gamma}_{2k+2}(\mathcal{F}_0)$ via the restriction of Σ; here $\mathcal{F}_0\colon \mathbf{Z}[\mathbf{Z}] \to \mathbf{Z}$ is the augmentation and $\tilde{\Gamma}_{2k+2}(\mathcal{F}_0)$ is the quotient of $\Gamma_{2k+2}(\mathcal{F}_0)$ by the image of $L_{2k+2}(\mathbf{Z})$.

Let $\mathcal{F}\colon \mathbf{Z}[\mathbf{Z}] \to \mathbf{Z}[\mathbf{Z}_s]$ be the natural map. Then, as in the usual surgery theory, there is a transfer map

$$\tau = \tau_k\colon \Gamma^h_{2k+2}(\mathcal{F}) \longrightarrow \Gamma_{2k+2}(\mathcal{F}_0) ;$$

τ can be given a purely algebraic description; geometrically it corresponds to passing to the s-fold cover. By $\tilde{\tau}$ we denote the map from $\tilde{\Gamma}^h_{2k+2}(\mathcal{F})$ to $\tilde{\Gamma}_{2k+2}(\mathcal{F}_0)$ induced by τ.

PROPOSITION 11.1. *An element $x \in \ker \rho_k$, $k \geqq 3$, admits a representative that is the fixed point of a semi-free action of $\mathbf{Z}_s$ that is locally linear if and only if $\Sigma(x)$ is in the image $\tilde{\tau}$.*

This is actually not difficult to verify.

Using the method of [14], [47] (essential fact is that $\Gamma_i(\mathcal{F}_0) = 0$, i odd), one may exhibit a knot $\alpha^k_\xi \in C^{\mathrm{TOP}}_{2k-1}$ with $\rho_k(\alpha^k_\xi) = \xi \in L_{2k}(e)$, for all ξ, $k \geqq 2$, that is the fixed point set of a semi-free action of $\mathbf{Z}_s$ which is locally linear.

THEOREM 11.2. *For $k \geqq 2$, the cosets of $\Sigma^{-1}\tilde{\tau}_k(\tilde{\Gamma}_{2k+2}(\mathcal{F}))$ containing the classes represented by the knots α^k_ξ, $\xi \in L_{2k}(e)$, are precisely the elements of C^{TOP}_{2k-1} admitting representatives that are fixed points of locally linear semi-free actions of $\mathbf{Z}_s$.*

Let $i: \mathcal{F} \to \mathcal{F}$ be defined by sending a generator to s times a generator, and let $i_*: \Gamma_{2k+2}(\mathcal{F}_0) \to \Gamma_{2k+2}(\mathcal{F})$ be induced by i.

LEMMA 11.3. *For $\eta \in \Gamma_{2k+2}(\mathcal{F}_0)$, $\tau i_*(\eta) = s\eta$.*

This can be proved in the same way as the analogous theorem for surgery groups.

COROLLARY 11.4. *The cosets of $s(\ker \rho) = \{sx \mid x \in \ker \rho\}$ containing the elements α^k_ξ, $k \geqq 2$, all admit representatives that are the fixed points locally linear semi-free actions of $\mathbf{Z}_s$. In particular, for all $y \in C^{\mathrm{TOP}}_{2k-1}$, sy admits such a representative.*

In particular, it follows from [32] that if s is odd, all torsion classes in C^{TOP}_{2k-1} admit representatives that are the fixed points of locally linear semi-free actions of $\mathbf{Z}_s$.

Chapter IV: Some global results

12. Close embeddings in codimension two

Let W^{n+2}, $n \geqq 5$, be a P.L. manifold, not necessarily compact, with a metric d for its topology.

THEOREM 12.1. *Let $f: M^n \to W$, M a closed, simply-connected* P.L. *manifold, be a locally flat* P.L. *embedding, with trivial normal bundle. Then there exists $\varepsilon > 0$ such that if $g: M \to W$ is a locally flat embedding with $d(f(x), g(x)) < \varepsilon$ for $x \in M$, then there exists a* P.L. *homeomorphism $\varphi: M \to M$, homotopic to the identity, so that g is (locally flat) concordant to $f \circ \varphi$, assuming n is even. If n is odd, the conclusion will still hold after replacing g by its connected sum with a knot.*

Remarks 1. If $s\colon [\Sigma M; G/\mathrm{PL}] \to L_{n+1}(e)$ is a monomorphism, then every P.L. homeomorphism of M homotopic to the identity is pseudo-isotopic to it. Therefore, for such an M, one may take φ to be the identity, in the conclusion of 12.1. For example, this is the case when M is a complex projective space CP^k.

2. The concordance of 12.1 may be to required take place within a given tubular neighborhood of $f(M)$.

3. In the paper [18], it will be shown that for n even the conclusion of Theorem 12.1 holds in the smooth, P.L., and topological categories without any restriction on the normal bundle.

Proof of 12.1. Identify M with $f(M)$, so that f becomes an inclusion $M \subset W$, and let $M \subset M \times D^2 \subset W$ be a closed tubular neighborhood. Choose ε so that $d(x, y) < \varepsilon$ for $y \in M$ implies $x \in M \times D^2$. Further, by a standard argument, we may choose $\varepsilon > 0$ so that if $d(g(x), f(x)) < \varepsilon$ for all $x \in M$, then $g\colon M \to M \times D^2$ is homotopic to f. Then if $d(g(x), f(x)) < \varepsilon$, $g(M)$ will lie in $M \times D^2$, and $g\colon M \to M \times D^2$ will be homotopic to the zero section.

If n is even, then $C_{\mathrm{PL}}(M) = 0$, by Theorem 6.6. So in this case g is cobordant to the zero-section. It follows easily that g is *concordant* to $\psi \mid M$, where ψ is a P.L. homeomorphism of $M \times D^2$ that is homotopic to the identity, as a map of manifold pairs. If n is odd, Theorem 6.6 asserts that $\alpha\colon C_{\mathrm{PL}}(S^n) \to C_{\mathrm{PL}}(M)$, given by connected sum of the zero section with a knot, is an isomorphism. Hence ψ will still exist for a P.L. embedding obtained from g by connected sum with a suitable knot.

Theorem 12.1 now follows easily from

LEMMA 12.2. *Let* $\psi\colon (M \times D^2, M \times S^1) \to (M \times D^2, M \times S^1)$ *be a* P.L. *homeomorphism homotopic to the identity. Then* ψ *is pseudo-isotopic to* $\varphi \times \mathrm{id}_{D^2}$, *for* $\varphi\colon M \to M$ *a* P.L. *homeomorphism homotopic to the identity.*

Proof. By a well-known argument (compare 4.4), it suffices to prove (in the notation of [59, § 10]) that the map

$$\mathcal{S}_{\mathrm{PL}}(M \times I, M \times \partial I) \longrightarrow \mathcal{S}_{\mathrm{PL}}(M \times D^2 \times I, M \times D^2 \times \partial I)\ ,$$

given by taking the product with the identity on D^2, is a surjective map. Let $E = M \times D^2$. Then the following diagram with exact rows commutes:

$$\begin{array}{ccccccc} L_{n+2}(e) & \longrightarrow & (M \times I, M \times \partial I) & \longrightarrow & [\Sigma M; G/\mathrm{PL}] & \longrightarrow & L_{n+1}(e) \\ \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_3} & & \downarrow{\scriptstyle \gamma_4} \\ L_{n+4}(\mathbf{Z} \to e) & \longrightarrow & \mathcal{S}(E \times I, E \times \partial I) & \longrightarrow & [\Sigma E; G/\mathrm{PL}] & \longrightarrow & L_{n+3}(\mathbf{Z} \to e)\ , \end{array}$$

where the γ_i are all given geometrically by crossing with D^2. The rows can be found in [58, § 10], for example. The map γ_3 can also be described as the natural map induced by the projection of E on M, and so is an isomorphism. Thus, to prove 12.2, it will suffice to show that γ_1 and γ_4 are isomorphisms.

To see this, consider the exact sequence

$$0 \longrightarrow L_i(\mathbf{Z} \to e) \xrightarrow{\partial} L_{i-1}(\mathbf{Z}) \xrightarrow{\delta} L_{i-1}(e) \longrightarrow 0 ;$$

the long exact sequence for Wall groups [58, § 3] splits in this case for functorial reasons. If i is odd, δ is an isomorphism [46], so $L_i(\mathbf{Z} \to e) = L_{i-2}(e) = 0$. If i is even, then ∂ is an isomorphism, and $\partial\gamma: L_{i-2}(e) \to L_{i-1}(\mathbf{Z})$ is clearly the isomorphism of [46] given by crossing with a circle, so that γ is also an isomorphism.

13. Knottings of $S^n \times M$ in $S^{n+2} \times M$ and knot periodicity

Let M be an oriented, closed P.L. manifold. We consider locally flat P.L. embeddings $f: S^n \times M \to S^{n+2} \times M$, homotopic to the product of a map from S^n to S^{n+2} with the identity of M, i.e., in the "usual" homotopy class. Two such, f and g, will be called *conjugate* if there are P.L. homeomorphisms ψ of $S^n \times M$ with itself and φ of $S^{n+2} \times M$ with itself, satisfying the following conditions:

(i) $\pi_2\psi$ is homotopic[1] to π_2, π_2, the natural projection onto M;

(ii) φ is orientation preserving (we suppose S^{n+2} and S^n have been given fixed orientations once and for all) and $\pi_2\varphi$ is homotopic to π_2.

(iii) $f = \varphi g \psi$.

As usual, we say f and g are *concordant* if there is a locally flat embedding of $S^n \times M \times I$ in $S^{n+2} \times M \times I$ that restricts to f on one end and g on the other. We say f and g are *cobordant* if they are conjugate to concordant embeddings. The cobordism classes of embeddings f in the homotopy class under study will be denoted $G_n(M)$. Clearly $G_n = G_n(pt)$ (C_n^{PL} in the notation of Chapter II) is the group of cobordism classes of knotted n-spheres in S^{n+2}, studied in [23], [25], [31].

When M is a closed topological manifold, the topological cobordism classes $G_n^{\mathrm{TOP}}(M)$ may also be defined.

Let $\alpha(M, n): G_{n+k} \to G_n(M)$, $k = \dim M$, be defined by taking connected sum of a knot with the embedding $f_0 \times \mathrm{id}_M$, $f_0: S^n \to S^{n+2}$ an unknotted embedding, e.g., an equator. Similarly, $\alpha^{\mathrm{TOP}}(M, n)$ is defined.

Let $\delta(M, n): G_n \to G_n(M)$ be defined as follows: If $x \in G_n$ is represented

[1] It is not hard to show that replacing (i) by the condition that ψ be homotopic to the identity yields the same set of cobordism classes.

by $f: S^n \to S^{n+2}$, then $\delta(M, n)(x)$ is represented by $f \times \mathrm{id}_M$. Similarly, define $\delta^{\mathrm{TOP}}(M, n)$.

THEOREM 13.1. *Assume $n \geqq 2$ and let M be a closed, simply-connected oriented* P.L. *manifold. Then $\alpha(M, n)$ is a bijection.*

The arguments of [14] show that for $n + k > 3$, $G_{n+k}(M) = G_{n+k}^{\mathrm{TOP}}(M)$. Hence 13.1 remains valid in the topological[1] case for $n \geqq 3$, there being nothing to prove when $n = 3$ and $k = 0$.

Recall from [14] that the natural map $G_3 \to G_3^{\mathrm{TOP}}$ is a monomorphism whose image $G_3 \subset G_3^{\mathrm{TOP}}$ has index 2. Non-smoothable knots are detected by an index invariant along a "suspended Seifert surface". Let CP^2 be complex projective 2-space.

THEOREM 13.2. *For $n \geqq 2$, but $n \neq 3$, $\delta(CP^2, n)$ is a bijection. For $n = 3$, $\delta^{\mathrm{TOP}}(CP^2, n)$ is a bijection.*

COROLLARY 13.3. *For $n > 3$ or $n = 2$,*

$$\alpha(CP^2, n)^{-1}\delta(CP^2, n): G_n \longrightarrow G_{n+4}$$

is an isomorphism. For $n = 3$, it is a monomorphism onto a subgroup of index two that extends to the isomorphism

$$\alpha^{\mathrm{TOP}}(CP^2, n)^{-1}\delta^{\mathrm{TOP}}(CP^2, n) \text{ of } G_n^{\mathrm{TOP}} \text{ with } G_{n+4}^{\mathrm{TOP}} .$$

For $n \neq 3$, the Corollary follows immediately from the theorems; for $n = 3$ one also has to use [14].

Corollary 13.3 gives a geometric description of the periodicity of knot cobordism: take the product with CP^2 and push the resulting knot of $S^n \times CP^2$ in $S^{n+2} \times CP^2$ into a cell. Using ideas of [14], in particular the definition of a Seifert matrix using the "suspended Seifert surface", one can show that the periodicity isomorphisms of 13.3 coincide with the (algebraic) isomorphisms resulting from Levine's calculation of the groups G_n in terms of cobordism classes of Seifert matrices.

Theorem 13.2 generalizes to any M of index one. For index $M \neq 1$, however, $\delta(M, n)$ will in general not be an isomorphism.

Recall that $G_n = 0$ if n is even [25]; this has already been reproved by us in Chapter II and will also follow from the proof of 13.1.[2] In fact, we will give a description (13.10) of knot groups, different from [31], but essentially identical to that of § 6.

[1] Of course, this also follows directly from the methods of the present paper.

[2] Except that we will take the result for granted for $n = 2$, in order to avoid the special arguments needed for this case.

If we altered the definition of *conjugacy* in defining $G_n(M)$, to require φ and ψ to be homotopic to the identity, the theorems would no longer be valid. The difference is precisely measured by the image of $[M; \mathrm{PL}_{n+3}]$ in $[M; G_{n+3}]$, G_{n+3} a component of $(S^{n+2})^{S^{n+2}}$.

To prove the theorems, let $f: S^n \times M \to S^{n+2} \times M$, M a closed simply-connected P.L. manifold, represent an element of $G_n(M)$.

LEMMA 13.4. *The normal bundle of f is trivial and has a unique trivialization (up to isotopy of bundle maps), determined by the orientations.*

Proof. Let T be a tubular neighborhood of $f(S^n \times M)$. Let W be the closure of $S^{n+2} \times M - T$. By Alexander duality, $H^1(W) = \mathbf{Z}$, and the orientations of $S^n \times M$ and $S^{n+2} \times M$ (and a choice of convention, fixed once and for all) determine a generator of $H^1(W)$. Let $\lambda: W \to S^1$ represent this generator. Since a fiber of T meets $f(S^n \times M)$ transversely in one point, it follows from Alexander duality that the restriction of λ to a fiber of the circle bundle ∂T is a homotopy equivalence. This implies that the normal bundle of f is fiber homotopically trivial, and hence trivial. The uniqueness follows from $H^1(S^n \times M) = 0$.

Thus there is an extension $\bar{f}: S^n \times M \times D^2 \to S^{n+2} \times M$ of f, unique up to ambient isotopy. Let $T = \bar{f}(S^n \times M \times D^2)$, with W the closure of its complement. Let $\pi_1: S^n \times M \times D \to S^n$ be the natural projection. The composite $\pi_1(f^{-1} \mid \partial T)$ obviously extends, uniquely up to homotopy, to $\omega: W \to D^{n+1}$. Define $F: W \to D^{n+1} \times M \times S^1$ by $F = (\omega k, \lambda)$, $k \mid \partial T = \pi_2 \bar{f}^{-1}$ and k homotopic to $\pi_2 \mid W$ (by homotopy extension), λ as in the proof of 13.4; we may also suppose $\lambda \bar{f} \mid S^n \times M \times S^1$ is the natural map. F will be called a *complementary map* for f.

Write $S^{n+2} \times M = (S^n \times M \times D^2) \cup (D^{n+1} \times M \times S^1)$, the standard decomposition, arising from the standard decomposition of S^{n+2} as a neighborhood of an unknotted n-sphere and its complement. We may extend F to $\hat{F}: S^{n+2} \times M \to S^{n+2} \times M$ by putting $\hat{F}(\bar{f}(x, y, z)) = (x, y, z)$ on T. We call $\hat{F}$ a *characteristic* map for f_i; it is unique up to homotopy and up to isotopy on T.

Note that the complementary and characteristic maps of a concordance can also be defined, and a relative version of the preceding discussion yields:

LEMMA 13.5. *Characterstic maps of concordant embeddings of $S^n \times M$ in $S^{n+2} \times M$ are homotopic via a homotopy that is a characteristic map of a concordance between them.*

LEMMA 13.6. *Let $f: S^n \times M \to S^{n+2} \times M$ be an embedding (in the usual homotopy class). Then the complementary map F is a homology equivalence,* *and the characteristic map* $\hat{F}$ *is a homotopy equivalence homotopic to a* P.L. *homeomorphism,* $n \geqq 2$.

Proof. The assertion about the complementary map follows easily from Alexander duality. That $\hat{F}$ is a homotopy equivalence then follows from standard arguments.

To prove the assertion about the characteristic map, it suffices [3], [51], [58] to show that it has vanishing normal invariant in $[S^{n+2} \times M; G/\mathrm{PL}]$, unless $n + k = 2$; in this case $\hat{F}$ is a degree one map of S^4 to itself and so homotopic to the id_{S^4}. Since the restriction $\hat{F}$ to $\hat{F}^{-1}(S^n \times M)$ is homotopic to a P.L. homeomorphism, the restriction of the normal invariant of $\hat{F}$ to $z \times M$, $z \in S^{n+2}$, will be trivial. On the other hand, $\hat{F}$ commutes with projection π_2 onto M, and hence extends to a map $G: D^{n+3} \times M \to D^{n+3} \times M$, easily seen to have degree one and hence to be a homotopy equivalence. In particular, the normal invariant of $\hat{F}$ is the restriction of an element in $[D^{n+3} \times M; G/\mathrm{PL}]$; hence it must be trivial.

Now let $f: S^n \times M \to S^{n+2} \times M$ represent an element $x \in G_n(M)$ and let $F: W \to D^{n+1} \times M \times S^1$ be a complementary map, defined using $\lambda: W \to S^1$ whose restriction to ∂T is the canonical projection, T a tubular neighborhood of the image of f. After a homotopy relative boundary, we may move F to F_1 transverse regular to $D^{n+1} \times M = D^{n+1} \times M \times z$, $z \in S^1$. Let $V = F_1^{-1}(D^{n+1} \times M)$. It is easy to see that there is a canonical bundle map (which we omit from the notation) covering $F_1 \mid V: V \to D^{n+1} \times M$, a degree one map inducing a homotopy equivalence of boundaries. We define $\rho(x) = \sigma(F_1 \mid V) \in L_{n+k+1}(e)$, $k = \dim M$.

Alternatively, $\rho(x)$ may be described as the image of $\sigma(F) \in L_{n+k+2}(\mathbf{Z})$ under the map $L_{n+k+2}(\mathbf{Z}) \to L_{n+k+1}(e)$ defined in [46]. This description (or a simple direct argument) makes it apparent that $\rho(x)$ is actually a well-defined invariant of the cobordism class x of f. For the case $n = 3$, $k = 0$, we may take this description as the definition of $\rho: G_3^{\mathrm{TOP}} \to L_4(e)$.

LEMMA 13.7. *Every element of* $G_n(M^k)$, $n \geqq 2$, *has a representative for which the complementary map induces an isomorphism of fundamental groups.*

For $n = 2$ and $k = 0$ this follows from the vanishing of G_2. In the case $n + k \geqq 3$, this result is proved quite like 4.3; hence the proof is omitted. We actually use this lemma only for technical convenience.

Next we define the action of $\Gamma_{n+k+3}(\mathcal{F})$ on $G_n(M^k)$, $\mathcal{F}: \mathbf{Z}[\mathbf{Z}] \to \mathbf{Z}$ the augmentation. Let $f: S^n \times M \to S^{n+2} \times M$ represent $x \in G_n(M^k)$, and let $F: W \to D^{n+1} \times M \times S$, which we assume to induce an isomorphism of fundamental groups, be a complementary map for f. Let $\gamma \in \Gamma_{n+k+3}(\mathcal{F})$. Clearly F is covered by a canonical bundle map, B say, the restriction of the one that covered $\hat{F}$. By the realization Theorems 1.8 and 2.2, and addenda, there exists a normal cobordism (h, b), relative boundary of (F, B) to (F', B'), $F'\colon W' \to D^{n+1} \times M \times S^1$ a homology equivalence inducing isomorphisms of fundamental groups. Since $\mathrm{id}_T \cup F'\colon T \cup_{\partial T} W' \to S^{n+2} \times M$ is a homotopy equivalence normally cobordant to the characteristic map $\hat{F}$, it has vanishing normal invariant, by 13.6. (Recall W was the closure of the complement of T). Hence $\mathrm{id}_T \cup F'$ is homotopic to a P.L. homeomorphism, g say, [3], [51]. We define $\gamma \cdot x$ to be the class of the composite

$$S^n \times M \xrightarrow{f} T \xrightarrow{g|T} S^{n+2} \times M .$$

It clearly does not depend on the choice of g.

To see that $\gamma \cdot x$ does not depend on the choice of normal cobordism (h, b) with $\sigma(h, b) = \gamma$, let (h_1, b_1) be another such, of (F, B) to $f_1'\colon W_1' \to D^{n+1} \times M \times S^1$. Let $c(v, t) = (v, -t)$, $v \in D^{n+1} \times M \times S^1$, $t \in I$. Let $(k, c) = (h, b) \cup (ch_1, b_1)$, a normal map from $V \cup_W V_1$ to $D^{n+1} \times M \times S^1 \times [-1, 1]$, where V and V_1 are the domains of h and h_1, respectively. By additivity, $\sigma(k, c) = 0$. Hence (k, c) is normally cobordant, relative boundary, to a homology equivalence,

$$H\colon Z \longrightarrow S^1 \times D^{n+1} \times M \times [-1, 1] ,$$

which induces an isomorphism of fundamental groups, by 1.7 or 2.1.

Let $R = T \times [-1, 1] \cup_{\partial T \times [-1,1]} Z$. Then $\hat{H} = H \cup \mathrm{id}_{T \times [-1,1]}$ is easily seen to be a homotopy equivalence, as R is an h-cobordism; R is easily seen to be simply-connected. Hence, by the h-cobordism theorem, there exists a P.L. homeomorphism $\beta\colon (T \cup W') \times [-1, 1] \to R$ with $\beta(y, 1) = y$ for $y \in T \cup W'$. Consider the composite

$$S^n \times M \times [-1, 1] \xrightarrow{f \times \mathrm{id}} T \times [-1, 1] \subset R \xrightarrow{\beta^{-1}} (T \cup W') \times [-1, 1]$$
$$\xrightarrow{g \times \mathrm{id}} S^{n+2} \times M[-1, 1] ,$$

a concordance. On $S^n \times M \times 1$, it evidently restricts to $(g \mid T) \circ f$, which is $\gamma \cdot x$ defined using (h, b). The composite $\hat{H}\beta$ is a homotopy of g with $(\mathrm{id}_T \cup F_1') \circ \beta \mid (T \cup W') \times (-1)$. Hence the above concordance on $S^n \times M \times -1$ is conjugate to the representative for $\gamma \cdot x$ defined above using (h_1, b_1); i.e., the two definitions are cobordant.

Using Lemma 13.5 and what has just been shown, it is easy to see that the cobordism class obtained is unchanged when f is altered by a concordance. It is easy to check that the cobordism class obtained remains unchanged when f is replaced by a conjugate embedding by considering the obvious relation of the complementary maps of conjugate embeddings. We leave the details to the reader.

As usual, additivity of surgery obstructions implies that $(\gamma + \gamma')\cdot(x) = \gamma\cdot(\gamma'\cdot(x))$. Also $0\cdot x = x$; this uses the fact that the characteristic map commutes up to homotopy with π_2, so that composition with a P.L. homeomorphism homotopic to it does not alter the cobordism class.

The next proposition is proved quite like 6.4.

PROPOSITION 13.8. *Let* $\gamma \in \Gamma_{n+k+3}(\mathcal{F})$; *let* $\xi \in G_{n+k}$. *Then*

$$\alpha(M^k, n)(\gamma\cdot\xi) = \gamma\cdot\big(\alpha(M^k, n)(\xi)\big) .$$

More generally, if $\eta \in G_n(M^k)$,

$$\gamma\cdot(\xi \# \eta) = (\gamma\cdot\xi) \# \eta = \xi \# (\gamma\cdot\eta) .$$

(Connected sum is denoted by "#".)

PROPOSITION 13.9. *Let* $\xi \in G_{n+k}$. *Then*

$$\rho\big(\alpha(M^k, n)\xi\big) = \rho(\xi) .$$

More generally, if $\eta \in G_n(M^k)$,

$$\rho(\xi \# \eta) = \rho(\xi) + \rho(\eta) .$$

Proof. It is easy to see that $\rho(\xi \# \eta)$ can be computed as a surgery obstruction of the boundary connected sum of a Seifert surface of a knot representing η, equipped with a normal map into a disk, with a normal map whose surgery obstruction is $\rho(\xi)$. The result follows from additivity of simply-connected surgery obstructions [3].

THEOREM 13.10. *Let* $n \geqq 2$. *Then the action of* $\Gamma_{n+3+k}(\mathcal{F})$ *on* $G_n(M^k)$ *induces a free action of* $\widetilde{\Gamma}_{n+k+3}(\mathcal{F})$. *Unless* $n = 3$ *and* $k = 0$, ρ *induces a bijection of the orbit set with* $L_{n+k+1}(e)$.

For $n = 3$ *and* $k = 0$, ρ *induces a bijection of the orbit set with even elements in* $L_4(e)$, *and extends to a bijection of the orbit space of* G^{TOP}, *under a similar action, with* $L_4(e)$.

Proof. For $n = 2$, $k = 0$, $G_2 = \Gamma_5(\mathcal{F}) = L_3(e) = 0$, so there is nothing to prove. So assume $n + k \geqq 3$.

That the image of ρ is as claimed follows from 13.9 and the fact (see for example [31]) that the image of ρ is as asserted for M a point. Here is an outline of a direct proof of this fact: By crossing with a circle, $L_{n+k+1}(e)$ can be viewed as the kernel of the natural map from $L_{n+k+2}(Z)$ to $L_{n+k+2}(e)$ [46]. By § 2, the realization theorem for surgery obstructions [58], and the fact that any homotopy $S^1 \times S^m$ is P.L. homeomorphic to $S^1 \times S^m$, $m \geqq 3$, [46], [38], any element in this kernel is the obstruction $\sigma(h, b)$, where

$$h\colon W \longrightarrow S^1 \times D^{n+k+1}$$

is a homology equivalence that induces an isomorphism on π_1 and a P.L. homeomorphism of boundaries. To obtain a knot representing an element in G_{n+k} with image $\sigma(h, b)$ under ρ, just attach $S^{n+k} \times D^2$ to W using $h \mid \partial W$.

The statement about G_3^{TOP} follows from the rest of the theorem and [14]; the extension of the action of $\Gamma_6(\mathcal{F})$ to G_3^{TOP} is straightforward, using topological surgery techniques [30], [27, 2].

Let $x \in G_n(M^k)$, and suppose to begin with, that $\rho(x) = 0$. Let $f\colon S^n \times M \rightarrow S^{n+2} \times M$ be a representative embedding, with a characteristic map $\hat{F}$. Let $H\colon S^{n+2} \times M \times I \rightarrow S^{n+2} \times M \times I$ be a homotopy of $\hat{F}$ to P.L. homeomorphism, g say, by 13.6. Note that g commutes with π_2 up to homotopy, so that $g^{-1} \mid S^n \times M\colon S^n \times M \rightarrow S^{n+2} \times M$, the restriction of g^{-1} to the standard inclusion, represents the same element as the standard inclusion.

Let D_0^{n+1} be a disk bounding the standard $S^n \subset S^{n+2}$, obtained from $D^{n+1} \times z \subset D^{n+1} \times S^1 \subset S^{n+2}$ by adding an extra collar of the boundary. We may suppose H transverse to $D_0^{n+1} \times M \times I$, without changing it on the boundary. Let $V = H^{-1}(D_0^{n+1} \times M \times I)$. Then (omitting bundle maps) we have a normal map

$$H \mid V\colon V \longrightarrow D_0^{n+1} \times M \times I\,.$$

The sum of the surgery obstructions of the various boundary components must vanish [3], [59, § 3,4]. Hence if $U = H^{-1}(S^n \times M \times I)$,

$$\sigma(H \mid U) + \rho(x) = 0\,.$$

Hence $\sigma(H \mid U) = 0$.

The "cobordism extension theorem" implies that, after changing H by a normal cobordism relative the boundary we may suppose $H \mid U\colon U \rightarrow S^n \times M \times I$ is a homotopy equivalence[1]; i.e., U is an h-cobordism, and so a product. Further, we may assume H is a bundle map on a neighborhood $R(U)$ of U such that H maps the closure of its complement, Q say, into $D^{n+1} \times M \times S^1$. But then $H \mid Q\colon Q \rightarrow D^{n+1} \times M \times S^1$ is a normal map that induces a homology equivalence of boundaries. Hence $\sigma(H) \in \Gamma_{n+k+3}(\mathcal{F})$ and it is quite easy to check that $x = \sigma(H)\cdot x_0$, where x_0 is represented by the standard embedding.

(*Note*: The fact that we had to use the h-cobordism theorem on U in

[1] For the case $n = 3$, $k = 0$ we observe that the surgery obstruction map $[\Sigma S^3; G/\text{PL}] \rightarrow L_4(e)$ is a monomorphism, so that its vanishing implies U is normally cobordant to a product $S^3 \times I$, relative boundary.

the last paragraph is what requires the introduction of the P.L. homeomorphism ψ in the definition of conjugacy.)

Now suppose $\rho(x) = \rho(x')$. Choose $\eta \in G_{n+k}$ with $\rho(\eta) = -\rho(x)$. Then $\rho(x \# \eta) = \rho(x' \# \eta) = 0$, so by what has just been shown, $x \# \eta$ and $x' \# \eta$ are in the same orbit; let $\gamma \in \Gamma_{n+k+3}(\mathcal{F})$ with $\gamma(x \# \eta) = x' \# \eta$. Let δ be the inverse of η in G_{n+k}. Applying 13.9, and associativity of $\#$,

$$\gamma \cdot x = (\gamma \cdot x) \# (\rho \# \delta) = \big(\gamma \cdot (x \# \eta)\big) \# \delta = (x' \# \eta) \# \delta = x' .$$

Thus x and x' are in the same orbit.

We leave the converse, that $\rho(\gamma \cdot x) = \rho(x)$, as an easy exercise in transversality.

Now consider the action of $\Gamma_{n+k+3}(\mathcal{F})$ on $G_n(M^k)$. It follows from the generalized Poincaré conjecture that the image of $L_{n+k+3}(e)$ acts trivially (compare [51], [3], for example). Since for $n + k + 3$ even the natural map $L_{n+k+3}(e) \to L_{n+k+3}(\mathbf{Z})$ is an isomorphism [46], and for $n + k + 3$ odd $\Gamma_{n+k+3}(\mathcal{F}) \subset L_{n+k+3}(e) = 0$, we have an induced action of $\widetilde{\Gamma}_{n+k+3}(\mathcal{F})$.

Suppose γ acts trivially on x_0, the class represented by the usual embedding. Then there is a normal cobordism with obstruction γ from the identity of $D^{n+1} \times M \times S^1$, relative boundary, to a homology equivalence $F: W \to D^{n+1} \times M \times S^1$ that is the complementary map to an embedding f representing x_0. Using the fact that f is cobordant to the usual embedding, it is easy to see that F is homology h-cobordant to the identity of $D^{n+1} \times M \times S^1$; gluing on this homology h-cobordism leads to a normal map Ξ, with obstruction γ, that induces a *homotopy* equivalence of boundaries. Clearly γ is the image of the usual surgery obstruction of Ξ in $L_{n+k+3}(\mathbf{Z})$ under the natural map. Thus $\widetilde{\Gamma}_{n+k+3}(\mathcal{F})$ acts freely on the orbit of x_0.

Let $x \in G_n(M^k)$, with $\gamma \cdot x = x$, $\gamma \in \Gamma_{n+k+3}(\mathcal{F})$. Let $\eta \in G_{n+k}$, with $\rho(\eta) = -\rho(x)$, so that $\rho(x \# \eta) = 0$, by 13.9. By 13.8, $\gamma(x \# \eta) = x \# \eta$. By what we have already shown, $x \# \eta$ is in the orbit of x_0. Hence γ represents zero in $\widetilde{\Gamma}_{n+k+3}(\mathcal{F})$, by the preceding argument. This completes the proof of 13.10.

COROLLARY 13.11. *$G_n(M^k) = 0$ if $n + k$ is even, $n \geqq 2$.*

Proof. $\Gamma_{n+k+3}(\mathcal{F}) = L_{n+k+1}(e) = 0$ in this case.

Proof of 13.1. We may assume $n + k \geqq 3$. To show that $\alpha(M, n)$ is onto, let $x \in G_n(M^k)$. Then, by 13.10, there exists $\eta \in G_{n+k}$ with $\rho(\eta) = -\rho(x)$, so by 13.9 $\rho(x \# \eta) = 0$. Hence by 13.10 $x \# \eta = \gamma \cdot x_0$, for some $\gamma \in \Gamma_{n+k+3}(\mathcal{F})$, x_0 the class of the usual embedding. Hence $x \# \eta = \alpha(M, n)(\xi)$, $\xi = \gamma\eta_0$, η_0 the trivial element of G_{n+k}. If δ is the inverse η in the group G_{n+k}, then clearly $x = \alpha(M, n)(\xi \# \delta)$.

Suppose $\alpha(M, n)\eta = \alpha(M, n)\xi$, η, $\xi \in G_{n+k}$. By 13.9, $\rho(\eta) = \rho(\xi)$. So $\eta = \gamma\cdot\xi$, by 13.10, some $\gamma \in \Gamma_{n+k+3}(\mathcal{F})$. By 13.8,

$$\alpha(M, n)\eta = \gamma(\alpha(M, n)\xi)\ .$$

Hence, by 13.10, γ represents zero in $\widetilde{\Gamma}_{n+k+3}(\mathcal{F})$, which implies that $\eta = \xi$.

Proof of 13.2. Theorem 13.2 follows directly from 13.10, 3.5, and the following two simple observations

(i) If $\gamma \in \Gamma_{n+3}(\mathcal{F})$ and $\eta \in G_n$, then $\delta(M, n)(\gamma\cdot\eta) = \bar{\gamma}\cdot(\delta(M, n)\eta)$, where in the right side, $\bar{\gamma}$ is the element $\gamma \times M \in \Gamma_{n+k+3}(\mathcal{F})$ obtained by taking the product of a suitable normal map with M;

(ii) $\rho(\delta(CP^2, n)\eta) = \rho(\eta)$ for $\eta \in G_n$; this uses periodicity of simply connected surgery obstructions [3], [51], [59]; note that the topological analogue of (ii) also holds for $n = 3$, $k = 0$.

Appendix I: An exact sequence

We have a natural map $\Gamma_n^s(\mathcal{F}) \to \Gamma_n^h(\mathcal{F})$, forgetting preferred bases. Let $A_n(\pi', w')$ be as in [46, § 4]. Composing the natural map $\Gamma_n^h(\mathcal{F}) \to L_n^h(\pi', w')$ with a suitable map in the Rothenberg sequence [46, 4.1] gives a map $\Gamma_n^h(\mathcal{F}) \to A_n(\pi', w')$. Finally, it is not hard to find a natural lift of the map $A_n(\pi', w')$ of the Rothenberg sequence to a map $A_n(\pi', w') \to \Gamma_{n-1}^s(\mathcal{F})$. For n even, $\Gamma_{n-1}^s(\mathcal{F}) \subset L_{n-1}^s(\pi', w')$, so it suffices to see, in this case, that the image of the map of 4.1 of [46] lies in $\Gamma_{n-1}^s(\mathcal{F})$; but this follows easily from the definition of [46] and the fact that a matrix over $\mathbf{Z}\pi'$ lifts to $\mathbf{Z}\pi$. For n odd, one uses elements of Wh (π') to change the preferred base of $\kappa_r \otimes \mathbf{Z}\pi'$, κ_r a kernel (of dimension $2r$) over $\mathbf{Z}\pi$, analogous to the definition for the case in [46, 4.1].

PROPOSITION. *The sequence*

$$\longrightarrow \Gamma_n^s(\mathcal{F}) \longrightarrow \Gamma_n^h(\mathcal{F}) \longrightarrow A_n(\pi', w') \longrightarrow \Gamma_{n-1}^s(\mathcal{F}) \longrightarrow$$

is exact.

This is not hard, using [46, 4.1]. At certain points, one uses arguments quite similar to the proof of 4.1 in [46]; at others, including exactness at $A_{2k}(\pi', w')$, one uses the exactness of the Rothenberg sequence and the commutativity of

$$\begin{array}{ccccccc} & \nearrow & \Gamma_n^s(\mathcal{F}) & \longrightarrow & \Gamma_n^h(\mathcal{F}) & \searrow & \\ A_{n+1}(\pi', w') & & \downarrow j_* & & \downarrow j_* & & A_n(\pi', w')\ . \\ & \searrow & L_n^s(\pi', w') & \longrightarrow & L_n^h(\pi', w') & \nearrow & \end{array}$$

We omit the details.

For example, if $\pi' = \mathbf{Z}_s$ is cyclic and w' is trivial, then $A_n(\pi', w') = 0$ for n odd, by 7.5 ff of Chapter 11 of [1] and 6.7 of [40]. Hence if $\pi = \mathbf{Z}$ and $\mathcal{F}$ is the natural map, $\Gamma^s_{2k}(\mathcal{F}) \to \Gamma^h_{2k}(\mathcal{F})$ is a monomorphism. Using the exact sequence 3.2, this implies the following, which was used in § 10.

COROLLARY. *Let ϕ be the square*

$$\begin{array}{ccc} \mathbf{Z}[\mathbf{Z}] & \xrightarrow{\text{id}} & \mathbf{Z}[\mathbf{Z}] \\ \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \mathcal{F}} \\ \mathbf{Z}[\mathbf{Z}] & \xrightarrow{\mathcal{F}} & \mathbf{Z}[\mathbf{Z}_s] \end{array},$$

$\mathcal{F}$ the natural map. Then $\Gamma^s_{2k}(\phi) \to \Gamma^h_{2k}(\phi)$, the natural homomorphism, is a monomorphism.

Appendix II: "Cracking"

This appendix gives an algebraic analogue and extension of the "cracking" process of S. Lopez de Medrano [36] for Z_2-actions (i.e., $\pi = Z_2$ in Theorem A.1). This result is used to prove a general codimension two splitting theorem of the case of cyclic fundamental groups, Corollary 8.4 above.

LEMMA. *Let π be a cyclic group generated by t, with $|\pi| \not\equiv 1 \pmod 4$. Then the ideal generated by 5 and $(t-3)$ in the group ring $\mathbf{Z}\pi$ is all of $\mathbf{Z}\pi$.*

Proof. Let $n = |\pi|$. We wish to solve

$$5\alpha + (t-3)\beta = 1.$$

Let $\alpha = a_0 + a_1 t + \cdots + a_{n-1}t^{n-1}$, $\beta = b_0 + \cdots + b_{n-1}t^{n-1}$. Then we must solve the system

$$\begin{aligned} 5a_0 - 3b_0 + b_{n-1} &= 1 . \\ 5a_j - 3b_j + b_{j-1} &= 0 , \qquad 1 \leqq j \leqq n-1 . \end{aligned}$$

First let $a_j = 0$, $1 \leqq j \leqq n-1$, so that $b_{j-1} = 3b_j$. Then the first equation becomes

$$5a_0 - (3^{n-1} - 1)b_{n-1} = 1 ,$$

which has an integral solution as 5 and $3^{n-1} - 1$ are relatively prime for $n \not\equiv 1 \pmod 4$.

THEOREM A.1. *Let G be infinite cyclic with generator T and π of even order with generator t. Let $\mathcal{F}: Z[G] \to Z[\pi]$ be induced by $\mathcal{F}[T] = t$. Let $w: \pi \to \{\pm 1\}$ with $w(t) = -1$, and let $Z[G]$ and $Z[\pi]$ have the involutions determined by $w \circ \mathcal{F}$ and w, respectively. Then the natural map*

$$L^e_{4k}(G, w \circ \mathcal{F}) \longrightarrow \Gamma^e_{4k}(\mathcal{F})$$

is trivial, $e = s, h$.

Proof. We need only consider the case $e = s$. $L_{4k}(G, w \circ \mathcal{F})$ is isomorphic to Z_2, generated by (H, ϕ, μ), where μ has matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix},$$

the Milnor matrix, with respect to a basis $e_1, \cdots, e_8$, say. Following Lopez de Medrano, let

$$e_5' = -e_1 + 2e_2 - 3e_3 + 4e_4 - 5e_5 + 4e_6 - 2e_7 + 3e_8 .$$

Then let $K \subset H$ be spanned by the elements

$$Te_1 + e_7,\ Te_2 + e_6,\ Te_3 + e_5',\ Te_4 + e_8 .$$

One checks easily that ϕ and (hence) μ vanish on K. For example,

$$\begin{aligned} &\phi(Te_2 + e_6, Te_3 + e_5') \\ &\quad = -\phi(e_2, e_3) + T\phi(e_2, e_5') - T^{-1}\phi(e_3, e_5') + \phi(e_6, e_5') \\ &\quad = -1 + 1 = 0 . \end{aligned}$$

On the other hand, by the lemma, there exist $\alpha, \beta \in Z\pi$ so that the matrix

$$\begin{pmatrix} t & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & t & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 2 & t-3 & 4 & -5 & 4 & -2 & 3 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 1 \\ 0 & 0 & \beta & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

will have determinant one. Hence, using the fact that $SK_1(Z\pi) = 0$ [1], K is a pre-subkernel of (H, ϕ, μ), viewed as a form over $\mathcal{F}$, which proves the theorem.

PRINCETON UNIVERSITY
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