the projected point of $\tilde\gamma(\bar t)$ to $T^*(\partial\Omega\times R)$ is $(\cos\theta_0, \sin\theta_0, \bar t, \varepsilon\cos\theta_0(-\sin\theta_0, \cos\theta_0), \varepsilon\alpha)$, where $\varepsilon^2=1$. This implies that $\tilde\gamma(t)=(\alpha/\beta, \alpha t, t, 0, \varepsilon, \varepsilon\alpha)$ for $0<t<\bar t$.

Let us compute the rotated angle A around the origin of the projected ray of $\{\gamma_g^{(p)}(t): 0\leqq t\leqq\tilde t\}$ to the (x, y) plane and the rotated angle B around the origin of the projected ray of $\{\tilde\gamma(t): 0\leqq t\leqq\tilde t\}$ to the (x, y) plane. Since the speed of the projected ray of $\gamma_g^{(p)}(t)$ to the (x, y) plane is β, $A=\beta(\tilde t-\bar t)+\theta_0$. We assume that $\tilde\gamma(t)$ is on S rays during the time $(2m+1)\bar t$ $(m\geqq 1)$ and $\tilde\gamma(t)$ is on gliding P rays during the time $\tilde t-(2m+1)\bar t$. The angle $<COD$, where $O=(0,0)$ and C, D are projected points of $\tilde\gamma^{(s)}(\tilde t; \tilde t)$ and $\tilde\gamma^{(s)}(\tilde t-2\bar t; \tilde t)$ to the (x, y) plane, respectively, is $2\theta_0$. So we have $B=(2m+1)\theta_0+(\tilde t-(2m+1)\bar t)\beta$. The relation between A and B is $A-B=2n\pi$ for some integer n. The time of passaging of a S ray on the line CD is $2\alpha^{-1}\sin\theta_0$ and the one of a P gliding ray on the arc CD is $2\beta^{-1}\theta_0$. By $2\beta^{-1}\theta_0<2\alpha^{-1}\sin\theta_0$, $n\leqq 0$. The relation is $\beta\alpha^{-1}(1-\beta^2/\alpha^2)^{1/2}-\theta_0=\tan\theta_0-\theta_0=n\pi/m\leqq 0$. This is a contradiction for $0<\theta_0<\pi/2$. The proof is completed.

Remark 3.5. – We explain the reason why we assume the dimension of R^n is 2. If $n=3$, the statement (i) of Theorem 2.2 is changed as follows (*see* Theorem 4.4 in [4]); if $\gamma_i^{(s)}(\omega)\subset WF_b(u)$ and $\gamma_{in}^{(p)}(\omega)\cap WF_b(u)=\varnothing$, then we have one of the following two reflective phenomena; (*a*) $\gamma_r^{(s)}(\omega)\cup\gamma_{tr}^{(p)}(\omega)\subset Wf_b(u)$, (*b*) $\gamma_r^{(s)}(\omega)\subset WF_b(u)$ and $\gamma_{tr}^{(p)}(\omega)\cap WF_b(u)=\varnothing$. In the case (*a*) or (*b*) the incident S waves are called SV waves or SH waves in seismology. Thus if $n=3$, we have a similar phenomenon to the case $n=2$ or have a simple reflective phenomenon of S ray. Unfortunatery we do not have a mathematical condition of separating SV singularities and SH singularities. For $n\geqq 4$ we have the same situation.

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(Manuscript received February 1990.)

K. Yamamoto
Department of Mathematics,
Nagoya Institute of Technology,
Nagoya, Gokiso-cho,
466, Japan

J. Math. Pures Appl.,
71, 1992, *p.* 429 *à* 453

THE STRUCTURE OF *q*-SYMPLECTIC GEOMETRY

By M. de GOSSON

Introduction

Felix Klein defined a geometry by specifying a manifold and a Lie group acting on that manifold. Jean Leray has shown in the first chapter of his treatise *Lagrangian Analysis and Quantum Mechanics* [L] that for every $q=1, 2, \ldots, +\infty$ the q-fold covering group $Sq_q(n)$ of the symplectic group $Sp(n)$ acts on the $2q$-fold covering space $\Lambda_{2q}(n)$ of the lagrangian Grassmannian $\Lambda(n)$; each of the groups $Sp_q(n)$ thus defines a geometry on $\Lambda_{2q}(n)$, which Leray calls *q-symplectic geometry*.

The aim of this article is to show that the algebraic and topological structures of $Sp_q(n)$ and $\Lambda_q(n)$ can be described by using a modified Maslov index, which will be defined as a function $\Lambda_\infty(n)\times\Lambda_\infty(n)\to\mathbb{Z}$, exempted of any transversality assumption. It will lead us ultimately to an explicit description of the action of $Sp_q(n)$ on $\Lambda_{2q}(n)$, that is, of the structure of q-symplectic geometry.

Some of the results contained in this paper have been announced in our *C. R. Acad. Sci. Paris*, Notes [G_1] and [G_2].

This article is divided into three chapters; each chapter is subdivided into sections.

CONTENTS AND MAIN RESULTS:

I. Preliminaries

In Section 1 we briefly review the properties of the covering groups $Sp_q(n)$ and of the covering spaces $\Lambda_q(n)$ that will be needed; the main result is Theorem 1 which describes the action of $Sp_q(n)$ on $\Lambda_{2q}(n)$ in terms of the generators α and β of $\pi_1(Sp(n))\simeq(\mathbb{Z}, +)$ and $\pi_1(\Lambda(n))\simeq(\mathbb{Z}, +)$ whose natural images in $\mathbb{Z}$ are $+1$; then if $s_q\in Sp_q(n)$ and $l_{2q}\in\Lambda_{2q}(n)$ we have (formula (1.9)):

(1) $$(\alpha s_q)\, l_{2q}=\beta^2(s_q\, l_{2q})=s_q(\beta^2\, l_{2q})$$

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES. – 0021-7824/1992/05 429 25/$ 4.50/

which can be considered as the definition of q-symplectic geometry.

In Section 2 we recall the definition and properties of the Maslov index: it is a $\mathbb{Z}$-valued function m of pairs $(l_\infty, l'_\infty) \in \Lambda_\infty(n) \times \Lambda_\infty(n)$ projecting onto pairs $(l, l') \in \Lambda(n) \times \Lambda(n)$ such that $l \cap l' = \{0\}$; Leray defines that function via the theory of chain intersection [definition (2.4)], following an idea of V. I. Arnold [A].

A fundamental property of the Maslov index is that it is the only function of pairs (l_∞, l'_∞) such that $l \cap l' = \{0\}$ which is locally constant on its domain and allows the following decomposition of the index of inertia of a triple $(l, l', l'') \in \Lambda(n) \times \Lambda(n) \times \Lambda(n)$ with $l \cap l' = l' \cap l'' = l'' \cap l = \{0\}$:

(2) $$m(l_\infty, l'_\infty) - m(l_\infty, l''_\infty) + m(l'_\infty, l''_\infty) = \operatorname{Inert}(l, l', l'')$$

[Theorem 1, formula (2.9)].

Two other properties of m will be essential in this article: m is invariant under the action of $\operatorname{Sp}(n)$, and:

(3) $$m(\beta^r l_\infty, \beta^{r'} l'_\infty) = m(l_\infty, l'_\infty) + r - r'$$

II. Definition and properties of the Maslov index without transversality assumptions

In Section 1 we expose Kashiwara's theory of the signature $\operatorname{sign}(l, l' l'')$ of a triple of lagrangian planes; we are following closely [L.V.], (p. 39-45), for the proofs of the properties of that index, which enjoys an essential cocycle property [Theorem 1, formula (1.7)]:

(4) $$\operatorname{sign}(l, l', l'') - \operatorname{sign}(l, l', l''') + \operatorname{sign}(l, l'', l''') - \operatorname{sign}(l', l'', l''') = 0$$

We show in Proposition (1.6) that the signature is related to Leray's index of inertia by the formula:

(5) $$\begin{cases} \operatorname{sign}(l, l', l'') = 2 \operatorname{Inert}(l, l', l'') - n \\ \text{when} \\ l \cap l' = l' \cap l'' = l'' \cap l = \{0\}. \end{cases}$$

In Section 2, Theorem 1 we show that formulae (4) and (5) hereabove make possible the definition of a variant of the Maslov index on $\Lambda_\infty(n) \times \Lambda_\infty(n)$, without any transversality assumption; more precisely:

(6) there exists a unique function

$$\mu: \quad \Lambda_\infty(n) \times \Lambda_\infty(n) \to \mathbb{Z}$$

such that

(7) $$\mu(l_\infty, l'_\infty) - \mu(l_\infty, l''_\infty) + \mu(l'_\infty, l''_\infty) = \operatorname{sign}(l, l', l''),$$

(8) $\mu(l_\infty, l'_\infty) - \operatorname{sign}(l, l', l'')$ is locally constant on the subset $\{(l_\infty, l'_\infty, l'');\ l \cap l'' = l' \cap l'' = \{0\}\}$ of $\Lambda_\infty(n) \times \Lambda_\infty(n) \times \Lambda(n)$.

That function μ, which will also be called Maslov index, is related to the function m by formula (2.5):

(9) $$\mu(l_\infty, l''_\infty) = 2m(l_\infty, l'_\infty) - n \quad \text{when } l \cap l' = \{0\}.$$

We then investigate the properties of that Maslov index μ; in particular μ is invariant under the action of $\operatorname{Sp}_\infty(n)$ and (3) implies that for every $(r, r') \in \mathbb{Z} \times \mathbb{Z}$:

(10) $$\mu(\beta^r l_\infty, \beta^{r'} l'_\infty) = \mu(l_\infty, l'_\infty) + 2r - 2r'.$$

In Section 3 we show that if $l_\infty \in \Lambda_\infty(n)$ has projection $l \in \Lambda(n)$, and $s_\infty \in \operatorname{Sp}_\infty(n)$, then the integer $\mu(s_\infty l_\infty, l_\infty)$ depends only on $(s_\infty, l) \in \operatorname{Sp}_\infty(n) \times \Lambda(n)$, hence definition (3.2) of the Maslov index μ_l on $\operatorname{Sp}_\infty(n)$:

(11) $$\mu_l(s_\infty) = \mu(s_\infty l_\infty, l_\infty).$$

We state and prove the properties of that index in theorems 1 and 2, and Proposition (3.13); in particular Proposition (3.13) shows that (10) implies:

(12) $$\mu_l(\alpha^r s_\infty) = \mu_l(s_\infty) + 4r$$

where α denotes as above a generator of $\pi_1(\operatorname{Sp}(n))$.

In Section 4 we apply the results of Section 2 and 3 to define Maslov indices on the q-fold coverings $\Lambda_q(n)$ and $\operatorname{Sp}_q(n)$ for $q \in \mathbb{N}^*$: let $(l_q, l'_q) \in \Lambda_q(n) \times \Lambda_q(n)$ be the projection of $(l_\infty, l'_\infty) \in \Lambda_\infty(n) \times \Lambda_\infty(n)$; in view of property (10) the following Definition (4.3) of the Maslov index $\mu_{2q}(l_q, l'_\infty)$ makes sense:

(13) $$\mu(l_q, l'_q) = \text{class of } \mu(l_\infty, l'_\infty) \text{ modulo } 2q.$$

Similarly, if $s_\infty \in \operatorname{Sp}_\infty(n)$ has projection $s_q \in \operatorname{Sp}_q(n)$, we may define, noting property (12):

(14) $$\mu_l[s_q]_{4q} = \text{class of } \mu_l(s_\infty) \text{ modulo } 4q.$$

The properties of the indices $\mu_{2q}(.,.)$ and $\mu_l[\cdot]_{4q}$ are then easily deduced from the properties of μ and μ_l.

III. The structure of q-symplectic geometry

In Section 1 we show, using the previous results, that to every $l' \in \Lambda(n)$ one can associate injective mappings $\Lambda_\infty(n) \to \Lambda(n) \times \mathbb{Z}$ and $\operatorname{Sp}_\infty(n) \to \operatorname{Sp}(n) \times \mathbb{Z}$ defined by:

(15) $\Lambda_\infty(n) \ni l_\infty \mapsto (l, \lambda) \in \Lambda(n) \times \mathbb{Z}$, where $\lambda \equiv n - \dim(l \cap l')$, modulo 2 (Theorem 1; 1.)

(16) $\operatorname{Sp}_\infty(n) \ni s_\infty \mapsto (s, \sigma) \in \operatorname{Sp}(n) \times \mathbb{Z}$, where $\sigma = n - \dim(sl' \cap l')$, mod 2;

Transporting the topology of $\Lambda_\infty(n)$ via the mapping (1.5), the subset $\{(l, \lambda);\ \lambda \equiv n - \dim(l \cap l'), \bmod 2\}$ thus becomes a topological space $[\Lambda(n) \times \mathbb{Z}]_{l'}$, which we identify with $\Lambda_\infty(n)$; in the same way, transporting both the topology and the group structure of $\mathrm{Sp}_\infty(n)$ via the mapping (16), the subset $\{(s, \sigma);\ \sigma \equiv n - \dim(sl' \cap l'), \bmod 2\}$ of $\mathrm{Sp}(n) \times \mathbb{Z}$ becomes a topological group $[\mathrm{Sp}(n) \times \mathbb{Z}]_{l'}$, which we identify to $\mathrm{Sp}_\infty(n)$; the group structure of $[\mathrm{Sp}(n) \times \mathbb{Z}]_{l'}$ is given by:

$$(17) \qquad (s, \sigma)(s', \sigma') = (ss', \sigma + \sigma' + \operatorname{sign}(l', sl', ss' l')).$$

Similarly (Theorem 1, § 2), $\Lambda_q(n)$ [resp. $\mathrm{Sp}_q(n)$] is identified with a subset of $\Lambda(n) \times \mathbb{Z}/2q\mathbb{Z}$ [resp. $\mathrm{Sp}(n) \times \mathbb{Z}/4q\mathbb{Z}$] transporting the topological and algebraical structures by the mappings induced by (15) and (16), leading to the identifications:

$$(18) \qquad \Lambda_q(n) = [\Lambda(n) \times \mathbb{Z}]_{l'}/2q\mathbb{Z},$$

$$(19) \qquad \mathrm{Sp}_q(n) = [\mathrm{Sp}(n) \times \mathbb{Z}]_{l'}/4q\mathbb{Z}.$$

Defining the topological spaces:

$$(20) \qquad [\Lambda(n) \times \mathbb{Z}/2q\mathbb{Z}]_{l'} = [\Lambda(n) \times \mathbb{Z}]_{l'}/2q\mathbb{Z},$$

$$(21) \qquad [\mathrm{Sp}(n) \times \mathbb{Z}/4q\mathbb{Z}]_{l'} = [\mathrm{Sp}(n) \times \mathbb{Z}]_{l'}/4q\mathbb{Z}.$$

We finally describe the action (1) of $\mathrm{Sp}_q(n)$ on $\Lambda_{2q}(n)$ in terms of the spaces (20), (21) (Theorem 4):

(22) For $l' \in \Lambda(n)$, the topological group $[\mathrm{Sp}(n) \times \mathbb{Z}_{4q}]_{l'}$ acts transitively on the topological space $[\Lambda(n) \times \mathbb{Z}_{4q}]_{l}$ by the law:

$$(s, \sigma_{4q}) \cdot (l, \lambda_{4q}) = (sl, \sigma_{4q} + \lambda_{4q} + \operatorname{sign}_{4q}(l', sl', sl)).$$

Remark. — G. Lion and M. Vergne have tried, in their monograph [L.V], to construct directly $[\Lambda(n) \times \mathbb{Z}]_l$ and $[\mathrm{Sp}(n) \times \mathbb{Z}]_l$ by equipping $\Lambda(n) \times \mathbb{Z}$ and $\mathrm{Sp}(n) \times \mathbb{Z}$ with topologies defined by using Kashiwara's signature, and to deduce the Maslov index from the properties of these spaces. As we showed in [6₁], [6₂], their attempt was not conclusive: the Maslov index cannot be trivially deduced from Kashiwara's signature.

I. Preliminaries

1. THE COVERING GROUPS OF $\mathrm{Sp}(n)$ AND THE COVERING SPACES OF $\Lambda(n)$. — We are reviewing here some results of symplectic geometry. Standard references are $[\text{G. S.}]_1$, Chap. IV, § 2; $[\text{G. S.}]_2$, Chap. I; [L], Chap. I, and the references therein.

Let $V = \mathbb{R}^n \times \mathbb{R}^n$ be equipped with its usual real vector space structure, and ω be the standard symplectic form on V:

$$(1.1) \qquad \omega(z, z') = \sum_1 y_i x'_i - y'_i x_i$$

for $z = ((x_i), (y_i))$ and $z' = ((x'_i), (y'_i))$ in V. The pair (V, ω) is called the standard $2n$-dimensional symplectic space.

The symplectic group $\mathrm{Sp}(n)$ consists of all linear mappings $s: V \to V$ such that $\omega(sz, sz') = \omega(z, z')$ for every $(z, z') \in V \times V$; $\mathrm{Sp}(n)$ is a closed subgroup of the linear group $\mathrm{Gl}(2n, \mathbb{R})$ [in fact of the special linear group $\mathrm{Sl}(2n, \mathbb{R})$] and is homeomorphic to $U(n) \times \mathbb{R}^{n(n+1)}$, $U(n)$ being the unitary group, from which follows that:

(1.2) *$\mathrm{Sp}(n)$ is a connected Lie group, and $\pi_1(\mathrm{Sp}(n))$ is isomorphic to $(\mathbb{Z}, +)$*
hence:

(1.3) *For every $q = 1, 2, \ldots, +\infty$ there exists a unique q-folded covering $\mathrm{Sp}_q(n)$ of $\mathrm{Sp}(n)$ and $\mathrm{Sp}_\infty(n)$ is the universal covering group of $\mathrm{Sp}(n)$;*

A subspace of V is called isotropic when the restriction of the symplectic form ω to that subspace is identically zero; the dimension of an isotropic subspace is inferior or equal to $1/2 \dim(V) = n$; the isotropic subspaces of maximal dimension n are called lagrangian planes. The set $\Lambda(n)$ of all lagrangian planes is called the *lagrangian grassmannian*; it is a connected submanifold of the Grassmannian of all the n-dimensional planes. Two lagrangian planes l and l' are said to be *transverse* if $l \cap l' = \{0\}$, or which amounts to the same, if $V = l \oplus l'$. The action of $\mathrm{Sp}(n)$ on V induces an action of $\mathrm{Sp}(n)$ of $\Lambda(n)$:

(1.4) *$\mathrm{Sp}(n)$ acts transitively on $\Lambda(n)$, and on the set $\{(l, l') \in \Lambda(n) \times \Lambda(n);\ l \cap l' = \{0\}\}$ of pairs of transverse lagrangian planes.*

Let $l_0 = \mathbb{R}^n \times \{0\}$, $l_0^* = \{0\} \times \mathbb{R}^n$; l_0 and l_0^* are transverse lagrangian planes. It is possible to choose a scalar product $(.|.)$ on V such that the associated Hermitian structure satisfies:

$$\omega(z, z') = \operatorname{Im}(z \mid z'), \qquad il_0 = l_0^*$$

and the unitary group $U(n)$ can then be identified with a subgroup of $\mathrm{Sp}(n)$, also denoted by $U(n)$, and that subgroup acts transitively on $\Lambda(n)$; moreover:

(1.5) *The mapping:*

$$\psi: \Lambda(n) \ni l = u l_0^* \mapsto {}^t u\, u \in U(n)$$

with $u \in U(n)$ is a homeomorphism of $\Lambda(n)$ onto the subset $W(n) = \{w \in U(n);\ w = {}^t w\}$ *of* $U(n)$ hence $\Lambda(n)$ is identified with the subset $W(n)$ of $U(n)$; now $W(n)$ is homeomorphic to $U(n)/0(n)$ since $0(n)$ is the stabilizer of l_0^* in $U(n) \subset \mathrm{Sp}(n)$; from this follows:

(1.6) *The lagrangian Grassmannian $\Lambda(n)$ is a connected submanifold of $\mathrm{Sp}(n)$, and $\pi_1(\Lambda(n))$ is isomorphic to $(\mathbb{Z}, +)$.*
which immediately implies:

(1.7) *For every $q = 1, 2, \ldots, +\infty$, $\Lambda(n)$ has a unique q-fold covering space $\Lambda_q(n)$ and $\Lambda_\infty(n)$ is the universal covering space of $\Lambda(n)$;*

(1.8) *Remark.* – *For two lagrangian planes l and l', the condition $l \cap l' = \{0\}$ is equivalent to*: $\psi(l) - \psi(l')$ *is invertible.*

The mapping $U(n) \ni u \mapsto u^t u \in W(n)$ induces a monomorphism

$$(\mathbb{Z}, +) \simeq \pi_1(U(n)) \to \pi_1(W(n)) \simeq (\mathbb{Z}, +)$$

which is multiplication by 2 on $\mathbb{Z}$; now $Sp(n)$ is homeomorphic to $U(n) \times \mathbb{R}^{n(n+1)}$ and $W(n)$ to $\Lambda(n)$, from this follows that the monomorphism $\pi_1(U(n)) \to \pi_1(w(n))$ hereabove induces a monomorphism $\pi_1(Sp(n)) \to \pi_1(\Lambda(n))$ which sends α on β^2. A consequence of this is ([L], Chap. I, § 2,3, Theorem 3):

THEOREM (Leray). – 1. α *acts on* $Sp_q(n)$, α^r *does not act as the identity on* $Sp_q(n)$ *unless* $r \equiv 0$, *mod q*; β *acts on* $\Lambda_q(n)$, β^r *does not act on* $\Lambda_q(n)$ *as the identity unless* $r \equiv 0$, mod q; 2. $Sp(n)$ *acts transitively on* $\Lambda_{2q}(n)$ *and:*

(1.9) $(\alpha s_q) l_{2q} = s_q(\beta^2 l_{2q}) = \beta^2 (s_q l_{2q})$ *for* (s_q, l_{2q}) *in* $Sp_q(n) \times \Lambda_{2q}(n)$.

That theorem defines q-symplectic geometry: relation (1.9) is essential.

(1.10) *Remark.* – It is the case $q=2$ which is essential in Lagrangian Analysis, *i.e.* in the theory of asymptotic solutions to partial differential equations: $Sp_2(n)$ has a unitary representation in $L^2(\mathbb{R}^1)$, the metaplectic group $Mp(n)$ (*see* [L], Chap. I, § 1,2; [G.S]$_1$, Chap. V, § 7; [G.S.]$_2$, Chap. I, § 11, [Se]), and thus 2-symplectic geometry describes the action of $Mp(n)$ on $\Lambda_4(n)$.

2. THE MASLOV INDEX. – For the results of this section we refer to [L], Chap. I, § 2, 3, 2,4 and 2,5.

J. M. Souriau [5] has given a variant of the definition of the Maslov index that is considered here.

Let $\Lambda_\infty(n)$ be the universal covering space of the lagrangian Grassmannian $\Lambda(n)$ [*see* (1.7)]; we denote by $l_\infty \mapsto l$ the natural projection of $l_\infty \in \Lambda_\infty(n)$ onto $l \in \Lambda(n)$.

(2.1) DEFINITION. – $\sum_\infty$ *is the subset of* $\Lambda^2_\infty(n) = \Lambda_\infty(n) \times \Lambda_\infty(n)$ *consisting of all pairs* (l_∞, l'_∞) *satisfying the condition* $l \cap l' \neq \{0\}$.

We will say that (l_∞, l'_∞) is a pair of transverse elements of $\Lambda_\infty(n)$ if $l \cap l' = \{0\}$, that is, if $(l_\infty, l'_\infty) \notin \sum_\infty$.

Let $C^p(S^1, \mathbb{Z})$ be the group of p-chains in the unit circle $S^1 = \{z \in \mathbb{C};\ |z| = 1\}$ with coefficients in $\mathbb{Z}$; one can attach to every pair $(l_\infty, l'_\infty) \in \Lambda^2_\infty(n)$ an element $sp(l_\infty, l'_\infty) \in C^0(S^1, \mathbb{Z})$ called the *spectrum* of (l_∞, l'_∞); this is done as follows: ψ denoting as in Section 1 the natural homeomorphism $\Lambda(n) \to W(n)$, the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ of $\psi(l)\psi(l')^{-1}$ have modulus one; denoting by $k_1, k_2, \ldots, k_m$ their respective multiplicities, $sp(l_\infty, l'_\infty)$ is defined by:

(2.2) $$sp(l_\infty, l'_\infty) = k_1(\lambda_1) + k_2(\lambda_2) + \ldots + k_m(\lambda_m).$$

Defining the support $|\sigma|$ of a 0-chain $\sigma = m_1(\lambda_1) + m_2(\lambda_2) + \ldots + m_p(\lambda_p)$ by $|\sigma| = \{\mu_i;\ m_i \neq 0\}$, Remark (1.8) implies:

(2.3) $(l_\infty, l'_\infty) \in \Lambda^2_\infty(n) \setminus \sum_\infty$ *if and only if* $(+1) \notin |sp(l_\infty, l'_\infty)|$.

Let us now consider two points $(l_\infty, l'_\infty), (m_\infty, m'_\infty)$ of $\Lambda^2_\infty(n)$, and let Γ be an arc in $\Lambda^2_\infty(n)$ joining (l_∞, l'_∞) to (m_∞, m'_∞); the mapping:

$$\Gamma \ni (n_\infty, n'_\infty) \mapsto sp(n_\infty, n'_\infty)$$

maps the arc Γ onto an element $Sp(\Gamma) \in C^1(S^1, Z)$. If now:

$$\partial\Gamma = (m_\infty, m'_\infty) - (l_\infty, l'_\infty)$$

then

$$\partial\, sp(\Gamma) = sp(m_\infty, m'_\infty) - sp(l_\infty, l'_\infty).$$

Choosing a pair $(l_{0,\infty}, l^*_{0,\infty}) \in \Lambda^2_\infty(n) \setminus \sum_\infty$ projecting onto $(l_0, l^*_0) \in \Lambda(n)$ and such that $l_{0,\infty}$ and $l^*_{0,\infty}$ may be joined by an arc γ in $\Lambda_\infty(n)$ whose spectrum $sp(\gamma)$ belongs to the upper half-circle $\{z \in S^1;\ \operatorname{Im} z \geqq 0\}$. Let Γ be an arc joining (l_∞, l'_∞) to $(l_{0,\infty}, l^*_{0,\infty})$; the Maslov index $m(l_\infty, l'_\infty)$ is defined by:

(2.4) DEFINITION

$$m(l_\infty, l'_\infty) = KI(sp(\Gamma), (+1))$$

where KI, the Kronecker index, is the function which to every pair $(\gamma^1, \gamma^0) \in C^1(S^1, \mathbb{Z}) \times C^0(S^1, \mathbb{Z})$ such that $|\gamma^1| \cap |\gamma^0| = \varnothing$ associates $KI(\sigma^1, \sigma^0) \in \mathbb{Z}$, and is characterized by:

(*a*) KI is linear in its arguments;

(*b*) $KI(\gamma^1, z_0) = +1$ (resp. 0) if γ^1 is a positively oriented arc in S^1 and z_0 an interior (resp. exterior) points of γ;

(*c*) $KI(\gamma, \partial\gamma') = -KI(\gamma', \partial\gamma)$; (*see* [Le], Chap. III, § 5, or any theory of chain intersection);

Remark 1. – Definition (2.4) makes sense since property (b) is indeed satisfied in view of (2.3).

Remark 2. – $m(l_\infty, l'_\infty)$ depends only on the homotopy class of Γ, that is on $\partial\Gamma = (l_\infty, l'_\infty) - (l_{0,\infty}, l^*_{0,\infty})$.

Before we state Leray's Theorem which characterizes in a very simple way the Maslov index, we have to recall the definition of the index of inertia of a triple of pairwise transverse lagrangian planes ([L], Chap. I, § 2,4).

Let $(l, l', l'') \in \Lambda^3(n)$ be such that $l \cap l' = l' \cap l'' = l'' \cap l = \{0\}$ then the conditions

(2.6) $$(z, z', z'') \in l \times l' \times l''; \qquad z + z' + z'' = 0,$$

define three isomorphisms:

$$\mathrm{P}: l \ni z \mapsto z' \in l', \qquad \mathrm{P}': l' \ni z' \mapsto z'' \in l'',$$

and

$$\mathrm{P}'': l'' \ni z'' \mapsto z \in l$$

such that PP″P′, P′PP″ and P″P′P are the identity. The quadratic forms on l, l' and l'' defined by:

$$(2.7)\quad \begin{cases} \mathrm{R}(z)=\omega(z,z')=\omega(z,\mathrm{P}z); \\ \mathrm{R}'(z')=\omega(z',z'')=\omega(z',\mathrm{P}'z') \\ \text{and} \\ \mathrm{R}''(z'')=\omega(z'',z)=\omega(z'',\mathrm{P}''z'') \end{cases}$$

are such that $\mathrm{R}(z)=\mathrm{R}'(z')=\mathrm{R}''(z'')$ in view of $(2.6)_2$, hence $\mathrm{R}=\mathrm{R}'\mathrm{P}, \mathrm{R}'=\mathrm{R}''\mathrm{P}'$ and $\mathrm{R}''=\mathrm{R}\mathrm{P}''$, thus these quadratic forms all have the same index of inertia, which is denoted by Inert (l, l', l'').

We then have ([L], Chap. I, § 2.5, theorem 5.1):

THEOREM (Leray). – *The Maslov index* (2.4) *is the only function*

$$(2.8)\quad m: \Lambda_\infty^2(n)\setminus\sum_\infty \to \mathbb{Z}$$

that is locally constant on its domain and such that:

$$m(l_\infty, l'_\infty)-m(l_\infty, l''_\infty)+m(l'_\infty, l''_\infty)=\mathrm{Inert}(l, l', l'').$$

(2.9) *Taking into account definition* (2.4), *the Maslov index has the following properties*:

$$(2.10)\quad \begin{cases} m(l_\infty, l'_\infty)+m(l'_\infty, l_\infty)=n; \qquad m(l_{0,\infty}, l^*_{0,\infty})=n \\ \text{and} \\ m(l_{0,\infty}, l^*_{0,\infty})=0 \end{cases}$$

(2.11) $m(\beta^r l_\infty, \beta^{r'} l'_\infty)=m(l_\infty, l'_\infty)+r-r'$ *for every* $(r, r')\in\mathbb{Z}^2$;

(2.12) $m(s_\infty l_\infty, s_\infty l'_\infty)=m(l_\infty, l'_\infty)$ *for every* $s_\infty\in\mathrm{Sp}_\infty(n)$.

Remark 3. – The formula in theorem (2.9) giving the decomposition of the index of inertia clearly implies:

$$(2.13)\quad \mathrm{Inert}(l, l', l'')-\mathrm{Inert}(l, l', l''')+\mathrm{Inert}(l, l'', l''')-\mathrm{Inert}(l', l'', l''')=0$$

hence the index of inertia can be viewed as a $\mathbb{Z}$-valued 2-cochain on $\{(l, l', l'')\in\Lambda^3(n);\ l\cap l'=l'\cap l''=l''\cap l=\{0\}\}$ whose coboundary is zero, hence it is a cocycle on this set. Defining, for a triple $(l_\infty, l'_\infty, l''_\infty)$ of pairwise transverse elements of $\Lambda_\infty(n)$:

$$(2.14)\quad \mathrm{Inert}_\infty(l_\infty, l'_\infty, l''_\infty)=\mathrm{Inert}(l, l', l'')$$

we can interpret, in view of (2.13), Inert_∞ as a cocycle on

$$\{(l_\infty, l'_\infty, l''_\infty);\ l\cap l'=l'\cap l''=l''\cap l=\{0\}\};$$

it then follows from Leray's theorem that $\mathrm{Inert}_\infty=\delta m$, hence the Maslov index is a coboundary.

II. Definition and properties of the Maslov index without transversality assumptions

1. THE SIGNATURE OF A TRIPLE OF LAGRANGIAN PLANES. – Let Q be a quadratic form on a finite dimensional vector space E. The matrix of Q has p (resp. q) positive (resp. negative) eigenvalues; the pair (p, q) is usually referred to as the "signature" of the quadratic form Q. We will slightly modify that classical terminology, by *defining* the signature of Q as being the integer $p-q\in\mathbb{Z}$; we will denote this integer sign (Q); this notation is consistent with the case $n=1$ since in that case we have either $p=1$ and $q=0$, or $p=0$ and $q=1$.

Let $(l, l', l'')\in\Lambda^3(n)$ be an arbitrary triple of lagrangian planes; the signature of the quadratic form Q on $l\times l'\times l''$:

$$(1.1)\quad \mathrm{Q}(z, z', z'')=\omega(z, z')+\omega(z', z'')+\omega(z'', z)$$

is called the signature (or Kashiwara index) of the triple (l, l', l''), and denoted by $\mathrm{sign}(l, l', l'')$ (the original notation used in [L.V.], $[\mathrm{G}]_1$, $[\mathrm{G}]_2$ is $\tau(l, l', l'')$; we have preferred to use the symbol "sign" since it emphasizes in a more convincing way its relationship (1.6) with Leray's index of inertia defined in Chap. I, § 2).

The two following properties of the signature are obvious, in view of the definition of Sp (n) and the antisymmetry of the symplectic form ω:

(1.2) $\mathrm{sign}(sl, sl', sl'')=\mathrm{sign}(l, l', l'')$ *for every* $s\in\mathrm{Sp}(n)$.

(1.3) $\mathrm{sign}(l, l', l'')$ *is unchanged* (*resp. changes* sign) *by any even* (*resp. odd*) *permutation of the triple* (l, l', l'').

The signature of a triple of lagrangian planes is expressed as the signature of a quadratic form Q in $3n$ variables; if a slight assumption of transversality is added, it can be expressed as the signature of a form in only n variables:

(1.4) PROPOSITION. – *Assume* $l\cap l''=\{0\}$, *then* $\mathrm{sign}(l, l', l'')$ *is the signature of the quadratic form* $\mathrm{Q}'(z')=\omega(z', \mathrm{P}(l'', l)z')=\omega(\mathrm{P}(l, l'')z')$ *on* l', *where* $\mathrm{P}(l, l'')$ *is the projection operator on* l *along* l'' *and* $\mathrm{P}(l'', l)=\mathrm{I}-\mathrm{P}(l, l'')$ *is the projection operator on* l'' *along* l;

Proof. — We have:

$$Q(z,z',z'')=\omega(z,z')+\omega(z',z'')+\omega(z'',z)$$
$$=\omega(z,P(l'',l)z')+\omega(P(l,l'')z',z'')+\omega(z'',z)$$
$$=\omega(P(l,l'')z',P(l'',l)z')-\omega(z-P(l,l'')z',z''-P(l'',l)z').$$

Let $u=z-P(l,l'')z'$, $u'=z'$, $u''=z''-P(l'',l)z'$, the signature of Q is then the signature of the quadratic form:

$$(u,u',u'')\mapsto\omega(P(l,l'')u',P(l'',l)u')-\omega(u,u'')$$

hence the result since the signature of the form $(u,u'')\mapsto\omega(u,u'')$ is equal to zero.

(1.5) COROLLARY. — *Let* $l_0=\mathbb{R}^n\times\{0\}$, $l_0^*=\{0\}\times\mathbb{R}^n$, $l=\{(x,Ax);x\in\mathbb{R}^n\}$, A *being a symmetric linear mapping* $\mathbb{R}^n\to\mathbb{R}^n$. *Then* $\operatorname{sign}(l_0^*,l,l_0)=\operatorname{sign}(A)$.

Proof. — In view of proposition (1.4) hereabove, $\operatorname{sign}(l_0^*,l,l_0)$ is the signature of the quadratic form Q′ on l given by

$$Q'(z)=(P(l_0^*,l_0)z,P(l_0,l_0^*)z)$$

hence $Q'(z)=\langle x,Ax\rangle$ and the corollary follows.

The signature of pairwise transverse lagrangian planes is related to their index of inertia:

(1.6) PROPOSITION. — Let $(l,l',l'')\in\Lambda^3(n)$ *be such that* $l\cap l'=l'\cap l''=l''\cap l=\{0\}$; *then*:

$$\operatorname{sign}(l,l',l'')=2\operatorname{Inert}(l,l',l'')-n$$

Proof. — By theorem (I.2.9) and property (I.2.12) of the Maslov index the index of inertia is invariant by the symplectic group: $\operatorname{Inert}(sl,sl',sl'')=\operatorname{Inert}(l,l',l'')$ for every $s\in Sp(n)$. It is hence sufficient in view of (1.3) and (I.1.4) to prove (1.6) for $l=l_0^*$, $l'=\{(x',Ax');{}^tA=A\}$, A being invertible and $l''=l_0$. Then in view of corollary (1.5) we have:

$$\operatorname{sign}(l_0^*,l,l_0)=p-q$$

where p (resp. q) is the number of positive (resp. negative) eigenvalues of the symmetric matrix A.

On the other hand, $\operatorname{Inert}(l,l',l'')$ is the index of inertia of the quadratic form R″ on l'' defined by (I.2.6), (I.2.7), that is, since condition (I.2.6) can be written in the present case $x'+x''=0$, $y+Ax''=0$, the index of inertia of the form $R''(z'')=\omega(z'',z)=\langle x'',Ax''\rangle$, hence $\operatorname{Inert}(l,l',l'')$ is the number q of negative eigenvalues of A; the result follows since $p+q=n$, A being invertible;

It immediately follows from (1.6) and (I.2.13) in Remark 3 that the signature $\operatorname{sign}(l,l',l'')$ is a cocycle on the set $\{(l,l',l'')\in\Lambda^3(n);l\cap l'=l'\cap l''=l''\cap l=\{0\}\}$; it is in fact a cocycle on $\Lambda^3(n)$:

THEOREM. — *For* $(l,l',l'',l''')\in\Lambda^4(n)$ *we have*:

(1.7) $$\operatorname{sign}(l,l',l'')-\operatorname{sign}(l,l',l''')+\operatorname{sign}(l,l'',l''')-\operatorname{sign}(l',l'',l''')=0.$$

Proof (Kashiwara). — In view of (1.3) it is equivalent to show that:

(1) $\operatorname{sign}(l,l',l'')=\operatorname{sign}(l,l',l''')+\operatorname{sign}(l',l'',l''')+\operatorname{sign}(l'',l,l''')$.

Assume first $l\cap l'''=l'\cap l'''=l''\cap l'''=\{0\}$.

By proposition (1.4) the right band side of (1) is the signature of the quadratic form Q″ on $l\times l'\times l''$ given by:

(2) $Q''(z,z',z'')=\omega(P(l,l''')z',z')+\omega(P(l',l''')z'',z'')+\omega(P(l'',l'''),z,z)$.

Now, the linear mapping $(z,z',z'')\mapsto(u,u',u'')$, given by

$$u=z+P(l,l''')z',\qquad u'=z'+P(l',l''')z''\qquad\text{and}\qquad u''=z''+P(l'',l''')z$$

is invertible; its inverse is given by

$$z=\frac{1}{2}(u-P(l,l''')u'+P(l,l''')u'')$$
$$z'=\frac{1}{2}(u'-P(l',l''')u''+P(l',l''',u)$$
$$z''=\frac{1}{2}(u''-P(l'',l''')u+P(l'',l'''')u').$$

We have:

(3) $\omega(z,z')=\omega(P(l,l''')u',u')+\omega(u,u')$
$$+\omega(u,P(l',l''')u'')+\omega(P(l,l''')u',P(l',l'''u)u)=0,$$

(4) $\omega(z',z'')=\omega(P(l',l''')u'',u'')+\omega(u',u'')$
$$+\omega(u',P(l'',l''')u)+\omega(P(l',l''')u'',P(l'',l'')u)=0,$$

(5) $\omega(z'',z)=\omega(P(l'',l''')u,u)+\omega(u'',u)$
$$+\omega(u'',P(l,l''')u')+\omega(P(l'',l''')u,P(l,l'')u')=0.$$

Noting that $u'=P(l,l''')u'+P(l''',l)u'$, we can write:

$$\omega(u,u')+\omega(u',P(l'',l''')u)+\omega(P(l'',l''')u,P(l,l''')u')$$
$$=\omega(u,P(l''',l)u')+\omega(P(l,l''')u',P(l'',l''')u)$$
$$+\omega(P(l''',l)u',P(l'',l''')u)+\omega(P(l'',l''')u,P(l,l''')u')$$
$$=\omega(u,P(l''',l)u')+\omega(P(l''',l)u',P(l'',l''')u)$$
$$=\omega(P(l''',l)u',P(l''',l'')u)=0$$

and similarly:

$$\omega(u', u'') + \omega(u'', \mathrm{P}(l, l''')\, u') + \omega(\mathrm{P}(l, l''')\, u', \mathrm{P}(l', l''')\, u'') = 0$$

$$\omega(u'', u) + \omega(u, \mathrm{P}(l', l''')\, u'') + \omega(\mathrm{P}(l', l''')\, u'', \mathrm{P}(l'', l''')\, u) = 0$$

hence, adding equalities (3), (4), (5):

$$\omega(z, z') + \omega(z', z'') + \omega(z'', z) = \omega(\mathrm{P}(l, l''')\, u', u') + \omega(\mathrm{P}(l', l''')\, u'', u'') + \omega(\mathrm{P}(l'', l''')\, u, u)$$

which shows that the quadratic forms Q in (1.1) and Q″ in (2) hereabove are equivalent, thus establishing the result in the considered case. To prove the theorem in the general case, let $m \in \Lambda(n)$ be such that:

$$m \cap l = m \cap l' = m \cap l'' = m \cap l''' = \{0\}.$$

We have, in view of the first case:

(6) $$\operatorname{sign}(l, l', l'') = \operatorname{sign}(l, l', m) + \operatorname{sign}(l', l'', m) + \operatorname{sign}(l'', l, m)$$

(7) $$\operatorname{sign}(l, l', l'') = \operatorname{sign}(l, l', m) + \operatorname{sign}(l', l'', m) + \operatorname{sign}(l'', l, m)$$

(8) $$\operatorname{sign}(l, l'', l''') = \operatorname{sign}(l, l'', m) + \operatorname{sign}(l'', l''', m) + \operatorname{sign}(l''', l, m)$$

(9) $$\operatorname{sign}(l', l'', l''') = \operatorname{sign}(l', l'', m) + \operatorname{sign}(l'', l''', m) + \operatorname{sign}(l''', l', m)$$

hence the result adding together (6) and (8), and substracting (7) and (9), and using the antisymmetry property (1.3) of the signature.

(1.8) PROPOSITION 1. *The signature is locally constant on the subset*

$$\{(l, l', l'');\ l \cap l' = l' \cap l'' = l'' \cap l = \{0\}\} \quad \textit{of} \quad \Lambda(n) \times \Lambda(n) \times \Lambda(n);$$

2. *For any triple* (l, l', l'') *of lagrangian planes we have*:

(1.9) $$\operatorname{sign}(l, l', l'') \equiv n + \dim(l \cap l') + \dim(l' \cap l'') + \dim(l'' \cap l), \qquad \bmod 2$$

Proof. — Let us first prove the following general lemma:

LEMMA. — *Let* $(l, l', l'') \in \Lambda(n) \times \Lambda(n) \times \Lambda(n)$. *The kernel of the quadratic form* Q *in* (1.1) *defining* $\operatorname{sign}(l, l', l'')$ *is isomorphic to* $(l \cap l') \times (l' \cap l'') \times (l'' \cap l)$.

Proof of the lemma. — Set $\mathrm{Z} = (z, z', z'') \in l \times l' \times l''$, and let A be a symmetric matrix such that $\mathrm{Q}(\mathrm{Z}) = \langle \mathrm{AZ}, \mathrm{Z} \rangle$, the brackets denoting the scalar product on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$. We have $\mathrm{Z} \in \operatorname{Ker}(\mathrm{A})$ if and only if $\langle \mathrm{AZ}, \mathrm{U} \rangle = 0$ for all $\mathrm{U} = (u, u', u'') \in l \times l' \times l''$. Now, that condition is equivalent to:

$$\mathrm{Q}(\mathrm{Z} + \mathrm{V}) - \mathrm{Q}(\mathrm{U}) = 0 \quad \text{for all } \mathrm{U} \in l \times l' \times l'',$$

that is to:

$$\omega(z, u') + \omega(z', u'') + \omega(z'', u) + \omega(u, z') + \omega(u', z'') + \omega(u'', z) = 0$$

which can be rewritten as:

$$\omega(z - z'', u') + \omega(z' - z, u'') + \omega(z'' - z', u) = 0$$

Since l, l' and l'' are lagrangian planes, that relation implies $z - z'' \in l'$, $z' - z \in l''$, $z'' - z' \in l$, and thus the relations:

$$u = z' + z'' - z \in l' \cap l''$$

$$u' = z'' + z - z' \in l \cap l''$$

$$u'' = z + z' - z'' \in l'' \cap l'$$

define an isomorphism of Ker(A) onto the product $(l' \cap l'') \times (l \cap l'') \times (l'' \cap l')$, hence the Lemma.

Proof of 1. — If $l \cap l' = l' \cap l'' = l'' \cap l = \{0\}$, the Lemma shows that the matrix A defining the quadratic form Q is invertible, and the result follows.

Proof of 2. — Let p (resp. q) be the number of positive (resp. negative) eigenvalues of A. In view of the Lemma we have:

$$\operatorname{rank}(\mathrm{A}) = p + q = 3n - \dim(l \cap l') - \dim(l' \cap l'') - \dim(l'' \cap l)$$

hence the result, since:

$$\operatorname{sign}(l, l', l'') = p - q = \operatorname{rank}(\mathrm{A}) - 2q$$

2. THE MASLOV INDEX ON $\Lambda_\infty(n)$. — At the end of Section 2, Chap. I, we showed that Leray's Maslov index $m : \Lambda^2_\infty(n) \to \mathbb{Z}$ could be viewed as the coboundary of the index of inertia. We are going to construct in this section an index $\mu : \Lambda^2_\infty(n) \to \mathbb{Z}$ closely related to m on their common domain, and which is a coboundary of the signature of a triple of lagrangian planes.

THEOREM 1. — *There exists a unique function*:

$$\mu : \Lambda^2_\infty(n) \to \mathbb{R}$$

having the two following properties:

(2.1) $$\mu(l_\infty, l'_\infty) - \mu(l_\infty, l''_\infty) + \mu(l'_\infty, l''_\infty) = \operatorname{sign}(l, l', l''').$$

(2.2) *The mapping*: $(l_\infty, l'_\infty, l'') \mapsto \mu(l_\infty, l'_\infty) - \operatorname{sign}(l, l', l'')$ *is locally constant on the set* $\{(l_\infty, l'_\infty, l'');\ l \cap l'' = l' \cap l'' = \{0\}\}$, *hence* μ *is locally constant on* $\Lambda^2_\infty(n) \setminus \sum_\infty$.

Proof. — Let us first show that there exists at most one function μ for which (2.1) and (2.2) hold; noting that the second assertion in (2.2) immediately follows from (1.8), it is therefore sufficient to prove:

(2.3) LEMMA 1. — *Let* (G, +) *be an abelian group. Every function* $\nu : \Lambda^2_\infty(n) \to G$ *locally constant on* $\Lambda^2_\infty(n) \setminus \sum_\infty$ *and such that* $\nu(l_\infty, l'_\infty) - \nu(l_\infty, l''_\infty) + \nu(l'_\infty, l''_\infty) = 0$ *is identically zero.*

Proof of (2.3). — Choose $l'' \in \Lambda(n)$ such that $l \cap l'' = l' \cap l'' = \{0\}$; writting:

$$(2.4) \qquad \nu(l_\infty, l'_\infty) = \nu(l_\infty, l''_\infty) - \nu(l'_\infty, l''_\infty)$$

and noting there exist neighborhoods or l_∞, l'_∞ not containing l''_∞, it follows that ν is locally constant on $\Lambda^2_\infty(n)$, hence constant since $\Lambda_\infty(n)$ is connected; Taking $l_\infty = l'_\infty$ in (2.4), the value of this constant is zero.

1. *Definition of* μ *for* $(l_\infty, l'_\infty) \in \Lambda^2_\infty(n) \setminus \sum_\infty$. — Setting in this case:

$$(2.5) \qquad \mu(l_\infty, l'_\infty) = 2m(l_\infty, l'_\infty) - n;$$

it is clear that (2.1) holds, using property (I.2.9) [Theorem (2.8)] of the function m and the relation (1.6) between the index of inertia and the signature.

2. *Definition of* μ *for* $(l_\infty, l'_\infty) \in \Lambda^2_\infty(n)$. — Let l''_∞ and l'''_∞ be two elements of $\Lambda_\infty(n)$ such that:

$$l \cap l'' = l \cap l''' = l' \cap l'' = l' \cap l''' = l'' \cap l''' = \{0\};$$

in view of property (2.1) established in the transversal case, we have:

$$\mu(l_\infty, l''_\infty) - \mu(l_\infty, l'''_\infty) + \mu(l''_\infty, l'''_\infty) = \operatorname{sign}(l, l'', l'''),$$
$$\mu(l'_\infty, l''_\infty) - \mu(l'_\infty, l'''_\infty) + \mu(l''_\infty, l'''_\infty) = \operatorname{sign}(l', l'', l'''),$$

hence, substracting both equalities:

$$(2.6) \quad \mu(l_\infty, l''_\infty) - \mu(l_\infty, l'''_\infty) - \mu(l'_\infty, l''_\infty) + \mu(l'_\infty, l'''_\infty) = \operatorname{sign}(l, l'', l''') - \operatorname{sign}(l', l'', l''').$$

Substracting the cocycle relation in theorem (1.7) from (2.6) we finally get:

$$\mu(l_\infty, l'''_\infty) - \mu(l'_\infty, l'''_\infty) + \operatorname{sign}(l, l', l''') = \mu(l_\infty, l''_\infty) - \mu(l'_\infty, l''_\infty) + \operatorname{sign}(l, l', l'').$$

That equality shows that the following definition of $\mu(l_\infty, l'_\infty)$ which is necessary for (2.1) to hold, is independent of the choice of $l''_\infty \in \Lambda_\infty(n)$ such that $l \cap l'' = l' \cap l'' = \{0\}$:

$$(2.7) \qquad \mu(l_\infty, l'_\infty) = \mu(l_\infty, l''_\infty) - \mu(l'_\infty, l''_\infty) + \operatorname{sign}(l, l', l'').$$

To prove property (2.2), choose again $l''_\infty \in \Lambda_\infty(n)$ such that $l \cap l'' = l' \cap l'' = \{0\}$. Definition (2.7) then yields in view of definition (2.5):

$$\mu(l_\infty, l'_\infty) - \operatorname{sign}(l, l', l'') = 2(m(l_\infty, l''_\infty) - m(l'_\infty, l''_\infty))$$

hence the result in view of (1.8) and since the function m is locally constant on its domain in view of the first part theorem (I.2.8).

From now on we will call the function μ characterized by Theorem 1 "Maslov index on $\Lambda_\infty(n)$", thus slightly altering the accepted terminology, as we already did in our papers $[G]_1$, $[G]_2$;

Let us next investigate the properties of the Maslov index.

(2.8) PROPOSITION

$$\mu(.,.) \in \mathbb{Z}; \qquad \mu(l_\infty, l'_\infty) + \mu(l'_\infty, l_\infty) = 0; \qquad \mu(l_{0,\infty}, l^*_{0,\infty}) = n; \qquad \mu(l^*_{0,\infty}, l_{0,\infty}) = -n.$$

Proof. — (2.8_1) follows from definitions (2.5), (2.7), and the fact that m and sign take their values in $\mathbb{Z}$.

(2.8_2) follows from definitions (2.5), (2.7), property (1.2) of sign and property $(\text{I}.2.10_1)$ of m. Property $(\text{I}.2.10_2)$ of m and definition (2.5) immediately imply (2.8_3), (2.8_4).

The following result describes the action of $\pi_1(\Lambda(n))$ on the Maslov index; it will be crucial for our constructions in Chapter II.

(2.9) PROPOSITION. — $\mu(\beta^r l_\infty, \beta^{r'} l'_\infty) = \mu(l_\infty, l'_\infty) + 2r - 2r'$, β *being the generator of* $\pi_1(\Lambda(n))$ *whose image in* $\mathbb{Z}$ *is* $+1$.

Proof. — The result is a straightforward consequence of definition (2.5) and of the property (I.2.11) of m when $l \cap l' = \{0\}$.

In the general case let again $l''_\infty \in \Lambda_\infty(n)$ be such that $l \cap l'' = l' \cap l'' = \{0\}$, then definition (2.7) yields, since $\beta^r l_\infty$ and $\beta^{r'} l'_\infty$ have respective projections l and l':

$$\mu(\beta^r l_\infty, \beta^{r'} l'_\infty) = \mu(\beta^r l_\infty, l''_\infty) - \mu(l''_\infty, \beta^{r'} l'_\infty) + \operatorname{sign}(l, l', l'')$$
$$= \mu(l_\infty, l''_\infty) + 2r - \mu(l''_\infty, l'_\infty) - 2r' + \operatorname{sign}(l, l', l'') = \mu(l_\infty, l'_\infty) + 2r - 2r'$$

(2.10) *Remark.* — Proposition (2.9) together with (2.8_3) shows that the range of μ is $\mathbb{Z}$.

(2.11) PROPOSITION. — $\mu(s_\infty l_\infty, s_\infty l') = \mu(l_\infty, l'_\infty)$ *for every* $s_\infty \in \mathrm{Sp}_\infty(n)$.

Proof. — The result immediately follows from definitions (2.5), (2.7) and the property (2.12) of m.

3. MASLOV INDICES ON $\mathrm{Sp}_\infty(n)$. — In [L], Chap. I, Sec. 2.7, Leray defined the Maslov index of an element $s_\infty \in \mathrm{Sp}_\infty(n)$ projecting onto $s \in \mathrm{Sp}(n)$ such that $s l_0 \cap l_0 = \{0\}$ by the formula $m(s_\infty) = m(s_\infty l_{0,\infty}, l_{0,\infty})$, $l_{0,\infty}$ being some element of $\Lambda_\infty(n)$ projection onto l_0.

We find it more convenient for our purposes not to single out any particular element of $\Lambda(n)$.

We will first prove that the Maslov index on $\mathrm{Sp}_\infty(n)$ only depends on the projection of the reference element of $\Lambda_\infty(n)$.

(3.1) LEMMA. — *Let* $s_\infty \in \mathrm{Sp}_\infty(n)$ *and* $l_\infty \in \Lambda_\infty(n)$. *The integer* $\mu(s_\infty l_\infty, l_\infty)$ *only depends on* s_∞ *and the projection* $l \in \Lambda(n)$ *of* l_∞.

Proof. – Let l_∞ and l'_∞ be two elements of $\Lambda_\infty(n)$; there exists $r \in \mathbb{Z}$ such that $l'_\infty = \beta^r l_\infty$, hence, using proposition (2.9) and formula (I.1.9) in Leray's theorem (Chap. I, § 1):

$$\mu(s_\infty l'_\infty, l'_\infty) = \mu(s_\infty(\beta^r l_\infty), \beta^r l_\infty) = \mu(\beta^r(s_\infty l_\infty), \beta^r l_\infty) = \mu(s_\infty l_\infty, l_\infty).$$

That Lemma justifies the following definition and notation:

(3.2) Definition. – *A lagrangian plane $l \in \Lambda(n)$ being given, we call Maslov index on $\mathrm{Sp}_\infty(n)$, and we denote by $\mu_l(.)$ the function:*

$$\mathrm{Sp}_\infty(n) \ni s_\infty \mapsto \mu_l(s_\infty) \in \mathbb{Z}$$

where $\mu_l(s_\infty) = \mu(s_\infty l_\infty, l_\infty)$.

The following relation gives an explicit formula for the change of l:

(3.3) $$\mu_l(s) - \mu_{l'}(s) = \mathrm{sign}(sl, l, l') - \mathrm{sign}(sl, sl', l')$$

Proof. – In view of formula (2.1) in theorem 1, Section 2 and proposition (2.11), we have:

$$\mu(s_\infty l_\infty, l_\infty) - \mu(s_\infty l_\infty, l'_\infty) + \mu(l_\infty, l'_\infty) = \mathrm{sign}(sl, l, l')$$

and

$$\mu(l_\infty, l'_\infty) - \mu(s_\infty l_\infty, l'_\infty) + \mu(s_\infty l'_\infty, l'_\infty) = \mathrm{sign}(sl, sl', l')$$

hence (3.3), by substracting those two equalities.

Let e (resp. e_∞) be the identity of $\mathrm{Sp}(n)$ [resp. $\mathrm{Sp}_\infty(n)$]. Let $(s_\infty, s'_\infty, s''_\infty)$ be a triple of elements of $\mathrm{Sp}_\infty(n)$ such that:

(3.4) $s_\infty s'_\infty s''_\infty = s''_\infty s_\infty s'_\infty = s'_\infty s''_\infty s_\infty = e_\infty$ and define:

(3.5) $$\mathrm{sign}_l(s, s', s'') = \mathrm{sign}(s^{-1} l, s' l, l) = \mathrm{sign}(s''^{-1} l, sl, l) = \mathrm{sign}(s'^{-1} l, s'' l, l).$$

We then have the analogue of Theorem 1, Section 2:

Theorem 1. – *The Maslov index μ_l is the only function*

$$\mathrm{Sp}_\infty(n) \to \mathbb{R}$$

such that:

(3.6) $(s_\infty, l, l'') \mapsto \mu_l(s_\infty) - \mathrm{sign}(sl, l, l'')$ *is locally constant on the set*

$$\{(s_\infty, l, l'');\ sl \cap l'' = l \cap l'' = \{0\}\},$$

hence $(l, s_\infty) \to \mu_l(s_\infty)$ *is locally constant on the subset* $\{(s_\infty, l);\ sl \cap l = \{0\}\}$ *of* $\mathrm{Sp}_\infty(n) \times \Lambda(n)$;

(3.7) $\mu_l(s_\infty) - \mu_l(s'^{-1}_\infty) + \mu_l(s''_\infty) = \mathrm{sign}_l(s, s', s'')$ *for every triple* $(s_\infty, s'_\infty, s''_\infty)$ *satisfying condition* (3.4).

Proof. – Let us first prove the following analogue of Lemma 1, Section 2:

Lemma. – *Every function ν_l on $\mathrm{Sp}_\infty(n)$ with values in a abelian group $(G, +)$, locally constant on $\{s_\infty \in \mathrm{Sp}_\infty(n);\ sl \cap l = \{0\}\}$ and such that:*

(3.8) $\nu_l(s_\infty) - \nu_l(s'^{-1}_\infty) + \nu_l(s''_\infty) = 0$ *for every triple $(s_\infty, s'_\infty, s''_\infty)$ of elements of $\mathrm{Sp}_\infty(n)$ verifying* (3.4), *is identically zero.*

Proof of the Lemma. – We first note that every element $s \in \mathrm{Sp}(n)$ is the product of s_1, s_2 in $\mathrm{Sp}(n)$ such that $s_1 l \cap l = s_2 l \cap l = \{0\}$: let $l' \in \Lambda(n)$ be such that $l \cap l' = sl \cap l' = \{0\}$; in view of (I.1.4) we can find $s_1 \in \mathrm{Sp}(n)$ such that $(sl, l') = s_1(l', l)$, hence $sl = s_1 l'$ and $s_1 l \cap l = \{0\}$. In view of (I.1.4) again, there exists $s'_2 \in \mathrm{Sp}(n)$ such that $l' = s'_2 l$, hence $sl = s_1 s'_2 l$, and $s = s_1 s_2$, where $s_2 = s'_2 r$, r belonging to the isotropy group of l in $\mathrm{Sp}(n)$; since $s'_2 rl \cap l = s'_2 l \cap l = l' \cap l = \{0\}$, we have indeed $s = s_1 s_2$, with $s_1 l \cap l = s_2 l \cap l = \{0\}$.

Let now $(s_\infty, s'_\infty, s''_\infty)$ be a triple of elements in $\mathrm{Sp}_\infty(n)$ projecting onto (s, s_2^{-1}, s_1^{-1}), then $s s' s'' = e$, and (3.8) can be written as:

$$\nu_l(s_\infty) = \nu_l(s'^{-1}_\infty) - \nu_l(s''_\infty).$$

Since $s'^{-1} l \cap l = s_2 l \cap l = \{0\}$, $s'' l \cap l = s_1 l \cap l = \{0\}$, that relation shows that ν_l is locally constant on $\mathrm{Sp}_\infty(n)$, hence constant, since $\mathrm{Sp}_\infty(n)$ is connected; the value of this constant is zero, taking $s_\infty = e_\infty$ in (3.9).

Now property (3.6) of μ_l immediately follows from property (2.1) in Theorem 1, Section 2, and property (3.7) from property (2.2) (*ibid*);

Theorem 2. – *The Maslov index μ_l has the following properties*:

(3.9) $$\mu_l(s_\infty) \equiv n - \dim(sl \cap l'), \quad \mod 2$$

(3.10) $$\mu_l(s^{-1}_\infty) = -\mu_l(s_\infty); \quad \mu_l(e_\infty) = 0$$

(3.11) $$\mu_l(s_\infty s'_\infty) = \mu_l(s_\infty) + \mu_l(s'_\infty) + \mathrm{sign}(l, sl, ss' l)$$

(3.12) $$\mu(s_\infty l_\infty, l'_\infty) = \mu_{l'}(s_\infty) + \mu(l_\infty, l'_\infty) + \mathrm{sign}(l', sl'; sl).$$

Proof of (3.9). – Formula (1.9) in proposition (1.8) together with proposition (2.9).

Proof of (3.10). – Choose $s_\infty = s'_\infty = s''_\infty = e$ in (3.4), then we have by (3.7) in Theorem 1:

$$\mu_l(e_\infty) - \mu_l(e_\infty) + \mu_l(e_\infty) = \mathrm{sign}_l(e, e, e)$$

that is, by definition (3.5) of sign_l:

$$\mu_l(e_\infty) = \mathrm{sign}(l, l, l) = 0$$

hence (3.10_1) is proven; to prove (3.10_2), choose $s'_\infty = e_\infty$, $s''_\infty = s^{-1}_\infty$ in (3.4), then (3.7) yields:

$$\mu_l(s_\infty) - \mu_l(e_\infty) + \mu_l(s^{-1}_\infty) = \mathrm{sign}_l(s, e, s^{-1})$$

that is:

$$\mu_l(s_\infty) = -\mu_l(s^{-1}_\infty) + \mathrm{sign}(s^{-1} l, l, l) = -\mu_l(s^{-1}_\infty).$$

Proof of (3.11). – Set $s'''_\infty = s_\infty s'_\infty$, then $s'''_\infty s'^{-1}_\infty s_\infty = e_\infty$ and (3.7) yields:

$$\mu_l(s'''_\infty) - \mu_l(s'_\infty) + \mu_l(s^{-1}_\infty) = \operatorname{sign}_l(s'''^{-1}, s'^{-1}, s^{-1})$$

hence, by (3.10$_2$) and definition (3.5):

$$\mu_l(s_\infty s'_\infty) = \mu_l(s_\infty) + \mu_l(s'_\infty) + \operatorname{sign}(s'^{-1} s^{-1} l, s'^{-1} l, l)$$

hence (3.11) in view of the invariance of the signature by $\mathrm{Sp}(n)$ [property (I.1.2)].

Proof of 3.12. – We have, in view of formula (2.1) in Theorem 1, Section 2:

$$\mu(s_\infty l_\infty, s_\infty l''_\infty) - \mu(s_\infty l_\infty, l'_\infty) + \mu(s_\infty l'_\infty, l'_\infty) = \operatorname{sign}(sl, sl', l')$$

hence (3.12) in view of the invariance of μ by $\mathrm{Sp}_\infty(n)$, proposition (2.11).

The action of $\pi_1(\mathrm{Sp}(n))$ on the Maslov index is described as follows, α denoting again the generator of $\pi_1(\mathrm{Sp}(n))$ whose natural image in $\mathbb{Z}$ is $+1$:

(3.13) PROPOSITION. – $\mu_l(\alpha^r s_\infty) = \mu_l(s_\infty) + 4r$ *for every* $r \in \mathbb{Z}$.

Proof. – In view of proposition (2.9) and formula (1.9) in Leray's theorem, Chap. I, Section 1, we have, taking into account definition (4.2) of μ_l:

$$\mu_l(\alpha^r s_\infty) = \mu((\alpha^r s_\infty) l_\infty, l_\infty) = \mu(\beta^{2r}(s_\infty l_\infty), l_\infty) = \mu(s_\infty l_\infty, l_\infty) + 4r = \mu_l(s_\infty) + 4r.$$

Proposition (3.13) hereabove will allow us in next paragraph to define the Maslov index on $\mathrm{Sp}_q(n)$.

4. THE MASLOV INDICES ON $\Lambda_q(n)$ AND $\mathrm{Sp}_q(n)$. – Let $q \in \mathbb{N}^*$. We will denote by $\mathbb{Z}_{2q}$ the quotient group $\mathbb{Z}/2q\mathbb{Z}$, and by x_{2q} the natural image in $\mathbb{Z}_{2q}$ of $x \in \mathbb{Z}$, with the convention $O_{2q} = 0$. Similarly, if f is a function $E \to \mathbb{Z}$, the induced function $E \to \mathbb{Z}_{2q}$ will be denoted by f_{2q}. The natural projection on $\Lambda(n)$ of $l_q \in \Lambda_q(n)$ will be denoted by l, and the natural projection onto $\Lambda_q(n)$ of $\sum_\infty \subset \Lambda_\infty(n) \times \Lambda_\infty(n)$ by $\sum_q$.

Let $(l_q, l'_q) \in \Lambda^2_q(n) = \Lambda_q(n) \times \Lambda_q(n)$, and (l_∞, l'_∞), (m_∞, m'_∞) be two pairs of elements of $\Lambda^2_\infty(n)$ with projection (l_q, l'_q); in view of Leray's theorem, Chap. I.1, there exists $(r, r') \in \mathbb{Z}^2$ such that

(4.1) $$l_\infty = \beta^r m_\infty, l'_\infty = \beta^{r'} m'_\infty, \quad \textit{with}: \ r \equiv r' \equiv 0, \mod q$$

hence, in view of propositon (2.9):

(4.2) *If* (l_∞, l'_∞) *and* (m_∞, m'_∞) *have same projection* $(l_q, l'_q) \in \Lambda_q(n)$, *then:*

$$\mu(l_\infty, l'_\infty) - \mu(m_\infty, m'_\infty) \equiv 0, \quad \mod 2q.$$

The following definition thus makes sense:

(4.3) DEFINITION. – *We call "Maslov index of* (l_q, l'_q) *and denote by* $\mu_{2q}(l'_q, l_q)$ *the class modulo* $2q$ of $\mu(l_\infty, l'_\infty)$, (l_∞, l'_∞) *being any element of* $\Lambda^2_\infty(n)$ *projecting onto* (l_q, l'_q).

The results of paragraph 2 enables us to prove the following:

THEOREM 1. – *The Maslov index* μ_{2q} *on* $\Lambda_q(n)$ *is the only function*

$$\Lambda^2_q(n) \to \mathbb{Z}_{2q}$$

having the two following properties:

(4.4) $$\mu_{2q}(l_q, l'_q) - \mu_{2q}(l_q, l''_q) + \mu_{2q}(l'_q, l''_q) = \operatorname{sign}_{2q}(l, l', l''),$$

(4.5) $\mu_{2q}(l_q, l'_q) - \operatorname{sign}_{2q}(l, l', l'')$ *is locally constant on the subset* $\{(l_q, l'_q, l''); l \cap l'' = l' \cap l'' = \{0\}\}$ *of* $\Lambda^2_q(n) \times \Lambda(n)$, (*hence* μ_{2q} *is locally constant on* $\Lambda^2_q(n) \setminus \sum_q$).

The Maslov index μ_{2q} *has the following properties*:

(4.6) $$\mu_{2q}(\beta^r l_q, \beta^{r'} l'_q) = \mu_{2q}(l_q, l'_q) + (2r - 2r')_{2q},$$

(4.7) $$\mu_{2q}(l_q, l'_q) + \mu_{2q}(l'_q, l_q) = 0.$$

Proof. – The uniqueness of a function $\Lambda^2_q(n) \to \mathbb{Z}_{2q}$ satisfying (4.4), (4.5) follows from Lemma (2.3). Properties (4.4), (4.5) follow from properties (2.1) and (2.2) of μ (theorem 1, §2); (4.6) is immediately deduced from properties (2.9); (4.7) from (2.8$_1$).

Recalling (Leray's theorem, Chap. I, § 1.2) that $\mathrm{Sp}_q(n)$ acts on $\Lambda_{2q}(n)$, we also have:

(4.8) PROPOSITION. – *For every pair* $(l_{2q}, l'_{2q}) \in \Lambda_{2q}(n)$ *and every* $s_q \in \mathrm{Sp}_q(n)$,

$$\mu_{4q}(s_q l_{2q}, s_q l'_{2q}) = \mu_{4q}(l_{2q}, l'_{2q}).$$

Proof. – Immediate in view of proposition (2.11).

Let us now discuss the Maslov index on the covering group $\mathrm{Sp}_q(n)$ of $\mathrm{Sp}(n)$. We denote $s_q \in \mathrm{Sp}_q(n)$ [resp. $s \in \mathrm{Sp}(n)$] the projection of $s_\infty \in \mathrm{Sp}_\infty(n)$ (resp. s_q). By the same argument as was used for the definition of the Maslov index μ_{2q} on $\Lambda_q(n)$, the following definition makes sense in view of proposition (3.13):

(4.9) DEFINITION. – *Let* $s_q \in \mathrm{Sp}_q(n)$, *we call Maslov index of* s_q *and denote by* $\mu_l[s_q]_{4q}$ *the class modulo* $4q$ *of* $\mu_l(s_\infty) \in \mathbb{Z}$, s_∞ *being any element of* $\mathrm{Sp}_\infty(n)$ *with projection* s_q on $\mathrm{Sp}_q(n)$.

Since $\mathrm{Sp}_1(n) = \mathrm{Sp}(n)$, we will use in the case $q = 1$ the notation $s_1 = s$, $\mu_l[.]_4 = \mu_l[.]$, and call $\mu_l[.]$ the Maslov index on $\mathrm{Sp}(n)$.

THEOREM 2. – 1. *The Maslov index* $\mu_l[.]_{4q}$ *is the only function*:

$$\mathrm{Sp}_q(n) \to \mathbb{Z}_{4q}$$

having the two following properties:

(4.10) $\mu_l[s_q]_{4q} - \mu_l[s'^{-1}_q]_{4q} + \mu_l[s''_q]_{4q} = (\operatorname{sign}(s, s', s''))_{4q}$ *when* $s_q s'_q s''_q = s'_q s''_q s_q = s''_q s_q s'_q = 1_q$,

(4.11) $(s_q, l, l'') \mapsto \mu_l[s_q]_{4q} - (\operatorname{sign}(sl, l, l''))_{4q}$ *is locally constant on the subset* $\{(s_q, l, l''); sl \cap l'' = l \cap l'' = \{0\}\}$ *of* $\mathrm{Sp}_q(n) \times \Lambda^2(n)$; *hence* $(l, s_q) \to \mu_l[s_q]$ *is locally constant on the subset* $\{(l, s_q); s_q l \cap l = \{0\}\}$ *of* $\Lambda(n) \times \mathrm{Sp}_q(n)$.

2. *The Maslov index* $\mu_l[\cdot]_{4q}$ *has furthermore the following properties*:

(4.12) $\mu_l[s_q]_{4q}$ *and* n-dim $(sl \cap l)$ *have the same image in* $\mathbb{Z}_2$;

(4.13) $$\mu_l[s_q]_{4q} - \mu_{l'}[s_q]_{4q} = (\operatorname{sign}(sl, l, l') - \operatorname{sign}(sl, sl', l'))_{4q},$$

(4.14) $$\mu_l[s_q^{-1}] = -\mu_l[s_q], \qquad \mu_l[e_q]_{4q} = 0,$$

(4.15) $$\mu_l[s_q s'_q]_{4q} = \mu_l[s_q]_{4q} + \mu[s'_q]_{4q} + (\operatorname{sign}(l, sl, ss'l))_{4q}.$$

Proof. — 1. is an immediate consequence of theorem 1, Section 3; 2. is an immediate consequence of theorem 2, *ibid.*

III. The structure of q-symplectic geometry

1. The relation between $\Lambda_\infty(n)$ and $\Lambda(n) \times \mathbb{Z}$, $\mathrm{Sp}_\infty(n)$ and $\mathrm{Sp}(n) \times \mathbb{Z}$. — The results of Chap. III, Sections 1 and 2 enable us to prove:

Theorem 1. — 1. *For every* $l'_\infty \in \Lambda_\infty(n)$, *the mapping*:

(1.1) $$\Lambda_\infty(n) \ni l_\infty \mapsto (l, \mu(l_\infty, l'_\infty)) \in \Lambda(n) \times \mathbb{Z}$$

is an injection which is a bijection:

(1.2) $$\begin{cases} h_{l'_\infty}: \ \Lambda_\infty(n) \to (\Lambda(n) \times \mathbb{Z})_{l'} \\ (\Lambda(n) \times \mathbb{Z})_{l'} = \{(l, \lambda) \in \Lambda(n) \times \mathbb{Z};\ \lambda \equiv n - \dim(l \cap l'), \mathrm{mod}\, 2\}. \end{cases}$$

2. *The restriction of this bijection*:

(1.3) $$\{l_\infty \in \Lambda_\infty(n);\ l \cap l' = \{0\}\} \to \{(l, \lambda) \in \Lambda(n) \times \mathbb{Z};\ l \cap l' = \{0\},\ \lambda \equiv n, \mathrm{mod}\, 2\}$$

is a homeomorphism when $\mathbb{Z}$ *is equipped with the discrete topology*.

Proof of 1. — If $(1, \mu(l_\infty, l'_\infty)) = (l'', \mu(l''_\infty, l'_\infty))$ then $l = l''$, hence there exists $r \in \mathbb{Z}$ such that $l_\infty = \beta^r l''_\infty$; since $\mu(l_\infty, l'_\infty) = \mu(\beta^r l_\infty, l'_\infty) = \mu(l_\infty, l'_\infty) + 2r$ in view of proposition (II.2.9), we have $r = 0$, hence $l_\infty = l''_\infty$, and (2.1) is an injection; the range of this injection is a subset of $(\Lambda(n) \times \mathbb{Z})_{l'} = \{(l, \lambda) \in \Lambda(n) \times \mathbb{Z};\ \lambda \equiv n - \dim(l \cap l'),\ \mathrm{mod}\, 2\}$ in view of (II.2.12); if conversely $(l, \lambda) \in (\Lambda(n) \times \mathbb{Z})_{l'}$ then $\lambda \equiv n - \dim(l \cap l')$, mod 2; let $l''_\infty \in \Lambda_\infty(n)$ have projection $l \in \Lambda(n)$, we have $\mu(l''_\infty, l'_\infty) = \lambda + 2r$ for some $r \in \mathbb{Z}$; let us set $l_\infty = \beta^{-r} l''_\infty$; in view of proposition (II.2.9) we have $\mu(l_\infty, l'_\infty) = \mu(\beta^{-r} l''_\infty, l_\infty) = \mu(l''_\infty, l_\infty) - 2r = \lambda$, hence (l, λ) is the image of l_∞, which shows that the range of h_{l_∞} is $(\Lambda(n) \times \mathbb{Z})_{l'}$.

Proof of 2. — In view of property (I.2.2) in Theorem 1, Chap. I, Section 2, μ is locally constant on the set $\Lambda^2_\infty(n) \setminus \sum_\infty = \{(l_\infty, l'_\infty) \in \Lambda^2_\infty(n);\ l \cap l' = \{0\}\}$; the result follows since $h_{l'_\infty}$ is a bijection.

Corollary 1. — *The set of all homeomorphisms* $h_{l'_\infty}$ *defined by* (1.3), *for* $l' \in \Lambda(n)$, $l' \cap l = \{0\}$, *is a system of local charts of the manifold* $\Lambda_\infty(n)$, *the transition functions*

being given by:

$$h_{l'_\infty} \circ h_{l''_\infty}^{-1} : (l, \mu(l_\infty, l'_\infty)) \mapsto (l, \mu(l_\infty, l''_\infty))$$

with

(1.4) $$\mu(l_\infty, l'_\infty) - \mu(l_\infty, l''_\infty) = \operatorname{sign}(l, l', l'') - \mu(l'_\infty, l''_\infty).$$

Proof. — Immediate by formula (I.2.1) in theorem 1, Chap. I, Section 2.

Next result is essential in q-geometry; it shows that one can identify the universal covering space $\Lambda_\infty(n)$ of $\Lambda(n)$ with the set $(\Lambda(n) \times \mathbb{Z})_{l'}$ equipped with an adequate topology;

Corollary 2. — 1. *For* $l' \in \Lambda(n)$, *the set* $(\Lambda(n) \times \mathbb{Z})_{l'}$ *defined by* (1.2) *can be equipped with a topology for which it becomes a topological space* $[\Lambda(n) \times \mathbb{Z}]_{l'}$, *and for which* $h_{l'_\infty}$ *is a homeomorphism*:

$$h_{l'_\infty}: \ \Lambda_\infty(n) \to [\Lambda(n) \times \mathbb{Z}]_{l'}.$$

2. *That topology is the topology characterized by the conditions*:

(i) *for every* $l'' \in \Lambda(n)$, *the mapping*

(1.5) $$[\Lambda(n) \times \mathbb{Z}]_{l'} \ni (l, \lambda) \mapsto \lambda - \operatorname{sign}(l, l', l'') \in \mathbb{Z}$$

is locally constant on the subset $\{(l, \lambda);\ l \cap l'' = \{0\}\}$ *of* $\Lambda(n) \times \mathbb{Z}$;

(ii) *the projection*:

$$[\Lambda(n) \times \mathbb{Z}]_{l'} \ni (l, \lambda) \mapsto l \in \Lambda(n)$$

is continuous.

Proof. — 1. is immediate by Theorem 1, 1. transporting the toplogy of $\Lambda_\infty(n)$ onto $(\Lambda(n) \times \mathbb{Z})$, via $h_{l'_\infty}$; 2. follows from the property (II.2.2), Theorem 1, Chap. II, Section 2 of the Maslov index and from Theorem 1.1.

We are next going to prove the analogues of theorem 1 and its corollaries 1 and 2 for the universal covering group $\mathrm{Sp}_\infty(n)$ of $\mathrm{Sp}(n)$.

Theorem 2. — 1. *For every* $l \in \Lambda(n)$, *the mapping*:

(1.6) $$\mathrm{Sp}_\infty(n) \ni s_\infty \mapsto (s, \mu_l(s_\infty)) \in \mathrm{Sp}(n) \times \mathbb{Z}$$

is an injection which is a bijection:

(1.7) $$H_l: \ \mathrm{Sp}_\infty(n) \to (\mathrm{Sp}(n) \times \mathbb{Z})_l$$

with $(\mathrm{Sp}(n) \times \mathbb{Z})_{l'} = \{(s, \sigma) \in \mathrm{Sp}(n) \times \mathbb{Z};\ \sigma \in \mu_l[s]\}$.

2. *The restriction of this bijection*:

(1.8) $$\{s_\infty \in \mathrm{Sp}_\infty(n);\ sl \cap l = \{0\}\} \to \{(s, \sigma) \in \mathrm{Sp}(n) \times \mathbb{Z};\ sl \cap l = \{0\};\ \sigma \in \mu_l[s]\}$$

is a homeomorphism when $\mathbb{Z}$ *is equipped with the discrete topology.*

Proof of 1. – If $(s, \mu_l(s_\infty)) = (s', \mu_l(s'_\infty))$, then $s = s'$ hence there exists $r \in \mathbb{Z}$ such that $s_\infty = \alpha^r s'_\infty$, and $\mu_l(s_\infty) = \mu_l(s'_\infty) + 4r$ in view of proposition (II.3.13), hence $r = 0$ and $s = s'$; this shows that the mapping (1.6) is an injection. By (II.3.9) in Theorem 2, Chap. II, Section 3, it is a bijection onto $(\mathrm{Sp}(n) \times \mathbb{Z})_{l'}$;

Proof of 2. – Immediate since μ_l is locally constant on the set $\{ s_\infty \in \mathrm{Sp}_\infty(n);\ sl \cap l = \{0\} \}$ (Theorem 1, Chap. II, § 3).

COROLLARY 3. – *The set of all homeomorphisms* $H_l (l \in \Lambda(n))$ *defined by* (1.7) *is a system of local charts of* $\mathrm{Sp}_\infty(n)$, *the transition functions being given by*:

$$H_l \circ H_{l'}^{-1} : (s, \mu_l(s)) \to (s, \mu_{l'}(s))$$

with:

$$\mu_l(s_\infty) - \mu_{l'}(s_\infty) = \mathrm{sign}(sl, l, l') - \mathrm{sign}(sl, sl', l'). \tag{1.6}$$

Proof. – Immediate by formula (3.7), Theorem 1, Chap. II, Section 3.

As we identified in corollary 2, $\Lambda_\infty(n)$ with a subset of $\Lambda(n) \times \mathbb{Z}$ by transporting the topology of $\Lambda_\infty(n)$ onto $(\Lambda(n) \times \mathbb{Z})_l$, we can identify $\mathrm{Sp}_\infty(n)$ with the set $(\mathrm{Sp}(n) \times \mathbb{Z})_l$ by transporting the topological group structure:

COROLLARY 4. – 1. *For every* $l \in \Lambda(n)$, *the set* $(\mathrm{Sp}(n) \times \mathbb{Z})_l$ *defined by* (1.7) *can be equipped with the structure of a topological group* $[\mathrm{Sp}(n) \times \mathbb{Z}]_l$ *for which*

$$H_l : \quad \mathrm{Sp}_\infty(n) \to [\mathrm{Sp}(n) \times \mathbb{Z}]_l$$

becomes an isomorphism of topological groups.

2. *The composition law of the group* $[\mathrm{Sp}(n) \times \mathbb{Z}]_l$ *is given by*:

(1.10) $(s, \sigma).(s', \sigma') = (ss', \sigma + \sigma' + \mathrm{sign}(l, sl, ss' l))$ with $\sigma \in \mu_l[s]$, $\sigma' \in \mu_l[s']$, *hence*: $\sigma + \sigma' + \mathrm{sign}(l, sl, ss' l)$ is in $\mu_l[s']$;

3. *The topology of* $[\mathrm{Sp}(n) \times \mathbb{Z}]_l$ *is characterized by the conditions*:

(i) *for every* $l'' \in \Lambda(n)$, *the mapping*:

$$[\mathrm{Sp}(n) \times \mathbb{Z}]_l \ni (s, \sigma) \mapsto \sigma - \mathrm{sign}(sl, l, l'') \in \mathbb{Z} \tag{1.11}$$

is locally constant on the subset:

$$\{ (s, \sigma);\ l \cap l'' = sl \cap l'' = \{0\} \} \text{ of } \mathrm{Sp}(n) \times \mathbb{Z},$$

(ii) *the projection*:

(1.12) $[\mathrm{Sp}(n) \times \mathbb{Z}]_l \ni (s, \sigma) \mapsto s \in \mathrm{Sp}(n)$ *is continuous.*

Proof of 1. – Immediate by theorem 2, (1.)

Proof of 2. – Immediate by formula (3.11)-(3.12) in theorem 2, Chap. II, § 3.

Proof of 3. – Immediate by property (3.6) in theorem 1, Chap. II, § 3.

The symplectic group $\mathrm{Sp}(n)$ acts transitively on the lagrangian Grassmanian $\Lambda(n)$; this action is covered by a transitive action of $\mathrm{Sp}_\infty(n)$ on $\Lambda_\infty(n)$ (*see* Leray's theorem, Chap. I, §1); the following result gives a precise description of that action in terms of the signature:

THEOREM 3. – *For* $l' \in \Lambda(n)$ *the topological group* $[\mathrm{Sp}(n) \times \mathbb{Z}]_{l'}$ *acts transitively on the topological space* $[\Lambda(n) \times \mathbb{Z}]_{l'}$ *by*:

$$(s, \sigma).(l, \lambda) = (sl, \sigma + \lambda + \mathrm{sign}(l', sl', sl)) \tag{1.13}$$

where

$$\begin{cases} \sigma \equiv n - \dim(l' \cap sl'), \qquad \lambda \equiv n - \dim(l \cap l'), \\ \mathrm{sign}(l', sl', sl) \equiv n + \dim(l' \cap sl') + \dim(l' \cap l) + \dim(l' \cap sl), \bmod 2; \end{cases} \tag{1.14}$$

hence also:

$$\sigma + \lambda + \mathrm{sign}(l', sl', sl) \equiv n - \dim(sl \cap l'), \bmod 2.$$

Proof. – Immediate in view of the definitions of $[\mathrm{Sp}(n) \times \mathbb{Z}]_{l'}$, $[\Lambda(n) \times \mathbb{Z}]_{l'}$ using formula (3.13) in Chapter II, Section 3, Theorem 2.

2. THE ACTION OF $\mathrm{Sp}_q(n)$ ON $\Lambda_{2q}(n)$. – In view of Leray's theorem, Chap. I, § 1, $\mathrm{Sp}_q(n)$ acts transitively on $\Lambda_{2q}(n)$ for $q \in \mathbb{N}^*$. The following results immediately deduced from Section 1 together with the description of $\mathrm{Sp}_q(n)$ and $\Lambda_{2q}(n)$ made in Chapter II, Section 4, describes this action.

THEOREM 1. – 1. *For every* $l'_q \in \Lambda_q(n)$, *the mapping*:

$$\Lambda_q(n) \ni l_q \mapsto (l, \mu_{2q}(l_q, l'_q)) \in \Lambda(n) \times \mathbb{Z}_{2q} \tag{2.1}$$

is an injection which is a bijection:

$$h_{l'_q} : \quad \Lambda_q(n) \to (\Lambda(n) \times \mathbb{Z}_{2q})_{l'} \tag{2.2}$$

where $(\Lambda(n) \times \mathbb{Z}_{2q})_{l'} = \{ (l, \lambda_{2q});\ \lambda \equiv n - \dim(l \cap l');\ \bmod 2 \}$.

2. *The restriction of this bijection*:

$$\{ l_q \in \Lambda_q(n);\ l \cap l' = \{0\} \} \to \{ (l, \lambda_{2q});\ l \cap l' = \{0\},\ \lambda \equiv n, \bmod 2 \} \tag{2.3}$$

is a homeomorphism when $\mathbb{Z}_{2q}$ *is equipped with the discrete topology.*

Proof. – Obvious, in view of theorem 1, Section 1, definition (4.3), Section 4, Chap. II of the Maslov index μ_{2q} on $\Lambda_q(n)$ and (4.5) (*ibid.*).

Exactly by the same argument as in the proof of the corollary 1 of theorem 1 of last section, one proves that the set of all homeomorphisms (2.3) is a system of local charts of $\Lambda_q(n)$, the transition functions being given by the formula:

$$\mu_{2q}(l_q, l'_q) - \mu_{2q}(l_q, l''_q) = \mathrm{sign}_{2q}(l, l', l'') - \mu_{2q}(l'_q, l''_q). \tag{2.4}$$

COROLLARY. — 1. *For* $l' \in \Lambda(n)$, *the set* $(\Lambda(n) \times \mathbb{Z}_{2q})_{l'}$ *can be equipped with a topology such that it becomes a topological space* $[\Lambda(n) \times \mathbb{Z}_{2q}]_{l'}$ *for which*

$$h_{l'_q}: \Lambda_\infty(n) \to [\Lambda(n) \times \mathbb{Z}_{2q}]_{l'}$$

is a homeomorphism;

2. *that topology is the topology characterized by the conditions:*

(i) *for every* $l'' \in \Lambda(n)$, *the mapping*

$$[\Lambda(n) \times \mathbb{Z}_{2q}]_{l'} \ni (l, \lambda_{2q}) \mapsto \lambda_{2q} - \operatorname{sign}_{2q}(l, l', l'') \in \mathbb{Z}_{2q} \tag{2.5}$$

is locally constant on the set $\{(l, \lambda_{2q});\ l \cap l'' = \{0\}\}$;

(ii) *the projection:*

$$[\Lambda(n) \times \mathbb{Z}_{2q}]_{l'} \ni (l, \lambda_{2q})) \mapsto l \in \Lambda(n)$$

is continuous.

Proof. — Similar, "mutatis mutandis", to the proof of corollary 2 of theorem 1 in last section.

Using the definition (4.9), Chap. II, Section 4 of the Maslov index on $\operatorname{Sp}_q(n)$ one proves exactly in the same way, using theorem 2 of last section and its corollary:

THEOREM 2. — 1. *For every* $l \in \Lambda(n)$, *the mapping*:

$$\operatorname{Sp}_q(n) \ni s_q \mapsto (s, \mu_l[s_q]_{4q}) \in \operatorname{Sp}(n) \times \mathbb{Z}_{4q} \tag{2.6}$$

is a bijection onto

$$(\operatorname{Sp}(n) \times \mathbb{Z}_{4q})_l = \{(s, \sigma_{4q});\ \sigma \in \mu_1[s]\}. \tag{2.7}$$

2. *the restriction of this bijection*:

$$\{s_{4q} \in \operatorname{Sp}_q(n);\ sl \cap l = \{0\}\} \mapsto \{(s, \sigma_{4q});\ sl \cap l = \{0\},\ \sigma \in \mu_l[s]_{4q}\} \tag{2.8}$$

is a homeomorphism when $\mathbb{Z}_{4q}$ *is equipped with the discrete topology.*

3. *the set of all those homeomorphisms* (2.8) *is a system of local charts of* $\operatorname{Sp}_q(n)$ *their transition functions being given by the relations:*

$$\mu_l[s_q]_{4q} - \mu_{l'}[s_q]_{4q} = \operatorname{sign}(sl, l, l') - \operatorname{sign}(sl, sl', l'). \tag{2.9}$$

Similarly, by theorem 2 in Chap. II, Section 2 we have:

THEOREM 3. — *For every* $l \in \Lambda(n)$, *the set* $(\operatorname{Sp}(n) \times \mathbb{Z}_{4q})_l$ *defined by* (2.7) *can be equipped with a topological group structure for which the homeomorphism* (2.8) *is an isomorphism of topological groups;*

$$(s, \sigma_{4q})(s', \sigma'_{4q}) = (ss', \sigma_{4q} + \sigma'_{4q} + \operatorname{sign}_{4q}(l, sl, ss'l) \tag{2.10}$$

with $\sigma \in \mu_l[s]$, $\sigma' \in \mu_l[ss']$, $\sigma + \sigma' + \operatorname{sign}_{4q}(l, sl, ss'l) \in \mu_l[ss']$,

3. *the topology of* $[\operatorname{Sp}(n) \times \mathbb{Z}_{4q}]_l$ *is characterized by the conditions*:

(i) *for every* $l'' \in \Lambda(n)$, *the mapping*

$$[\operatorname{Sp}(n) \times \mathbb{Z}_{4q}] \ni (s, \sigma_{4q}) \mapsto \sigma_{4q} - \operatorname{sign}_{4q}(sl, l, l'') \in \mathbb{Z}_{4q} \tag{2.11}$$

is locally constant on the set $\{(s, \sigma_{4q});\ l \cap l'' = sl \cap l'' = \{0\}\}$,

(ii) *the projection*:

(2.12) $[\operatorname{Sp}(n) \times \mathbb{Z}_{4q}]_l \ni (s, \sigma_{4q}) \mapsto s \in \operatorname{Sp}(n)$ *is continuous.*

The topological space $[\Lambda(n) \times \mathbb{Z}_{4q}]_l$ and the topological group $[\operatorname{Sp}(n) \times \mathbb{Z}_{4q}]_l$ being thus identified with respectively $\Lambda_{2q}(n)$ and $\operatorname{Sp}_q(n)$, the structure of q-geometry is then described by:

THEOREM 4. — *For* $l' \in \Lambda(n)$, *the topological group* $[\operatorname{Sp}(n) \times \mathbb{Z}_{4q}]_{l'}$ *acts transitively on the topological space* $[\Lambda(n) \times \mathbb{Z}_{4q}]_{l'}$ *by*:

$$(s, \sigma_{4q}) \cdot (l, \lambda_{4q}) = (sl, \sigma_{4q} + \lambda_{4q} + \operatorname{sign}_{4q}(l', sl', sl)). \tag{2.13}$$

Proof. — Immediate by Theorem 3, Chap. II, Section 1, and theorems 1, 2, 3 hereabove.

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(Manuscript received November 1989.)

Maurice de GOSSON
Faculté Libre des Sciences,
13, rue de Toul,
F-59046 Lille (Cedex)