A Geometric Interpretation of Siebenmann's Periodicity Phenomenon

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1. STATEMENT OF MAIN THEOREM

If M is a manifold, $\mathscr{S}(M)$, the structure set of M is defined, following Sullivan and Wall [11], as the set of pairs (N,f) consisting of a manifold N and a simple homotopy equivalence $f : N \to M$ that restricts to a homeomorphism on the boundary. Two pairs, (N_1, f_1) and (N_2, f_2), represent the same element if there is a homeomorphism $H : N_1 \to N_2$ such that $f_1 \sim f_2 H$ rel ∂. One of the most beautiful results in the theory is Siebenmann's periodicity theorem that $\mathscr{S}(M) \cong \mathscr{S}(M \times D^4)$ for most M, e.g. all M with nonempty boundary (see [7] and the paper of Nicas [4] for a correction). Siebenmann's proof was rather indirect; it proceeded by constructing a simplicial set whose π_0 was $\mathscr{S}(M)$ and which was the fiber of a fibration which had periodicity properties. While this is enough for Siebenmann's (and others') applications, (such as providing a group structure on $\mathscr{S}(M)$ via the obvious one on $\mathscr{S}(M \times D^4)$ analogous to the definition of π_4), its indirectness is mysterious (see [7]). (This map is <u>not</u> just crossing with D^4; one does not then get a homeomorphism on the boundary.) In this paper we shall give a geometrically defined map $\mathscr{S}(M) \to \mathscr{S}(M \times D^4)$ (or actually to $\mathscr{S}(E)$ for a class of total spaces of four-plane bundles) that has all the properties of Siebenmann periodicity, by means of embedding theory. In addition to whatever aesthetic advantages there may be in the gained geometricity, there is also at least one practical pay-off. As an application we shall show that many homotopy $\mathbb{C}P^n$'s have locally smooth S^1-actions.

Before proceeding it is important to note that periodicity is very much a topological phenomenon that reflects both the periodicity of surgery groups $L_n(\pi) = L_{n+4}(\pi)$ and of the classifying space $\mathbb{Z} \times G/Top \approx \Omega^4 G/Top$. (The $\mathbb{Z}$ factor accounts for the exceptions to periodicity among closed manifolds.) In particular, it fails even in the PL category since G/PL has one twisted k-invariant at the prime 2. On the other hand, that is the whole (small) difference. Moreover, the tools that we will use include embedding theory, block bundles, transversality to subpolyhedra which are only available in the PL category. Nonetheless, the experts should be able to use mapping cylinders, approximate fibrations, and torus tricks to make these ideas (although not the details) work topologically.

Our construction is an analogue of the classical notion of the branched cyclic cover of a manifold W branched over (or along) a codimension two (locally flat) submanifold M. Here one starts with a k-fold cyclic regular cover on W-M. The well known fact is that if the restriction of the cover to a circle meridionally linking M is the usual $z \to z^k$ cover of S^1 to itself, then there is a canonical manifold compactification of the cover, obtained by appropriately "filling in" M. The covering translates extend to a $\mathbb{Z}_k$ action on the whole branched cover with fixed set M, and the quotient is W. (One way to do this is to remove the interior of any regular neighborhood of M; the boundary has the structure of an S^1-block bundle. The k-fold cover restricts to a k-fold cover on each block resulting in a new S^1-block bundle over M. One can now (inductively over a triangulation, as usual) cone down to obtain a manifold. The key to doing this is that the total space of the fibration restricted to a (linking meridinal) sphere is again a sphere.

There are, of course, other fibrations over spheres with total spaces spheres - namely the Hopf fibrations $S^1 \to S^3 \to S^2$ and $S^3 \to S^7 \to S^4$ and one can use these to get notions of branched fibrations. Only the first is relevant to this paper (although everything we say applies equally to the second). To repeat, in other words, whenever $M \subset W$ is codimension three and one has a principal S^1-bundle on W-M which restricts to the Hopf bundle on a linking S^2 (i.e., the first Chern class evaluated on $[S^2]$ is ± 1) then the total space can be canonically compactified by "filling in" M in such a way that M is now a codimension four submanifold, and the S^1 action (fiberwise, by rotation) on the (open) principal bundle piece extends to one on the compactification with M as fixed set. We call this space the branched S^1-fibration of W along M.

One way to analyze branched S^1-fibrations is to make use of the S^1 action and the relations that exist between (Mischenko-Ranicki symmetric or higher) signatures of a manifold and the fixed set of any S^1 action on it. (see [11]). There are unfortunately some thorny transversality problems which make the discussion at the prime 2 complicated. We shall be slightly more indirect, motivated by [2].

We are now ready to give the construction of the periodicity map. The fundamental theorems of embedding theory (see eg. [10, §11]) reduce PL embedding problems in codimension at least three to problems in the homotopy category; in particular, if $(N,f) \in \mathcal{S}(M)$ then one can homotop the composite $i \circ f : N \xrightarrow{f} M \xrightarrow{i} M \times D^3$ to an embedding. Since $i \circ f$ and i are homotopic the S^1-branched fibrations of $M \times D^3$ along them are homotopy equivalent, and since the only difference between the branch fibrations is what goes on in the neighborhood of $M \times \{0\}$ in which N embeds, there is no change on the boundary. It is an enlightening exercise in Whitehead torsions and this construction to calculate the torsion of the homotopy equivalence between branched S^1-fibrations in terms of $\pi(f)$. Since f is simple, the branched fibrations are in fact simple homotopy equivalent and one obtains an element of $\mathcal{S}(M \times D^4)$.

THEOREM. Branched S^1-fibrations define a homomorphism so that one has an exact sequence if $\dim M \geq 5$ or if $\pi_1 M$ is "small" ([3]) and $\dim M = 4$

$$0 \to \mathcal{S}(M) \to \mathcal{S}(M \times D^4) \to L_0(0) = \mathbb{Z}\ .$$

The last arrow is trivial if $\partial M \neq \emptyset$.

This interprets the corrected formulation of [7] given in [4].

2. PROOF OF THEOREM

An embedding, as in §1, $e : M' \to M \times D^3$, which is a simple homotopy equivalence, gives rise to, by consideration of the complement of the interior of a regular neighborhood of $e(M')$, an s-cobordism between $\partial(M \times D^3) = M \times S^2$ and the boundary of that regular neighborhood. (This equation should be reinterpreted in the obvious way if M itself has boundary.) S-cobordisms have product structures and boundaries of regular neighborhoods have block (sphere) bundle structures. As a result $M \times S^2$ has been canonically given the structure of an S^2-block bundle over M'. One can now take fiberwise Hopf bundles to produce $\mathring{\mathbb{C}P}^2$ block bundle structure over M' on $M \times \mathring{\mathbb{C}P}^2$ (where $\mathring{\mathbb{C}P}^2$ is the complex projective plane with an open disk removed). Let us summarize this in the homotopy commutative diagram

$$\begin{array}{ccc} E(\mathring{\mathbb{C}P}^2) & \xrightarrow{\text{homeo}} & M \times \mathring{\mathbb{C}P}^2 \\ \downarrow & & \downarrow \\ M' & \xrightarrow{h} & M \end{array}$$

(A symbol E(F) is the total space of an F-block bundle.) Now glue in the D^4 bundle over M' constructed in §1 to $E(\mathring{\mathbb{C}P}^2)$ and $M \times D^4$ to $M \times \mathring{\mathbb{C}P}^2$ (recall that these bundles are simple homotopy equivalent rel a homeomorphism on the boundary). This produces a diagram

$$\begin{array}{ccc} E(\mathbb{C}P^2) & \longrightarrow & M \times \mathbb{C}P^2 \\ \downarrow & & \downarrow \pi \\ M' & \longrightarrow & M \end{array}$$

Using this we shall verify injectivity of the branched cover map $\mathcal{S}(M) \to \mathcal{S}(M \times D^4)$. Very similar considerations yield the whole result.

Therefore, suppose now that $[h] \in \mathcal{S}(M)$ vanishes in $\mathcal{S}(M \times D^4)$, then the D^4 bundle over M' is homotopic rel ∂ to a homeomorphism to $M \times D^4$ so that $E(\mathbb{C}P^2) \to M \times \mathbb{C}P^2$ is homotopic (even rel $E(\mathring{\mathbb{C}P}^2)$) to a homeomorphism. We now essentially consider the block fibering obstructions [6] [1] for $M \times \mathbb{C}P^2 \xrightarrow{\pi} M \xrightarrow{h^{-1}} M'$ to deduce first that M' is normally cobordant to M.

Notice that $M \times \mathbb{C}P^2$ over M' is fiber homotopy trivial. Let $K \subset M'$ be part of a characteristic variety [8] for M'. Assume $h^{-1} \pitchfork K$. Now consider $(h^{-1}\pi)^{-1}K \to K \times \mathbb{C}P^2$. Since $h^{-1}\pi$ is homotopic to a block fibration the surgery obstruction is zero. On the other hand $(h^{-1}\pi)^{-1}K = \pi^{-1}(h^{-1})^{-1}K$ $= ((h^{-1})^{-1}K) \times \mathbb{C}P^2$ so the surgery obstruction is that of $(h^{-1})^{-1}K \to K$, which must therefore vanish.

Since the surgery obstructions vanish for all pieces of the characteristic variety [8], h^{-1} has trivial normal invariant. Hence h is normally cobordant to the identity as well.

Now we must show that h is, in fact, homotopic to a homeomorphism. Consider any normal cobordism N from M' to M. Cross it with D^3. The π-π

theorem applies to show that $M' \times D^3$ is homeomorphic to $M \times D^3$. Therefore all the block bundles considered throughout are trivial. In particular, $M \times \mathbb{C}P^2 = E(\mathbb{C}P^2) = M' \times \mathbb{C}P^2$. Consider now the surgery exact sequences

$$\ldots \to [\Sigma M : G/Top] \to L_{n+1}(\pi) \to \mathcal{S}(M)$$

$$\ldots \to [\Sigma(M \times \mathbb{C}P^2) : G/Top] \to L_{n+5}(\pi) \to \mathcal{S}(M \times \mathbb{C}P^2)$$

We know that the obstruction for $N \times \mathbb{C}P^2$ lies in the image of $[\Sigma(M \times \mathbb{C}P^2) : G/Top]$ and want to deduce that it actually lies in the image of $[\Sigma M : G/Top]$. This is true because they have the same image : indeed, according to [9] the image only depends on the images of $\mathbb{H}(M \times \mathbb{C}P^2; \mathbb{L}_0)$ and $\mathbb{H}(M; \mathbb{L}_0)$ in $\mathbb{H}(B\pi; \mathbb{L}_0)$ which are visibly identical.

3. AMPLIFICATION AND APPLICATIONS

3.1. Periodicity holds equally for other D^4 bundles over M. Our methods interpret this if the bundle has a semifree S^1-action fixing the zero-section.

3.2. Branched S^3-fibrations interpret the periodicity $\mathcal{S}(M) \to \mathcal{S}(M \times D^8)$.

3.3. One can use the theorem to construct group actions. Consider the S^1 action on $\mathbb{C}P^n$ given by $\theta(z_1 \cdots z_{n+1}) = (\theta z_1, \theta z_2, z_3, z_4, \cdots, z_{n+1})$. It has fixed set $\mathbb{C}P^1 \sqcup \mathbb{C}P^{n-2}$. Taking the quotient, embedding a homotopy $\mathbb{C}P^{n-2}$ in it and taking the branched S^1-cover produces a locally linear S^1 action on a homotopy $\mathbb{C}P^n$. Its splitting invariants [8] can be computed using the theorem (and remark 1) from those of the homotopy $\mathbb{C}P^{n-2}$. In particular if the bottom two splitting invariants of a homotopy $\mathbb{C}P^n$ vanish then it has a locally linear S^1 action equivariantly homotopy equivalent to this linear one. Compare [5] for smooth results and conjectures.

3.4. However, if the fixed set is nullhomotopic in the ambient manifold, the group actions constructed will be on that manifold. For example $\mathcal{S}(S^4 \times S^4) = \mathbb{Z} \oplus \mathbb{Z}$ and each element is the fixed set of a locally linear S^1 action on $S^6 \times S^6$.

3.5. These results can be extended, however not from the embedding theory point of view, to other codimensions. This will be the subject of another paper [2].

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