J. fixed point theory appl. 3 (2008), 79–93

1661-7738/010079-15, *published online* 19.4.2008
DOI 10.1007/s11784-008-0065-6

Journal of Fixed Point Theory and Applications

Degree of rational mappings, and the theorems of Sturm and Tarski

A. Khovanskii and Y. Burda

To V. I. Arnold on the occasion of his 70th birthday

Abstract. We provide an explicit algorithm of computing the mapping degree of a rational mapping from the real projective line to itself. As a corollary we prove Sturm's theorem and a number of its generalizations. These generalizations are used to prove Tarski's theorem about real semialgebraic sets. Similarly a version of Tarski's theorem can be proved for an arbitrary algebraically closed field.

Mathematics Subject Classification (2000). 14P10.

Keywords. Mapping degree, semialgebraic set, Tarski's theorem.

This paper discusses a topological proof of the classical theorems of Sturm and Tarski. It also contains new results. A rational function of one real variable defines a mapping of the real projective line (which is homeomorphic to a circle) to itself. How the degree of such a mapping can be computed for a given rational function $f = P/Q$? We suggest a simple explicit formula for the degree, using a representation of f as a continued fraction. Such a representation can be found using Euclid's algorithm (that computes the greatest common divisor of P and Q). Thus a new application of this ancient algorithm is established.

The fundamental theorem of algebra (about the number of complex roots of a polynomial) has a topological explanation: the topological degree of a polynomial mapping of the complex projective line to itself is equal to the algebraic degree of the polynomial that defines this mapping. We show that Sturm's problem of counting the number of real roots of a real polynomial reduces to finding the topological degree of a mapping of the real projective line to itself, given by a certain rational function. This observation allows us to find a new solution of Sturm's problem and also to solve more general problems. For instance, it allows one to find the number of real roots satisfying $Q > 0$, where Q is a given real polynomial. These one-dimensional generalizations of Sturm's theorem allow us to prove Tarski's theorem, which is a wide multidimensional generalization of Sturm's theorem. Tarski's theorem states that the image of a real semialgebraic set under a polynomial mapping is a real semialgebraic set.

Tarski's theorem has a complex counterpart. It is significantly simpler than the real version and was known long before. It is interesting that our topological proof of Tarski's theorem extends almost word for word to the complex version. One only has to use the topological degree of a complex rational mapping instead of that of a real rational mapping. Because of the fundamental theorem of algebra this topological degree can be defined in purely algebraic terms. Thus our proof of Tarski's theorem extends to any algebraically closed field.

To make our paper self-contained, we recall some classical facts that we will need.

In Sections 1 and 2 auxiliary material is presented: in Section 1 we recall the notion of degree of a mapping from a circle to itself, and in Section 2 we recall the representation of a rational function as a continued fraction. In Section 3, which is central to this paper, we compute the topological mapping degree of a rational mapping of the projective line to itself. In Section 4 we show that this topological degree depends "constructively" on the coefficients of the rational function. In Section 5 we use the results of Section 4 to prove Tarski's theorem. In Section 7 we discuss the complex counterpart of that theorem.

The material of this note was presented by Khovanskii in his lectures on differential topology in the University of Toronto. Burda wrote up the notes of these lectures, which served as the first version of this paper.

1. Degree of a mapping from a circle to itself

For a continuous mapping $f : M_1 \to M_2$ of an oriented compact manifold M_1 to a connected oriented compact manifold M_2 of the same dimension, the mapping degree is defined. For us the main example will be the mapping degree of a mapping from a circle to itself, $M_1 = S^1_1$, $M_2 = S^1_2$. The theory of mapping degree is especially simple in this case (see for example [1]) and in this section we present its outline.

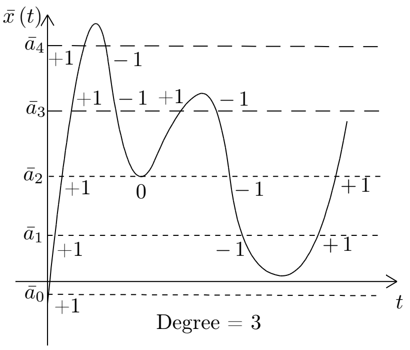
Let the circle S^1_1 be represented by the interval $I = [0, 1]$ with ends identified, $S^1_1 = I/\partial I$. Let also S^1_2 be represented as the real line $\mathbb{R}$ factored by the lattice $\mathbb{Z}$. A continuous mapping $x : I/\partial I \to \mathbb{R}/\mathbb{Z}$ can be lifted to a continuous mapping $\bar{x} : I \to \mathbb{R}$. The mapping $\bar{x}$ is defined up to an additive constant. The difference $\bar{x}(1) - \bar{x}(0)$ is thus a well defined integer, called the *mapping degree* of $x : I/\partial I \to \mathbb{R}/\mathbb{Z}$. *The mapping degree is invariant under continuous homotopy.* Indeed, from the definition one sees that under a continuous change of x the mapping degree changes continuously. But the degree is an integer, hence it is in fact constant under homotopy.

The mapping degree is a complete homotopy invariant: any two mappings having the same degree are homotopic. Indeed, for a mapping x of degree k the function $\bar{x}$ is of the form $\bar{x}(t) = kt + f(t)$, where $f(t)$ is a continuous periodic function with period 1. The homotopy $\bar{x}(t,\tau) = kt + \tau f(t)$ shows that x can be continuously deformed to a mapping x_0 which lifts to the mapping $\bar{x}_0(t) = kt$. *For every integer k there is a mapping of degree k*: an example is the mapping x_0, which lifts to kt.

We will need the following geometric way of computing the degree. We will say that a value $a \in S^1_2$ is *almost regular* for the continuous mapping $x : S^1_1 \to S^1_2$ if it has finitely many preimages. Let b be a preimage of a, i.e. $x(b) = a$. Let t be a local coordinate in a small neighborhood of b (with $t(b) = t_0$) on the circle S^1_1, let u be a local coordinate in a small neighborhood of a (with $u(a) = u_0$) on S^1_2 and suppose that these coordinates are compatible with the orientations of the circles. The function $F_b(t) = u(x(t)) - u_0$ has a well defined sign on each of the two connected components $t < t_0$ and $t > t_0$ of the punctured neighborhood of b. We define the *index* of the function x at the point b depending on the change of the sign of F_b from the component $t < t_0$ to $t > t_0$. The index at b is $+1$ if the sign changes from minus to plus; it is -1 if the sign changes from plus to minus; and it is 0 if the sign does not change.

Claim 1. *The degree of a continuous mapping from a circle to itself is equal to the number of preimages of any almost regular value, counted with the corresponding indices.*

Proof. Let $\bar{a}_i \in \mathbb{R}$ denote all the lifts of $a \in \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$ (the difference between any two lifts $\bar{a}_i$ is an integer). The set of preimages of a under x consists of the preimages of the points $\bar{a}_i$ under $\bar{x}$. The degree k of x is $\bar{x}(1) - \bar{x}(0)$. Suppose for definiteness that $k = \bar{x}(1) - \bar{x}(0) > 0$. Then there are exactly k numbers $\bar{a}_i \in \mathbb{R}$ satisfying the inequalities $\bar{x}(0) \le \bar{a}_i < \bar{x}(1)$. For every such $\bar{a}_i$, the number of preimages $\bar{x}^{-1}(\bar{a}_i)$, counted with indices, is readily seen to be one. For every $\bar{a}_i$ not satisfying any of the inequalities, the number of preimages $\bar{x}^{-1}(\bar{a}_i)$ counted with indices is zero. This proves Claim 1.

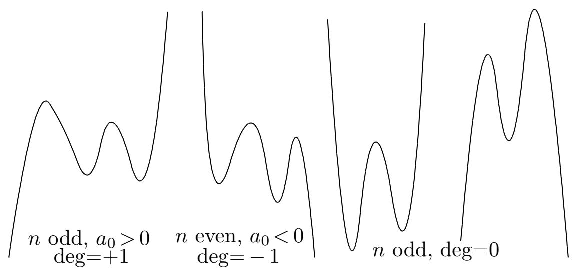


In the following we will consider the circle as the real projective line (topologically $\mathbb{R}P^1$ is just a circle). We will think of $\mathbb{R}P^1$ as the standard line $\mathbb{R}^1$, compactified by one point at infinity and oriented in the usual way.

Example. Consider a real polynomial $P = a_0x^n + a_1x^{n-1} + \cdots + a_n$ as a mapping $P : \mathbb{R}P^1 \to \mathbb{R}P^1$. The degree $[P]$ of this mapping is

$$[P] = \begin{cases} +1 & \text{if } n \text{ is odd and } a_0 > 0, \\ -1 & \text{if } n \text{ is even and } a_0 < 0, \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Indeed, when $x \to \pm\infty$ a nonconstant polynomial P tends to plus or minus infinity. The sign of the infinity depends on the parity of the degree of and on the sign of the leading coefficient a_0. By considering all possible combinations we get the above formula.



2. Representation of a rational function as a continued fraction

A very nice introduction to the theory of continued fractions can be found in [2]. Every rational function f can be represented as a finite continued fraction

$$f = P_0 + \cfrac{1}{P_1 + \ddots + \cfrac{1}{P_k}}, \tag{1}$$

where $P_0, \ldots, P_k$ are polynomials. We recall how this can be done.

To represent $f = P/Q$ as a continued fraction one has to repeat the following two operations:

1) *Division with remainder.* This operation is performed if the degree of Q is smaller than that of P. This operation results in representing P/Q as $P_0 + \tilde{P}/Q$, where P_0 is the quotient and $\tilde{P}$, with $\deg \tilde{P} < \deg Q$, is the remainder of division of the polynomial P by the polynomial Q.
2) *Taking inverse of fraction.* This operation is performed if the degree of P is smaller than that of Q. This operation results in representation of the fraction P/Q as $1/\frac{Q}{P}$.

Consider the pair P, Q. If $\deg P \geq \deg Q$, by applying the first operation we can represent P/Q as $P_0 + \tilde{P}/Q$. Thus we get a new pair of polynomials $\tilde{P}, Q$ that has smaller sum of degrees than the pair P, Q. If $\deg P < \deg Q$, then we apply the second operation to represent P/Q as $1/F$, where $F = Q/P$. Then we can apply the first operation to F, and represent it as $Q/P = Q_0 + \tilde{Q}/P$. We get a new pair of polynomials $\tilde{Q}, P$, which has a smaller sum of degrees than the initial pair P, Q. Now the problem of representing P/Q as a continued fraction is reduced to the problem of representing $\tilde{P}/Q$ or $\tilde{Q}/P$ as a continued fraction, which is simpler, since these functions have a smaller sum of degrees of the numerator and denominator.

If we continue this process we will represent the rational function P/Q as a continued fraction. *The algorithm described above in fact coincides with Euclid's algorithm for finding the greatest common divisor of P, Q.*

3. Degree of a rational mapping from the real projective line to itself

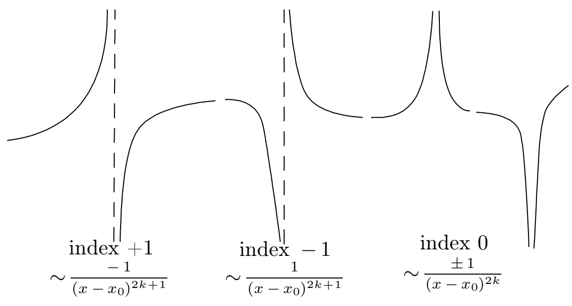
Let f be a real rational function. We will consider it as a mapping $f : \mathbb{R}P^1 \to \mathbb{R}P^1$ of the real projective line to itself. We will answer the following problem:

Problem 1. For given real polynomials P, Q compute the degree of the mapping $f : \mathbb{R}P^1 \to \mathbb{R}P^1$, where $f = P/Q$.

A nonconstant rational function attains each of its values finitely many times. Hence we can use Claim 1 to compute the mapping degree.

Claim 2. *If rational functions f_1, f_2 do not have common poles on $\mathbb{R}P^1$, then* $\deg(f_1 + f_2) = \deg f_1 + \deg f_2$.

Proof. We will compute the degree of a rational function as the number of preimages of the point ∞ counted with indices. Every pole b of $f_1 + f_2$ is either a pole of f_1, or a pole of f_2. Suppose b is a pole of f_1. Then the assumption tells us that f_2 is regular near b. Hence f_1 and $f_1 + f_2$ behave in the same manner near b. This means that b has the same index as the preimage of ∞ under f_1 and under $f_1 + f_2$. The same considerations apply to poles of f_2, which are regular points for f_1. This proves Claim 2.



If f_1 and f_2 have common poles, in general $\deg(f_1 + f_2) \neq \deg f_1 + \deg f_2$. For example. if $f_1 = f_2$ and $\deg f_1 \neq 0$, then $\deg(f_1 + f_2) = \deg f_1 \neq \deg f_1 + \deg f_2$.

Claim 3. *For a rational function f that is not identically zero,* $\deg f^{-1} = -\deg f$.

Proof. We can assume that f is not identically 1. We will compute the degree of the rational functions f and f^{-1} as the number of preimages of the point 1 counted with indices. It is clear that the preimages of 1 for f and f^{-1} are the same. Also the index of b as a preimage of f is minus the index of b, considered as a preimage of 1 for f^{-1}. This proves Claim 3.

To solve Problem 1 we will need to represent the rational function f as a continued fraction. The following theorem is one of the main results of this paper.

Theorem 4. *Let a rational function f be represented as a continued fraction* (1). *Then the degree* $\deg f$ *of the mapping* $f : \mathbb{R}P^1 \to \mathbb{R}P^1$ *is given by*

$$\deg f = [P_0] - [P_1] + \cdots + (-1)^k [P_k].$$

Proof. Induction on the number $k+1$ of polynomials appearing in the representation (1) of f as a continued fraction. Theorem 4 was already considered in the Example of Section 1 for functions f that are polynomials. We can assume that the polynomial P_0 from the representation (1) of f is not zero. Indeed, if P_0 is zero, then we can compute the degree of f^{-1} instead of the degree of f (see Claim 3), for which the polynomial P_1 plays the role of P_0.

If for $f = P/Q$ the polynomial P_0 is not zero, then $P/Q = P_0 + \tilde{P}/Q$, where $\tilde{P}$ is the remainder of division of P by Q. The only pole of P_0 is ∞, where $\tilde{P}/Q$ is zero. Claims 2 and 3 show that in this case $\deg(P/Q) = \deg P_0 + \deg(\tilde{P}/Q) = [P_0] - \deg(Q/\tilde{P})$. The representation of $Q/\tilde{P}$ as a continued fraction is

$$Q/\tilde{P} = P_1 + \cfrac{1}{P_2 + \ddots + \cfrac{1}{P_k}}.$$

It contains k polynomials. By induction hypothesis, $\deg(Q/\tilde{P}) = [P_1] - [P_2] + \cdots + (-1)^{k-1}[P_k]$. Hence $\deg(P/Q) = [P_0] - [P_1] + \cdots + (-1)^k [P_k]$, which proves the theorem.

4. Constructivity of the degree of a rational function as a function of its coefficients

In the previous section we have computed explicitly the degree of a rational mapping of the projective line to itself. Here we will prove that our computation was "constructive". The function sign on the real line there has value $+1$ at positive arguments, 0 at the point 0, and -1 at negative arguments. In this section we will compute $\sum_{Q(a)=0} \operatorname{sign}(P(a))$ for a pair of polynomials P, Q. This number is closely related to the mapping degree and is important for what follows.

Definition. A *basic semialgebraic set* in $\mathbb{R}^n$ is a subset defined by a system of algebraic equations and inequalities $R_1 = 0, \ldots, R_\mu = 0$, $Q_1 > 0, \ldots, Q_\nu > 0$, where R_i, Q_j are real polynomials. A *semialgebraic set* is a finite union of basic semialgebraic sets.

Lemma 5. *Finite unions, finite intersections and complements of semialgebraic sets are semialgebraic. The preimage of a semialgebraic set under a polynomial mapping is a semialgebraic set.*

The proof follows directly from definitions (the last property relies on the fact that superposition of polynomials is a polynomial).

Definition. A real function defined on a real semialgebraic set in $\mathbb{R}^n$ is called *constructive* if it attains finitely many values and all its level sets are real semialgebraic.

Lemma 6. *Constructive functions defined on a given semialgebraic set form a real algebra* (i.e. the set of constructive functions is closed under arithmetic operations and multiplication by real constants).

We will identify a pair of polynomials $P = a_0x^N + \cdots + a_N$ and $Q = b_0x^M + \cdots + b_M$ with the point $(a_0, \ldots, a_N, b_0, \ldots, b_M)$ in the space $\mathbb{R}^{N+1} \times \mathbb{R}^{M+1}$. Let $f = P/Q$, where the polynomial Q is not identically zero. Consider the degree of the mapping $f : \mathbb{R}P^1 \to \mathbb{R}P^1$ as a function $\deg : \mathbb{R}^{N+1} \times (\mathbb{R}^{M+1} \setminus \{0\}) \to \mathbb{Z}$.

Theorem 7. *The function* deg *defined above is constructive.*

Proof. Induction on the sum $N + M$ of the degrees of P and Q. Let $P = a_0x^N + \cdots + a_N, Q = b_0x^M + \cdots + b_M$. If $N = M = 0$, then deg is identically zero. We will present the level sets of deg as unions of semialgebraic sets as follows.

If P is identically zero (which defines a semialgebraic condition on the coefficients), then $\deg(P/Q) = 0$. If $P \neq 0$, we will subdivide further. On the sets $a_0 = 0$ and $b_0 = 0$, the function deg is constructive by the induction hypothesis.

If $a_0 \neq 0$ and $b_0 \neq 0$, we can assume that $M \geq N$, for otherwise we can use the fact that $\deg(P/Q) = -\deg(Q/P)$ (the polynomial P is not identically zero). Divide P by Q with remainder: $P/Q = P_0 + R/Q$, where P_0, R are polynomials with coefficients that are rational functions on $\mathbb{R}^{N+1} \times \mathbb{R}^{M+1}$, whose denominators are powers of the variable b_0 (e.g. $P_0 = \frac{a_0}{b_0}x^{M-N} + \frac{a_1b_0 - b_1a_0}{b_0^2}x^{M-N-1} + \cdots$). We have $\deg(P/Q) = \deg P_0 + \deg(R/Q)$. Divide the space of coefficients into the half-spaces $b_0 > 0$ and $b_0 < 0$. In the half-space $b_0 > 0$ multiply P_0 and R by a power of b_0 large enough to make the coefficients of P_0 and R polynomials in $a_0, \ldots, a_N, b_0, \ldots, b_M$. When we multiply the polynomials by a positive constant, the degrees of the mappings P_0 and R/Q do not change and we can apply the inductive hypothesis. In the half-space $b_0 < 0$ we can achieve the same by multiplying P_0 and R by a large enough power of $-b_0$. By the induction hypothesis the function $\deg(R/Q)$ is constructive. The function $\deg P_0$ is clearly also constructive (see the Example from Section 1). Hence the function $\deg(P/Q)$, which is equal to $\deg P_0 + \deg(R/Q)$, is constructive. This proves Theorem 7.

We will need the following simple observation.

Lemma 8. *Every zero of a polynomial P is a pole of the function P'/P, having index -1.*

Proof. If near the zero x_0 the polynomial P is equal to $a_k(x-x_0)^k+\cdots$, where $k>0$, then $P'/P=k/(x-x_0)+\cdots$.

In the same manner we can prove the following lemma:

Lemma 9. *Every zero x_0 of the polynomial P which is not a zero of the polynomial Q is a pole of the function QP'/P, of index -1 if $Q(x_0)>0$, and $+1$ if $Q(x_0)<0$.*

Let P,Q be real polynomials and set $f=QP'/P$. Let (1) be the representation of f as a continued fraction.

Lemma 10. *The sum $\sum_{P(x)=0}\operatorname{sign}(Q(x))$ is equal to $\deg(QP'/P)-\deg P_0$. Using the representation* (1)*, this sum can also be written as*

$$\sum_{P(x)=0}\operatorname{sign}(Q(x))=[P_1]-[P_2]+\cdots+(-1)^{k+1}[P_k].$$

Proof. Consider the mapping $f=QP'/P:\mathbb{R}P^1\to\mathbb{R}P^1$. The finite poles of f are exactly at the zeros of P where Q does not vanish. According to Lemma 9, the poles have the following indices. If $P(x_0)=0$ and $Q(x_0)>0$, then x_0 is a pole of f of index -1. If $P(x_0)=0$ and $Q(x_0)<0$, then x_0 is a pole of f of index $+1$. If $P(x_0)=0$ and $Q(x_0)=0$, then x_0 is not a pole of f. We have not yet considered the point at infinity. To do so, divide QP' by P with remainder and represent f as a continued fraction: $f=P_0+R/P$. The difference $f-P_0$ is continuous at infinity. Hence the index of ∞ as a pole of f is $[P_0]$. This computation implies the first statement of Lemma 10. The second statement follows from the first and from Theorem 4.

Corollary 11. *Let $X=(\mathbb{R}^{N+1}\smallsetminus\{0\})\times\mathbb{R}^{M+1}$ be the subset of the space of coefficients of polynomials P,Q where P does not vanish identically. The sum $\sum_{P(x)=0}\operatorname{sign}(Q(x))$ is a constructive function on X.*

5. Sturm theorem and its generalizations

We will start from the following question.

Problem 2. *How many different real roots does a given real polynomial P have?*

The number of different zeroes of P is equal to the sum

$$\sum_{P(x)=0}\operatorname{sign}(Q(x))$$

for Q identically equal to 1. We can compute this sum explicitly. Thus we have an explicit way of answering Problem 2. Here is one of the formulations for the answer: *the number of different real roots of the polynomial P is equal to minus the degree of the mapping $f : \mathbb{R}P^1 \to \mathbb{R}P^1$, where $f = P'/P$. If $f = P'/P$ is represented as a continued fraction* (1), *then the number of real roots of P is* $-[P_0] + [P_1] - \dots + (-1)^{k+1}[P_k]$.

Let us now consider a more general problem.

Problem 3. *How many different real roots does a given real polynomial P have in the domain $Q > 0$, where Q is another polynomial?*

Denote by Q_+ the number of zeroes of P in the domain $Q > 0$, and by Q_- the number of zeroes of P in $Q < 0$. Lemma 10 allows us to compute $Q_+ - Q_- = \sum_{P(x)=0} \operatorname{sign}(Q(x))$. Applying Lemma 10 to the function Q^2 instead of Q, we get $Q_+ + Q_- = \sum_{P(x)=0} \operatorname{sign}(Q(x)^2)$. This allows us to find Q_+, namely

$$Q_+ = \frac{1}{2} \sum_{P(x)=0} \operatorname{sign}(Q) + \frac{1}{2} \sum_{P(x)=0} \operatorname{sign}(Q^2).$$

Problem 3 is now solved. We see that to solve Problem 3 it is enough to compute the representations of QP'/P and Q^2P'/P as continued fractions.

Corollary 12. *The number of different zeroes of a real polynomial P in a given interval (a, b) can be computed explicitly.*

Indeed, the interval (a, b) has the form $Q > 0$ for the polynomial $Q(x) = (b - x)(x - a)$.

The famous theorem of Sturm gives another way of proving Corollary 12 (Sturm's theorem provides the solution to a very special case of Problem 3).

Problem 4. *How many different real roots does a given real polynomial P have in the set defined by the conditions $R_1 = 0, \dots, R_\mu = 0$, $Q_1 > 0, \dots, Q_\nu(x) > 0$, where $P, R_1, \dots, R_\mu, Q_1, \dots, Q_\nu$ are given real polynomials?*

In the computation of the number Q_+ (see Problem 3) we encountered the formula $\frac{1}{2}(\operatorname{sign}(Q) + \operatorname{sign}(Q^2))$. This formula is related to the polynomial L of one variable q, given by $L(q) = \frac{1}{2}(q^2 + q)$. This polynomial has the following property: it vanishes when $q = -1$ or $q = 0$, and it is equal to 1 when $q = 1$. Hence $L(\operatorname{sign}(Q(x))) = 1$ if and only if $Q(x) > 0$.

To solve problem 4 we will generalize this polynomial. Define the polynomial L of the variables $(r_1, \dots, r_\mu, q_1, \dots, q_\nu)$ by the following formula:

$$\begin{aligned} L &= (1 - r_1)(1 + r_1) \dots (1 - r_\mu)(1 + r_\mu) \frac{q_1(1 + q_1)}{2} \dots \frac{q_\nu(1 + q_\nu)}{2} \\ &= \frac{1}{2^\nu} \sum_{\substack{\varepsilon_i = 0 \text{ or } 1 \\ \sigma_i = 1 \text{ or } 2}} (-1)^{\varepsilon_1 + \dots + \varepsilon_\mu} r_1^{2\varepsilon_1} \dots r_\mu^{2\varepsilon_\mu} q_1^{\sigma_1} \dots q_\nu^{\sigma_\nu}. \end{aligned}$$

The polynomial L has the following property. Let Y be the finite subset in the domain of the polynomial L, consisting of the points where every coordinate is either -1, 0 or $+1$. The polynomial L vanishes at all points of Y except the point

$r_1 = \cdots = r_\mu = 0, q_1 = \cdots = q_\nu = 1$, where it is 1. Hence the number of interest in Problem 4 can be expressed as

$$\sum_{P(x)=0} L(\mathrm{sign}(R_1(x)), \ldots, \mathrm{sign}(R_\mu(x)), \mathrm{sign}(Q_1(x)), \ldots, \mathrm{sign}(Q_\nu(x))).$$

This sum can be rewritten as

$$\frac{1}{2^\nu} \sum_{\substack{\varepsilon_i=0 \text{ or } 1 \\ \sigma_i=1 \text{ or } 2}} (-1)^{\varepsilon_1+\cdots+\varepsilon_\mu} \sum_{P(x)=0} \mathrm{sign}(R_1(x)^{2\varepsilon_1} \ldots R_\mu(x)^{2\varepsilon_\mu} Q_1(x)^{\sigma_1} \ldots Q_\nu(x)^{\sigma_\nu}).$$

Every summand

$$\sum_{P(x)=0} \mathrm{sign}(R_1(x)^{2\varepsilon_1} \ldots R_\mu(x)^{2\varepsilon_\mu} Q_1(x)^{\sigma_1} \ldots Q_\nu(x)^{\sigma_\nu})$$

can be computed explicitly by using Lemma 10.

This solves Problem 4.

Theorem 13. *Let $\mathbb{R}^N$ be the space of coefficients of all the polynomials appearing in Problem 4. Let X be the semialgebraic subset corresponding to the coefficients for which P does not vanish identically. The function f which assigns to $x \in X$ the number of roots of P in Problem 4 is constructive.*

Proof. This follows from the explicit formula for the number of roots in Problem 4 and from Corollary 11.

Problem 5. *Determine whether the system of inequalities $Q_1 > 0, \ldots, Q_\nu > 0$, has a real solution, where $Q_1, \ldots, Q_\nu$ are given real polynomials.*

It is enough to consider the case when none of the polynomials $Q_1, \ldots, Q_\nu$ is identically zero: otherwise the problem reduces to the same problem with a smaller number of inequalities. Consider the polynomial $Q = Q_1 \ldots Q_\nu$. Let N be an integer greater than the sum of the degrees of $Q_1, \ldots, Q_\nu$. The rational function $R = Q/(1 + x^N)$ cannot be constant. The zeroes of its derivative are the roots of the nonzero polynomial $P = Q'(1 + x^N) - NQx^{N-1}$. By Rolle's theorem, R has a zero of its derivative in every connected component of the set $Q_1 > 0, \ldots, Q_\nu > 0$. Thus the system of inequalities $Q_1 > 0, \ldots, Q_\nu > 0$ has a solution if and only if the system $P = 0, Q_1 > 0, \ldots, Q_\nu > 0$, containing the nontrivial equation $P = 0$, has a solution. According to Theorem 13, the number of solutions of this system is a constructive function in the space of its coefficients. Thus Problem 5 has a constructive solution. We have proved the following theorem.

Theorem 14. *Let $\mathbb{R}^N$ be the space of all the coefficients of polynomials from Problem 5. The subset of $\mathbb{R}^N$ corresponding to the systems having a solution is semialgebraic.*

6. Tarski's theorem

We will prove the following theorem.

Tarski's theorem ([3]–[6]). *The image of a semialgebraic set $X \subset \mathbb{R}^m$ under a polynomial mapping $T = (T_1, \dots, T_k) : \mathbb{R}^m \to \mathbb{R}^k$ is a semialgebraic set.*

Proof. Step 1 (reduction). It is enough to prove the theorem for the projection $\pi_0 : \mathbb{R}^{n+1} \to \mathbb{R}^n$, forgetting one coordinate: $\pi_0(x_0, \dots, x_n) = (x_1, \dots, x_n)$. Indeed, let X be a semialgebraic set. Define the semialgebraic set Y in $\mathbb{R}^{m+k}$ with coordinates $(x_1, \dots, x_m, y_1, \dots, y_k)$, by the conditions

$$(x_1, \dots, x_m) \in X, \quad y_1 = T_1(x_1, \dots, x_m), \ \dots, \ y_k = T_k(x_1, \dots, x_m).$$

The set Y is the graph of the mapping T. The image of Y under the projection $\pi : \mathbb{R}^{m+k} \to \mathbb{R}^k$, $\pi(x_1, \dots, x_m, y_1, \dots, y_k) = (y_1, \dots, y_k)$, coincides with $T(X)$. Hence by proving the theorem for projections, we will prove it for every polynomial mapping. The projection π can be written as the composition of projections, each forgetting only one coordinate. Thus it is enough to prove the theorem for π_0.

We can assume that X is a basic semialgebraic set (every semialgebraic set is a union of basic semialgebraic sets, and the image of a union is the union of images). So let $X = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid R_1(\mathbf{x}) = 0, \dots, R_\mu(\mathbf{x}) = 0, Q_1(\mathbf{x}) > 0, \dots, Q_\nu(\mathbf{x}) > 0\}$. We want to prove that $\pi_0(X)$ is semialgebraic.

Step 2: Reduction of Tarski's theorem to Problems 4–5. Write every polynomial defining X as a polynomial of one variable x_0 with coefficients which are polynomials in $x_1, \dots, x_n$, for example $R_1(x_0, \dots, x_n)$ will be considered as $a_0(x_1, \dots, x_n)x_0^{N_1} + \dots + a_{N_1}(x_1, \dots, x_n)$. The question whether $(x_1, \dots, x_n)$ belongs to $\pi_0(X)$ is equivalent to the question whether the system $R_1 = 0, \dots, R_\mu = 0$, $Q_1 > 0, \dots, Q_\nu > 0$ has a solution (for the coefficients evaluated at $(x_1, \dots, x_n)$).

Let X_i denote the subset in the space of coefficients where $R_1, \dots, R_{i-1}$ are identically zero, but R_i is not. According to Theorem 13, on X_i the number of different solutions of the system $R_i = 0, \dots, R_\mu = 0$, $Q_1 > 0, \dots, Q_\nu > 0$ is a constructive function. The union of non-zero level sets of this function is a semialgebraic set, i.e. the intersection of $\pi_0(X)$ with X_i is semialgebraic.

Let Y denote the subset of the space of coefficients where all $R_1, \dots, R_\mu$ are identically zero. According to Theorem 14, the subset of Y where the system $Q_1 > 0, \dots, Q_\nu > 0$ has a solution, is semialgebraic. This is the intersection of $\pi_0(X)$ with Y. As $\pi_0(X)$ is the union of its intersections with the sets X_i and Y, it is semialgebraic. This proves Tarski's theorem.

Tarski was interested in the following problem: *does any formula that uses semialgebraic sets, finite unions, intersections and symbols $\neg, \exists, \forall$ define a semialgebraic set?* For example does the expression $(y \in \mathbb{R} \mid \forall \varepsilon > 0 \ \exists x \in X$ such that $-\varepsilon < x - y < \varepsilon)$, which gives the closure of the semialgebraic set $X \subset \mathbb{R}$, always define a semialgebraic set in $\mathbb{R}$? Tarski proved that the answer to this question is positive. Indeed, finite intersections, unions and complements of semialgebraic sets are semialgebraic, the $\exists$ quantifier corresponds to projection (the formula $y \mid \exists x \in X$ means that y belongs to the projection of X along the coordinate x), and the $\forall$ quantifier is equivalent to the formula $(\neg(\exists(\neg)))$. In particular, the closure of a semialgebraic set is semialgebraic.

7. Complex version of Tarski's theorem

A *basic complex semialgebraic set* in $\mathbb{C}^n$ is defined by a system $R_1 = \cdots = R_\mu = 0$, $Q_1 \neq 0, \ldots, Q_\nu \neq 0$ of polynomial equalities and inequalities. A *complex semialgebraic set* is a finite union of basic complex semialgebraic sets.

Complex semialgebraic sets differ strongly from real semialgebraic sets. For instance, a complex semialgebraic set in $\mathbb{C}^n$ either covers almost all of it, or only a very small part of it. More precisely, for a complex semialgebraic set $F \subset \mathbb{C}^n$ in $\mathbb{C}^n$ the following holds.

Claim 15. *Either F is contained in some algebraic hypersurface $\Sigma \subset \mathbb{C}^n$, or F contains some Zariski open set $U \subset \mathbb{C}^n$ (i.e. $U = \mathbb{C}^n \setminus \Sigma$, where Σ is an algebraic hypersurface).*

Proof. The first case occurs when in the definition of every basic complex semialgebraic set in the definition of F there are nontrivial equations, and the second when at least one of the basic sets is defined only by inequalities.

Corollary 16. (1) *If a complex semialgebraic set F is not a subset of measure zero, then it contains some Zariski open set.*

(2) *On the complex line $\mathbb{C}$ the only complex semialgebraic sets are finite sets and their complements.*

However, complex and real semialgebraic sets share some properties. Most significant of them is Tarski's theorem:

Complex version of Tarski's theorem. *The image of the complex semialgebraic set $X \subset \mathbb{C}^m$ under a polynomial mapping $T = (T_1, \ldots, T_k) : \mathbb{C}^m \to \mathbb{C}^k$ is a complex semialgebraic set.*

The complex version of Tarski's theorem is simpler than the real one and was known long before. Surprisingly, our topological proof of Tarski's theorem also applies to the complex case. We will show how this can be realized.

The degree $\deg f$ of a complex rational function $f = P/Q$ is the degree of the mapping $f : \mathbb{C}P^1 \to \mathbb{C}P^1$. It is clear that if the polynomials P and Q have no common divisors, then the degree of f is the maximum of the degrees of the polynomials P and Q. If P and Q do have a common divisor, then for the computation of the degree of f we have to cancel the common divisor first. To do this we apply Euclid's algorithm. By using it we can represent f as a continued fraction (1).

Claim 17. *For the function f represented as a continued fraction* (1)*, the following equality holds:* $\deg f = \deg P_0 + \deg P_1 + \cdots + \deg P_k$.

Proof. If f is a polynomial, then its degree is equal to the algebraic degree of f. For every rational function f we have $\deg f = \deg f^{-1}$. If f and g are complex rational functions with no common poles, then $\deg(f + g) = \deg f + \deg g$. The proof is now the same as the proof of Theorem 4.

A function defined on a complex semialgebraic set is *complex constructive* if it attains only finitely many values and all its level sets are complex semialgebraic. Consider $\deg(P/Q)$ as a function on the space $\mathbb{C}^{N+1} \times (\mathbb{C}^{M+1} \setminus \{0\})$ of coefficients of polynomials P of degree $\leq N$ and non-zero polynomials Q of degree $\leq M$.

Corollary 18. *The function* $\deg : \mathbb{C}^{N+1} \times (\mathbb{C}^{M+1} \setminus \{0\}) \to \mathbb{Z}$ *is complex constructive.*

Corollary 18 follows from Claim 17 (cf. the proof of Theorem 7).

Define $\mathrm{sn} : \mathbb{C} \to \mathbb{Z}$ to be 0 at zero and 1 for all other arguments. For a pair of polynomials P, Q consider the function $f = QP'/P$ and the representation of f as a continued fraction (1).

Lemma 19. *The sum* $\sum_{P(x)=0} \mathrm{sn}(Q(x))$ *is equal to* $\deg(QP'/P) - \deg P_0$*. Using the representation* (1) *of f, this sum can also be represented as*

$$\sum_{P(x)=0} \mathrm{sn}(Q(x)) = \deg P_1 + \deg P_2 + \cdots + \deg P_k.$$

Lemma 19 can be proved in the same manner as Lemma 10. This lemma immediately implies the following corollary:

Corollary 20. *Let* $X = (\mathbb{C}^{N+1} \setminus \{0\}) \times \mathbb{C}^{M+1}$ *be the subset of the space of coefficients of polynomials P, Q where P is not identically zero. Then the sum* $\sum_{P(x)=0} \mathrm{sn}(Q(x))$ *is a constructive function on X.*

Problem $4'$. *How many different zeroes does a polynomial P have on the set defined by $R_1 = 0, \ldots, R_\mu = 0$, $Q_1 \neq 0, \ldots, Q_\nu \neq 0$, where $P, R_1, \ldots, R_\mu, Q_1, \ldots, Q_\nu$ are given complex polynomials?*

Define the polynomial $\tilde{L}$ of the variables $(r_1, \ldots, r_\mu, q_1, \ldots, q_\nu)$ by the following formula:

$$\tilde{L} = (1 - r_1) \ldots (1 - r_\mu) q_1 \ldots q_\nu = \sum_{\varepsilon_i = 0 \text{ or } 1} (-1)^{\varepsilon_1 + \cdots + \varepsilon_\mu} r_1^{\varepsilon_1} \ldots r_\mu^{\varepsilon_\mu} q_1 \ldots q_\nu.$$

It has the following property. Let $\tilde{Y}$ be the finite subset in the domain of $\tilde{L}$, consisting of the points where every coordinate is either 0 or +1. The polynomial $\tilde{L}$ vanishes at all points of $\tilde{Y}$ except the point $r_1 = \cdots = r_\mu = 0, q_1 = \cdots = q_\nu = 1$, where $\tilde{L}$ is 1. Hence the number of solutions of Problem $4'$ at a point x in the space of parameters is

$$\sum_{P(x)=0} \tilde{L}(\mathrm{sn}(R_1(x)), \dots, \mathrm{sn}(R_\mu(x)), \mathrm{sn}(Q_1(x)), \dots, \mathrm{sn}(Q_\nu(x))).$$

This sum can be rewritten as

$$\sum_{\varepsilon_i=0 \text{ or } 1} (-1)^{\varepsilon_1+\dots+\varepsilon_\mu} \sum_{P(x)=0} \mathrm{sn}(R_1(x)^{\varepsilon_1} \dots R_\mu(x)^{\varepsilon_\mu} Q_1(x) \dots Q_\nu(x)).$$

Every summand

$$\sum_{P(x)=0} \mathrm{sn}(R_1(x)^{\varepsilon_1} \dots R_\mu(x)^{\varepsilon_\mu} Q_1(x) \dots Q_\nu(x))$$

in this formula can be computed explicitly by using Lemma 19.

This solves Problem $4'$.

Corollary 21. *Let $\mathbb{C}^N$ be the space of coefficients of polynomials appearing in Problem $4'$. Let X be the semialgebraic subset corresponding to the coefficients for which P does not vanish identically. The function f which assigns to $x \in X$ the number of solutions of Problem $4'$ is constructive.*

Proof. This follows from the explicit formula for the number of solutions of Problem $4'$ and from Corollary 20.

The complex analogue of Problem 5 is very easy: *the system of polynomial inequalities $Q_1 \neq 0, \dots Q_\nu \neq 0$ has a solution if and only if at least one of the polynomials Q_i is nonzero.*

The proof of the complex version of Tarski's theorem now repeats the proof of the real version.

Corollary 22. *Any bounded complex semialgebraic set in $\mathbb{C}^n$ consists of finitely many points.*

Proof. The only complex semialgebraic sets on the complex line $\mathbb{C}$ are finite sets or their complements. The projection of a bounded complex semialgebraic set on any coordinate line is a bounded complex semialgebraic set (complex Tarski theorem) on the complex line, hence is a finite set. Thus the original set is also finite.

The proof of the complex Tarski theorem given above extends automatically to any algebraically closed field. For such fields the degree of a rational function $f = P/Q$, where P and Q are polynomials having no common divisors, should be *defined* as the maximum of the degrees of P and Q. In the case of arbitrary algebraically closed fields the topological definition of the degree of a rational function as the topological degree of a mapping from the projective line to itself is impossible. However, all the arguments that we used to prove the complex version can be repeated also in the case of arbitrary algebraically closed fields.

Acknowledgments

This research was partially supported by OGP grant 0156833 (Canada).

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A. Khovanskii and Y. Burda
Department of Mathematics
University of Toronto
Toronto, Ontario
Canada M5S 2E4
e-mail: askold@math.toronto.edu, yburda@math.toronto.edu