The Fundamental Group of the Circle Is Trivial

Florian Deloup

1. INTRODUCTION. The somewhat provocative title of this article does not refer to an actual refutation of a basic theorem of algebraic topology. Rather, it points to an experiment I would like to suggest to teachers of introductory courses in algebraic topology.

When I was a undergraduate student, I was often puzzled by the apparent arbitrariness of the notions that were introduced. At the same time, like many students, I was amazed at how those notions would fit together perfectly to produce nice theories. Understanding why one definition rather than another one is "right" is a fine art, and there is much room for argument about it. However, this kind of understanding lies at the core of doing mathematics.

I will illustrate this point with three closely related examples, all at the undergraduate level. Each relies on the difference, for a function of two variables, between continuity (respectively, smoothness) in each variable separately and continuity (respectively, smoothness) in both variables together. The idea of linking the second to the third example via an inversion in the unit sphere was suggested by one of the referees of this article.

2. WHY IS THE FUNDAMENTAL GROUP FUNDAMENTAL? It is reasonable to assume that the typical introductory course in algebraic topology includes the definitions of homotopy and homotopy groups. A *homotopy* between two continuous maps $f, g : X \to Y$ of topological spaces X and Y is usually defined as a continuous family $(f_t)_{t\in[0,1]}$ of continuous maps from X to Y such that $f_0 = f$ and $f_1 = g$. The problem with this definition is that it requires one to define beforehand what is meant by a "continuous family of continuous maps." This, of course, is not a problem if we know what topology to put on the space $\mathcal{C}(X, Y)$ of continuous maps from X to Y. Or, we may argue, we can just define a homotopy F between f and g as a continuous map $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all x in X, where $X \times [0, 1]$ is endowed with the product topology. This definition makes perfect sense. But then, we may ask—and in fact our students should—why is it that we require global continuity (or why is it that we choose *this* particular topology for $\mathcal{C}(X, Y)$)?

Let us relax slightly the condition of global continuity and define a *pseudo-homotopy* between $f, g : X \to Y$ to be a family $(f_t)_{t\in[0,1]}$ of continuous maps from X to Y such that the map $(x, t) \mapsto f_t(x)$ is continuous in each variable separately (i.e., for fixed x in X the map $t \mapsto f_t(x)$ is continuous on $[0, 1]$ and for fixed t in $[0, 1]$ the map $x \mapsto f_t(x)$ is continuous on X). From the usual argument that one presents in class (or leaves as an exercise for one's students), there is a well-defined *not-so-fundamental group* for any pathwise-connected topological space X and any specified basepoint x in X. Its elements are the pseudo-homotopy classes of continuous maps $\gamma : [0, 1] \to X$ such that $\gamma(0) = \gamma(1) = x$. (Such maps are called *loops* at x.) Just as for the fundamantal group, the group operation is induced by the concatenation of loops. We make the following bold assertion:

Theorem 1. *The not-so-fundamental group of the circle* $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ *is trivial.*

Proof. We take $x = 1$ to be the basepoint. What could be a pseudo-homotopy between an arbitrary loop γ_0 and the constant loop γ_1 at $x = 1$ (defined by $\gamma_1(s) = 1$)? The idea is that, as the pseudo-homotopy evolves, we can stay longer and longer at the point 1, then go along (the image of) the loop γ_0 faster and faster, until in the limit we truly stay at the point 1. We look for a pseudo-homotopy of the type $\gamma_t(x) = \gamma_0(f(s,t))$, where $f : [0,1] \times [0,1] \to [0,1]$ is continuous in each variable separately. When $0 \le s \le t$, we want to sit at the point 1; as s grows from t to 1, we want to go along (the image of) the loop γ_0. Looking for $f(s,t)$ as an affine function of s on $(t, 1]$, we find that

$$f(s,t) = \frac{s-t}{1-t} \quad (t < x \le 1)$$

is an appropriate choice. Hence the formula

$$\gamma_t(s) = \begin{cases} 1 & \text{if } 0 \le s \le t, \\ \gamma_0\left(\dfrac{s-t}{1-t}\right) & \text{if } t < s \le 1, \end{cases}$$

defines a suitable pseudo-homotopy. It is clear that the map

$$(s,t) \mapsto \gamma_t(s)$$

is (jointly) continuous on $[0,1] \times [0,1)$. In addition, we have

$$\lim_{t \to 1^-} \gamma_t(s) = 1$$

for each s in $[0,1]$, so the partial maps $t \mapsto \gamma_t(s)$ (s fixed) and $s \mapsto \gamma_t(s)$ (t fixed) are both continuous on $[0,1]$ and we are done. ∎

Since an arbitrary loop $\gamma : [0,1] \to X$ in a pathwise-connected topological space X can be written as a composite map $[0,1] \to S^1 \to X$, we deduce the following (rather annoying for the theory of pseudo-homotopy) fact:

Corollary 2. *The not-so-fundamental group is always trivial.*

However, as we all know, the fundamental group of the circle is *not* trivial. It may be worthwhile for the reader to reconsider the proof of this fact and discover where global continuity is used. Here we merely check that the pseudo-homotopy that occurs in the proof of Theorem 1 is *not* in general a homotopy. Consider the nontrivial loop γ_0 that goes around the circle once and is defined on $[0,1]$ by $\gamma_0(s) = \exp(2\pi i s)$. We show that the pseudo-homotopy γ_t from γ_0 to γ_1 constructed in the proof of Theorem 1 is not a homotopy. The reason is that the convergence of $\gamma_t(s)$ is not uniform in s as $t \to 1$. Indeed, the quantity

$$|\gamma_t(s) - 1| = \left|\exp\left(2\pi i \frac{s-t}{1-t}\right) - 1\right| = 2\left|\sin\left(\pi\left(\frac{s-t}{1-t}\right)\right)\right|$$

does not converge to 0 as $(s,t) \to (1,1)$. In fact, the right-hand side has no limit. To see this, let α be an arbitrary real number such that $\alpha > 1$. If we choose

$$s_n = 1 - \frac{1}{n}, \qquad t_n = 1 - \frac{\alpha}{n},$$

so that $s_n - t_n = (\alpha - 1)/n > 0$ and $1 - t_n = \alpha/n$, we find that

$$\left|\sin\left(\pi \frac{s_n - t_n}{1 - t_n}\right)\right| = \left|\sin\left(\pi \frac{\alpha - 1}{\alpha}\right)\right|$$

for each n. Noting that $\alpha = 1$ and $\alpha = 2$ produce different values for

$$\left|\sin\left(\pi \frac{\alpha - 1}{\alpha}\right)\right|,$$

we see that the expression

$$\left|\sin\left(\pi \frac{s - t}{1 - t}\right)\right|$$

has no limit as (s, t) tends to $(1, 1)$.

3. WHY MUST AN AMBIENT ISOTOPY BE AMBIENT? We now consider a second example. Again, it is safe to assume that any introductory course in knot theory will introduce the notions of isotopy and ambient isotopy. An *isotopy* from a topological space X to a topological space Y is "a continuous one-parameter family of embeddings." It is formally defined as a continuous map $F : X \times [0, 1] \to Y$,

$$(x, t) \mapsto F(x, t) = f_t(x),$$

such that for each t in $[0, 1]$ the map $f_t : X \to Y$ is an embedding (i.e., f_t is a homeomorphism between X and $f_t(X)$, the latter endowed with the relative topology it inherits from Y). Classically, a *knot* is defined as a smooth (infinitely differentiable) embedding of S^1 in $\mathbb{R}^3$. Equivalently, a knot is a smooth loop $\gamma : [0, 1] \to \mathbb{R}^3$ whose image is a closed curve with no self-intersection (a simple closed curve). Sometimes, a knot is taken to be the image of such an embedding. Rather than giving away the accepted definition in class, I suggest proceeding differently once more and asking the following question:

Question. What is the "good definition" of knot isotopy?

A first attempt at answering this question would be to define an isotopy between two knots $f_0 : S^1 \to \mathbb{R}^3$ and $f_1 : S^1 \to \mathbb{R}^3$ to be an isotopy $(f_t)_{t\in[0,1]}$ from S^1 to $\mathbb{R}^3$, that begins and ends with the given knots. We also require the mappings f_t $(0 \le t \le 1)$ to be smooth. Classically, one tells students that this attempt will not work because of the following observation:

Theorem 3. *With the foregoing definition, any two knots are isotopic.*

Theorem 3 is usually used as a pretext for introducing another notion of isotopy (that of ambient isotopy). We will challenge this notion soon (keep on reading), but we first give an idea of the proof of Theorem 3.

It is not hard to see that the relation "being isotopic to" is an equivalence relation. It is therefore sufficient to show that any knot is isotopic to some specified knot. A knot is called an *unknot* if its image in $\mathbb{R}^3$ lies in a fixed plane. It follows from a classical result in planar topology that any two unknots are isotopic. Hence it is sufficient to show that any knot is isotopic to an unknot.

The idea of a possible isotopy is described by the picture in Figure 1: it consists in shrinking the knotted part to a point. Though this idea works (details are left to the

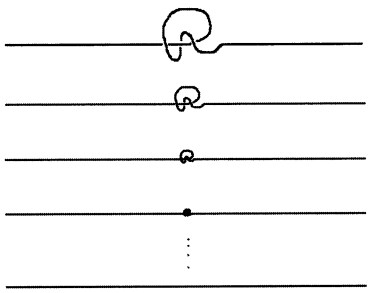


Figure 1. Moments in the isotopy of a (part of a) knot.

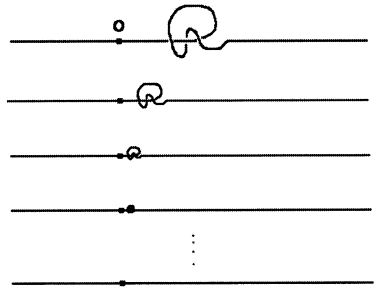


Figure 2. Moments in another isotopy of a (part of a) knot.

interested reader as an exercise), we proceed in a slightly different fashion. We keep the idea of shrinking the knot to a point of $\mathbb{R}^3$, but we simultaneously move the knot towards the origin, as shown in Figure 2.

Proof. We prove that any knot is isotopic to an unknot. Represent the given knot as a smooth map $\gamma : [0, 1] \to \mathbb{R}^3$, say $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ with $\gamma(0) = \gamma(1)$. Without loss of generality, we can assume that $\gamma(0) = (0, 0, 0)$. Furthermore, since γ is smooth, subjecting γ to an isotopy if necessary, we may assume that the following properties hold (see Figure 3):

(1) There exist s_1 and s_2 with $0 < s_1 < s_2 < 1$ such that $\gamma(s) = (s, 0, 0)$ whenever $s_1 \leq s \leq s_2$. (In particular, the restriction of γ to $[s_1, s_2]$ describes a straight segment.)

(2) There is a closed ball B such that $B \cap \gamma([0, 1]) = \gamma([0, s_1])$ and $(\mathbb{R}^3 - B) \cap \gamma([0, 1])$ is planar.

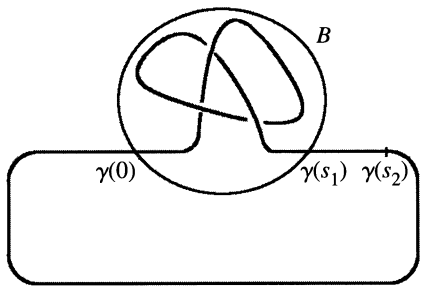


Figure 3. A smooth knot satisfying conditions (1) and (2).

We now work out the idea of shrinking the knot towards the origin from the right side (first coordinate positive). Since all the "knottedness" occurs between $s = 0$ and $s = s_1$, we are led to the formula

$$f_t(s) = \begin{cases} (1-t)\,\gamma\left(\dfrac{s}{1-t}\right) & \text{if } 0 \leq s \leq (1-t)s_1, \\ (s, 0, 0) & \text{if } (1-t)s_1 \leq s \leq s_2, \\ \gamma(s) & \text{if } s_2 \leq s \leq 1, \end{cases}$$

when $0 \leq t < 1$, while at the end of the isotopy (i.e., for $t = 1$), we set

$$f_1(s) = \begin{cases} (s, 0, 0) & \text{if } 0 \leq s \leq s_2, \\ \gamma(s) & \text{if } s_2 \leq s \leq 1. \end{cases}$$

Let us verify that the function $(s, t) \mapsto f_t(s)$ is continuous. From the formulas, this is clear at any point (s, t) with $t < 1$. We must check global continuity when $t = 1$. Let $0 \leq s \leq 1$. When $s > 0$, continuity at $(s, 1)$ still follows from the formulas. So the real issue is continuity at $(s, t) = (0, 1)$. For s satisfying $0 \leq s \leq s_2$, we have

$$\|f_t(s)\| \leq \max\left(|s|, \left\|(1-t)\,\gamma\left(\frac{s}{1-t}\right)\right\|\right) \leq \max\left(|s|, |1-t| \sup_{s\in[0,1]} \|\gamma(s)\|\right).$$

It follows that

$$\|f_t(s) - f_1(0)\| = \|f_t(s)\| \longrightarrow 0$$

as $(s, t) \to (0, 1)$. The conclusion follows. ∎

In one respect, the proof is incomplete: it should explain why the smoothness of the curve implies the existence of an isotopy transforming the given knot to a knot of the kind depicted in Figure 3, where the unknotted portion curve lies in a ball disjoint from (the interior of a ball containing) the knotted part. Can we imagine a knot so badly knotted or so pathological that we cannot straighten out *any* subarc without creating intersections, or even so bad that the complement of any proper subarc is not simply connected? The latter can happen if the embedding is a topological embedding (i.e., is continuous but not smooth). For this, see [**9**].

Here lies a plausible motivation for the "classic" definition of a *tame* knot: there exists a plane P in $\mathbb{R}^3$ such that the projection of the (image of the) knot onto P has no multiple points apart from a *finite* number of double points. Such a definition could be classified as "strategic withdrawal" in Lakatos's terminology [**5**]. We (innocently, of course) avoided this problem by decreeing that our embeddings be smooth. I do not know whether Theorem 3 holds in general for nontame knots.[1]

There are, of course, other motivations for the classical definition of tame knots:

1. (Combinatorial motivation) Knots should be "finite" objects. Two equivalent knots should be seen as equivalent after a *finite* number of combinatorial operations (called *Reidemeister moves*). Nontame knots (also called "wild" knots) involve an infinite amount of knotting, as evidenced by the fact that *any* planar diagram of a wild knot has an infinite number of double points (crossings).
2. (Motivation from physics) Our concept of a knot comes from physical knots, made of rope or wire, having nonzero thickness. Such physical knots are in fact solid tori. But tameness of a knot K specifically implies the existence of a solid torus tubular neigborhood of K.
3. (Thomian motivation) Knottedness should be a global concept rather than a local one. Knots should be locally unknotted.

The reader who is familiar with isotopies will recognize that the isotopy that appears in the proof of Theorem 3 bears some ressemblance to the one appearing in Alexander's trick.

[1] Specifically for knots K in $\mathbb{R}^3$ such that the complement in $\mathbb{R}^3$ of *any* proper subarc in K is not simply connected.

Theorem 4 (Alexander's Trick). *Any self-homeomorphism $\phi : D \to D$ of a disk D that fixes the boundary ∂D pointwise is isotopic to the identity through self-homeomorphisms of D fixing the boundary pointwise.*

Proof. Define an isotopy $(\phi_t)_{t\in[0,1]}$ by

$$\phi_t(x) = \begin{cases} t\phi\left(\frac{1}{t}x\right) & \text{if } \|x\| \le t, \\ x & \text{if } \|x\| \ge t \end{cases}$$

for $0 < t \le 1$ and $\phi_0 = \mathrm{id}_D$. This is the required isotopy between $\phi_0 = \mathrm{id}_D$ and $\phi_1 = \phi$. ∎

Now students are told that "in view of Theorem 3," a finer notion should be defined, that of *ambient isotopy*. Two embeddings $f_0, f_1 : X \to Y$ are *ambient isotopic* in Y if there is an isotopy $(F_t)_{t\in[0,1]}$ from Y to itself such that $F_0 = \mathrm{id}_Y$, the identity mapping of Y, and $f_1 = F_1 \circ f_0$. The main point is that F_t is a map from the whole space Y to itself. The map $F_1|_{Y-f_0(X)}$ sends the complement of $f_0(X)$ homeomorphically to the complement of $f_1(X)$.

This notion applies in particular to knots, and it leads to the usual notion of equivalence for knots: two knots are *equivalent* if they are ambient isotopic in $\mathbb{R}^3$ (for convenience, we do not take orientation into account here).

However, one can be psychologically impressed by Theorem 3. It says that the relation "is isotopic to" is not very useful in classifying knots, though we might have easily thought otherwise (after all, Alexander's trick plays a fundamental rôle in both high- and low-dimensional topology, in particular in the dynamical study of self-homeomorphisms). What is wrong with the usual notion of isotopy? On the one hand, we can say that, unlike isotopy, ambient isotopy takes into account the space surrounding the knots; this is reflected in the fact that the topological type of the complement is preserved by ambient isotopies. But it is not so clear a priori why isotopies *fail* to preserve the topological type of the complement. On the other hand, we may account for the failure in a different way. Our intuition tells us that an isotopy of the kind occurring in the proof of Theorem 3 cannot be smooth at every point. A second look at the proof shows, in fact, that the isotopy $(s, t) \mapsto f_t(s)$ is *not* smooth at $(1, 0)$ (provided that the knot is indeed nontrivial). Nontriviality of the knot implies that there exists s_* with $0 < s_* < s_1$ such that $d\gamma/ds(s_*) \neq (1, 0, 0)$. We have

$$\left.\frac{\partial f_t(s)}{\partial s}\right|_{s=(1-t)s_*} = \frac{d\gamma}{ds}(s_*) = (1,0,0) \longrightarrow (1,0,0)$$

as $t \to 1^-$, whereas

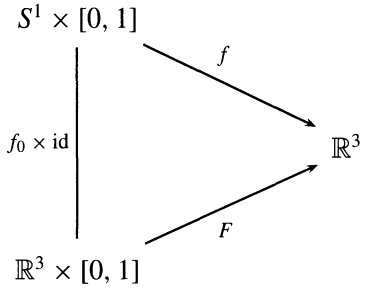
$$\left.\frac{\partial f_t(s)}{\partial s}\right|_{s=(1-t)s_*} = \frac{d\gamma}{ds}(s_*) \longrightarrow \frac{d\gamma}{ds}(s_*) \neq (1,0,0)$$

as $t \to 1^-$. It follows that $\lim_{(s,t)\to(1,0)} \partial f_t(s)/\partial s$ does not exist.

Given the idea that led us to the isotopy in the proof of Theorem 3, one suspects that the isotopy in question cannot be made smooth in general (unless the knots are really equivalent). This is indeed true. Here a knot isotopy $(f_t)_{t\in[0,1]}$ is *smooth* if the map $(x, t) \mapsto f_t(x)$ from $S^1 \times [0, 1]$ to $\mathbb{R}^3$ is (jointly) smooth.

Let $(f_t)_{t\in[0,1]}$ be an isotopy between two knots $f_0 : S^1 \to \mathbb{R}^3$ and $f_1 : S^1 \to \mathbb{R}^3$. We say that an ambient isotopy $(F_t)_{t\in[0,1]}$ of $\mathbb{R}^3$ *extends* the isotopy $(f_t)_{t\in[0,1]}$ provided that

$f_t = F_t \circ f_0$ for each t in $[0, 1]$. In other words, the diagram



is commutative.

Theorem 5. *Any smooth isotopy $(s, t) \mapsto f_t(s)$ between two knots $f_0, f_1 : S^1 \to \mathbb{R}^3$ extends to an ambient isotopy of $\mathbb{R}^3$.*

This is a particular case of the "Isotopy Extension Theorem." See [**4**, Theorem 1.3, p.180] for a proof (which definitely uses differentiability) accessible to an undergraduate. The fact that S^1 is compact plays a key role. In particular, Theorem 5 says what our intuition suggests, namely, that smoothly isotopic knots are ambient isotopic in $\mathbb{R}^3$ (i.e., equivalent).

So if *any* two knots were isotopic through a *smooth* isotopy $(f_t)_{t\in[0,1]}$, then the given isotopy would extend to an ambient isotopy of $\mathbb{R}^3$. Hence the knots would be equivalent ! But of course there exist nonequivalent knots—although students usually have to wait a while before they see an actual example with a complete proof.

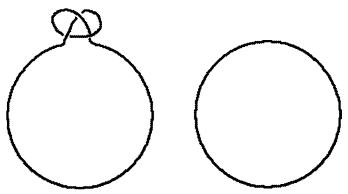


Figure 4. Two nonequivalent knots.

In particular, a smooth isotopy satisfying the conditions of Theorem 5 *does* preserve the topological type of the complement! This point seems to be overlooked in all introductory books on knot theory that I checked,[2] whereas it is often emphasized that a general (i.e., not smooth) isotopy does not invariably have that property (Theorem 3). In fact, the intuition that the notion of equivalence between knots coincides with isotopy is correct! (Intuition involves only smooth isotopies.)

An introductory course in knot theory might well begin with the usual notion of isotopy. Is it the right definition (with respect to what is expected: for instance, the knots in Figure 4 should be nonequivalent)? On intuitive grounds, it is. Why not even work with this definition until one is forced by the mathematics to introduce further the smooth (or piecewise linear) category? There is some justification to the fact that smoothness is a notion that our intuition often takes for granted!

[2]See the references at the end of the article. References [**2**] and [**6**] use Theorem 3 to "justify" the notion of ambient isotopy; [**10**] does not motivate the notion of ambient isotopy; [**8**] does not discuss it; interestingly, [**7**, p. 9] defines an ambient isotopy as a smooth isotopy, which is closer to the view we are advocating here.

There is another notion that our intuition usually takes for granted, especially when a smooth isotopy is involved: compactness. We illustrate this in the next section.

4. RUNNING OFF TO INFINITY. When isotopies involve noncompact spaces, ambient isotopy and smooth isotopy may be two distinct notions. We illustrate this with Theorem 6.

Consider a *long knot* L in $\mathbb{R}^3$: it is (the image of a) smooth embedding $t \mapsto \gamma(t) = (x(t), y(t), z(t))$ from the *open* interval $(0, 1)$ to $\mathbb{R}^3$ such that y and z are compactly supported, $x(t) \to -\infty$ as $t \to 0^+$, and $x(t) \to +\infty$ as $t \to 1^-$. In particular, there exists $\epsilon > 0$ such that both $\gamma(0, \epsilon)$ and $\gamma(1 - \epsilon, 1)$ are contained in the x-axis. An example is shown in Figure 5. (We are really thinking of $\gamma : [0, 1] \to S^3 = \mathbb{R}^3 \cup \{\infty\}$, with $\gamma(0) = \gamma(1) = \infty$ supplying the missing point from the knot.)

Figure 5. A long knot in $\mathbb{R}^3$.

We say that a long knot is a *planar* if its image lies in some plane. It follows from the definition that a planar long knot must lie in a plane containing the x-axis. It is a classical exercise to show that any two planar long knots are smoothly isotopic. Actually, a stronger statement is true (see [**4**, Exercise 9, p. 183]):

Theorem 6. *Any two long knots are smoothly isotopic.*

Proof. It is sufficient to prove that any two long knots are smoothly isotopic to the *trivial long knot* whose image is the x-axis. Here the isotopy we are looking for is *smooth*, which seems to preclude the kind of solution developed in the proof of Theorem 3. We can follow the original suggestion of M. Hirsch ("roll the knot to infinity") and build the isotopy by hand. However, it is possible to deduce Theorem 6 from Theorem 3 as follows. First notice that the isotopy $(f_t)_{t\in[0,1]}$ in the proof of Theorem 3 is smooth everywhere except at the point $(0, 1)$, where it satisfies $f_1(0) = (0, 0, 0)$, the origin of $\mathbb{R}^3$. Now we can use the inversion I in the unit sphere of $\mathbb{R}^3$, the map $\mathbb{R}^3 \cup \{\infty\} \to \mathbb{R}^3 \cup \{\infty\}$ given by

$$z \mapsto I(z) = \frac{z}{\|z\|^2}$$

for $z \notin \{0, \infty\}$, with $I(0) = \infty$ and $I(\infty) = 0$. Notice that I transforms long knots into knots that lie on the x-axis in a neighborhood of the origin (and vice versa). Thus each knot $I \circ f_t$ is a long knot, a well-defined map from the open interval $(0, 1)$ to $\mathbb{R}^3$. In particular, the image of $I \circ f_1$ is easily seen to be planar, hence smoothly isotopic to the trivial long knot. Moreover, it is not hard to see that any long knot is smoothly isotopic to a long knot that is the image under I of a knot satisfying the conditions (1)–(2) of Lemma 3. Since $I_{|\mathbb{R}^3-\{0\}} : \mathbb{R}^3 - \{0\} \to \mathbb{R}^3$ is smooth, the map from $[0, 1] \times [0, 1]$ to $\mathbb{R}^3$ given by

$$(s, t) \mapsto I \circ f_s(t)$$

is the required smooth isotopy. ∎

Basically the idea consists in pushing off to infinity the nonsmoothness of the isotopy $(f_t)_{t\in[0,1]}$. The reader can check that the isotopy $(s,t) \mapsto I \circ f_t(s)$ does not have bounded velocity at infinity. Running off to infinity is sometimes a way to escape problems of regularity.

5. CONCLUSION. As a matter of fact, no student has ever complained to the author—nor to any teacher whom he knows—that the not-so-fundamental group could be ill-defined or trivial. But he wishes he had heard more complaints about the right (accepted) definition.

Questioning the right definition is the only way to understand it. And if it's not the right definition, you may discover a better one.

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