A Short Topological Proof of Cohn's Theorem

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§ I Introduction

In 1964 P. M. Cohn introduced a *generalized Euclidean algorithm* for the group ring QF of the free group (or monoid) over a field Q [Cohn, FIR]. He deduced that QF is a *fir* which means that all of its ideals[2] are free as modules[2] over QF. In particular, QF_m is *locally free*, that is to say, finitely generated submodules of free modules over the group ring of the free group on m generators with coefficients in Q are free themselves[3]. This is the theorem to which the title refers[4].

Topologists are interested in this last result because it yields an isomorphism between the chain complex with coefficients in a field Q of the universal covering of any compact connected 2-complex with free fundamental group to the chain complex with coefficients in Q of the universal covering of the wedge of one- and two-dimensional spheres. Shortly after Cohn presented his theorem, Bass [Bass] extended an argument of Seshadri to prove that then, in fact, such an isomorphism exists with integer coefficients. This in turn implies that the homotopy type of a compact connected 2-complex with free fundamental group is completely determined by the number of free generators of its fundamental group and its Euler characteristic [Wall, prop. 3.3].

The proof of Cohn's Theorem is lengthy and leads to technical difficulties[5], but it has the merit that it can be generalized to study the theory of modules over the group ring of an arbitrary free product with field coefficients [Bergman]. However, for the theorem in its original statement there is a short proof by geometric methods, which works with an algorithmic reduction of the *diameter* of generators of an ideal. This will be done in this paper.

The idea of this proof was first discussed during a seminar organized by Wolfgang Metzler in 1987 in Southern Tyrol (Italy). I want to thank the participants of my working section Paul Latiolais, Martin Lustig and Wolfgang Metzler for their stimulating suggestions and comments. Also I want to thank Leonid Vaserstein for his interest and encouragement, and George Bergman for his helpful correspondence.

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1 Partially supported by a DFG-grant.

2 Throughout this paper, module means left module and ideal means left ideal.

3 However, in contrast to what happens over Euclidean rings, a free module might contain free submodules of higher rank. For example, the augmentation ideal of QF_m is a free QF_m-module of rank m.

4 Actually the methods presented here also reveal QF being a fir but we do not want to distract the reader's attention from what we are essentially interested in.

5 Warren Dicks pointed out to me that the first correct proof was given by Jacques Lewin [Lewin] five years later.

§ 2 The Geometry of QF_m

In what follows, let Q be a field, F_m the free group on m generators, and $E_n(QF_m)$ the group generated by the elementary n×n matrices over the group ring QF_m. Here an elementary matrix differs from the unit matrix only by one entry. If this happens to be on the diagonal, only a group element or -1 is allowed.

Elements of the group ring QF_m can be interpreted geometrically as 0-dimensional chains in the Cayley-graph Γ of F_m. It will be crucial for the following observations that Γ is a tree.

For $0 \neq x = \sum_{w \in F_m} q_w w \in QF_m$, look at the set of $w \in F_m$ with $q_w \neq 0$. Define *dist (x)* to be the maximum of lengths (number of edges) of reduced edge paths from **1** to such a w. With dist (0) := -1 we have

$$\text{dist}\,(x) = 0 \iff x \in Q \setminus \{0\}$$

This distance from the origin, however, is not invariant under the group operation. In contrast, *diam (x)* is invariant under the group action where diam (x) denotes the maximum of lengths of reduced edge paths between two elements u and v with nonzero q_u and q_v in $0 \neq x = \sum_{w \in F_m} q_w w$; diam (0) := -1.

So diam (x) is the diameter[6] of the tree spanned by those group elements for which x has a nonzero coefficient. This tree also has a well defined *barycenter* $\hat{x} \in \Gamma$: If the midpoints p,q of two diameters of a tree were distinct, then from the midpoint of an arc from p to q one could run more than half a diameter in both directions, which is a contradiction.

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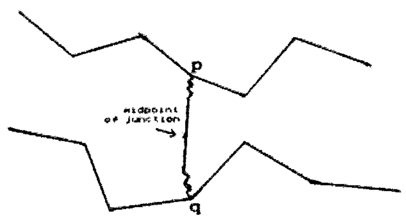
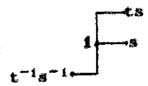


<u>Example:</u> Let s and t be generators of F_2 and take $x := q_{ts}ts + q_s s + q_{t^{-1}s^{-1}}t^{-1}s^{-1}$ with all coefficients in $Q \setminus \{0\}$. The tree spanned by x has the shape:

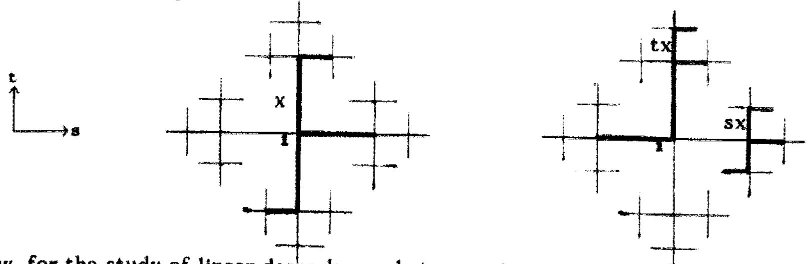


<u>We read off</u> dist (x) = 2, diam (x) = 4, $\hat{x} = \mathbf{1} \in F_2$.

6 A diameter in a finite tree is a longest reduced edge path in it as well as the length of the path.

7 The reader should figure out, in which pieces of the figure identifications are possible.

The group F_2 acts by translation:



Now, for the study of linear dependences between given $x_1,\dots,x_n \in QF_m$, consider n-tuple $a_1,\dots,a_n \in QF_m$ and form the linear combination $\sum_{\nu=1}^{n} a_\nu x_\nu$. Writing the a_ν $\sum_{w\in F_m} q_w^\nu w$, we obtain $\sum_{\substack{w\in F_m\\ 1\le\nu\le n}} q_w^\nu w x_\nu$. Among the summands look what the maximal lue of dist $(q_w^\nu w x_\nu)$ is.

Points at that distance from $\mathbb{1}$ are called *extreme points* of the linear mbination. For example, the picture above to the right can be interpreted as a ear combination having just one summand ax, where $a = s+t$. All points at tance 3 from $\mathbb{1}$ form the set of extreme points of this linear combination.

We shall call $x_1,\dots,x_n \in QF_m$ *weakly linearly dependent*, if there exist $a_1,\dots,a_n \in QF_m$: all zero such that dist $(\sum_{\nu=1}^{n} a_\nu x_\nu)$ < dist (extreme points of $\sum_{\nu=1}^{n} a_\nu x_\nu$), i.e. if contributions to extreme points cancel.

Note that if $x_1,\dots,x_n \in QF_m$ are not weakly linearly dependent, the nonzero x_ν linearly independent.

§ 3 The Algorithm

orem: If $x_1,\dots,x_n \in QF_m$ are weakly linearly dependent then there exists $E_n(QF_m)$ such that if

$$\begin{pmatrix} y_1\\ \vdots\\ y_n\end{pmatrix} := \mathcal{G}\begin{pmatrix} x_1\\ \vdots\\ x_n\end{pmatrix}$$

then $\sum_{\nu=1}^{n}$ diam $(y_\nu) < \sum_{\nu=1}^{n}$ diam (x_ν).

of: Let $\sum_{\nu=1}^{n} a_\nu x_\nu$ be a weak linear dependence with $a_\nu = \sum_{w\in F_m} q_w^\nu w$ and extreme its being all points at distance r_0 from $\mathbb{1}$. Let w_0x_1 (if needs be, renumber) be maximal diameter d among those wx_ν containing extreme points. Abbreviate $d/2$.

Consider the set $\mathfrak{N}$ of wx_ν with $q_w^\nu \neq 0$ such that wx_ν contains an extreme it p for which $w_0\hat{x}_1$ lies on the reduced path from $\mathbb{1}$ to p. p then is called a cial extreme point. By definition of $\mathfrak{N}$, the contributions of $\sum_{\mathfrak{N}} q_w^\nu w x_\nu$ to special eme points must also cancel.

Now observe that $w_0x_1 \in \mathfrak{N}$[8]:

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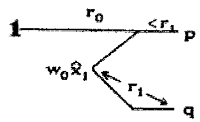

Otherwise one could join $w_0\hat{x}_1$ to a reduced path from $\mathbb{1}$ to an extreme point p which is contained in w_0x_1. There must be a radius starting at $w_0\hat{x}_1$ and going in a different direction ending in some q. That radius adds to the reduced path from $\mathbb{1}$ to $w_0\hat{x}_1$ proving dist (q) to be bigger than r_0. This is a contradiction.

Observe also that points of any $wx_\nu \in \mathfrak{N}$ can be no further away from $w_0\hat{x}_1$ than special extreme points are:

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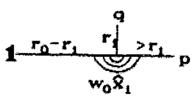


Otherwise put a reduced path $\mathfrak{p}$ from such a point p to $w_0\hat{x}_1$ together with the reduced path from $w_0\hat{x}_1$ to $\mathbb{1}$. The latter has been shown in the last paragraph to have length r_0-r_1. As the length of the whole is not allowed to exceed r_0, there must be cancelling edges at $w_0\hat{x}_1$. But then $\mathfrak{p}$ together with a reduced path from $w_0\hat{x}_1$ to some special extreme point q of wx_ν yields a reduced path from p to q showing diam $wx_\nu > d$. This is a contradiction.

The subtree formed by points at distance at most r_1 from $w_0\hat{x}_1$ has barycenter $w_0\hat{x}_1$. A diameter of $wx_\nu \in \mathfrak{N}$ runs completely in that subtree. Therefore, if its length takes the maximal possible value d, its midpoint is $w_0\hat{x}_1$. Hence, its barycenter $w\hat{x}_\nu$ falls upon $w_0\hat{x}_1$. But for a fixed ν there exists at most one $u \in F_m$ carrying $\hat{x}_1$ into $\hat{x}_\nu$ such that w is completely determined as w_0u^{-1}. In particular $wx_1 \in \mathfrak{N}$ implies $w = w_0$.

By the same argument, if diam $(\sum_{\mathfrak{N}} q_w^\nu w x_\nu)$ still was d, any diameter of it would have $w_0\hat{x}_1$ as its midpoint. (At least) one of the two radii of this diameter combines with the reduced path from $\mathbb{1}$ to $w_0\hat{x}_1$ to form a reduced path. Its endpoint has distance $r_1+(r_0-r_1)$ from $\mathbb{1}$ and is therefore a special extreme point of $\sum_{\nu=1}^{n} a_\nu x_\nu$. This contradicts the fact that $\sum_{\mathfrak{N}} q_w^\nu w x_\nu$ was supposed to have coefficient 0 at special extreme points.

Therefore with $y_1 := x_1 + \frac{1}{q_{w_0}^1} w_0^{-1} \sum_{\substack{\mathfrak{N}\\ \nu\neq 1}} q_w^\nu w x_\nu$, and $y_2 = x_2, \dots, y_n = x_n$

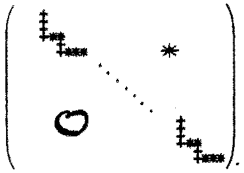
we have $$\text{diam } y_1 = \text{diam } (\sum_{\mathfrak{N}} q_w^\nu w x_\nu) < \text{diam } (x_1) \qquad \triangledown$$

Corollary: Finitely generated submodules of free modules over QF_m are free.

Proof: Let M be a finitely generated submodule of a free QF_m-module. Since a finite system of generators of M can only use a finite number of basis elements, M is, without loss of generality, a submodule of a free module of finite rank. Let $\mathcal{A}$ be the QF_m-matrix with rows consisting of a system of generators of M expressed in a basis of the free module. Using row-operations as in the theorem

8 In fact, note that all extreme points of w_0x_1 are special.

(change of generators of M), row-permutations (renumbering of the generators of M) and cancelling zero-rows if possible, $\mathcal{A}$ can be brought into the form



In each row the first nonzero element is marked with a +, and those elements of a fixed column are not weakly linearly dependent and nonzero. In particular, these elements are linearly independent so that M is free with the row vectors of the transformed matrix as a basis. ▽

We conclude with an outline of how these results have been used for the homotopy classification of compact connected 2-complexes with free fundamental group:

Arguing similarly as in the proof of the Corollary, one can deduce from the Theorem[9] that $GL_n(QF_m)$ is generated by $E_n(QF_m)$ and $GL_n(Q)$-matrices. This property when joined to the Corollary form the hypotheses which imply that finitely generated modules over free groups are free [Bass]. At this step then, the required homotopy classification follows easily [Wall].

Bibliography

[Bass] H. Bass — Projective Modules over Free Groups are Free. Journal of Algebra 1 (1964), 367-373.

[Bergm, corresp] G. Bergman — correspondence (1987).

[Bergman] G. Bergman — Modules over Coproducts of Rings. Transactions of the American Mathematical Society 200 (1974), 1-33.

[Cohn, FIR] P.M. Cohn — Free Ideal Rings. J.Algebra 1 (1964), 47-69 Corrigendum ibid 6 (1967), 410.

[Cohn, FP I] P.M. Cohn — On the Free Product of Associative Rings I. Math. Zeitschr. 71 (1959), 380-398.

[Cohn, FP II] P.M. Cohn — On the Free Product of Associative Rings II. Math. Zeitschr. 73 (1960), 433-456.

[9] A good exercise is the following: Show that a unit $u=\sum q_w w$ fulfills $q_w \neq 0$ for exactly one $w \in F_m$.

[Cohn, FP III] P.M. Cohn — On the Free Product of Associative Rings III. J. Algebra 8 (1968), 376-383. Corrigendum ibid 10 (1968),123.

[D S] M. Dyer, A. Sieradski — Trees of Homotopy Types of Two-Dimensional CW-Complexes. Comm. Math. Helvetici 1, 48 (1973), 31-44

[Hog Diss] C. Hog-Angeloni — Beiträge zum (einfachen) Homotopietyp zweidimensionaler Komplexe zu freien Produkten und anderen gruppentheoretischen Konstruktionen. Dissertation, Frankfurt 1988.

[Hog 2] C. Hog-Angeloni — On the Homotopy Type of 2-Complexes with a Free Product of Cyclic Groups as Fundamental Group. This volume.

[Lewin] J. Lewin — Free Modules over Free Algebras and Free Group Algebras: The Schreier Technique. Transactions of the American Mathematical Society 145 (1969), 455-465.

[Seshadri] C. Seshadri — Triviality of Vector Bundles over the Affine Space K^2. Proc. Natl. Acad. Sci., U.S. 44 (1958), 456-458.

[Wall] C. Wall — Finiteness Conditions for CW-Complexes. Ann. Math. 81 (1965), 56-69