On the Pontryagin-Viro Form

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This paper is dedicated to the memory of V. A. Rokhlin

ABSTRACT. A new invariant, the Pontryagin-Viro form, of algebraic surfaces is introduced and studied. It is related to various Rokhlin–Guillou–Marin forms and generalizes Mikhalkin's complex separation. The form is calculated for all real Enriques surfaces for which it is well defined; in most cases it distinguishes deformation classes of surfaces with homeomorphic real parts.

Introduction

In this paper we introduce a new invariant, the so-called *Pontryagin-Viro form* of a real algebraic surface or, more generally, of a closed smooth $4n$-manifold X with involution $c\colon X \to X$. The invariant, which is only well defined in certain special cases, is a quadratic function $\mathcal{P}\colon \mathcal{F} \to \mathbb{Z}/4$, where $\mathcal{F} \subset H_*(\operatorname{Fix} c; \mathbb{Z}/2)$ is a subgroup of the total homology of the fixed point set of c (or, in the case of an algebraic surface, of the real part of the surface). We mainly concentrate on the case $\dim X = 4$; in this case $\mathcal{P}$ turns out to be closely related to the *Rokhlin–Guillou–Marin forms* (see 2.2) of various characteristic surfaces in X and X/c and, thus, is a direct generalization of the notion of *complex separation* introduced by Mikhalkin [**Mik**]. (Mikhalkin's complex separation is defined when $H_1(X; \mathbb{Z}/2) = 0$.) The relation to the Rokhlin–Guillou–Marin form gives a number of congruences that the Pontryagin-Viro form must satisfy (see 4.2).

This work was mainly inspired by our study of real Enriques surfaces (joint work with I. Itenberg and V. Kharlamov). Recall that an *Enriques surface* is a complex analytic surface E with $\pi_1(E) = \mathbb{Z}/2$ and $2c_1(E) = 0$. Such a surface is called *real* if it is supplied with an anti-holomorphic involution $\operatorname{conj}\colon E \to E$; the fixed point set $E_{\mathbb{R}} = \operatorname{Fix} \operatorname{conj}$ is called the *real part* of E. The set of components of the real part of a real Enriques surface naturally splits into two disjoint *halves* $E_{\mathbb{R}}^{(1)}$, $E_{\mathbb{R}}^{(2)}$ (see 5.1); this splitting is a deformation invariant of pair $(E; \operatorname{conj})$.

1991 *Mathematics Subject Classification.* 57S25, 14J28, and 14P25.

Key words and phrases. involution on manifold, Enriques surface, real algebraic surface.

An essential part of this work was done during my visits to Université Louis Pasteur, Strasbourg, Universitá di Trento, and Université de Rennes I.

The topology of real Enriques surfaces is studied in [**DK1**] and [**DK2**], where they are classified up to homeomorphism of the triad $(E_{\mathbb{R}}; E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$. Currently, we know the classification up to deformation equivalence (which is the strongest equivalence relation from the topological point of view); a preliminary report is found in [**DK3**]; details will appear in [**DIK**]. (In [**DK3**] there are a few erroneous statements, which are corrected here.) For a technical reason, real Enriques surfaces are divided in [**DK3**] into three types, *hyperbolic*, *parabolic*, and *elliptic*, according to whether the minimal Euler characteristic of the components of $E_{\mathbb{R}}$ is negative, zero, or positive, respectively. It turns out that in most cases a real Enriques surface is determined up to deformation equivalence by such classical invariants as the homeomorphism type of the triad $(E_{\mathbb{R}}; E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$ and whether the fundamental classes $[E_{\mathbb{R}}]$ and $[E_{\mathbb{R}}^{(i)}]$, $i = 1, 2$, vanish or are characteristic in the homology of E or some auxiliary manifolds. The few exceptions, mainly M-surfaces of parabolic and elliptic types, differ by the Pontryagin-Viro form.

In this paper the Pontryagin-Viro form is calculated for all real Enriques surfaces for which it is well defined (see Section 7). There is a necessary condition ($\chi(E_{\mathbb{R}}) = 0 \bmod 8$) and certain sufficient conditions (Lemma 5.2.1) for $\mathcal{P}$ to be well defined, and, when defined, $\mathcal{P}$ must satisfy certain congruences (Proposition 5.2.2), which follow from the general congruences in 4.2. The result of the calculation can be roughly stated as follows (see Theorems 7.1.1, 7.2.1, and 7.3.1 for the precise statements): *Consider a triad* $(E_{\mathbb{R}}; E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$ *with* $\chi(E_{\mathbb{R}}) = 0 \bmod 8$. *Any (partial) quadratic form* $\mathcal{P}\colon H_*(E_{\mathbb{R}}^{(1)}) \oplus H_*(E_{\mathbb{R}}^{(2)}) \to \mathbb{Z}/4$ *satisfying the congruences of Proposition* 5.2.2 *can be realized as the Pontryagin-Viro form of a real Enriques surface. If* $(E_{\mathbb{R}}; E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$ *does not satisfy the sufficient conditions of Lemma* 5.2.1, *it can also be realized by a real Enriques surface not admitting a Pontryagin-Viro form.* Note that, in fact, the deformation type of a surface admitting Pontryagin-Viro form is determined by the topology of $(E_{\mathbb{R}}; E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$ and the isomorphism type of $\mathcal{P}\colon H_*(E_{\mathbb{R}}^{(1)}) \oplus H_*(E_{\mathbb{R}}^{(2)}) \to \mathbb{Z}/4$ (see [**DIK**]).

Originally, in order to distinguish nonequivalent real Enriques surfaces, we calculated the Pontryagin-Viro form by explicitly constructing membranes in E/conj. In Section 6 I develop a different approach, which facilitates the calculation and, on the other hand, covers all real Enriques surfaces that are of interest. The approach is applicable to a specific construction (which, as is shown in [**DIK**], produces all M-surfaces of elliptic and parabolic types): the surface in question is constructed starting from a pair of real curves P, Q on a real rational surface Z, and the Pontryagin-Viro form is given in terms of the topology of their real parts $(Z_{\mathbb{R}}; P_{\mathbb{R}}, Q_{\mathbb{R}})$. The fact that $\mathcal{P}$ is related to the complex orientation of the branch curve was indicated to me by G. Mikhalkin.

Contents. Section 1 introduces the primary tool, the so-called Kalinin spectral sequence. Section 2 recalls the basic notions related to quadratic forms and to the Rokhlin–Guillou–Marin form of a characteristic surface. The Pontryagin-Viro form is introduced in Section 3, and its basic properties, including the congruences, are studied in Section 4. In Section 5 the general results are carried over to real Enriques surfaces. In Section 6 we calculate the Pontryagin-Viro form of a real Enriques surface obtained by a specific construction, using the so-called Donaldson trick; these results are applied in Section 7 to produce the complete list of possible values of the Pontryagin-Viro form on a real Enriques surface.

Acknowledgements. I am thankful to I. Itenberg and V. Kharlamov for many hours of fruitful discussions of the subject; the results of this paper depend essentially upon our joint work on real Enriques surfaces. I am also thankful to G. Mikhalkin for his helpful remarks. My deepest gratitude is to the late V. A. Rokhlin, who influenced my whole mathematical career a great deal. In particular, it was he and his school who attracted my attention to the topology of real algebraic varieties.

Notation. Unless stated otherwise, all homology and cohomology groups are with coefficients in $\mathbb{Z}/2$. We freely denote by $2\colon \mathbb{Z}/2 \to \mathbb{Z}/4$ the nontrivial homomorphism, as well as the induced homomorphisms $H_*(\,\cdot\,;\mathbb{Z}/2) \to H_*(\,\cdot\,;\mathbb{Z}/4)$, etc.

Given a vector bundle ξ, we denote by $w_i(\xi)$ and $u_i(\xi)$ its Stiefel–Whitney and Wu classes, respectively. $w = 1+w_1+\ldots$ and $u = 1+u_1+\ldots$ are the corresponding total classes. If X is a smooth manifold and τX its tangent bundle, we abbreviate $w_i(\tau X) = w_i(X)$ and $u_i(\tau X) = u_i(X)$. For a smooth submanifold $V \subset X$ we denote by $\nu V = \nu_X V$ its normal bundle in X.

Let X be a closed manifold of dimension n. Then $[X] \in H_n(X)$ is its fundamental class and $\langle X\rangle \in H_0(X)$ is the 0-class defined by the union of points, one in each component of X. (Certainly, the latter definition applies to any polyhedron.) Poincaré duality $\cap[X]\colon H^i(X) \to H_{n-i}(X)$ is denoted by $D_X = D$.

If X is a complex manifold, $c_i(X) \in H_{2i}(X;\mathbb{Z})$ stand for its Chern classes and K_X, for both the canonical class in $\operatorname{Pic}(X)$ and its image $D_X c_1(K_X)$ in $H_{2n-2}(X)$ (so that we can write $[D] = K_X$ for a divisor D).

1. Kalinin's spectral sequence

1.1. Basic concepts. Let X be a good topological space (say, a finite-dimensional CW-complex) and $c\colon X \to X$ an involution. Unless stated otherwise, we assume X to be connected. Denote by F the fixed point set $\operatorname{Fix} c$ and by $\bar X$, the orbit space X/c. Let $\mathrm{pr}\colon X \to \bar X$ be the projection and $\mathrm{in}\colon F \hookrightarrow X$ and $\overline{\mathrm{in}}\colon F \hookrightarrow \bar X$ the inclusions.

Recall that the *Borel–Serre spectral sequences* ${}^rE_{p,q}$ and ${}^rE^{p,q}$ are the Serre spectral sequences of the fibration $S^\infty \times_c X \to \mathbb{R}\mathrm{p}^\infty$, where $S^\infty \times_c X$ is the *Borel construction* $S^\infty \times X/(\mathbf{s},x) \sim (-\mathbf{s},cx)$. As is shown in [**Ka**], multiplication by the generator $h \in H^1(\mathbb{R}\mathrm{p}^\infty)$ establishes isomorphisms ${}^rE_{p,q+1} \to {}^rE_{p,q}$ and ${}^rE^{p,q} \to {}^rE^{p,q+1}$ for $p \gg 0$ and thus produces stabilized spectral sequences $({}^rH_*, {}^rd_*)$ and $({}^rH^*, {}^rd^*)$, which we call *Kalinin's spectral sequences* of (X,c), so that

(1) ${}^1H_* = H_*(X)$ and ${}^1H^* = H^*(X)$,
(2) ${}^1d_* = (1+c_*)$ and ${}^1d^* = (1+c^*)$,
(3) ${}^rH_* \Rightarrow H_*(F)$ and ${}^rH^* \Rightarrow H^*(F)$.

An alternative, geometrical, description of Kalinin's spectral sequences and related objects is found in [**DK2**].

The convergence in (3) means that there is an increasing filtration $\{\mathcal{F}^p\}$ on $H_*(F)$, a decreasing filtration $\{\mathcal{F}_p\}$ on $H^*(F)$, and homomorphisms $\mathrm{bv}_p\colon \mathcal{F}^p \to {}^\infty H_p$ and $\mathrm{bv}^p\colon {}^\infty H^p \to H^*(F)/\mathcal{F}_{p-1}$ that establish isomorphisms of the graded groups. (Note that in general the filtrations do not respect the grading on $H_*(F)$ and $H^*(F)$.) We will call bv_p and bv^p the *Viro homomorphisms*; often they will be considered as additive relations (partial homomorphisms) $H_*(F) \dashrightarrow H_p(X)$ and $H^p(X) \dashrightarrow H^*(F)$.

1.2. Multiplicative structures. The homology and cohomology versions of Kalinin's spectral sequences are dual to each other, i.e., ${}^rH^p = \operatorname{Hom}({}^rH_p, \mathbb{Z}/2)$ and ${}^rd^p = \operatorname{Hom}({}^rd_p, \mathrm{id}_{\mathbb{Z}/2})$. The cup- and cap-products convert ${}^rH^*$ and ${}^rH_*$ to a graded $\mathbb{Z}/2$-algebra and a graded ${}^rH^*$-module, respectively, so that all the differentials except ${}^1d$ are differentiations. Furthermore, if X is a closed n-manifold and $F \neq \varnothing$, the Poincaré duality D_X induces isomorphisms $D\colon {}^rH^p \to {}^rH_{n-p}$, and in the usual way one can define intersection pairings $*\colon {}^rH_p \otimes {}^rH_q \to {}^rH_{p+q-n}$. The induced (via bv_*) pairing on the graded group $\mathrm{Gr}^*_{\mathcal{F}} H_*(F)$ is called *Kalinin's intersection pairing.* The ordinary intersection pairing on $H_*(F)$ will be denoted by $\circ$.

1.2.1. THEOREM ([**DK2**]). *Let X be a smooth closed n-manifold and $c\colon X \to X$ a smooth involution. Then for $a \in \mathcal{F}^p$ and $b \in \mathcal{F}^q$, one has $w(\nu F) \cap (a \circ b) \in \mathcal{F}^{p+q-n}$ and*

$$\mathrm{bv}_p\, a \circ \mathrm{bv}_q\, b = \mathrm{bv}_{p+q-n}[w(\nu F) \cap (a \circ b)].$$

1.3. Relation to the Smith exact sequence. Recall that the *Smith exact sequences* of (X, c) are the exact sequences

$$\to H_{p+1}(\bar{X}, F) \xrightarrow{\Delta} H_p(\bar{X}, F) \oplus H_p(F) \xrightarrow{\mathrm{tr}_*} H_p(X) \xrightarrow{\mathrm{pr}_*} H_p(\bar{X}, F) \to$$

$$\to H^p(\bar{X}, F) \xrightarrow{\mathrm{pr}^*} H^p(X) \xrightarrow{\mathrm{tr}^*} H^p(\bar{X}, F) \oplus H^p(F) \xrightarrow{\Delta} H^{p+1}(\bar{X}, F) \to .$$

The connecting homomorphisms Δ are given by $x \mapsto \omega \cap x \oplus \partial x$ (in homology) and $x \oplus f \mapsto \omega \cup x + \delta f$ (in cohomology), where $\omega \in H^1(\bar{X} \setminus F)$ is the characteristic class of the double covering $X \setminus F \to \bar{X} \setminus F$. In [**DK2**] it is shown that Kalinin's spectral sequences can be derived from the Smith exact sequences. In this paper we only need the corresponding description of the differentials and Viro homomorphisms:

1.3.1. THEOREM ([**DK2**]). *The differentials ${}^rd_p$ and ${}^rd^p$, considered as additive relations $H_p(X) \dashrightarrow H_{p+r-1}(X)$ and $H^p(X) \dashrightarrow H^{p-r+1}(X)$, are given by*

$${}^rd_* = \mathrm{tr}_* \circ \iota \circ (\Delta^{-1} \circ \iota)^{r-1} \circ \mathrm{pr}_*, \qquad {}^rd^* = \mathrm{pr}^* \circ (\pi \circ \Delta^{-1})^{r-1} \circ \pi \circ \mathrm{tr}^*,$$

where $\iota\colon H_p(\bar{X}, F) \to H_p(\bar{X}, F) \oplus H_p(F)$ and $\pi\colon H^p(\bar{X}, F) \oplus H^p(F) \to H^p(\bar{X}, F)$ are, respectively, the inclusion and the projection.

A (nonhomogeneous) class $x = \sum_{i \leqslant p} x_i$, $x_i \in H_i(F)$, belongs to $\mathcal{F}^p$ if and only if there are elements $y_i \in H_i(\bar{X}, F)$ such that $\Delta(y_{i+1}) = y_i \oplus x_i$ for $i < p$. In this case $\mathrm{bv}_p\, x = \mathrm{tr}_*(y_p \oplus x_p)$ (modulo the indeterminacy subgroup).

A class $x \in H^p(X)$ survives to ${}^\infty H^p$ if and only if $\mathrm{tr}^* x = y^p \oplus x^p$ extends to a sequence $y^i \oplus x^i \in H^i(\bar{X}, F) \oplus H^i(F)$, $i \leqslant p$, such that $y^{i+1} = \Delta(y^i \oplus x^i)$ for $i < p$. In this case $\mathrm{bv}^p\, x = \sum_{i \leqslant p} x^i \bmod \mathcal{F}_{p-1}$.

1.4. The groups ${}^rB$ and ${}^rZ$. Let ${}^rB_p \subset {}^rZ_p \subset H_p(X)$ be the pullbacks of $\operatorname{Im}{}^{r-1}d_p$ and $\operatorname{Ker}{}^{r-1}d_p$, respectively, so that ${}^rH_p = {}^rZ_p/{}^rB_p$. Denote ${}^\infty B_p = \bigcup_r {}^rB_p$ and ${}^\infty Z_p = \bigcap_r {}^rZ_p$. Then ${}^\infty H_p = {}^\infty Z_p/{}^\infty B_p$. There are obvious cohomology analogs ${}^rB^p \subset {}^rZ^p \subset H^p(X)$, and ${}^rH^p = {}^rZ^p/{}^rB^p$ for $1 \leqslant p \leqslant \infty$.

1.4.1. PROPOSITION. *One has ${}^\infty Z_p = \operatorname{Ker}[\mathrm{pr}_*\colon H_p(X) \to H_p(\bar{X}, F)]$ and ${}^\infty B^p = \operatorname{Im}[\mathrm{pr}^*\colon H^p(\bar{X}, F) \to H^p(X)]$.*

PROOF. The statement follows from Theorem 1.3.1. Since all the spaces involved are finite-dimensional, both Smith exact sequences terminate. Hence, an element $x \in H_p(X)$ is annihilated by all ${}^rd_p$, $p > 0$, if and only if $(\iota \circ \mathrm{pr}_*)(x) = 0$. Similarly, for any element $x \in H^p(\bar{X}, F)$, the multiple image $\Delta^{r-1}(x)$ belongs to $\operatorname{Im}(\pi \circ \mathrm{tr}^*)$ for $r \gg 0$. $\square$

1.4.2. COROLLARY. *If X is a closed smooth manifold, c is a smooth involution, and $F \neq \emptyset$, then*

$$^{\infty}B_* = \operatorname{Im}[\operatorname{tr}_*\colon H_*(\bar X \setminus F) \to H_*(X)], \qquad ^{\infty}Z^* = \operatorname{Ker}[\operatorname{tr}^*\colon H^*(X) \to H^*(\bar X \setminus F)].$$

PROOF. This follows from Proposition 1.4.1 and Poincaré duality. □

1.5. Miscellaneous statements. In this section we state several simple results needed in the sequel.

1.5.1. PROPOSITION ([**DK2**]). *Denote by* Sq_1 *the homology Bockstein homomorphism. Then for any class* $x = \sum_{i \leqslant p} x_i \in \mathcal{F}^p$, $x_i \in H_i(F)$, *one has*

$$\operatorname{Sq}_1 \operatorname{bv}_p x = \operatorname{bv}_{p-1}\Big(\operatorname{Sq}_1 x + \sum_i i x_i\Big) = \operatorname{bv}_{p-1}\Big(\operatorname{Sq}_1 x + \sum_i (i+1) x_i\Big).$$

(*In particular, the classes in parentheses belong to* $\mathcal{F}^{p-1}$.)

1.5.2. PROPOSITION. *Let X be an oriented closed smooth n-dimensional manifold, $H_1(X) = 0$, and $B \subset X$ a c-invariant oriented closed smooth submanifold of pure codimension 2 such that $[B] = 0$ in $H_{n-2}(X)$. Assume that c reverses the co-orientation of B. Let, further, $p\colon Y \to X$ be the (unique) double covering of X branched over B and ω_p its characteristic class. Then for $x \in H_1(F \setminus B)$ one has $\langle \omega_p, x\rangle = \operatorname{bv}_2 x \circ \frac{1}{2}[B]$, where $\frac{1}{2}[B]$ is obtained by dividing the integral class $[B] \in H_{n-2}(X;\mathbb{Z})$ by 2.*

PROOF. Realize x by an oriented simple loop $\mathfrak{l}$. After multiplying it by an odd integer, we may assume that $\mathfrak{l}$ bounds an oriented membrane $\mathfrak{M}$ in X, which may be chosen transversal to B. Then $\langle \omega_p, x\rangle = \operatorname{Card}(\mathfrak{M} \cap B) \bmod 2$. On the other hand, $[\mathfrak{M} \cup c(\mathfrak{M})] = \operatorname{bv}_2 x$ and the statement follows from $\operatorname{Card}(\mathfrak{M} \cap B) = \frac{1}{2}[\mathfrak{M} \cup c(\mathfrak{M})] \circ [B]$. (Note that c reverses the orientation of $\mathfrak{M}$.) □

1.5.3. PROPOSITION. *Assume that X is a closed 4-manifold, $H_1(X;\mathbb{Z}) = 0$, and F is a surface. Then $\bar X$ is a $\mathbb{Z}$-homology 4-sphere if and only if c is an M-involution (i.e., $\dim H_*(F) = \dim H_*(X)$) and F is connected. If this is the case, c_* acts as multiplication by (-1) on $H_2(X;\mathbb{Z})$.*

PROOF. Assume that $F \neq \varnothing$ (since otherwise $H_1(\bar X) = \mathbb{Z}/2$). Then $H_1(\bar X;\mathbb{Z}) = 0$ and the first statement follows if we compare the Euler characteristics using the Riemann–Hurwitz formula. For the second statement, observe that $H_*(\bar X;\mathbb{Q})$ is the c_*-skew-invariant part of $H_*(X;\mathbb{Q})$; this determines the action of c_* on $H_*(X;\mathbb{Z}) \subset H_*(X;\mathbb{Q})$. □

2. Rokhlin–Guillou–Marin congruence

2.1. Quadratic forms and Brown invariant. The results of this section can be found in most textbooks in arithmetics; see also [**vdB, Br, GM, KV**].

Let V be a $\mathbb{Z}/2$-vector space and $\circ\colon V \otimes V \to \mathbb{Z}/2$ a symmetric bilinear form. A function $q\colon V \to \mathbb{Z}/4$ is called a *quadratic extension* of $\circ$ if $q(x+y) = q(x)+q(y)+2(x \circ y)$ for all $x, y \in V$. The pair (V, q) is called a *quadratic space.* (Obviously, $\circ$ is recovered from q.) A quadratic space is called *nonsingular* if the bilinear form is nonsingular, i.e., $V^{\perp} = 0$; it is called *informative* if $q|_{V^{\perp}} = 0$. The following is straightforward:

2.1.1. PROPOSITION. *Let V be a $\mathbb{Z}/2$-vector space and $\circ\colon V \otimes V \to \mathbb{Z}/2$ a symmetric bilinear form. Then*

(1) $q(x) \equiv x^2 \bmod 2$ *for any* $x \in V$ *and any quadratic extension* q *of* $\circ$;
(2) *quadratic extensions of* $\circ$ *form an affine space over* $V^\vee = \operatorname{Hom}(V, \mathbb{Z}/2)$ *via* $(q+l)(x) = q(x) + 2l(x)$ *for* $l \in V^\vee$ *and* $x \in V$;
(3) *if* $\circ$ *is nonsingular, its quadratic extensions form an affine space over* V *via* $(q+v)(x) = q(x) + 2(v \circ x)$ *for* $v, x \in V$.

The *Brown invariant* $\operatorname{Br}(V,q)$ (or just $\operatorname{Br} q$) of a nonsingular quadratic space is the (mod 8)-residue defined by

$$\exp\left(\frac{1}{4} i\pi \operatorname{Br} q\right) = 2^{\frac{1}{2}\dim V} \sum_{x \in V} \exp\left(\frac{1}{2} i\pi q(x)\right).$$

This notion extends to informative spaces: since q vanishes on $V^\perp$, it descends to a quadratic form $q'\colon V/V^\perp \to \mathbb{Z}/4$, and one lets $\operatorname{Br} q = \operatorname{Br} q'$.

2.1.2. PROPOSITION. *For any informative quadratic space* (V,q), *one has*:

(1) $\operatorname{Br} q \equiv \dim(V/V^\perp) \bmod 2$;
(2) $\operatorname{Br} q \equiv q(u) \bmod 4$ *for any characteristic element* $u \in V$;
(3) $\operatorname{Br}(q+v) = \operatorname{Br} q - 2q(v)$ *for any* $v \in V$ *(see Proposition* 2.1.1(3));
(4) $\operatorname{Br} q = 0$ *if and only if* (V,q) *is null cobordant; i.e., there is a subspace* $H \subset V$ *such that* $H^\perp = H$ *and* $q|_H = 0$.

The Brown invariant is additive: *for any pair* (V_i, q_i), $i = 1, 2$, *of quadratic spaces one has* $\operatorname{Br}(V_1 \oplus V_2, q_1 \oplus q_2) = \operatorname{Br}(V_1, q_1) + \operatorname{Br}(V_2, q_2)$.

2.1.3. PROPOSITION. *Let L be a unimodular integral lattice (i.e., a free Abelian group with a nonsingular symmetric bilinear form $L \otimes L \to \mathbb{Z}$). Let $V = L \otimes \mathbb{Z}/2$ and define a quadratic form $q\colon V \to \mathbb{Z}/4$ via* $q(x) = \bar{x}^2 \bmod 4$ *for* $x \in V$ *and* $\bar{x} \in L$ *such that* $\bar{x} \equiv x \bmod 2L$. *Then* $\operatorname{Br}(V,q) \equiv \sigma(L) \bmod 8$.

A subspace W of an informative quadratic space (V,q) is called *informative* if $W^\perp \subset W$ and $q|_{W^\perp} = 0$. (Clearly, an informative subspace is an informative space; hence, its Brown invariant is well defined.)

2.1.4. PROPOSITION. *If W is an informative subspace of an informative quadratic space* (V,q), *then* $\operatorname{Br}(W, q|_W) = \operatorname{Br}(V,q)$.

REMARK. The notion of informative subspace still makes sense if the quadratic form q is defined only on W. Proposition 2.1.4 can then be interpreted as follows: the Brown invariant of any extension of q to a quadratic form on V equals $\operatorname{Br} q$.

2.2. Rokhlin–Guillou–Marin congruence (see [GM]). Let Y be an oriented closed smooth 4-manifold and U a *characteristic surface* in Y, i.e., a smooth closed 2-submanifold (not necessarily orientable) with $[U] = u_2(Y)$ in $H_2(Y)$. Denote by $i\colon U \hookrightarrow Y$ the inclusion and let $K = \operatorname{Ker}[i_*\colon H_1(U) \to H_1(Y)]$. Then there is a natural function $\mathfrak{q}\colon K \to \mathbb{Z}/4$, which is a quadratic extension of the intersection index form on $H_1(U)$. We call it the *Rokhlin–Guillou–Marin form* of (Y,U). It can be defined as follows: pick a class $x \in K$ and realize it by a union $\mathfrak{l}$ of disjoint simple closed smooth loops in U. It spans an immersed surface $\mathfrak{M}$ in Y, which can be chosen normal to U along $\mathfrak{l} = \partial\mathfrak{M}$ and transversal to U at its inner points. (Such a surface is called a *membrane*.) Consider a normal line field ξ on $\mathfrak{l}$ tangent to U and define the *index* $\operatorname{ind}\mathfrak{M} \in \frac{1}{2}\mathbb{Z}$ as one half of the obstruction to extending ξ

to a normal line field on $\mathfrak{M}$. (Since $\tau\mathfrak{M} \oplus \nu\mathfrak{M}$ is an oriented vector bundle, the obstruction is well defined as an integer. If $\mathfrak{l}$ is two-sided in U, the index is usually defined using vector fields instead of line fields; this explains the factor 1/2.) Then $\mathfrak{q}(x) = 2 \operatorname{ind} \mathfrak{M} + 2 \operatorname{Card}(\operatorname{int} \mathfrak{M} \cap F) \bmod 4$.

2.2.1. THEOREM ([GM]). *Let Y, U, and $(K, \mathfrak{q})$ be as above. Then $(K, \mathfrak{q})$ is an informative subspace of $H_1(U)$ and* $2 \operatorname{Br} \mathfrak{q} = \sigma(Y) - U \circ U \bmod 16$, *where $U \circ U$ stands for the normal Euler number of U in Y.*

REMARK. There is an alternative construction of the Rokhlin–Guillou–Marin form. Since U is characteristic, $Y \smallsetminus U$ admits a Spin-structure which does not extend through any component of U. Its restriction to the boundary of a tubular neighborhood of U induces in a natural way a Pin^--structure on U (cf. [**Fin**]), which defines a quadratic form q on $H_1(U)$. It is not difficult to see that q is well defined up to adding elements of $\operatorname{Im}[i^* \colon H^1(Y) \to H^1(U)]$ (see Theorem 2.1.1(2)) and, hence, its restriction to K does not depend on the choice of the Spin-structure; it coincides with $\mathfrak{q}$.

3. Pontryagin-Viro form

3.1. Definition of the Pontryagin-Viro form. The *Pontryagin square* is the cohomology operation $P^{2n} \colon H^{2n}(X) \to H^{4n}(X; \mathbb{Z}/4)$ uniquely defined by the following properties (see, e.g., [**EM**]):

(1) $P^{2n}(x + y) = P^{2n}(x) + P^{2n}(y) + 2(x \cup y)$ for any $x, y \in H^{2n}(X)$;
(2) $P^{2n}(x) \equiv x^2 \bmod 2$ for any $x \in H^{2n}(X)$;
(3) $P^{2n}(\bar{x} \bmod 2) = \bar{x}^2$ for any $\bar{x} \in H^{2n}(X; \mathbb{Z}/4)$.

Constructively P^{2n} can be defined via $P^{2n}(x) = (\bar{x} \cup_0 x + \bar{x} \cup_1 \delta\bar{x}) \bmod 4$, where $\bar{x} \in C^{2n}(X; \mathbb{Z})$ is an integral cochain representing x and $\cup_i$ are the cup-i-products used in one of the definitions of Steenrod squares.

From now on we assume that X is a connected oriented closed smooth manifold of dimension $4n$ and c is a smooth involution. Denote by $P_{2n} \colon H_n(X) \to \mathbb{Z}/4$ the composition

$$H_{2n}(X) \xrightarrow{D_X^{-1}} H^{2n}(X) \xrightarrow{P^{2n}} H^{4n}(X; \mathbb{Z}/4) \xrightarrow{\cap[X]} \mathbb{Z}/4.$$

3.1.1. PROPOSITION–DEFINITION. *If $P_{2n}({}^{\infty}B_{2n}) = 0$, then P_{2n} descends to a well-defined quadratic function ${}^{\infty}H_{2n} \to \mathbb{Z}/4$. The composition of this function and the Viro homomorphism* $\operatorname{bv}_{2n} \colon \mathcal{F}^{2n} \to {}^{\infty}H_{2n}$ *is denoted by $\mathcal{P}$ and is called the Pontryagin-Viro form. It is a quadratic extension of Kalinin's intersection form $* \colon \mathcal{F}^{2n} \otimes \mathcal{F}^{2n} \to \mathbb{Z}/2$, i.e., $\mathcal{P}(x + y) = \mathcal{P}(x) + \mathcal{P}(y) + 2(x * y)$ for any $x, y \in \mathcal{F}^{2n}$.*

PROOF. The statement follows immediately from the fact that ${}^{\infty}B_* \circ {}^{\infty}Z_* = 0$ (where $\circ$ stands for the intersection pairing) and property 3.1(1) above. □

3.1.2. PROPOSITION. *$\mathcal{P}$ is well defined if and only if $u_{2n}(\bar{X} \smallsetminus F) = 0$.*

PROOF. The statement is a consequence of Corollary 1.4.2 and the obvious relation $P_{2n}(\operatorname{tr}_* x) = 2(x \circ x) = 2\langle u_{2n}(\bar{X} \smallsetminus F), x\rangle$ for $x \in H_{2n}(\bar{X} \smallsetminus F)$. The latter follows from properties 3.1(1) and (2) of Pontryagin squares and the fact that tr_*, when restricted to the manifold $\bar{X} \smallsetminus F$, coincides with the inverse Hopf homomorphism $\operatorname{pr}^!$. □

3.1.3. COROLLARY. *Assume that F has pure dimension* $\dim X - 2$, *so that $\ddot{X}$ is a manifold. Then $\mathcal{P}$ is well defined if and only if $D_{\ddot{X}}u_{2n}(X)$ belongs to the image of* $\overline{\text{in}}_*\colon H_{2n}(F) \to H_{2n}(X)$.

3.1.4. PROPOSITION. *If $\mathcal{P}$ is well defined, then* $\operatorname{Br}\mathcal{P} \equiv \sigma(X) \bmod 8$.

PROOF. By definition, $\mathcal{P}$ is well defined if and only if ${}^{\infty}Z_{2n}$ is an informative subspace of $(H_{2n}(X), P_{2n})$. (Due to the Poincaré duality ${}^{\infty}Z_{\overline{2n}} = {}^{\infty}B_{2n}$; see [**DK2**].) Hence, $\operatorname{Br}\mathcal{P} = \operatorname{Br}P_{2n}$. On the other hand, $H_{2n}(X;\mathbb{Z}) \otimes \mathbb{Z}/2$ is also an informative subspace, and the congruence follows from Proposition 2.1.3 applied to $H_{2n}(X;\mathbb{Z})/\operatorname{Tors}$. □

3.2. Some sufficient conditions.

3.2.1. PROPOSITION. *Let c be orientation preserving. Then P_{2n} descends to ${}^{2}H_{2n}$ if and only if the $2n$-dimensional component of* $\text{in}_!(u(\tau F)u^{-1}(\nu_X F))$ *equals* $u_{2n}(X)$.

PROOF. As known, the $2n$-dimensional component of $\text{in}_!(u(\tau F)u^{-1}(\nu_X F))$ coincides with the characteristic class θ_{2n} of the twisted intersection form $(x, y) \mapsto x \circ c_* y$ (see, e.g., [**CM**]). On the other hand, for $x \in H_{2n}(X)$ one has

$$P_{2n}({}^{1}d_{2n}x) = P_{2n}(x + c_*x) = 2P_{2n}(x) + 2(x \circ c_*x) = 2\langle \theta_{2n} + u_{2n}, x\rangle$$

(since $P_{2n}(x) \equiv x^2 \bmod 2$), and the statement follows. □

3.2.2. COROLLARY. *A necessary condition for $\mathcal{P}$ to be well defined is that θ_{2n}, the $2n$-dimensional component of* $\text{in}_!(u(\tau F)u^{-1}(\nu_X F))$, *coincides with* $u_{2n}(X)$. *The following are sufficient conditions*:

(1) *c is an M-involution* (*i.e.*, ${}^{r}d_* = 0$ *for* $r \geqslant 1$);
(2) $\theta_{2n} = u_{2n}(X)$ *and c is $\mathbb{Z}/2$-Galois maximal* (*i.e.*, ${}^{r}d_* = 0$ *for* $r \geqslant 2$);
(3) $\theta_{2n} = u_{2n}(X)$ *and* $H_i(X) = 0$ *for* $0 < i < 2n$.

REMARK. If $\dim F \leqslant 2n$, the condition of Proposition 3.2.1 (and, hence, the necessary condition of Corollary 3.2.2) reduces to $[F]_{2n} = Du_{2n}(X)$ in $H_{2n}(X)$, where $[F]_{2n}$ is the fundamental class of the union of $2n$-dimensional components of F. In this case Proposition 3.2.1 can be proved using the following observation:

3.2.3. PROPOSITION (V. Arnold). *If* $\dim X = 2k$ *is even and* $\dim F \leqslant k$, *then the fundamental class $[F]_k$ of the union of k-dimensional components of F realizes in $H_k(X)$ the characteristic class of the twisted intersection form* $(x, y) \mapsto x \circ c_* y$.

REMARK. If $\dim X = 4$ and F is a surface, Proposition 3.2.1 follows also from the projection formula $D_X u_2(X) = \operatorname{tr}_* D_{\ddot{X}} u_2(\ddot{X}) + [F]$: since $D_X u_2(X)$ comes from F, its pullback in $\ddot{X}$ is zero.

REMARK. Note that conditions (1), (2) in Corollary 3.2.2 do not require actual calculation of the differentials. Indeed, from Kalinin's spectral sequence it follows that c is an M-involution if and only if $\dim H_*(F) = \dim H_*(X)$ (and in this case $\theta_{2n} = u_{2n}(X)$, as $c_* = \text{id}$ and the twisted intersection form coincides with the ordinary one). Furthermore, c is $\mathbb{Z}/2$-Galois maximal if and only if $\dim H_*(F) = \dim {}^{1}H_*$, the latter group being equal to $\operatorname{Ker}(1 + c_*)/\operatorname{Im}(1 + c_*)$.

3.3. Membranes. The first statement below, which is a direct consequence of the definitions, allows us to calculate the Pontryagin square in a 4-manifold.

3.3.1. LEMMA. *Let X be an oriented closed smooth 4-manifold and $\mathfrak{M} \to X$ an immersed closed surface. Then $P_2[\mathfrak{M}] = \mathfrak{M} \circ \mathfrak{M} + 2\chi(\mathfrak{M}) \bmod 4$, where $\mathfrak{M} \circ \mathfrak{M} = e(\mu\mathfrak{M}) + 2i \bmod 4$ is the normal Euler number of $\mathfrak{M}$ plus twice the number of its self-intersection points $\bmod 4$. (If $\mathfrak{M}$ is oriented, $\mathfrak{M} \circ \mathfrak{M} = [\mathfrak{M}]_{\mathbb{Z}}^2 \bmod 4$, where $[\mathfrak{M}]_{\mathbb{Z}} \in H_2(X;\mathbb{Z})$ is the integral class realized by $\mathfrak{M}$.)*

The next two statements provide for geometrical means of calculating bv_2 and, hence, the Pontryagin-Viro form in a 4-manifold.

3.3.2. LEMMA. *Let $\mathfrak{M}$ be a closed surface with involution c so that $\operatorname{Fix} c$ consists of several two-sided circles $l_1, \dots, l_p$, several one-sided circles $n_1, \dots, n_q$, and several simple isolated points $P_1, \dots, P_r$. Then $[\mathfrak{M}] = \mathrm{bv}_2\, \varkappa$, where*

$$\varkappa = \sum [l_i] + \sum [n_j] + \sum [P_k] + \sum \langle n_i \rangle .$$

PROOF. Without loss of generality we may assume that $\mathfrak{M}$ is connected. Then, due to Proposition 3.2.3, $\mathrm{bv}_1\big(\sum[l_i] + \sum[n_j]\big)$ equals $w_1(\mathfrak{M})$ in ${}^\infty H_1$; hence, $q \equiv \dim H_*(\mathfrak{M}) \bmod 2$. Due to the Smith congruence, we have $\chi(\operatorname{Fix} c) \equiv \chi(\mathfrak{M}) \bmod 2$, and also $r \equiv \dim H_*(\mathfrak{M}) \bmod 2$. Thus, $\mathrm{bv}_0\, \varkappa = 0$ and $\mathrm{bv}_1\, \varkappa$ is well defined. Now one can easily check that $\mathrm{bv}_1\, \varkappa$ annihilates ${}^\infty H_1$ (which is generated by the images under bv_1 of $[l_i]$, $[n_j]$, and elements of the form $\langle Q_1 - Q_2 \rangle$, $Q_1, Q_2 \in \operatorname{Fix} c$). Since Kalinin's intersection form is nondegenerate, $\mathrm{bv}_1\, \varkappa = 0$. Hence, $\mathrm{bv}_2\, \varkappa$ is well defined, and it must coincide with the only nontrivial element $[\mathfrak{M}] \in {}^\infty H_2$. ❑

3.3.3. COROLLARY. *Let $\mathfrak{M}$ and $\varkappa$ be as in Lemma 3.3.2. If $\mathfrak{M}$ is equivariantly immersed in a topological space X with involution, then $[\mathfrak{M}]$ realizes $\mathrm{bv}_2\, \varkappa$. If, further, $\dim X = 4$ and $\mathcal{P}$ is well defined, then $\mathcal{P}(\varkappa) = \mathfrak{M} \circ \mathfrak{M} + 2\chi(\mathfrak{M}) \bmod 4$.*

4. Congruences

4.1. Characteristic surfaces in $\bar{X}$. Let us assume that X is an oriented closed smooth 4-manifold, $c\colon X \to X$ is a smooth orientation-preserving involution, and $F = \operatorname{Fix} c \neq \varnothing$ has pure dimension 2. Under these assumptions $\bar{X}$ is also an oriented closed manifold.

We keep the notation introduced in Section 1. In addition, we denote by $\mathcal{F}^p_{[i]}$ and $\widehat{\mathcal{F}}^p_{[i]}$, respectively, the intersection $\mathcal{F}^p \cap H_i(F)$ and the projection of $\mathcal{F}^p$ to $H_i(F)$. Recall that the connecting homomorphism Δ of the homology Smith exact sequence is given by $y \mapsto \omega \cap y \oplus \partial y$, where $\omega \in H^1(\bar{X} \smallsetminus F)$ is the characteristic class of the covering $X \smallsetminus F \to \bar{X} \smallsetminus F$. Since the covering $X \to \bar{X}$ is branched along F, one has $\partial D_{\bar{X}} \omega = [F]$.

4.1.1. LEMMA. $\mathrm{bv}_2\, \mathcal{F}^2_{[0]} = \mathrm{tr}_* H_2(\bar{X}) \bmod {}^\infty B_2$. *Furthermore, for $y \in H_2(\bar{X})$ one has $\mathrm{tr}_* y = \mathrm{bv}_2(\mathrm{in}^! y) \bmod {}^\infty B_2$; in particular, $\mathcal{P}(\mathrm{in}^! y) = 2y^2 \bmod 4$ provided that $\mathcal{P}$ is well defined.*

PROOF. It follows from Theorem 1.3.1 that the image $\mathrm{bv}_2\, \mathcal{F}^2_{[0]}$ consists of all elements of the form $\mathrm{tr}_* y_2$, where $y_2 \in H_2(\bar{X}, F)$ extends to a sequence $y_i \in H_i(\bar{X}, F)$, $i = 0, 1, 2$, such that $\Delta(y_2) = y_1$ and $\Delta(y_1) = y_0 \oplus x_0$ for some $x_0 \in H_0(F)$. Thus, y_1 may be an arbitrary element, and the only restriction to y_2 is

$\partial y_2 = 0$, i.e., $y_2 = \operatorname{rel} y$ for some $y \in H_2(\bar{X})$. For the 'furthermore' part observe that $\operatorname{tr}_* y = \operatorname{tr}_* y_2 = \operatorname{bv}_2 x_0$ and

$$x_0 = \partial(\omega \cap \operatorname{rel} y) = \partial(D\omega \cap D^{-1}y) = [F] \cap \bar{\operatorname{in}}^* D^{-1}y = D_F \bar{\operatorname{in}}^* D^{-1}y = \bar{\operatorname{in}}^! y,$$

where $D = D_{\bar{X}}$. □

4.1.2. LEMMA. $\operatorname{bv}_1 \mathcal{F}^1_{[0]} = \operatorname{tr}_* H_1(\bar{X}, F) \bmod {}^\infty B_1$.

4.1.3. LEMMA. $\widehat{\mathcal{F}}^2_{[1]} = \operatorname{Ker}[\bar{\operatorname{in}}_*\colon H_1(F) \to H_1(\bar{X})]$.

PROOF OF LEMMAS 4.1.2 AND 4.1.3. The statements follow directly from Theorem 1.3.1. □

4.1.4. COROLLARY. $\operatorname{Ker}[\bar{\operatorname{in}}_*\colon H_2(F) \to H_2(\bar{X})]$ *is the annihilator of* $\mathcal{F}^2_{[0]}$ *with respect to the intersection index pairing* $H_2(F) \otimes H_0(F) \to \mathbb{Z}/2$ *(or, equivalently, Kalinin's intersection pairing* $H_2(F) \otimes \mathcal{F}^2_{[0]} \to \mathbb{Z}/2$*).*

PROOF. Let $K = \operatorname{Ker}[\bar{\operatorname{in}}_*\colon H_2(F) \to H_2(\bar{X})]$. Since the restrictions of the ordinary intersection index pairing and Kalinin's intersection pairing to $H_2(F) \otimes \mathcal{F}^2_{[0]}$ coincide (see Theorem 1.2.1), it suffices to verify that $u \in K$ if and only if $\operatorname{bv}_2 u$ annihilates $\operatorname{bv}_2 \mathcal{F}^2_{[0]} = \operatorname{tr}_* H_2(\bar{X})$ in ${}^\infty H_2$. For $y \in H_2(\bar{X})$ one has $\operatorname{in}_* u \circ \operatorname{tr}_* y = \bar{\operatorname{in}}_* u \circ y$; this product vanishes for all $y \in H_2(\bar{X})$ if and only if $\bar{\operatorname{in}}_* u = 0$. □

4.1.5. COROLLARY. *Assume that* $\mathcal{P}$ *is well defined. Then an element* $u \in H_2(F)$ *realizes* $Du_2(\bar{X})$ *if and only if* $\mathcal{P}(x) = 2(u \circ x) \bmod 4$ *for all* $x \in \mathcal{F}^2_{[0]}$.

4.2. Pontryagin-Viro form and Rokhlin–Guillou–Marin forms. We still assume that X is an oriented closed smooth 4-manifold, c is smooth and orientation preserving, and $F \neq \varnothing$ has pure dimension 2. Assume also that $\mathcal{P}$ is well defined; due to Corollary 3.1.3 this implies that $u_2(\bar{X})$ is realized by a union of components of F.

4.2.1. PROPOSITION. *Let* $F' \subset F$ *be a union of components of* F *such that* $\mathcal{P}(x) = 2([F'] \circ x) \bmod 4$ *for all* $x \in \mathcal{F}^2_{[0]}$. *Let* $H' = H_1(F') \cap \widehat{\mathcal{F}}^2_{[1]}$ *and define a quadratic function* $\mathcal{P}'\colon H' \to \mathbb{Z}/4$ *via* $x_1 \mapsto \mathcal{P}(x_1 + x_0) + 2([F'] \circ x_0)$, *where* $x_0 \in H_0(F)$ *is any element such that* $x_1 + x_0 \in \mathcal{F}^2$. *Then* $\mathcal{P}'$ *coincides with the Rokhlin–Guillou–Marin form* $\mathfrak{q}'$ *of the characteristic surface* F' *in* $\bar{X}$. *In particular,* $(H', \mathcal{P}')$ *is an informative subspace of* $H_1(F')$.

PROOF. First notice that q' is well defined and, due to Lemma 4.1.3, its domain coincides with that of $\mathfrak{q}'$. Choose an element $x \in H'$ and consider a membrane $\mathfrak{M}$ as in 2.2. Let $\mathfrak{M}' = \operatorname{pr}^{-1}\mathfrak{M}$; it is a closed c-invariant surface in X. The index $\operatorname{ind}\mathfrak{M}$ (see 2.2) equals the normal Euler number $\mathfrak{M}' \circ \mathfrak{M}'$. (Indeed, $2 \operatorname{ind}\mathfrak{M}$ is the obstruction to the existence of a normal line field on $\mathfrak{M}'$; it is twice as big as the obstruction to the existence of a normal vector field.) The intersection points of $\operatorname{int}\mathfrak{M}$ and F correspond to isolated fixed points of $c|_{\mathfrak{M}'}$, and all the 1-dimensional components of $\operatorname{Fix} c|_{\mathfrak{M}'}$ are two-sided in $\mathfrak{M}'$. The statement follows now from comparing the definitions of $\mathfrak{q}'$ and q' and Lemma 3.3.3. (Note that the total number of intersection points of $\operatorname{int}\mathfrak{M}$ and F is even and, hence, so is $\chi(\mathfrak{M}')$.) □

4.2.2. THEOREM. *If* F' *and* $\mathcal{P}'$ *are as in Proposition* 4.2.1, *then*

$$F' \circ F' + \operatorname{Br}\mathcal{P}' \equiv \frac{1}{4}\big[F \circ F + \sigma(X)\big] \bmod 8.$$

PROOF. The statement follows from Proposition 4.2.1, Theorem 2.2.1 applied to $F' \subset \bar{X}$, and the well-known calculation of the ingredients of Theorem 2.2.1: the self-intersection numbers of F' in X and $\bar{X}$ are related via $(F' \circ F')_X = 2(F' \circ F')_{\bar{X}}$, and the signature of $\bar{X}$ is given by the Hirzebruch formula $\sigma(X) = 2\sigma(\bar{X}) - F \circ F$. □

4.2.3. THEOREM. *The restriction $\mathcal{P}_{[1]}$ of $\mathcal{P}$ to $\mathcal{F}^2_{[1]}$ coincides with the Rokhlin–Guillou–Marin form $\mathfrak{q}$ of the characteristic surface F in X. In particular, $(\mathcal{F}^2_{[1]}, \mathcal{P}_{[1]})$ is an informative subspace of $H_1(F)$ and*

$$F \circ F + 2 \operatorname{Br} \mathcal{P}_{[1]} \equiv \sigma(X) \bmod 16.$$

PROOF. The two forms are compared as in the previous proof: $\mathfrak{q}$ is calculated via a generic membrane $\mathfrak{M}$ as in 2.2 and $\mathcal{P}_{[1]}$, via $\mathfrak{M}' = \mathfrak{M} \cup c(\mathfrak{M})$. We may assume that $\mathfrak{M}'$ is an immersed surface. It realizes $\operatorname{bv}_2(\partial\mathfrak{M})$, as all the components of $\partial\mathfrak{M}$ are two-sided in $\mathfrak{M}'$ and the intersection points of $\operatorname{int}\mathfrak{M} \cap F$ are *not* fixed points of the lift of c to the normalization of $\mathfrak{M}'$. Note also that the self-intersection points of $\mathfrak{M}'$ that are not on F appear in pairs and thus do not contribute to $\mathfrak{M}' \circ \mathfrak{M}'$. □

REMARK. Let X be the complexification of a real algebraic surface and let $c = \operatorname{conj}$ be the Galois involution on X. (More generally, one can assume that X is a compact smooth complex analytic surface and c is an anti-holomorphic involution.) Then $\operatorname{Fix} c = X_{\mathbb{R}}$ is the real part of X and multiplication by $\sqrt{-1}$ establishes an isomorphism $\tau X_{\mathbb{R}} = \nu X_{\mathbb{R}}$. In particular, for any component F_i of $X_{\mathbb{R}}$ one has $F_i \circ F_i = -\chi(F_i)$, and the congruences of Theorems 4.2.2 and 4.2.3 take the form

$$\chi(F') \equiv \frac{1}{4}\left[\chi(X_{\mathbb{R}}) - \sigma(X)\right] + \operatorname{Br} \mathcal{P}' \bmod 8, \tag{4.1}$$

$$\chi(X_{\mathbb{R}}) \equiv 2 \operatorname{Br} \mathcal{P}_{[1]} - \sigma(X) \bmod 16. \tag{4.2}$$

Since $\chi(F') \equiv \operatorname{Br} \mathcal{P}' \bmod 2$, (4.1) implies

$$\chi(X_{\mathbb{R}}) \equiv \sigma(X) \bmod 8. \tag{4.3}$$

(Certainly, (4.3) follows also from the Arnold congruence for real algebraic surfaces with $[X_{\mathbb{R}}] = D_X w_2(X)$ in $H_2(X)$.)

5. Real Enriques surfaces

5.1. Real Enriques surfaces. Recall that an algebraic surface X is called a *K3-surface* if $\pi_1(X) = 0$ and $c_1(X) = 0$. An algebraic surface E is called an *Enriques surface* if $\pi_1(E) = \mathbb{Z}/2$ and the universal covering X of E is a $K3$-surface. (The classical definition of Enriques surfaces is $c_1(E) \neq 0$, $2c_1(E) = 0$, and the relationship to $K3$-surfaces follows from the standard classification.) All $K3$-surfaces form a single deformation family; they are all diffeomorphic to a degree 4 surface in P^3. Similarly, all Enriques surfaces form a single deformation family and are all diffeomorphic to each other. The intersection forms of $K3$- and Enriques surfaces are, respectively, $H_2(X; \mathbb{Z}) \cong 3E_8 \oplus 2U$ and $H_2(E; \mathbb{Z}) \cong E_8 \oplus U$, where E_8 is the even unimodular form of signature -8 and U is the hyperbolic plane.

A *real Enriques surface* is an Enriques surface E supplied with an anti-holomorphic involution $\operatorname{conj}: E \to E$, known as its *real structure*. The fixed point set $E_{\mathbb{R}} = \operatorname{Fix} \operatorname{conj}$ is called the *real part* of E. (Obviously, these definitions apply to any algebraic variety.)

Fix a real Enriques surface E and denote by $p\colon X \to E$ its universal covering and by $\tau\colon X \to X$, the *Enriques involution* (i.e., deck translation of p). The real structure conj on E lifts to two real structures $t^{(1)}, t^{(2)}\colon X \to X$, which commute with each other and with τ. Let $X_{\mathbb{R}}^{(i)} = \operatorname{Fix} t^{(i)}$, $i = 1, 2$, be their real parts. The projections $E_{\mathbb{R}}^{(i)} = p(X_{\mathbb{R}}^{(i)})$ are called *halves* of $E_{\mathbb{R}}$. It is easy to see that $E_{\mathbb{R}}^{(1)}$ and $E_{\mathbb{R}}^{(2)}$ are disjoint, $E_{\mathbb{R}} = E_{\mathbb{R}}^{(1)} \cup E_{\mathbb{R}}^{(2)}$, and both $E_{\mathbb{R}}^{(i)}$ consist of whole components of $E_{\mathbb{R}}$. Furthermore, two components $F_1, F_2 \subset E_{\mathbb{R}}$ belong to the same half if and only if $\mathrm{bv}_1\langle F_1 - F_2\rangle = 0$ (see [**DK2**]).

A real Enriques surface is said to be of *hyperbolic*, *parabolic*, or *elliptic type* if the minimal Euler characteristic of the components of $E_{\mathbb{R}}$ is negative, zero, or positive, respectively.

5.2. The Pontryagin-Viro form on a real Enriques surface. Fix a real Enriques surface E. For the topological types of the connected components of $E_{\mathbb{R}}$, we will use the notation $S = S^2$, $S_p = \#_p(S^1 \times S^1)$, and $V_p = \#_p\mathbb{R}\mathrm{p}^2$. The decomposition of $E_{\mathbb{R}}$ into two halves will be written as $E_{\mathbb{R}} = \{E_{\mathbb{R}}^{(1)}\} \sqcup \{E_{\mathbb{R}}^{(2)}\}$.

E is said to be of *type* I if $[E_{\mathbb{R}}] = 0$ in $(H_2(E;\mathbb{Z})/\operatorname{Tors}) \otimes \mathbb{Z}/2$ or, equivalently, $[X_{\mathbb{R}}^{(1)}] + [X_{\mathbb{R}}^{(2)}] = 0$ in $H_2(X)$; otherwise E is said to be of *type* II. Type I is further subdivided into I_0 and I_u, depending on whether $[E_{\mathbb{R}}] = 0$ or $Du_2(E)$ in $H_2(E)$.

5.2.1. LEMMA. *The following are sufficient conditions for the existence of the Pontryagin-Viro form $\mathcal{P}\colon \mathcal{F}^2 \to \mathbb{Z}/4$ on a real Enriques surface E:*

(1) *E is an M-surface;*
(2) *E is of type I_u and either $E_{\mathbb{R}}$ is nonorientable or both $E_{\mathbb{R}}^{(1)}$ and $E_{\mathbb{R}}^{(2)}$ are nonempty;*
(3) *E is of type I, $E_{\mathbb{R}}$ is nonorientable, and either both $E_{\mathbb{R}}^{(1)}$ and $E_{\mathbb{R}}^{(2)}$ are nonempty or $E_{\mathbb{R}}$ contains a nonorientable component of odd genus.*

PROOF. As is shown in [**DK2**], the subgroup $\mathcal{F}^2 \subset H_*(E_{\mathbb{R}})$ is generated by elements of the form $[F_0]$ and $\langle F_1 - F_2\rangle \oplus x_1$, where F_0, F_1, F_2 are components of $E_{\mathbb{R}}$, $x_1 \in H_1(E_{\mathbb{R}})$, and either $x_1^2 = 1$ and F_1, F_2 are in distinct halves, or $x_1^2 = 0$ and F_1, F_2 are in the same half. In particular, E is Galois maximal if and only if $E_{\mathbb{R}}$ is nonorientable or both $E_{\mathbb{R}}^{(1)}$, $E_{\mathbb{R}}^{(2)}$ are nonempty. If E satisfies the hypotheses of (3), then $Du_2(E) \neq 0$ in ${}^{\infty}H_2$. Since $[E_{\mathbb{R}}] = Du_2(E)$ in ${}^{\infty}H_2$, type I implies I_u. All the statements follow now from Corollary 3.2.2. □

From now on we assume that $\mathcal{P}$ is defined. Two components F_1, F_2 of the same half are said to be in one *quarter* if $\mathcal{P}\langle F_1 - F_2\rangle = 0$. Since $\mathcal{P}$ is linear on $\mathcal{F}^2_{[0]}$, each half $E_{\mathbb{R}}^{(i)}$ splits into two quarters, which consist of whole components of $E_{\mathbb{R}}^{(i)}$. We denote this by $E_{\mathbb{R}}^{(i)} = (\text{quarter } 1) \sqcup (\text{quarter } 2)$. Following [**Mik**], the decomposition of $E_{\mathbb{R}}$ into four quarters is called *complex separation.* Due to Corollary 4.1.5, it has the following geometrical meaning: a subsurface $F' \subset E_{\mathbb{R}}$ is characteristic in E/conj if and only if it is the union of two quarters which belong to distinct halves.

Let $E_{\mathbb{R}} = \{(Q_1^{(1)}) \sqcup (Q_2^{(1)})\} \sqcup \{(Q_1^{(2)}) \sqcup (Q_2^{(2)})\}$ be the decomposition of $E_{\mathbb{R}}$ into quarters. If both halves are nonempty, denote by $\mathfrak{q}_{ij}^{(1)}$ and $\mathfrak{q}_{ji}^{(2)}$ the restriction to $H_1(Q_i^{(1)})$ (respectively, $H_1(Q_j^{(2)})$) of the Rokhlin–Guillou–Marin form of the characteristic surface $Q_i^{(1)} \cup Q_j^{(2)}$. As follows from Proposition 4.2.1,

$$\mathfrak{q}_{i1}^{(1)} = \mathfrak{q}_{i2}^{(1)} + Dw_1(Q_i^{(1)}) \quad \text{and, hence, } \operatorname{Br}\mathfrak{q}_{i1}^{(1)} = -\operatorname{Br}\mathfrak{q}_{i2}^{(1)},$$
$$\mathfrak{q}_{j1}^{(2)} = \mathfrak{q}_{j2}^{(2)} + Dw_1(Q_i^{(2)}) \quad \text{and, hence, } \operatorname{Br}\mathfrak{q}_{j1}^{(2)} = -\operatorname{Br}\mathfrak{q}_{j2}^{(2)}$$

(see Propositions 2.1.2 and 2.1.1). If one of the halves, say $E_{\mathbb{R}}^{(2)}$, is empty, denote by $\mathfrak{q}_i^{(1)}$ the restriction of $\mathcal{P}$ to $H_2(Q_i^{(1)})$; this form is defined on the annihilator of $w_1(Q_i^{(1)})$ and is informative. In this notation congruence (4.1) takes the following form:

5.2.2. PROPOSITION. *If both the halves are nonempty, then for $i, j = 1, 2$*

$$\chi(Q_i^{(1)}) + \chi(Q_j^{(2)}) \equiv 2 + \tfrac{1}{4}\chi(E_{\mathbb{R}}) + \operatorname{Br}\mathfrak{q}_{ij}^{(1)} + \operatorname{Br}\mathfrak{q}_{ji}^{(2)} \bmod 8.$$

If $E_{\mathbb{R}}^{(2)} = \varnothing$, then for $i = 1, 2$

$$\chi(Q_i^{(1)}) \equiv 2 + \tfrac{1}{4}\chi(E_{\mathbb{R}}) + \operatorname{Br}\mathfrak{q}_i^{(1)} \bmod 8.$$

Another invariant used in the classification is the value $\mathcal{P}(w_1)$, where w_1 is the characteristic class of a nonorientable component of $E_{\mathbb{R}}$ of even Euler characteristic.

5.2.3. PROPOSITION. *Let $F_1, F_2 \subset E_{\mathbb{R}}$ be two nonorientable components of even Euler characteristic. Then $\mathcal{P}(w_1(F_1)) = \mathcal{P}(w_1(F_2))$.*

PROOF. As is shown in [**DK1**], if $E_{\mathbb{R}}$ has two nonorientable components of even Euler characteristic, it has no other nonorientable components. Hence, $w_1(E_{\mathbb{R}}) = w_1(F_1) + w_1(F_2)$. On the other hand, $w_1(E_{\mathbb{R}})$ is a characteristic element in $\mathcal{F}^2$ and, due to Propositions 2.1.2(2) and 3.1.4, $\mathcal{P}(w_1(E_{\mathbb{R}})) = 0$. □

6. The Pontryagin-Viro form via Donaldson's trick

6.1. Constructing surfaces via Donaldson's trick. Let Z be a rational surface with real structure $c\colon Z \to Z$ and nonempty real part, and $P, Q \subset Z$ a pair of nonsingular real curves.

6.1.1. ASSUMPTION. In this and next sections we assume that

(1) $[P]$ and $[Q]$ are even in $H_2(Z;\mathbb{Z})$ and $[P] + [Q] = -2K_Z$,
(2) $\dim\operatorname{Ker}[\text{inclusion}_*\colon H_2(P) \to H_2(Z)] = 1$,
(3) the multiplicity of each intersection point of P and Q is at most 2.

Since $[P]$ is even, $P_{\mathbb{R}}$ divides $Z_{\mathbb{R}}$ into two parts with common boundary $P_{\mathbb{R}}$. Denote their closures by $Z^{\pm P} = Z^{\pm}$. Similarly, introduce two parts $Z^{\pm Q}$ with common boundary $Q_{\mathbb{R}}$. Let $Z^{\delta\varepsilon} = Z^{\delta P} \cap Z^{\delta Q}$ for $\delta, \varepsilon = \pm$ and assume that

(4) $Z^{++} = \varnothing$, i.e., $P_{\mathbb{R}} \subset Z^{-Q}$ and $Q_{\mathbb{R}} \subset Z^{-P}$.

Consider the double covering $Y \to Z$ branched over P and denote by B the pullback of Q. Let Y' be the minimal resolution of singularities of B (which occur at the tangency points of P and Q and are all nondegenerate double points) and B' the proper transform of B. The real structure c on Z lifts to two real structures $c^{\pm}$ on Y'. With respect to one of them, c^{+}, the real part $Y'_{\mathbb{R}}$ projects to Z^{+} and $B'_{\mathbb{R}} = \varnothing$.

6.1.2. PROPOSITION. *The surfaces Y and Y' above are rational, $[B] = -2K_Y$, and $[B'] = -2K_{Y'}$. The double covering $X \to Y'$ of Y' branched over B' is a $K3$-surface.*

PROOF. The relations $[B] = -2K_Y$ and $[B'] = -2K_{Y'}$ follow from the projection formula and Assumption 6.1.1(1). Thus, the anti-bicanonical class of Y is effective and, hence, Y is either rational or ruled. On the other hand, from the Smith exact sequence and (2) it follows that $H_1(Y) = 0$. Hence, Y is rational. Now the projection formula gives $2K_X = 0$; i.e., X is a minimal surface of Kodaira dimension 0. Using the Riemann–Hurwitz and adjunction formulas, one obtains $\chi(X) = 2\chi(Y') + 2K^2_{Y'} = 24$. Hence, X is a $K3$-surface. □

Denote by $t^{(1)}$ the deck translation of $X \to Y'$. Due to (4) above, one of the two lifts of c^+ to X is fixed point free; denote it by τ. One can now apply to $(t^{(1)}, \tau)$ the following equivariant version of Donaldson's trick:

6.1.3. PROPOSITION (cf. [**DK3**]). *Let X be a $K3$-surface and (c_h, c_a) a pair of commuting involutions on X, one holomorphic and one anti-holomorphic. Then there is a complex structure on X with respect to which c_h is anti-holomorphic and c_a is holomorphic.*

Let $\widetilde{X}$ be the resulting $K3$-surface. Then the quotient $E = \widetilde{X}/\tau$ is an Enriques surface and $t^{(1)}$ descends to a real structure conj on E. Clearly, $E^{(1)}_{\mathbb{R}} = B'/c^+$ and $E^{(2)}_{\mathbb{R}} = Y'_{\mathbb{R}}$, and there is a projection $\pi\colon E_{\mathbb{R}} \to \bar{Q} \cup Z^+$, which is a branched double covering outside the tangency points of $P_{\mathbb{R}}$ and $Q_{\mathbb{R}}$. The pullback $\pi^{-1}(T)$ of each tangency point T consists of a one-sided loop in $E^{(2)}_{\mathbb{R}}$ and a point in $E^{(1)}_{\mathbb{R}}$. Let $\rho\colon E_{\mathbb{R}} \to E_{\mathbb{R}}$ be the deck translation of π.

6.1.4. NOTATION. For a subset $S \subset \bar{Q} \cup Z^+$, we denote by $\langle\pi^{-1}(Q)\rangle \in H_0(E_{\mathbb{R}})$ the class generated by the connected components of $\pi^{-1}(S)$, and by $[\pi^{-1}(S)] \in H_*(E_{\mathbb{R}})$ the fundamental class of its components of highest dimension (provided that they are all closed manifolds).

6.2. Existence of $\mathcal{P}$ and complex separation. Let E be a real Enriques surface obtained as in 6.1 from a configuration $(Z; P, Q)$. For all intermediate objects we keep the notation introduced in 6.1.

A nonsingular real curve $C \subset Z$ is said to be of *type* I, or *separating*, if C/c is orientable. The real part of each real component C_i of C has a distinguished pair of opposite orientations, called *complex orientations*, which are induced from an orientation of C_i/c. The complement $C_i \smallsetminus C_{i,\mathbb{R}}$ consists of two components $C_i^{\pm}$ with common boundary $C_{i,\mathbb{R}}$; their natural orientations induce the two complex orientations of $C_{i,\mathbb{R}}$.

A triple $(Z; P, Q)$ as in 6.1 is said to be of *type* I if P is of type I and $Z^- \smallsetminus Q_{\mathbb{R}}$ is orientable. Let P_i, $i = 1, \dots, p$, and Q_j, $j = 1, \dots, q$, be the real components of the curves; Z_k^-, $k = 1, \dots, z^-$, the connected components of Z^{-+} and Z^{--}; and Z_l^+, $l = 1, \dots, z^+$, the connected components of Z^+. Fix some orientations of Z_k^- and some complex orientations of $P_{i,\mathbb{R}}$; this determines an orientation of $\bar{P}_i$ and a distinguished half P_i^+ for each real component P_i. Thus, the fundamental classes $[\bar{P}_i]$ and $[Z_k^-]$ are well defined mod 4 in all homology groups where they make sense. The classes $[\bar{Q}_j]$ and $[Z_l^+]$ are defined mod 2.

6.2.1. DEFINITION. We say that a triple $(Z; P, Q)$ of type I *admits a fundamental cycle* if there are some odd integers λ_i, $i = 1, \dots, p$, and $\varkappa_k^-$, $k = 1, \dots, z^-$, and some integers μ_j, $j = 1, \dots, q$, and $\varkappa_l^+$, $l = 1, \dots, z^+$, such that

$$\sum \lambda_i[P_{i,\mathbb{R}}] + \sum \varkappa_k^-[\partial Z_k^-] = 2\sum \mu_j[Q_{j,\mathbb{R}}] + 2\sum \varkappa_l^+[\partial Z_l^+] \tag{6.1}$$

in $H_1(P_{\mathbb R} \cup Q_{\mathbb R}; \mathbb Z/4)$. The combination

$$\mathfrak{C} = \sum \lambda_i[\bar P_i] + \sum \varkappa_k^-[Z_k^-] + 2\sum \mu_j[\bar Q_j] + 2\sum \varkappa_l^+[Z_l^+], \tag{6.2}$$

which is a (mod 4)-cycle in $\bar Z$, is called a *fundamental cycle*. It is called *proper* if $[\mathfrak{C}] = 2Dw_2(\bar Z)$ in $H_2(\bar Z; \mathbb Z/4)$.

6.2.2. PROPOSITION. *A triple $(Z; P, Q)$ of type* I *admits a fundamental cycle if and only if $\sum[P_{i,\mathbb R}]$ belongs to the subgroup in $H_1(Z^-; \mathbb Z/4)$ spanned by the classes $2[P_{i,\mathbb R}]$, $2[Q_{j,\mathbb R}]$, and $2\partial[Z_l^+]$. If Z is an M-surface with $Z_{\mathbb R}$ connected, any fundamental cycle is proper.*

PROOF. The first part follows from the exact sequence of pair $(Z^-, P_{\mathbb R} \cup Q_{\mathbb R})$. If Z is an M-surface with $Z_{\mathbb R}$ connected, then $\bar Z$ is a $\mathbb Z$-homology sphere (see Proposition 1.5.3) and $[\mathfrak{C}] = 2w_2(\bar Z)$ holds trivially. □

The Pontryagin-Viro form on a real Enriques surface E is said to have *ρ-invariant complex separation* if all quarters are fixed by ρ. Since $\mathcal P$ is obviously ρ_*-invariant, for each half $E_{\mathbb R}^{(i)}$, $i = 1, 2$, one has either $\rho(Q_j^{(i)}) = Q_j^{(i)}$, $j = 1, 2$, or $\rho(Q_1^{(i)}) = Q_2^{(i)}$. This remark is sufficient to exclude the possibility of noninvariant complex separation in all cases considered in Section 7 below.

6.2.3. THEOREM. *The real Enriques surface resulting from a triple $(Z; P, Q)$ has Pontryagin-Viro form with ρ-invariant complex separation if and only if all tangency points of P and Q are real and $(Z; P, Q)$ is of type* I *and admits a proper fundamental cycle. If this is the case, the complex separation is determined by a proper fundamental cycle* (6.2):

(1) *the components $\pi^{-1}(\bar Q_a)$, $\pi^{-1}(\bar Q_b)$ corresponding to real components Q_a, Q_b of Q belong to the same quarter if and only if $\mu_a - \mu_b \equiv 0 \bmod 2$;*
(2) *the components $\pi^{-1}(Z_a^+)$, $\pi^{-1}(Z_b^+)$ corresponding to $Z_a^+, Z_b^+ \subset Z^+$ belong to the same quarter if and only if $\varkappa_a^+ - \varkappa_b^+ \equiv 0 \bmod 2$.*

6.2.4. COROLLARY. *A proper fundamental cycle* (6.2) *is unique* mod 4 *up to* $2[Z_{\mathbb R}]$, $2([Z^+] + [\bar P])$, *and* $2([Z^{-+}] + [\bar Q])$.

6.3. Proof of Theorem 6.2.3. Note that the construction of 6.1 still works if Assumption 6.1.1(1) and (2) are replaced by the weaker assumption

(1′) $[P]$ and $[Q]$ are divisible by 2 in $H_2(Z; \mathbb Z)$.

Certainly, Proposition 6.1.2 does not hold in this case and 6.1.3 does not apply; thus, E is just a 4-manifold with orientation-preserving involution. In view of Corollary 3.1.3, Theorem 6.2.3 would follow from the following more general result:

6.3.1. THEOREM. *Let (E, conj) be the manifold with involution resulting from a triple $(Z; P, Q)$ satisfying* (1′) *above and Assumption* 6.1.1(3), (4). *Then $E_{\mathbb R} \subset E/\mathrm{conj}$ contains a ρ-invariant characteristic surface if and only if all tangency points of P and Q are real and $(Z; P, Q)$ is of type* I *and admits a proper fundamental cycle. If this is the case, the ρ-invariant characteristic subsurfaces of $E_{\mathbb R}$ are those of the form $\sum \mu_j[\pi^{-1}(\bar Q_j)] + \sum \varkappa_l^+[\pi^{-1}(Z_l^+)]$, where μ_j and $\varkappa_l^+$ are the coefficients of a proper fundamental cycle.*

In order to prove Theorem 6.3.1, we replace E/conj by $\bar Y' = Y'/\mathrm{c}^+$. In the construction Y' is obtained from Y by blow-ups of the tangency points of P and Q. If such a point is real, the blow-up results in connected summation of $\bar Y$ and

$\overline{\mathbb{C}p^2}$/conjugation $\cong S^4$ and thus does not affect the topology. A pair of conjugate tangency points results in the (topological) blow-up of their common image in $\bar{Y}$. This produces a (-1)-sphere in $\bar{Y}'$ which is (mod 2)-orthogonal to all pull-backs $[\pi^{-1}(\bar{P}_j)]$ and $[\pi^{-1}(Z_l^+)]$, which shows that $E_{\mathbb{R}}$ does not contain a ρ-invariant characteristic surface. Thus, we can assume that P and Q do not have imaginary tangency points and replace $\bar{Y}'$ by $\bar{Y}$.

Clearly, $\bar{Y}$ is the double covering of $\bar{Z} = Z/c$ branched over the *Arnold surface* $\mathfrak{A}^- = \bar{P} \cup Z^-$. Denote by $\mathrm{pr}\colon \bar{Y} \to \bar{Z}$ the projection and by $\omega \in H^1(\bar{Z} \smallsetminus \mathfrak{A}^-)$ its characteristic class. Since $\partial\colon H_3(\bar{Z}, \mathfrak{A}^-) \to H_2(\mathfrak{A}^-)$ is a monomorphism, ω is uniquely characterized by the property $\partial D\omega = [\mathfrak{A}^-]$. Note also that, since $\bar{Z}$ is orientable, $\omega \cap D\omega = \mathrm{Sq}_1\, D\omega \in H_2(\bar{Z}, \mathfrak{A}^-)$ and $\partial\, \mathrm{Sq}_1\, D\omega = \mathrm{Sq}_1 [\mathfrak{A}^-] = Dw_1(\mathfrak{A}^-)$.

6.3.2. LEMMA. *$\sum \mu_j[\pi^{-1}(\bar{Q}_j)] + \sum \varkappa_l^+[\pi^{-1}(Z_l^+)] = Dw_2(\bar{Y})$ in $H_2(\bar{Y})$ if and only if the following relation holds in $H_2(\bar{Z}, \mathfrak{A}^-)$:*

$$\sum \mu_j[\bar{Q}_j] + \sum \varkappa_l^+[Z_l^+] = \mathrm{Sq}_1\, D\omega + Dw_2(\bar{Z}). \tag{6.3}$$

PROOF. The statement follows immediately from the projection formula and the Smith exact sequence. □

6.3.3. LEMMA. *The linear combination* (6.2) *is a fundamental cycle if and only if $\sum \mu_j[\partial\bar{Q}_j] + \sum \varkappa_l^+[\partial Z_l^+] = Dw_1(\mathfrak{A}^-)$ in $H_1(\mathfrak{A}^-)$.*

PROOF. This is a direct consequence of the definition of Bockstein homomorphism via chains. □

From Lemma 6.3.3 it follows that a necessary condition for (6.3) to hold is that μ_j, $\varkappa_l^+$ must be coefficients of a fundamental cycle; in particular, this implies that $(Z; P, Q)$ must be of type I. If this is the case, $\mathrm{Sq}_1\, D\omega$ is the relativization of a class $x \in H_2(\bar{Z}, \bar{P}_{\mathbb{R}} \cup \bar{Q}_{\mathbb{R}})$, which is well defined up to the image of $H_2(\mathfrak{A}^-)$ and has the property $2x = \sum[\bar{P}_i] + \sum[Z_k^-] \bmod 2H_2(\mathfrak{A}^-)$ in $H_2(\bar{Z}, \bar{P}_{\mathbb{R}} \cup \bar{Q}_{\mathbb{R}}; \mathbb{Z}/4)$. Since both $\times 2\colon H_2(\bar{Z}, \bar{P}_{\mathbb{R}} \cup \bar{Q}_{\mathbb{R}}) \to H_2(\bar{Z}, \bar{P}_{\mathbb{R}} \cup \bar{Q}_{\mathbb{R}}; \mathbb{Z}/4)$ and the relativization $H_2(\bar{Z}; \mathbb{Z}/4) \to H_2(\bar{Z}, \bar{P}_{\mathbb{R}} \cup \bar{Q}_{\mathbb{R}}; \mathbb{Z}/4)$ are monomorphisms, (6.3) is equivalent to the fact that μ_j and $\varkappa_l^+$ are coefficients of a proper fundamental cycle. □

6.4. Values on 1-dimensional classes. Fix a triple $(Z; P, Q)$ satisfying the conditions of Theorem 6.2.3, so that the Pontryagin-Viro form $\mathcal{P}$ on E is well defined and the complex separation is ρ-invariant.

Denote $\mathfrak{S} = Z_{\mathbb{R}} \cup \bar{P} \cup \bar{Q}$ and for an immersed (in the obvious sense) loop $\mathfrak{l} \subset \mathfrak{S}$ transversal to $P_{\mathbb{R}}$ and $Q_{\mathbb{R}}$ define its 'normal Euler number' $e(\mathfrak{l}) \in \mathbb{Z}/2$ to be 1 or 0 mod 2 depending on whether $\mathfrak{l}$ is disorienting or not. (If $\mathfrak{l}$ passes through an isolated intersection point of $\bar{P}$ and $\bar{Q}$, the orientation is transferred so that the point has intersection index $+1$.) The following obvious observation is helpful in evaluating $e(\mathfrak{l})$: *let an oriented arc $\mathfrak{l}'$ belong to a half C^+ of a type* I *curve C (which, in our case, can be the union of real components of P and separating real components of Q) so that $\partial\mathfrak{l}' \subset C_{\mathbb{R}}$ and $\mathfrak{l}'$ is normal to $C_{\mathbb{R}}$. Then the coorientation induced from the complex orientation of $C_{\mathbb{R}}$ at the initial point of $\mathfrak{l}'$ is transferred by $\mathfrak{l}'$ to the coorientation opposite to the complex orientation of $C_{\mathbb{R}}$ at the terminal point of $\mathfrak{l}'$.*

6.4.1. PROPOSITION. *For $\mathfrak{l} \subset \mathfrak{S}$ as above one has*

$$\mathcal{P}([\pi^{-1}(\mathfrak{l})] \oplus \langle \pi^{-1}(\mathfrak{l}) \rangle) = 2e(\mathfrak{l}) + 2i_Q(\mathfrak{l}) + 2i^+(\mathfrak{l}) + i_{P\cap Q}(\mathfrak{l}) \bmod 4,$$

where $i_Q(\mathfrak{l})$ and $i^+(\mathfrak{l})$ are the numbers of isolated intersection points of $\mathfrak{l}$ with $\bar{Q}$ and Z^+, respectively, and $i_{P\cap Q}(\mathfrak{l})$ is the number of intersection points $\bar{P} \cap \bar{Q}$ through which $\mathfrak{l}$ passes.

PROOF. Let $\mathfrak{M}$ be a membrane in $\bar{Z}$ normal along $\partial\mathfrak{M} = \mathfrak{l}$ and transversal in $\operatorname{int}\mathfrak{M}$ to all strata of $\mathfrak{S}$. Then the value in question can be found by using $\mathfrak{M}_Y = \operatorname{pr}^{-1}(\mathfrak{M})$ (cf. Proposition 4.2.1). Clearly, $\operatorname{ind}\mathfrak{M}_Y = 2\operatorname{ind}\mathfrak{M} + \frac{1}{2}i_{P\cap Q}(\mathfrak{l})$. (To define $\operatorname{ind}\mathfrak{M}$, we use a normal line field on $\mathfrak{l}$ tangent to all strata of $\mathfrak{S}$ and patch it at the points of $\bar{P} \cap \bar{Q}$ using local orientations.) Further,

$$\operatorname{Card}(\operatorname{int}\mathfrak{M}_Y \cap E_{\mathbb{R}}^{(1)}) = i_Q(\mathfrak{l}) + 2\operatorname{Card}(\operatorname{int}\mathfrak{M} \cap \bar{Q}),$$

$$\operatorname{Card}(\operatorname{int}\mathfrak{M}_Y \cap E_{\mathbb{R}}^{(2)}) = i^+(\mathfrak{l}) + 2\operatorname{Card}(\operatorname{int}\mathfrak{M} \cap Z^+).$$

It remains to notice that $2\operatorname{ind}\mathfrak{M} = 0$ or $1 \bmod 2$ depending on whether the line field is orientable or not. □

6.4.2. PROPOSITION. *Given an intersection point $T \in P_{\mathbb{R}} \cap Q_{\mathbb{R}}$, for its pullback one has $\mathcal{P}([\pi^{-1}(T)] + \langle \pi^{-1}(T) \rangle) = 1$.*

Let S be a connected component of one of $\bar{Q}$, $\bar{P}$, or $Z_{\mathbb{R}} \smallsetminus (P_{\mathbb{R}} \cup Q_{\mathbb{R}})$. Assume that ∂S does not contain a tangency point of P and Q. Then ∂S is a loop in $E_{\mathbb{R}}$.

6.4.3. PROPOSITION. *Let Z_k^- be a connected component of Z^- with ∂Z_k^- disjoint from $P_{\mathbb{R}} \cap Q_{\mathbb{R}}$. Then $\mathcal{P}[\partial Z_k^-] = 2\chi(Z_k^-) \bmod 4$.*

6.4.4. PROPOSITION. *Let P_i be a real component of P with $P_{i,\mathbb{R}}$ disjoint from $P_{\mathbb{R}} \cap Q_{\mathbb{R}}$. Then $\mathcal{P}([P_{i,\mathbb{R}}] + \langle \pi^{-1}(\bar{P}_i \cap \bar{Q}) \rangle) = \frac{1}{2}[P_i]^2 \bmod 4$.*

PROOF OF PROPOSITIONS 6.4.2, 6.4.3, AND 6.4.4. We lift Z_k^- (respectively, P_i or the exceptional curve appearing when T is blown up in Y) to X and then project the result to E. This gives a conj-invariant closed surface in E, and the value in question is found via Corollary 3.3.3. □

In the rest of this section we consider the case in which S is either $\bar{Q}_j$ for a real component Q_j of Q or a component Z_l^+ of Z^+. Fix a proper fundamental cycle $\mathfrak{C}$ and denote by U the corresponding characteristic surface in $\bar{Y}$. Assume that U contains $\pi^{-1}(S) \subset E_{\mathbb{R}}$ and denote by $\mathfrak{q}_{\mathfrak{C}}$ the Rokhlin–Guillou–Marin form of U. The choice of $\mathfrak{C}$ determines a preferred orientation of $\bar{P}$ (which induces $\sum \lambda_i[P_{i,\mathbb{R}}] \bmod 4$ on $\partial\bar{P}$) or, equivalently, a half P^+ of $P \smallsetminus P_{\mathbb{R}}$.

Let $\omega \in H^1(Z_{\mathbb{R}})$ be the class Poincaré dual to $[C_{\mathbb{R}}]$, where $C \subset Z$ is a real curve with $2[C] = [P]$ in $H_2(Z;\mathbb{Z})$. The restriction of ω to $Z_{\mathbb{R}} \smallsetminus P_{\mathbb{R}}$ is the characteristic class of the restricted covering $Y \to Z$ (see Proposition 1.5.2). For each real component Q_j of Q denote by $\omega_j \in H^1(\bar{Q}_j \smallsetminus \bar{P})$ the characteristic class of the covering $\bar{Y} \to \bar{Z}$ restricted to $\bar{Q}_j \smallsetminus \bar{P}$. ($\omega_j$ can be interpreted as the linking number with $\mathfrak{A}^-$ in $\bar{Z}$.)

Assume that $\langle \omega, [\mathfrak{l}] \rangle = 0$ for each boundary component $\mathfrak{l} \subset \partial S$ and define the 'linking number' $\operatorname{lk}_{\mathfrak{C}} w_1(S) \in \mathbb{Z}/4$ of the characteristic class of S with the Arnold surface. Fix some orientations of the boundary components and, if $S = \bar{Q}_j$, some local orientations at the intersection points $S \cap \bar{P}$. This defines a lift of $w_1(S)$ to a class $w_1' \in H^1(S, \partial S \cup (S \cap \bar{P}))$ (which can be defined as the obstruction to extending the chosen orientations to the whole surface). We let

$$\operatorname{lk}_{\mathfrak{C}} w_1(S) = 2\langle \omega, Dw_1' \rangle \text{ if } S = Z_l^+ \text{ or } 2\langle \omega_j, Dw_1' \rangle - (\operatorname{int} S \circ \bar{P}) \text{ if } S = \bar{Q}_j.$$

(In the latter case the intersection index is defined mod 4 using the chosen local orientations of S; the condition $\langle \omega_j, [\partial S] \rangle = 0$ implies that ∂S is not linked with $\mathfrak{A}^-$ and, hence, $\operatorname{int} S \circ \bar{P} = 0 \bmod 2$.)

In the case $S = \bar{Q}_j$, for a real component Q_j of Q the above definition is cumbersome and not 'seen' in the real part. If Q_j is of type I, it can be simplified: $\operatorname{lk}_{\mathfrak{C}} w_1(S) = \operatorname{Card}(Q_j^- \cap P^-) - \operatorname{Card}(Q_j^+ \cap P^-) \bmod 4$ for any half Q_j^- of Q_j.

6.4.5. Proposition. *Let S be either $\bar{Q}_j$ for a real component Q_j of Q, or a component Z_l^+ of Z^+. Assume that ∂S is disjoint from $P_{\mathbb{R}} \cap Q_{\mathbb{R}}$ and $\langle \omega, [\mathfrak{l}] \rangle = 0$ for each boundary component $\mathfrak{l} \subset \partial S$. Then $\mathfrak{q}_{\mathfrak{C}}[\partial S] = \operatorname{lk}_{\mathfrak{C}} w_1(S)$.*

Remark. Proposition 6.4.5 applies to the generalized construction described in 6.3. In the case of Enriques surfaces, due to Corollary 6.2.4, the preferred orientation of P is defined up to total reversing and $\operatorname{lk}_{\mathfrak{C}} w_1(S)$ does not depend on the choice of $\mathfrak{C}$. Furthermore, due to the assumption made, ∂S is a collection of two-sided circles in $E_{\mathbb{R}}$. Hence, $\mathfrak{q}_{\mathfrak{C}}[\partial S] = \mathcal{P}[\partial S]$.

Proof. Let $\mathfrak{M}_Y \subset \bar{Y}$ be an oriented membrane normal along $\partial \mathfrak{M}_Y = \partial S$ and transversal in $\operatorname{int} \mathfrak{M}$ to both U and $\mathfrak{A}^-$. Let $\mathfrak{M}$ be its projection to Z. Clearly, $\operatorname{ind} \mathfrak{M}_Y = \operatorname{ind} \mathfrak{M}$ and the intersection points of $\operatorname{int} \mathfrak{M}_Y$ and U project one-to-one. Furthermore, $\mathfrak{M}$ is tangent to $\mathfrak{A}^-$ at its inner points; hence,

$$2\operatorname{Card}(\operatorname{int} \mathfrak{M}_Y \cap U) = (\operatorname{int} \mathfrak{M} \circ \mathfrak{C}) - 2(\operatorname{int} \mathfrak{M} \circ \mathfrak{A}^-) \bmod 4.$$

(Recall that U consists precisely of those components of $E_{\mathbb{R}}$ whose coefficients in $\mathfrak{C}$ are 2 mod 4.) Since $[\mathfrak{C}] = 2w_2(Z) \bmod 4$ and $[\mathfrak{A}^-] = 0 \bmod 2$ in Z, the expressions $2\operatorname{ind} \mathfrak{M} + (\operatorname{int} \mathfrak{M} \circ \mathfrak{C}) \bmod 4$ and $(\operatorname{int} \mathfrak{M} \circ \mathfrak{A}^-) \bmod 2$ do not depend on the choice of $\mathfrak{M}$; one can replace $\mathfrak{M}$ by another membrane, which does not have to lift to $\bar{Y}$. (Strictly speaking, to claim this, one should fix the orientation of $\partial \mathfrak{M}$ induced from $\mathfrak{M}$; however, it is chosen arbitrarily for $\mathfrak{M}_Y$ and does not affect the result.)

For the new membrane take S shifted along a normal vector field. To make it orientable, fix some choices used to define $\operatorname{lk}_{\mathfrak{C}} w_1(S)$, cut S along a simple loop $\mathfrak{l}$ representing w_1', pick a generic orientable membrane $\mathfrak{N}$ spanned by $\mathfrak{l}$, and attach $2\mathfrak{N}$ to the cut. Let $\mathfrak{M}$ be the result. Then $\operatorname{int} \mathfrak{M} \circ \mathfrak{A}^- = \operatorname{Card}(S \cap \bar{P}) \bmod 2$ and

$$\operatorname{int} \mathfrak{M} \circ \mathfrak{C} = 2\operatorname{ind} S + (\operatorname{int} S \circ \mathfrak{A}^-) + 2\operatorname{Card}(\mathfrak{N} \cap \mathfrak{A}^-) \bmod 4.$$

(The first term here is due to the original shift of S along a normal field: recall that S has coefficient 2 mod 4 in $\mathfrak{C}$. The second term stands for the intersection index of P and the cut of S, which are both oriented.) Since $\operatorname{ind} \mathfrak{M} = \operatorname{ind} S \bmod 2$, one obtains $\mathfrak{q}_{\mathfrak{C}}(\partial S) = 2\operatorname{Card}(\mathfrak{N} \cap \mathfrak{A}^-) - (\operatorname{int} S \circ P) = \operatorname{lk}_{\mathfrak{C}} w_1(S) \bmod 4$. □

6.5. Resolving singularities of P and Q. Let $(Z'; P', Q')$ be a triple satisfying Assumption 6.1.1(1)–(4), except that P' and Q' may be singular. Assume that the curve $P' + Q'$ has at most simple singular points (i.e., those of type A_p, D_q, E_6, E_7, or E_8). Then there is a sequence of blow-ups which converts $(Z'; P', Q')$ to a triple $(Z; P, Q)$ with P and Q nonsingular and satisfying Assumption 6.1.1(1)–(4). More precisely, the singularities of $P' + Q'$ can be resolved by blowing up double or triple points. Let O be a singular point of, say, P'. Blow it up and denote by e the exceptional divisor and by $\tilde{P}$, the proper transform of P'. Then the new pair (P, Q)

on the resulting surface is constructed as follows: Q is the full transform of Q' and P is either $\tilde{P}$ or $\tilde{P}+e$, depending on whether O is a double or triple point of P'. The singular points of Q' are resolved similarly, with P' and Q' interchanged. If the resulting curves (P, Q) are still singular, the procedure is repeated.

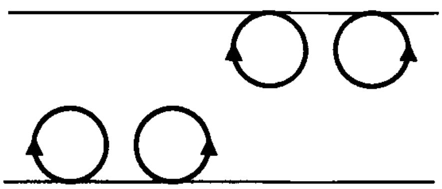
7. Calculation for real Enriques surfaces

7.1. M-surfaces of elliptic and parabolic type.

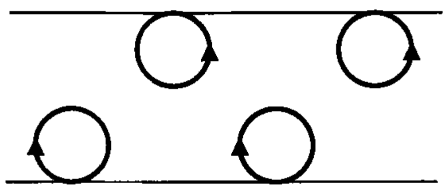
TABLE 1. M-surfaces of elliptic type ($E_{\mathbb{R}} = 4V_1 \sqcup 2S$)

$(2V_1 \sqcup S) \sqcup (2V_1 \sqcup S)$	$\varnothing$	$(V_1 \sqcup S) \sqcup (S)$	$(2V_1) \sqcup (V_1)$
$(4V_1) \sqcup (2S)$	$\varnothing$	$(V_1) \sqcup (2S)$	$(2V_1) \sqcup (V_1)$
$(2V_1 \sqcup S) \sqcup (V_1 \sqcup S)$	(V_1)	$(3V_1) \sqcup (S)$	$(V_1) \sqcup (S)$
$(3V_1) \sqcup (2S)$	(V_1)	$(2V_1) \sqcup (V_1 \sqcup S)$	$(V_1) \sqcup (S)$
$(2V_1 \sqcup S) \sqcup (S)$	$(V_1) \sqcup (V_1)$	$(2V_1) \sqcup (V_1 \sqcup S)$	$(V_1 \sqcup S)$
$(V_1 \sqcup S) \sqcup (V_1 \sqcup S)$	$(V_1) \sqcup (V_1)$	$(S) \sqcup (S)$	$(2V_1) \sqcup (2V_1)$
$(2V_1) \sqcup (2S)$	$(V_1) \sqcup (V_1)$	$(2S)$	$(2V_1) \sqcup (2V_1)$
$(V_1 \sqcup S) \sqcup (V_1 \sqcup S)$	$(2V_1)$	$(2V_1) \sqcup (S)$	$(V_1 \sqcup S) \sqcup (V_1)$
$(2V_1) \sqcup (2S)$	$(2V_1)$	$(V_1 \sqcup S) \sqcup (V_1)$	$(V_1 \sqcup S) \sqcup (V_1)$
$(3V_1) \sqcup (V_1 \sqcup S)$	(S)	$(2V_1) \sqcup (S)$	$(2V_1) \sqcup (S)$

7.1.1. THEOREM (**[DIK]**). *A real Enriques M-surface E of parabolic or elliptic type is determined up to deformation equivalence by its complex separation and the value $\mathcal{P}(w_1)$ of $\mathcal{P}$ on the characteristic element of a nonorientable component of $E_{\mathbb{R}}$ of even Euler characteristic (if such a component exists). The deformation types of such surfaces are given in Tables 1 and 2, which list the separations of the two halves and the possible values of $\mathcal{P}(w_1)$.*

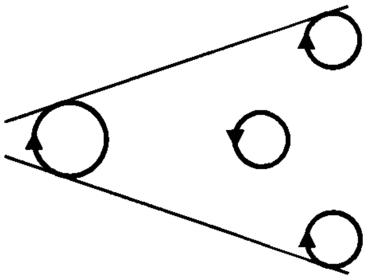


$\{(2V_2) \sqcup (\varnothing)\} \sqcup \{(2S) \sqcup (2S)\}$, $\mathcal{P}(w_1) = 0$

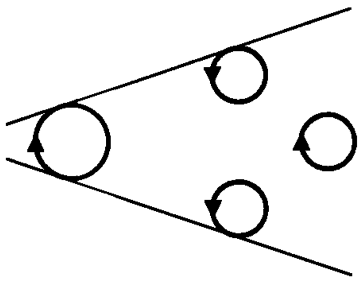


$\{(2V_2) \sqcup (\varnothing)\} \sqcup \{(2S) \sqcup (2S)\}$, $\mathcal{P}(w_1) = 2$

(Z' is a hyperboloid $\mathbb{R}\mathrm{p}^1 \times \mathbb{R}\mathrm{p}^1$)



$\{(V_2) \sqcup (V_2)\} \sqcup \{(2S) \sqcup (2S)\}$, $\mathcal{P}(w_1) = 0$



$\{(V_2) \sqcup (V_2)\} \sqcup \{(3S) \sqcup (S)\}$, $\mathcal{P}(w_1) = 2$

(Z' is a real projective plane $\mathbb{R}\mathrm{p}^2$)

FIGURE 1. Models of real Enriques surfaces with the real part $E_{\mathbb{R}} = \{2V_2\} \sqcup \{4S\}$

TABLE 2. M-surfaces of parabolic type

Case $E_{\mathbb{R}} = S_1 + V_2 + 4S$		
$(V_2 + 2S) + (2S)$	$(S_1) + (\varnothing)$	0
Case $E_{\mathbb{R}} = 2V_2 + 4S$		
$(V_2) + (V_2)$	$(2S) + (2S)$	0
$(V_2) + (V_2)$	$(3S) + (S)$	2
$(2V_2) + (\varnothing)$	$(2S) + (2S)$	0, 2
$(V_2 + 2S) + (2S)$	$(V_2) + (\varnothing)$	0
$(V_2 + S) + (2S)$	$(V_2 + S) + (\varnothing)$	2
$(V_2 + 2S) + (S)$	$(V_2) + (S)$	2
$(V_2 + S) + (S)$	$(V_2 + S) + (S)$	0
Case $E_{\mathbb{R}} = V_2 + 2V_1 + 3S$		
$(V_2 + 2S) + (2V_1 + S)$	$\varnothing$	0
$(V_2 + 2V_1 + S) + (2S)$	$\varnothing$	0, 2
$(V_2 + V_1 + S) + (V_1 + S)$	$(S) + (\varnothing)$	0, 2
$(V_2 + S) + (2V_1)$	$(S) + (S)$	0
$(V_2 + S) + (2V_1)$	$(2S) + (\varnothing)$	2
$(V_2 + 2V_1) + (S)$	$(S) + (S)$	0, 2
$(V_2 + V_1) + (V_1)$	$(2S) + (S)$	0, 2
$(V_2 + 2S) + (V_1 + S)$	$(V_1) + (\varnothing)$	0
$(V_2 + V_1 + S) + (2S)$	$(V_1) + (\varnothing)$	0, 2
$(V_2 + S) + (V_1 + S)$	$(V_1) + (S)$	0
$(V_2 + S) + (V_1 + S)$	$(V_1 + S) + (\varnothing)$	2
$(V_2 + V_1 + S) + (S)$	$(V_1) + (S)$	0, 2
$(V_2 + S) + (V_1)$	$(V_1 + S) + (S)$	0
$(V_2 + S) + (V_1)$	$(V_1) + (2S)$	2
$(V_2 + V_1) + (S)$	$(V_1 + S) + (S)$	0, 2
$(V_2) + (V_1)$	$(V_1 + S) + (2S)$	0
$(V_2) + (V_1)$	$(V_1 + 2S) + (S)$	2
$(V_2 + V_1) + (\varnothing)$	$(V_1 + S) + (2S)$	0, 2
$(V_2 + S) + (2S)$	$(V_1) + (V_1)$	0
$(V_2 + S) + (2S)$	$(2V_1) + (\varnothing)$	2
$(V_2 + 2S) + (S)$	$(V_1) + (V_1)$	0
$(V_2 + S) + (S)$	$(2V_1) + (S)$	0
$(V_2 + S) + (S)$	$(V_1 + S) + (V_1)$	2
$(V_2) + (S)$	$(V_1 + S) + (V_1 + S)$	0
$(V_2) + (S)$	$(2V_1 + S) + (S)$	2
$(V_2 + S) + (\varnothing)$	$(V_1 + S) + (V_1 + S)$	0
$(V_2 + S) + (\varnothing)$	$(2V_1) + (2S)$	2
$(V_2) + (\varnothing)$	$(2V_1 + S) + (2S)$	0
$(V_2) + (\varnothing)$	$(V_1 + 2S) + (V_1 + S)$	2

REMARK. In [Kü] it is shown that in all cases listed in the tables $\mathcal{P}$ is uniquely recovered (up to autohomeomorphism of $E_{\mathbb{R}}$ preserving the complex separation) from the complex separation and $\mathcal{P}(w_1)$ via Proposition 5.2.2 and, moreover, all forms satisfying the congruences of Proposition 5.2.2 are realized by real Enriques surfaces.

The calculation of Pontryagin-Viro forms is based on the results of Section 5 and the following statement, which gives explicit models of M-surfaces of elliptic and parabolic types:

7.1.2. THEOREM ([**DIK**]). *Up to deformation, any real Enriques M-surface of parabolic or elliptic type can be obtained by the construction of* 6.1 *and* 6.5 *from a triple* $(Z'; P', Q')$, *where either*

(1) $Z' = \mathbb{R}p^2$, *P' is an M-curve of degree* 4, *and Q' is a pair of lines; or*
(2) *Z' is a hyperboloid* $\mathbb{R}p^1 \times \mathbb{R}p^1$, *$P'$ is a nonsingular M-curve of bi-degree* $(4, 2)$, *and Q' is a pair of generatrices of bi-degree* $(0, 1)$.

Figure 1 illustrates the construction of the four nonequivalent surfaces with $E_{\mathbb{R}} = \{2V_2\} \sqcup \{4S\}$ (see Table 2). To emphasize the difference, the linear components of Q' (the lines) are shown tangent to P'; in reality they must be shifted away from P'.

7.2. M-surfaces of hyperbolic type.

TABLE 3. M-surfaces of hyperbolic type

Case $E_{\mathbb{R}} = V_3 + V_1 + 4S$		Other cases		
$(V_3 \sqcup V_1) \sqcup (\varnothing)$	$(2S) \sqcup (2S)$	$(V_4 \sqcup S) \sqcup (\varnothing)$	$(2S) \sqcup (2S)$	0
$(V_3 \sqcup S) \sqcup (\varnothing)$	$(V_1 \sqcup S) \sqcup (2S)$	$(V_{11} \sqcup V_1) \sqcup (\varnothing)$	$\varnothing$	
$(V_3 \sqcup S) \sqcup (V_1)$	$(2S) \sqcup (S)$	$(V_{11}) \sqcup (\varnothing)$	$(V_1) \sqcup (\varnothing)$	
$(V_3 \sqcup S) \sqcup (S)$	$(V_1 \sqcup S) \sqcup (S)$	$(V_{10}) \sqcup (\varnothing)$	$(V_2) \sqcup (\varnothing)$	0
$(V_3 \sqcup V_1 \sqcup S) \sqcup (S)$	$(S) \sqcup (S)$	$(V_9) \sqcup (\varnothing)$	$(V_3) \sqcup (\varnothing)$	
$(V_3 \sqcup 2S) \sqcup (S)$	$(V_1) \sqcup (S)$	$(V_8) \sqcup (\varnothing)$	$(V_4) \sqcup (\varnothing)$	2
$(V_3 \sqcup 2S) \sqcup (V_1 \sqcup S)$	$(S) \sqcup (\varnothing)$	$(V_7) \sqcup (\varnothing)$	$(V_5) \sqcup (\varnothing)$	
$(V_3 \sqcup 2S) \sqcup (2S)$	$(V_1) \sqcup (\varnothing)$	$(V_6) \sqcup (\varnothing)$	$(V_6) \sqcup (\varnothing)$	0
$(V_3 \sqcup V_1 \sqcup 2S) \sqcup (2S)$	$\varnothing$	$(V_{10}) \sqcup (\varnothing)$	$(S_1) \sqcup (\varnothing)$	0

7.2.1. THEOREM ([**DK3**]). *A real Enriques surface of hyperbolic type is determined up to deformation by the decomposition* $E_{\mathbb{R}} = \{E_{\mathbb{R}}^{(1)}\} \sqcup \{E_{\mathbb{R}}^{(2)}\}$. *The realized decompositions are listed in Table* 3.

As one can easily see, the Pontryagin-Viro form of an M-surface of hyperbolic type is uniquely recovered from Proposition (5.2.2). The corresponding complex separations and values $\mathcal{P}(w_1)$ are given in Table 3.

7.3. Other surfaces with Pontryagin-Viro form. Below we consider the remaining cases, i.e., those real Enriques surfaces admitting Pontryagin-Viro form that are not M-surfaces.

7.3.1. THEOREM. *Nonmaximal real Enriques surfaces admitting Pontryagin-Viro form are those and only those listed in Table* 4. *The Pontryagin-Viro form of such a surface is determined, via Proposition* (5.2.2), *by the decomposition* $E_{\mathbb{R}} = \{E_{\mathbb{R}}^{(1)}\} \sqcup \{E_{\mathbb{R}}^{(2)}\}$. *Table* 4 *lists the complex separations and values* $\mathcal{P}(w_1)$ *on the characteristic class of a nonorientable component of even genus, if such a component is present. An asterisk * marks the decompositions which are also realized by a real Enriques surface of type* II; *a double asterisk ** marks the decompositions which are also realized by a real Enriques surface of type* I *not admitting a Pontryagin-Viro form.*

TABLE 4. Other surfaces with Pontryagin-Viro form

$(M-2)$-surfaces				$(M-2)$-surfaces (continued)		
$(V_4)\sqcup(\varnothing)$	$(V_1)\sqcup(V_1)$	0	*	$(V_1)\sqcup(V_1)$	$(2S)\sqcup(S)$	
$(V_4)\sqcup(V_1)$	$(V_1)\sqcup(\varnothing)$	0	*	$(2V_1)\sqcup(S)$	$(S)\sqcup(S)$	
$(V_4)\sqcup(2V_1)$	$\varnothing$	0	*	$(V_1\sqcup S)\sqcup(V_1\sqcup S)$	$(S)\sqcup(\varnothing)$	
$(V_3)\sqcup(V_1)$	$(V_2)\sqcup(\varnothing)$	2	*	$(2V_1\sqcup S)\sqcup(2S)$	$\varnothing$	
$(V_3)\sqcup(\varnothing)$	$(V_2)\sqcup(V_1)$	2	*	$(V_1\sqcup S)\sqcup(S)$	$(V_1)\sqcup(S)$	
$(V_6)\sqcup(\varnothing)$	$(S)\sqcup(S)$	0	*	$(V_1\sqcup S)\sqcup(2S)$	$(V_1)\sqcup(\varnothing)$	
$(V_5)\sqcup(\varnothing)$	$(V_1)\sqcup(S)$		**	$(V_2\sqcup 2S)\sqcup(2S)$	$\varnothing$	0
$(V_5)\sqcup(V_1)$	$(S)\sqcup(\varnothing)$			$(V_2\sqcup S)\sqcup(S)$	$(S)\sqcup(S)$	0
$(V_5)\sqcup(S)$	$(V_1)\sqcup(\varnothing)$			$(V_2)\sqcup(\varnothing)$	$(2S)\sqcup(2S)$	0
$(V_5\sqcup V_1)\sqcup(S)$	$\varnothing$		**	$(S_1)\sqcup(\varnothing)$	$(2S)\sqcup(2S)$	
$(V_4)\sqcup(\varnothing)$	$(V_2)\sqcup(S)$	2	**	$(V_{10})\sqcup(\varnothing)$	$\varnothing$	0
$(V_4)\sqcup(S)$	$(V_2)\sqcup(\varnothing)$	0		$(M-4)$-surfaces		
$(V_3)\sqcup(S)$	$(V_3)\sqcup(\varnothing)$		**	$(V_4)\sqcup(S)$	$\varnothing$	0
$(V_4)\sqcup(S)$	$(S_1)\sqcup(\varnothing)$	0	**	$(2S)\sqcup(2S)$	$\varnothing$	
$(V_2)\sqcup(V_2)$	$(S_1)\sqcup(\varnothing)$	2		$(S)\sqcup(S)$	$(S)\sqcup(S)$	

Proof of Theorem 7.3.1 is based on the classification of real Enriques surfaces, which will appear in full in [**DIK**] (see also [**DK1, DK2, DK3**]). The necessary partial results are cited below.

The fact that in all cases listed in Table 4 the Pontryagin-Viro form is determined by Proposition (5.2.2) is straightforward. Thus, it remains to enumerate the surfaces for which the Pontryagin-Viro form is well defined. In view of (4.3), for such a surface E one has $\chi(E_{\mathbb{R}}) = 8$, 0, or -8. If $b_0(E_{\mathbb{R}}) = 1$, Proposition 5.2.2 applied to an empty quarter gives $\chi(E_{\mathbb{R}}) = -8$. Thus, it suffices to consider $(M-d)$-surfaces with either $\chi(E_{\mathbb{R}}) = 8$ and $d = 2, 4$, or $\chi(E_{\mathbb{R}}) = 0$ and $d = 2, 4$, or $\chi(E_{\mathbb{R}}) = -8$ and $d = 2$.

Case 1: $\chi(E_{\mathbb{R}}) = -8$, $d = 2$. The only topological type $E_{\mathbb{R}} = V_{10}$. *There are two deformation families of real Enriques surfaces E with $E_{\mathbb{R}} = V_{10}$; they are both of type* I *and differ by whether $w_2(E/\mathrm{conj})$ is or is not* 0 (see [**DK3**] and [**DIK**]). Since $E_{\mathbb{R}}$ has a single component, such a surface admits the Pontryagin-Viro form if and only if $w_2(E/\mathrm{conj}) = 0$.

Case 2: $\chi(E_{\mathbb{R}}) = 0$, $d = 2$. All such surfaces are of type I (see [**DK2**]); hence, they satisfy the hypotheses of Lemma 5.2.1(3) and the Pontryagin-Viro form is well defined.

Case 3: $\chi(E_{\mathbb{R}}) = 0$, $d = 4$. Among the five topological types, with $E_{\mathbb{R}} = 2S_1$, $S_1\sqcup V_2$, $2V_2$, $V_3\sqcup V_1$, and $V_4\sqcup S$ (see [**DK1**]), only the last one can satisfy Proposition 5.2.2. (Recall that an S_1 component must form a separate half.) Furthermore, the complex separation must be $\{(V_4)\sqcup(S)\}\sqcup\{\varnothing\}$; in particular, both the components of $E_{\mathbb{R}}$ are in one half. *There are two deformation families of real Enriques surfaces E with $E_{\mathbb{R}} = \{V_4\sqcup S\}\sqcup\{\varnothing\}$* (see [**DIK**]).[1] *They can be obtained by the construction of* 6.1 *from a triple $(Z; P, Q)$, where $Z = \Sigma_4$ is a rational ruled surface with a (-4)-section, $P \in |2e_\infty|$, and $Q \in |e_0 + e_\infty|$. (Here e_0 is the exceptional (-4)-section and e_∞ is the class of a generic section.) One type is obtained when $Z_{\mathbb{R}} = S_1$ and $P_{\mathbb{R}} = \varnothing$; the other one, when $Z_{\mathbb{R}} = \varnothing$.* In the former case, $Z_{\mathbb{R}} = S_1$, one can apply Theorem 6.2.3: since P is of type II, the Pontryagin-Viro form is not

[1] In [**DK3**] it is erroneously stated that there is one family with $E_{\mathbb{R}} = \{V_4\sqcup S\}\sqcup\{\varnothing\}$.

defined. Thus, it suffices to construct a surface with $E_{\mathbb{R}} = V_4 \sqcup S$ and well-defined Pontryagin-Viro form. This can be done as in 6.1 and 6.5, where $Z' = \mathbb{R}p^1 \times \mathbb{R}p^1$, P' is a pair of conjugate generatrices of bi-degree $(0,1)$, and Q' is the union of a pair of generatrices of bi-degree $(1,0)$ and a nonsingular curve of bi-degree $(2,2)$.

Case 4: $\chi(E_{\mathbb{R}}) = 8$, $d = 2$. There are three topological types, with $E_{\mathbb{R}} = 2V_1 \sqcup 3S$, $V_2 \sqcup 4S$, and $S_1 \sqcup 4S$ (see [**DK1**]), which we consider separately.

Each decomposition $E_{\mathbb{R}} = \{E_{\mathbb{R}}^{(1)}\} \sqcup \{E_{\mathbb{R}}^{(2)}\}$ *of* $E_{\mathbb{R}} = 2V_1 \sqcup 3S$ *is realized by two deformation families of real Enriques surfaces, one of type* I, *and one of type* II (see [**DIK**]). The surfaces of type I satisfy the hypotheses of Lemma 5.2.1(3) and, hence, have well-defined Pontryagin-Viro forms.

There is one deformation family of real Enriques surfaces E *with* $E_{\mathbb{R}} = \{V_2\} \sqcup \{4S\}$, *one family of surfaces with* $E_{\mathbb{R}} = \{V_2 \sqcup 2S\} \sqcup \{2S\}$, *and two families of surfaces with* $E_{\mathbb{R}} = \{V_2 \sqcup 4S\} \sqcup \{\varnothing\}$ (see [**DK3**] and [**DIK**]).[2] *All these surfaces are of type* I. In the first two cases the surfaces satisfy the hypotheses of Lemma 5.2.1(3) and, hence, have well-defined Pontryagin-Viro forms. In the last case the surfaces can be obtained by the construction of 6.1 and 6.5: one takes for Z' the projective plane $\mathbb{R}p^2$; for P', the union of two conics with two conjugate tangency points (so that $P_{\mathbb{R}} = \varnothing$); and for Q', the union of a generic real line and the line through the singular points of P'. The conics of Q' may be either both real or complex conjugate: in the former case Q' is of type II and the Pontryagin-Viro form is not defined; in the latter case Q' is of type I and the Pontryagin-Viro form is defined due to Theorem 6.2.3.

There are two deformation families of real Enriques surfaces E *with* $E_{\mathbb{R}} = \{S_1\} \sqcup \{4S\}$ (see [**DK3**] and [**DIK**]). *They are obtained by the construction of 6.1 and 6.5 from a triple* $(Z'; P', Q')$, *where* Z' *is the plane* $\mathbb{R}p^2$ (*or hyperboloid* $\mathbb{R}p^1 \times \mathbb{R}p^1$), P' *is a nonsingular* M*-curve of degree* 4 (*respectively, bi-degree* $(4,2)$), *and* Q' *is a pair of conjugate lines* (*respectively, generatrices of bi-degree* $(0,1)$). From Theorem 6.2.3 it follows that the Pontryagin-Viro form is well defined in the latter case and is not defined in the former case (since $Z^- \setminus Q_{\mathbb{R}}$ is nonorientable).

Case 5: $\chi(E_{\mathbb{R}}) = 8$, $d = 4$, i.e., $E_{\mathbb{R}} = 4S$. Only the decompositions $\{4S\} \sqcup \{\varnothing\}$ and $\{2S\} \sqcup \{2S\}$ can satisfy Proposition 5.2.2 (and, in fact, only these surfaces are of type I). Consider the two cases separately.

There are four deformation families of real Enriques surfaces E *with* $E_{\mathbb{R}} = \{4S\} \sqcup \{\varnothing\}$ (see [**DIK**]). *They differ by the classes realized by the image of* $X_{\mathbb{R}}^{(1)}$ *in* $X/\tau = E$ *and* $X/t^{(2)}$ (*see 5.1*); *the four possibilities are* (w_2, w_2), $(w_2, 0)$, $(0, w_2)$, *and* $(0,0)$. (Note that $X/t^{(2)}$ is diffeomorphic to an Enriques surface and $w_2(X/t^{(2)}) \neq 0$.) Since E/conj can as well be represented as the quotient space of $X/t^{(2)}$ by an involution whose fixed point set is $X_{\mathbb{R}}^{(1)}/t^{(2)}$, only the first of the four families may possess Pontryagin-Viro form. Such a surface can be obtained by the construction of 6.1 and 6.5. Take for Z' the hyperboloid $\mathbb{R}p^1 \times \mathbb{R}p^1$. Let (L_1, L_2) and (M_1, M_2) be two pairs of conjugate generatrices of bi-degree $(1,0)$ and (N_1, N_2) a pair of conjugate generatrices of bi-degree $(0,1)$. Pick a generic pair (C_1, C_2) of conjugate members of the pencil generated by $L_1 + M_1 + N_1$ and $L_2 + M_2 + N_2$ and let $P' = C_1 + C_2$ and $Q' = N_1 + N_2$. (P' is a real curve of type I with four nodes, which lie on Q'.) The existence of the Pontryagin-Viro form follows from Theorem 6.2.3.

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[2] In [**DK3**] it is erroneously stated that there is one family with $E_{\mathbb{R}} = \{V_2 \sqcup 4S\} \sqcup \{\varnothing\}$.

There is one deformation family of real Enriques surfaces E with $E_{\mathbb{R}} = \{2S\} \sqcup \{2S\}$ (see [**DIK**]). A surface with Pontryagin-Viro form is constructed similarly to the previous case. One takes for Z' the hyperboloid $\mathbb{R}p^1 \times \mathbb{R}p^1$; for P', a real M-curve of bi-degree $(4, 2)$ with two conjugate double points; and for Q', the union of two conjugate generatrices through the singular points of P'. □

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