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Quadratic functions and complex spin structures on three-manifolds

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Abstract

We show how the space of complex spin structures of a closed oriented three-manifold embeds naturally into a space of quadratic functions associated to its linking pairing. Besides, we extend the Goussarov–Habiro theory of finite type invariants to the realm of compact oriented three-manifolds equipped with a complex spin structure. Our main result states that two closed oriented three-manifolds endowed with a complex spin structure are undistinguishable by complex spin invariants of degree zero if, and only if, their associated quadratic functions are isomorphic.

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Complex spin structures, or Spin^c-structures, are additional structures with which manifolds may be equipped. They are needed to define the Seiberg–Witten invariants of 4-manifolds, as well as the

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Heegaard–Floer homologies of 3-manifolds by Ozsváth and Szabó. Any closed oriented 3-manifold M can be endowed with a Spin^c-structure and, in that case, Spin^c-structures are in canonical correspondence with Euler structures. The latter are classes of nonsingular vector fields on M which have been introduced by Turaev in order to refine Reidemeister torsion.

In this paper, we investigate the rôle played by quadratic functions in the topology of closed oriented 3-manifolds equipped with a Spin^c-structure or, equivalently, an Euler structure.

Extending constructions from [18,19,24], we associate, to any closed oriented 3-manifold M with a Spin^c-structure σ, its *linking quadratic function*

$$H_2(M;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\phi_{M,\sigma}} \mathbb{Q}/\mathbb{Z}.$$

The function $\phi_{M,\sigma}$ is quadratic in the sense that the symmetric pairing defined by $(x, y) \mapsto \phi_{M,\sigma}(x + y) - \phi_{M,\sigma}(x) - \phi_{M,\sigma}(y)$ is bilinear. Moreover, this symmetric bilinear pairing coincides with $L_M := \lambda_M \circ (B \times B)$ where

$$\mathrm{Tors}H_1(M;\mathbb{Z}) \times \mathrm{Tors}H_1(M;\mathbb{Z}) \xrightarrow{\lambda_M} \mathbb{Q}/\mathbb{Z}$$

is the linking pairing of M and B denotes the Bockstein homomorphism associated to the short exact sequence of coefficients $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. In contrast with $\phi_{M,\sigma}$, the bilinear pairing L_M does not depend on σ. Spin^c-structures on a given manifold M are determined by their corresponding quadratic functions.

Theorem 1. *Let M be a closed connected oriented 3-manifold. The map $\sigma \mapsto \phi_{M,\sigma}$ defines a canonical embedding*

$$\mathrm{Spin}^c(M) \overset{\phi_M}{\hookrightarrow} \mathrm{Quad}(L_M)$$

from the set of Spin^c*-structures on M to the set of quadratic functions with L_M as associated bilinear pairing.*

Via the map ϕ_M, topological notions can be put in correspondence with algebraic ones. For instance, the Chern class $c(\sigma) \in H^2(M)$ of the Spin^c-structure σ corresponds to the homogeneity defect $d_{\phi_{M,\sigma}} : H_2(M;\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ of the quadratic function $\phi_{M,\sigma}$, which is defined by $d_{\phi_{M,\sigma}}(x) = \phi_{M,\sigma}(x) - \phi_{M,\sigma}(-x)$.

When the Chern class $c(\sigma)$ is torsion, $\phi_{M,\sigma}$ happens to factor through B to a quadratic function

$$\mathrm{Tors}H_1(M;\mathbb{Z}) \xrightarrow{\phi_{M,\sigma}} \mathbb{Q}/\mathbb{Z}$$

with λ_M as associated bilinear pairing and is equivalent to the quadratic function constructed by Looijenga and Wahl [19] (see also [4,9]). In particular, the Spin^c-structure may arise from a classical spin structure, or Spin-structure. In that case, which is detected by the vanishing of $c(\sigma)$, the quadratic function $\phi_{M,\sigma}$ is homogeneous and coincides with yet earlier constructions due to Lannes and Latour [18], as well as Morgan and Sullivan [24] (see also [17,27]).

The linking quadratic function is used here to solve a problem related to the theory of finite type invariants by Goussarov and Habiro. Their theory [8,11,12] deals with compact oriented 3-manifolds and is based on an elementary move called Y-surgery. The Y-equivalence, which is defined to be the

equivalence relation among such manifolds generated by this move, has been characterized by Matveev in the closed case [22]. This characterization amounts to recognize the degree 0 invariants of the theory. His result, anterior to the work of Goussarov and Habiro, can be re-stated as follows: two closed oriented 3-manifolds M and M' are Y-equivalent if and only if they have isomorphic pairs (homology, linking pairing). A Spin-refinement of the Goussarov–Habiro theory (the possibility of which was announced in [11] and [12]) has also been considered in [21], where Matveev's theorem is extended to closed oriented 3-manifolds equipped with a Spin-structure.

We show that the Y-surgery move makes sense for closed oriented 3-manifolds equipped with a Spin^c-structure as well. The equivalence relation generated by this move among such manifolds is called, here, *Y^c-equivalence*. It follows that there exists a Spin^c-refinement of the Goussarov–Habiro theory. Our main result is a characterization of the Y^c-equivalence relation in terms of the linking quadratic function. In order to state this more precisely, let us fix a few notations.

Given an isomorphism $\psi : H_1(M;\mathbb{Z}) \to H_1(M';\mathbb{Z})$, the dual isomorphism to ψ by the intersection pairings is denoted by $\psi^\sharp : H_2(M';\mathbb{Q}/\mathbb{Z}) \to H_2(M;\mathbb{Q}/\mathbb{Z})$:

$$\forall x \in H_1(M;\mathbb{Z}),\ \forall y' \in H_2(M';\mathbb{Q}/\mathbb{Z}),\ x \bullet \psi^\sharp(y') = \psi(x) \bullet y' \in \mathbb{Q}/\mathbb{Z}.$$

Also, given sections s and s' of the surjections $B : H_2(M;\mathbb{Q}/\mathbb{Z}) \to \mathrm{Tors}\, H_1(M;\mathbb{Z})$ and $B : H_2(M';\mathbb{Q}/\mathbb{Z}) \to \mathrm{Tors}\, H_1(M';\mathbb{Z})$ respectively, we say that s and s' are ψ-compatible if the diagram

$$\begin{array}{ccc} H_2(M';\mathbb{Q}/\mathbb{Z}) & \xleftarrow{s'} & \mathrm{Tors}\, H_1(M';\mathbb{Z}) \\ {\scriptstyle \psi^\sharp}\downarrow \simeq & & \uparrow{\scriptstyle |\psi|} \simeq \\ H_2(M;\mathbb{Q}/\mathbb{Z}) & \xleftarrow[s]{} & \mathrm{Tors}\, H_1(M;\mathbb{Z}) \end{array}$$

commutes. We denote by P a Poincaré isomorphism and we recall that the Gauss sum of a quadratic function $q : G \to \mathbb{Q}/\mathbb{Z}$, defined on a finite Abelian group G, is the complex number $\sum_{x\in G} \exp(2i\pi q(x))$.

Theorem 2. *Let (M,σ) and (M',σ') be two closed connected oriented* 3*-manifolds with* Spin^c*-structure. The following assertions are equivalent*:

(1) *The* Spin^c*-manifolds (M,σ) and (M',σ') are Y^c-equivalent.*
(2) *There is an isomorphism $\psi : H_1(M;\mathbb{Z}) \to H_1(M';\mathbb{Z})$ such that $\phi_{M',\sigma'} = \phi_{M,\sigma} \circ \psi^\sharp$.*
(3) *There is an isomorphism $\psi : H_1(M;\mathbb{Z}) \to H_1(M';\mathbb{Z})$ such that*

- $\lambda_M = \lambda_{M'} \circ (\psi| \times \psi|)$,
- $\psi(P^{-1}c(\sigma)) = P^{-1}c(\sigma')$,
- *forsome ψ-compatible sections s and s' of the Bockstein homomorphisms, $\phi_{M,\sigma} \circ s$ and $\phi_{M',\sigma'} \circ s'$ have identical Gauss sums.*

Two special cases deserve to be singled out. First, consider manifolds whose first homology group is torsion free. The following result is deduced from Theorem 2.

Corollary 1. *Let (M,σ) and (M',σ') be two closed connected oriented* 3*-manifolds with* Spin^c*-structure, such that $H_1(M;\mathbb{Z})$ and $H_1(M';\mathbb{Z})$ are torsion free. The following assertions are*

equivalent:

(1) *The* Spin^c*-manifolds* (M, σ) *and* (M', σ') *are* Y^c*-equivalent.*
(2) *There is an isomorphism* $\psi : H_1(M; \mathbb{Z}) \to H_1(M'; \mathbb{Z})$ *such that* $\psi(P^{-1}c(\sigma)) = P^{-1}c(\sigma')$.

Second, consider the case of rational homology 3-spheres. According to what has been said above, if M is an oriented rational homology 3-sphere, then $\phi_{M,\sigma}$ can be regarded as a quadratic function $H_1(M; \mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ with λ_M as associated bilinear pairing. In that case, Theorem 2 specializes to the following corollary.

Corollary 2. *Let* (M, σ) *and* (M', σ') *be two oriented rational homology* 3*-spheres with* Spin^c*-structure. The following assertions are equivalent*:

(1) *The* Spin^c*-manifolds* (M, σ) *and* (M', σ') *are* Y^c*-equivalent.*
(2) *There is an isomorphism* $\psi : H_1(M; \mathbb{Z}) \to H_1(M'; \mathbb{Z})$ *such that* $\phi_{M,\sigma} = \phi_{M',\sigma'} \circ \psi$.
(3) *There is an isomorphism* $\psi : H_1(M; \mathbb{Z}) \to H_1(M'; \mathbb{Z})$ *such that*
   - $\lambda_M = \lambda_{M'} \circ (\psi \times \psi)$,
   - $\psi(P^{-1}c(\sigma)) = P^{-1}c(\sigma')$,
   - $\phi_{M,\sigma}$ *and* $\phi_{M',\sigma'}$ *have identical Gauss sums.*

The paper is organized as follows. In Section 1, we briefly review Spin^c-structures from a general viewpoint. Next, we restrict ourselves to the dimension 3, in which case one can work with Euler structures as well. At the end of the section, the technical problem of gluing Spin^c-structures is considered. This is needed to define the Y-surgery move in the setting of manifolds equipped with a Spin^c-structure, since this move is defined as a "cut and paste" operation. Our gluing lemma involves Spin^c-structures, on a compact oriented 3-manifold with boundary, which are relative to a fixed Spin-structure on the boundary.

Section 2 is devoted to the construction and study of the linking quadratic function. First, we give a combinatorial description of the Spin^c-structures of a given closed oriented 3-manifold presented by surgery along a link in $\mathbf{S}^3$. This leads to a Spin^c-refinement of Kirby's theorem. Next, we define the quadratic function $\phi_{M,\sigma}$ associated to a closed 3-dimensional Spin^c-manifold (M, σ): this is done essentially by defining a cobordism invariant of singular 3-dimensional Spin^c-manifolds over $K(\mathbb{Q}/\mathbb{Z}, 1)$. The quadratic function $\phi_{M,\sigma}$ can be computed combinatorially as soon as (M, σ) is presented by surgery along a link in $\mathbf{S}^3$. We prove Theorem 1 and some other basic properties of the map ϕ_M. Lastly, regarding σ as an Euler structure, we give for $\phi_{M,\sigma}$ an intrinsic formula that does not make reference to the dimension 4 anymore. This is obtained by presenting, à la Sullivan, elements of $H_2(M; \mathbb{Q}/\mathbb{Z})$ as immersed surfaces with n-fold boundary.

In Section 3, the Y^c-surgery move is defined using the above-mentioned gluing lemma. Next, Theorem 2 is proved working with surgery presentations of Spin^c-manifolds. We use the material of the previous section and a result due to Matveev, Murakami and Nakanishi [22,25] on ordered oriented framed links having the same linking matrix. Some algebraic ingredients about quadratic functions on torsion Abelian groups are needed as well. Those results, some of them well-known in the case of finite Abelian groups, have been proved aside in [5]. We conclude this paper by giving some applications of Theorem 2 and stating some problems.

1. Complex spin structures on three-manifolds

In this section, we review Spin^c-structures and other related structures, with special emphasis on the dimension 3. We also give a gluing lemma for Spin^c-structures.

1.1. Some conventions

In this paper, any manifold M is assumed to be compact, smooth and oriented. We denote by $-M$ the manifold obtained from M by reversing its orientation. If M has non-empty boundary, ∂M has the orientation given by the "outward normal vector first" rule. The oriented tangent bundle of M is denoted by TM.

Vector bundles will be stabilized from the left side. A section of a vector bundle is said to be *nonsingular* if it does not vanish at any point.

If G is an Abelian group, a *G-affine space* A is a set A on which G acts freely and transitively. The affine action is denoted additively; thus, for $a, a' \in A$, the unique element $g \in G$ satisfying $a' = a + g$ will be written $a' - a$.

Unless otherwise specified, all (co)homology groups are assumed to be computed with integer coefficients.

1.2. Complex spin structures

In this subsection, we consider a n-manifold M. We recall basic facts about Spin^c-structures on M, adopting a viewpoint which is analogous to that used in [3] for Spin-structures.

1.2.1. *From* Spin^c *onto* SO

Let $n \geqslant 1$ be an integer. The group Spin(n) is the 2-fold covering of the special orthogonal group SO(n):

$$1 \longrightarrow \mathbb{Z}_2 \to \mathrm{Spin}(n) \longrightarrow \mathrm{SO(n)} \longrightarrow 1.$$

The group Spin$^c(n)$ is defined by

$$\mathrm{Spin}^c(n) = \frac{\mathrm{Spin}(n) \times \mathrm{U}(1)}{\mathbb{Z}_2},$$

where $\mathbb{Z}_2$ is generated by $[(-1,-1)]$, hence the following short exact sequence of groups:

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{Spin}^c(n) \xrightarrow{\pi} \mathrm{SO(n)} \longrightarrow 1,$$

where the first map sends z to $[(1, z)]$ and where π is induced by the projection of Spin(n) onto SO(n).

The inclusion of SO(n) into SO$(n+1)$, defined by $A \mapsto (1) \oplus A$, induces a monomorphism Spin$^c(n) \rightarrowtail$Spin$^c(n+1)$ such that the diagram

$$\begin{array}{ccc} \mathrm{Spin}^c(n) & \rightarrowtail & \mathrm{Spin}^c(n+1) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathrm{SO}(n) & \rightarrowtail & \mathrm{SO}(n+1). \end{array}$$

commutes, hence a diagram at the level of classifying spaces:

$$\begin{array}{ccc} \mathrm{BSpin}^c(n) & \longrightarrow & \mathrm{BSpin}^c(n+1) \\ {\scriptstyle \mathrm{B}\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{B}\pi} \\ \mathrm{BSO}(n) & \longrightarrow & \mathrm{BSO}(n+1). \end{array} \tag{1.1}$$

Here, we take $\mathrm{BSO}(n)$ to be the Grassman manifold of oriented n-planes in $\mathbb{R}^\infty$ and the map $\mathrm{BSO}(n) \to \mathrm{BSO}(n+1)$ to be the usual one. We fix the classifying spaces $\mathrm{BSpin}^c(n)$ (in their homotopy equivalence classes) and, next, we fix the maps $\mathrm{B}\pi : \mathrm{BSpin}^c(n) \to \mathrm{BSO}(n)$ (in their homotopy classes) to be fibrations. Then, the map from $\mathrm{BSpin}^c(n)$ to $\mathrm{BSpin}^c(n+1)$ is choosen (in its homotopy class) to make diagram (1.1) *strictly* commute.

We denote by $\gamma_{\mathrm{SO}(n)}$ the universal n-dimensional oriented vector bundle over $\mathrm{BSO}(n)$. Let $\gamma_{\mathrm{Spin}^c(n)}$ be the pull-back of $\gamma_{\mathrm{SO}(n)}$ by $\mathrm{B}\pi$. Thanks to (1.1), there is a well-defined morphism between $(n+1)$-dimensional oriented vector bundles $\mathbb{R} \oplus \gamma_{\mathrm{Spin}^c(n)} \to \gamma_{\mathrm{Spin}^c(n+1)}$ induced by the usual one $\mathbb{R} \oplus \gamma_{\mathrm{SO}(n)} \to \gamma_{\mathrm{SO}(n+1)}$.

1.2.2. Rigid Spin^c*-structures*

Recall that M is a n-manifold to which some conventions, stated in Section 1.1, apply.

Definition 1.1. A *rigid* Spin^c*-structure* on M is a morphism $\mathrm{T}M \to \gamma_{\mathrm{Spin}^c(n)}$ between n-dimensional oriented vector bundles. A Spin^c*-structure* (or *complex spin structure*) on M is a homotopy class of rigid Spin^c-structures on M. We denote by $\mathrm{Spin}^c_r(M)$ the set of rigid Spin^c-structures on M, and by $\mathrm{Spin}^c(M)$ the set of its Spin^c-structures.

Obviously, a different choice of the classifying space $\mathrm{BSpin}^c(n)$ (in its homotopy type) or a different choice of the map $\mathrm{B}\pi$ (in its homotopy class) would lead to a different notion of rigid Spin^c-structure, but would not affect the definition of a Spin^c-structure. Rigid structures will be used later to define gluing maps.

Let β be the Bockstein homomorphism associated to the short exact sequence of coefficients

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

The fibration $\mathrm{B}\pi : \mathrm{BSpin}^c(n) \to \mathrm{BSO}(n)$ has fiber $\mathrm{BU}(1) \simeq K(\mathbb{Z}, 2)$ and, indeed, is a principal fibration with characteristic class $w := \beta w_2 \in H^3(\mathrm{BSO}(n))$, where w_2 is the second Stiefel–Whitney class. Then, by obstruction theory, we obtain the following well-known fact about existence and parametrization of Spin^c-structures.

Proposition 1.1. *The manifold M can be given a* Spin^c*-structure if and only if the cohomology class $\beta w_2(M) \in H^3(M)$ vanishes. In that case,* Spin$^c(M)$ *is a $H^2(M)$-affine space.*

One may easily verify that the homotopy-theoretical definition of a Spin^c-structure, which we have adopted here, agrees with the usual one.

Lemma 1.1. *Suppose that M is equipped with a Riemannian metric and denote by* $\mathrm{SO}(\mathrm{T}M)$ *the bundle of its oriented orthonormal frames. A* Spin^c*-structure on M is equivalent to an isomorphism class of pairs*

(η, H), *where* η *is a principal* $\mathrm{Spin}^c(n)$*-bundle over* M *and where* $H : \eta/\mathrm{U}(1) \to \mathrm{SO}(\mathrm{T}M)$ *is a principal* $\mathrm{SO}(n)$*-bundle isomorphism.*

To go to the point, we have only defined (rigid) Spin^c-structures on the manifold M. Nevertheless, the notion of a (rigid) Spin^c-structure obviously extends to any oriented vector bundle over any base space.

Remark 1.1. Thanks to the map $\mathbb{R} \oplus \gamma_{\mathrm{Spin}^c(n)} \to \gamma_{\mathrm{Spin}^c(n+1)}$ constructed at the end of Section 1.2.1, a rigid Spin^c-structure on $\mathrm{T}M$ gives rise to one on $\mathbb{R} \oplus \mathrm{T}M$. This induces a canonical map

$$\mathrm{Spin}^c(M) = \mathrm{Spin}^c(\mathrm{T}M) \to \mathrm{Spin}^c(\mathbb{R} \oplus \mathrm{T}M),$$

which is $H^2(M)$-equivariant and, so, bijective. Thus, a Spin^c-structure on M is equivalent to a Spin^c-structure on its stable oriented tangent bundle.

1.2.3. Orientation reversal

The *time-reversing* map is the orientation-reversing automorphism of $\mathbb{R} \oplus \mathrm{T}M$ defined by $(t, v) \mapsto (-t, v)$. Composition with that map transforms a rigid Spin^c-structure on $\mathbb{R} \oplus \mathrm{T}M$ to one on $\mathbb{R} \oplus \mathrm{T}(-M)$. So, by Remark 1.1, we get a canonical $H^2(M)$-equivariant map

$$\mathrm{Spin}^c(\mathrm{M}) \xrightarrow{-} \mathrm{Spin}^c(-M).$$

1.2.4. Relative Spin^c*-structures*

Suppose that M has some boundary and fix a rigid structure $s \in \mathrm{Spin}^c_r(\mathrm{T}M|_{\partial M})$ over ∂M.

Definition 1.2. A Spin^c*-structure on* M *relative to* s is a homotopy class rel ∂M of rigid Spin^c-structures on M that extend s. We denote by $\mathrm{Spin}^c(M, s)$ the set of such structures.

The following relative version of Proposition 1.1 is also proved by obstruction theory applied to the fibration $\mathrm{B}\pi$.

Proposition 1.2. *There exists a rigid* Spin^c*-structure on* M *that extends* s *if and only if a certain cohomology class*

$$w(M, s) \in H^3(M, \partial M)$$

vanishes. In that case, $\mathrm{Spin}^c(M, s)$ *is a* $H^2(M, \partial M)$*-affine space.*

1.2.5. Restriction to the boundary

Suppose that M has some boundary. Observe that there is a well-defined homotopy class of isomorphisms between the oriented vector bundles $\mathbb{R} \oplus \mathrm{T}\partial M$ and $\mathrm{T}M|_{\partial M}$, which is defined by any section of $\mathrm{T}M|_{\partial M}$ transverse to ∂M and directed outwards.

In particular, a Spin^c-structure on $\mathrm{T}M|_{\partial M}$ can be identified without ambiguity to a Spin^c-structure on ∂M. Thus, we get a canonical *restriction* map

$$\mathrm{Spin}^c(M) \longrightarrow \mathrm{Spin}^c(\partial M),$$

which is affine over the homomorphism $H^2(M) \to H^2(\partial M)$ induced by inclusion.

1.2.6. From Spin *to* Spin^c

Proceeding as in Section 1.2.2, we define the set $\mathrm{Spin}_r(M)$ of *rigid* Spin-*structures* on M and the set $\mathrm{Spin}(M)$ of Spin-*structures* on M. The latter is a $H^2(M;\mathbb{Z}_2)$-affine space as soon as $w_2(M)$ vanishes. The reader is referred to [3] for details.[1] The group homomorphism

$$\mathrm{Spin}(n) \xrightarrow{\beta} \mathrm{Spin}^c(n)$$

defined by $\beta(x) = [(x, 1)]$, makes the two projections onto $\mathrm{SO}(n)$ agree. This allows us to define a morphism $\gamma_{\mathrm{Spin(n)}} \to \gamma_{\mathrm{Spin}^c\mathrm{(n)}}$ between oriented n-dimensional vector bundles, the composition with which transforms a rigid Spin-structure u to a rigid Spin^c-structure denoted by $\beta(u)$. Thus, we get a canonical map

$$\mathrm{Spin}(M) \xrightarrow{\beta} \mathrm{Spin}^c(M)$$

which is affine over the Bockstein homomorphism $\beta : H^1(M;\mathbb{Z}_2) \to H^2(M)$.

If M has some boundary, we define *relative* Spin-*structures* on M as well. Their construction goes as in Section 1.2.4. Thus, for a fixed $s \in \mathrm{Spin}_r(\mathrm{T}M|_{\partial M})$, we get a map

$$\mathrm{Spin}(M, s) \xrightarrow{\beta} \mathrm{Spin}^c(M, \beta s),$$

which is affine over the Bockstein homomorphism $\beta : H^1(M, \partial M;\mathbb{Z}_2) \to H^2(M, \partial M)$.

1.2.7. From U *to* Spin^c

Let m be an integer such that $n \leqslant 2m$. We take $\mathrm{BU}(m)$ to be the Grassman manifold of complex m-planes in $\mathbb{C}^\infty$. The map $\mathrm{BU}(m) \to \mathrm{BSO}(2m)$, which consists in forgetting the complex structure on a complex m-plane, represents the usual inclusion of $\mathrm{U}(m)$ into $\mathrm{SO}(2m)$. We define $\gamma_{\mathrm{U(m)}}$ to be the pull-back of $\gamma_{\mathrm{SO}(2m)}$ by this map $\mathrm{BU}(m) \to \mathrm{BSO}(2m)$, which can be identified with the $2m$-dimensional oriented vector bundle underlying the universal m-dimensional complex vector bundle. Then, as we did in the Spin and Spin^c cases, we could define a "rigid U-structure" on $\mathbb{R}^{2m-n} \oplus \mathrm{T}M$ to be a morphism $\mathbb{R}^{2m-n} \oplus \mathrm{T}M \to \gamma_{\mathrm{U(m)}}$ between $2m$-dimensional oriented vector bundles. Such a morphism induces a complex structure on $\mathbb{R}^{2m-n} \oplus \mathrm{T}M$ by pulling back the canonical one on $\gamma_{\mathrm{U(m)}}$ and, conversely, any complex structure on $\mathbb{R}^{2m-n} \oplus \mathrm{T}M$ inducing the given orientation arises that way. Then, a "U-structure" on $\mathbb{R}^{2m-n} \oplus \mathrm{T}M$ is equivalent to a homotopy class of complex structures on $\mathbb{R}^{2m-n} \oplus \mathrm{T}M$ compatible with the given orientation.

There is a canonical way to embed $\mathrm{U}(m)$ into $\mathrm{Spin}^c(2m)$: see, for instance, [10, Proposition D.50]. This inclusion

$$\mathrm{U}(m) \overset{\omega}{\rightarrowtail} \mathrm{Spin}^c(2m)$$

makes the two maps to $\mathrm{SO}(2m)$ commute. This allows us to define a morphism $\gamma_{\mathrm{U}(m)} \to \gamma_{\mathrm{Spin}^c(2m)}$ between oriented $2m$-dimensional vector bundles, the composition with which transforms a "rigid U-structure" on $\mathbb{R}^{2m-n} \oplus \mathrm{T}M$ to a rigid Spin^c-structure on it. As a consequence of Remark 1.1, we get a

[1] In [3], rigid Spin-structures are called "w_2-structures" and are defined on the stable oriented tangent bundle. An observation similar to that given in Remark 1.1 for Spin^c-structures applies to Spin-structures.

canonical map

$$\mathrm{U}^s(M) \xrightarrow{\omega} \mathrm{Spin}^c(M)$$

from the set of stable complex structures on TM compatible with the orientation to the set of Spin^c-structures on M. (See [10, Proposition D.57] for a construction of ω involving the usual definition of a Spin^c-structure.)

1.2.8. Chern class

A Spin^c-structure α on M induces an isomorphism class of principal Spin$^c(n)$-bundles over M and, so, an isomorphism class of principal U(1)-bundles thanks to the homomorphism $\mathrm{Spin}^c(n) \to \mathrm{U}(1)$ defined by $[(x, y)] \mapsto y^2$. The first Chern class of the latter is denoted by $c(\alpha)$. We get a *Chern class* map

$$\mathrm{Spin}^c(M) \xrightarrow{c} H^2(M)$$

which is affine over the doubling map defined by $x \mapsto 2x$. When $c(\alpha)$ belongs to Tors $H^2(M)$, the Spin^c-structure α is said to be *torsion*.

1.3. Complex spin structures in dimension 3

In this subsection, we turn to 3-manifolds which, by Section 1.1, are assumed to be compact smooth and oriented. The preliminary remark is that any 3-manifold M can be endowed with a Spin^c-structure, since $w_2(M)$ is well-known to vanish.

We start by removing the rigidity of relative Spin^c-structures which is still remaining along the boundary. Next, we recall Turaev's observation that Spin^c-structures can be regarded as classes of vector fields. This holds true in the relative case as well.

1.3.1. Relative Spin^c*-structures*

Let M be a 3-manifold with boundary and let σ be a Spin-structure on ∂M. We define Spin^c-structures on M which are relative to σ. Note that, thanks to the observation initiating Section 1.2.5, one can identify $\sigma \in \mathrm{Spin}(\partial M)$ to a Spin-structure on $\mathrm{T}M|_{\partial M}$.

Lemma 1.2. *For any rigid* Spin*-structure s on $\mathrm{T}M|_{\partial M}$ representing σ (which we denote by $s \in \sigma$), the rigid* Spin^c*-structure $\beta(s)$ can be extended to M. Moreover, for any $s, s' \in \sigma$, there exists a canonical $H^2(M, \partial M)$-equivariant bijection*

$$\mathrm{Spin}^c(M, \beta s) \xrightarrow{\rho_{s,s'}} \mathrm{Spin}^c(M, \beta s').$$

Lastly, for any $s, s', s'' \in \sigma$, we have that $\rho_{s',s''} \circ \rho_{s,s'} = \rho_{s,s''}$.

Definition 1.3. A Spin^c*-structure on M relative to σ* is a pair (u, s) where $s \in \sigma$ and $u \in \mathrm{Spin}^c(M, \beta s)$, two such pairs (u, s) and (u', s') being considered as equivalent when $u' = \rho_{s,s'}(u)$. The set of such structures is denoted by $\mathrm{Spin}^c(M, \sigma)$ and can naturally be given the structure of a $H^2(M, \partial M)$-affine space.

Remark 1.2. There is an analogue to Lemma 1.1 that formulates what a Spin^c-structure on M relative to σ is in terms of principal bundles.

Example 1.1. Suppose that ∂M is a disjoint union of tori. The 2-torus has a distinguished Spin-structure σ^0 that is induced by its Lie group structure. Using the previous remark, it can be verified that a Spin^c-structure on M relative to the union of copies of σ^0 is equivalent to a relative Spin^c-structure in the sense of Turaev [30, Section 1.2].

Proof of Lemma 1.2. Let $w_2(M,s) \in H^2(M,\partial M;\mathbb{Z}_2)$ denote the obstruction to extend s to a rigid Spin-structure on M. We have that

$$\beta(w_2(M,s)) = w(M,\beta s) \in H^3(M,\partial M).$$

Thus, $w(M,\beta s)$ is of order at most 2 and, so, vanishes.

We now prove the second statement. Let $\varphi : [-1,0]\times\partial M \hookrightarrow M$ be a collar neighborhood of ∂M. In particular, φ induces a specific isomorphism between $\mathbb{R}\oplus \mathrm{T}\partial M$ and $\mathrm{T}M|_{\partial M}$: the rigid Spin-structures on $\mathbb{R}\oplus\mathrm{T}\partial M$ corresponding to s and s' are denoted by s_0 and s_1, respectively. By assumption, s_0 and s_1 are homotopic: let $S=(s_t)_{t\in[0,1]}$ be such a homotopy. This defines a rigid Spin-structure S on $[0,1]\times\partial M$ by identifying, at each time t, $\mathbb{R}\oplus\mathrm{T}\partial M$ with the restriction of $\mathrm{T}([0,1]\times\partial M)$ to $t\times\partial M$. The same collar neighborhood allows us to define a smooth gluing $M\cup([0,1]\times\partial M)$, as well as a positive diffeomorphism $\tilde{\varphi} : M \to M\cup([0,1]\times\partial M)$ (based on the affine identification between $[-1,0]$ and $[-1,1]$). Consider the map

$$\mathrm{Spin}^c(M,\beta s) \xrightarrow{\rho_S} \mathrm{Spin}^c(M,\beta s')$$

defined, for any $u\in \mathrm{Spin}^c_r(M)$ extending $\beta(s)$, by $\rho_S([u]) = [(u\cup\beta(S))\circ \mathrm{T}\tilde{\varphi}]$.

The map ρ_S is $H^2(M,\partial M)$-equivariant and is independent of the choice of φ. So, we are left to prove that ρ_S does not depend on the choice of the homotopy S between s_0 and s_1, which will allow us to set $\rho_{s,s'}=\rho_S$. To see that, consider the map β constructed in Section 1.2.6 from $\mathrm{Spin}([0,1]\times\partial M, 0\times(-s_0)\cup 1\times s_1)$ to $\mathrm{Spin}^c([0,1]\times\partial M, 0\times(-\beta s_0)\cup 1\times(\beta s_1))$, where $-s_0\in\mathrm{Spin}_r(\mathbb{R}\oplus\mathrm{T}(-\partial M))$ is obtained from s_0 by time-reversing. The Bockstein homomorphism β from $H^1([0,1]\times\partial M, \partial[0,1]\times\partial M;\mathbb{Z}_2)$ to $H^2([0,1]\times\partial M,\partial[0,1]\times\partial M)$ is trivial, since its codomain is isomorphic to the free Abelian group $H_1(\partial M)$. It follows that the former map β collapses, and the conclusion follows. □

Remark 1.3. The set of Spin-*structures on M relative to σ* is defined to be

$$\mathrm{Spin}(M,\sigma) = \{\alpha\in\mathrm{Spin}(M) \ : \ \alpha|_{\partial M}=\sigma\},$$

which may be empty. One can construct a canonical map

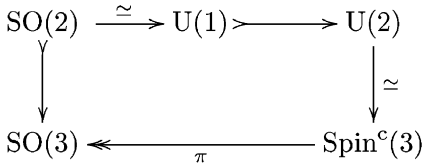
$$\mathrm{Spin}(M,\sigma) \xrightarrow{\beta} \mathrm{Spin}^c(M,\sigma)$$

by means of a rigid Spin-structure s on $\mathrm{T}M|_{\partial M}$ representing σ and the map β defined in Section 1.2.6 from $\mathrm{Spin}(M,s)$ to $\mathrm{Spin}^c(M,\beta s)$.

1.3.2. Spin^c*-structures as vector fields: the closed case*

Let M be a closed 3-manifold. We recall Turaev's definition [28] of an Euler structure on M, and how this corresponds to a Spin^c-structure on M.

Lemma 1.3. *The group* $\mathrm{Spin}^c(3)$ *can be identified with* $\mathrm{U}(2)$ *in such a way that the diagram*



commutes. Here, π *is the canonical projection,* $\mathrm{SO}(2)$ *is identified with* $\mathrm{U}(1)$ *in the usual way and is embedded into* $\mathrm{SO}(3)$ *by* $A \mapsto (1) \oplus A$*, whereas* $\mathrm{U}(1)$ *is embedded into* $\mathrm{U}(2)$ *by* $A \mapsto A \oplus (1)$.

Proof. There is a well-known way to construct a 2-fold covering from SU(2) onto SO(3), which consists in identifying SU(2) with the group of unit quaternions, $\mathbb{R}^3$ with the space of pure quaternions and making the former act on the latter by conjugation. Thus, we get a unique group isomorphism $\mathrm{SU}(2) \xrightarrow{\simeq} \mathrm{Spin}(3)$ which makes the two projections onto SO(3) commute. Then, the isomorphism

$$\frac{\mathrm{SU}(2) \times \mathrm{U}(1)}{\mathbb{Z}_2} \xrightarrow{\simeq} \mathrm{U}(2)$$

sending $[(A, z)]$ to zA induces a group isomorphism $\mathrm{U}(2) \xrightarrow{\simeq} \mathrm{Spin}^c(3)$. The reader may easily verify the commutativity of the above diagram. □

Definition 1.4. An *Euler structure* on M is a punctured homotopy class of nonsingular vector fields on M. Precisely, two nonsingular vector fields v and v' on M are considered as equivalent, when there exists a point $x \in M$ such that the restrictions of v and v' to $M \backslash x$ are homotopic among nonsingular vector fields on $M \backslash x$. The set of Euler structures on M is denoted by $\mathrm{Eul}(M)$.

If a cellular decomposition of M is given, punctured homotopy coincides with homotopy on the 2-skeleton of M. Then, obstruction theory applied to the bundle of non-zero vectors tangent to M says that Euler structures do exist (Poincaré–Hopf theorem: $\chi(M) = 0$) and that they form a $H^2(M; \pi_2(\mathrm{T}_y M \backslash 0))$-affine space (where $y \in M$). Since M has come with an orientation, $\mathrm{Eul}(M)$ is naturally a $H^2(M)$-affine space.

Lemma 1.4 (*Turaev [29]*). *There exists a canonical* $H^2(M)$*-equivariant bijection*

$$\mathrm{Eul}(M) \xrightarrow{\mu} \mathrm{Spin}^c(M).$$

Proof. Let v be a nonsingular vector field on M. We are going to associate to v a Spin^c-structure in the usual sense (see Lemma 1.1) and, for this, we need to endow M with a metric. Orient $\langle v \rangle^\perp$, the orthogonal complement of $\langle v \rangle$ in $\mathrm{T}M$, with the "right hand" rule (v being taken as right thumb). Then, $\mathrm{SO}(\langle v \rangle^\perp)$ is a reduction of $\mathrm{SO}(\mathrm{T}M)$ with respect to the inclusion of SO(2) into SO(3) defined by $A \mapsto (1) \oplus A$. The bundle $\mathrm{SO}(\langle v \rangle^\perp)$, together with the homomorphism $\mathrm{SO}(2) \simeq \mathrm{U}(1) \to \mathrm{U}(2)$ defined by $A \mapsto A \oplus (1)$, induces a principal U(2)-bundle η. According to Lemma 1.3, η can be declared to be a principal $\mathrm{Spin}^c(3)$-bundle and can be accompanied with an isomorphism $H : \eta/\mathrm{U}(1) \to \mathrm{SO}(\mathrm{T}M)$. The Spin^c-structure $[(\eta, H)]$ on M only depends on the punctured homotopy class of v, and we set $\mu([v]) = [(\eta, H)]$. The map μ can be verified to be $H^2(M)$-equivariant. □

Remark 1.4. Let $[v]$ be an Euler structure on M. The isomorphism class of principal U(1)-bundles induced by the Spin^c-structure $\mu([v])$ in Section 1.2.8 is represented by $\mathrm{SO}(\langle v^{\perp}\rangle)$, since the homomorphism $\mathrm{Spin}^c(3) \to \mathrm{U}(1)$ used there corresponds to the determinant map through the isomorphism $\mathrm{Spin}^c(3) \simeq \mathrm{U}(2)$ of Lemma 1.3. Consequently, the Chern class of $\mu([v])$ is the Euler class $e(\mathrm{T}M/\langle v\rangle)$, i.e., the obstruction to find a nonsingular vector field on M transverse to v.

According to the previous remark, Spin^c-structures arising from Spin-structures correspond to nonsingular vector fields on M which can be completed.

More precisely, let a *parallelization* of M be a punctured homotopy class of trivializations $t = (t_1, t_2, t_3)$ of the oriented vector bundle TM, and denote the set of such structures by $\mathrm{Parall}(M)$. Obstruction theory applied to the bundle of oriented frames of M says that parallelizations do exist (Stiefel theorem: $w_2(M) = 0$) and that they form a $H^1(M;\mathbb{Z}_2)$-affine space. (In the case of trivializations of TM, homotopy on the 2-skeleton coincides with homotopy on the 1-skeleton since $\pi_2(\mathrm{GL}_+(3)) = 0$.) Thus, one obtains the following well-known fact [16,23].

Lemma 1.5. *There exists a canonical $H^1(M;\mathbb{Z}_2)$-equivariant bijection*

$$\mathrm{Parall}(M) \xrightarrow{\mu} \mathrm{Spin}(M).$$

Define a map $\beta : \mathrm{Parall}(M) \to \mathrm{Eul}(M)$ by $\beta([t]) = [t_1]$ for any trivialization $t = (t_1, t_2, t_3)$ of TM. The next lemma follows from the definitions.

Lemma 1.6. *The following diagram is commutative*:

$$\begin{array}{ccc} \mathrm{Parall}(M) & \xrightarrow[\simeq]{\mu} & \mathrm{Spin}(M) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} \\ \mathrm{Eul}(M) & \xrightarrow[\simeq]{\mu} & \mathrm{Spin}^c(M). \end{array}$$

1.3.3. Spin^c*-structures as vector fields: the boundary case*

Let M be a 3-manifold with boundary. We define Euler structures on M which are relative to a homotopy class of trivializations of $\mathbb{R} \oplus \mathrm{T}\partial M$. We start with a preliminary observation.

What has been done in Section 1.3.2 for the oriented tangent bundle of a closed 3-manifold works for any 3-dimensional oriented vector bundle. In particular, if S is a closed surface, Section 1.3.2 can be repeated for $\mathbb{R} \oplus \mathrm{T}S$. This repetition ends with the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Parall}\,(\mathbb{R} \oplus \mathrm{T}S) & \xrightarrow[\simeq]{\mu} & \mathrm{Spin}(\mathbb{R} \oplus \mathrm{T}S) & \xleftarrow{\simeq} & \mathrm{Spin}(S) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & \\ \mathrm{Eul}\,(\mathbb{R} \oplus \mathrm{T}S) & \xrightarrow[\simeq]{\mu} & \mathrm{Spin}^c(\mathbb{R} \oplus \mathrm{T}S) & \xleftarrow{\simeq} & \mathrm{Spin}^c(S). \end{array}$$

The only change is that, because the base space S is now 2-dimensional, homotopies are not punctured anymore. An *Euler structure* on $\mathbb{R} \oplus \mathrm{T}S$ is defined to be a homotopy class of nonsingular sections of this

vector bundle and, similarly, a *parallelization* on $\mathbb{R} \oplus TS$ is a homotopy class of trivializations of this oriented vector bundle.

Example 1.2. Thus, the section $v = (1, 0)$ of $\mathbb{R} \oplus TS$ determines a Spin^c-structure $\mu([v])$ on the surface S. By Remark 1.4, the Chern class of $\mu([v])$ coincides with the Euler class $e(TS)$ of the surface S.

In the sequel, we fix a parallelization τ on $\mathbb{R} \oplus T\partial M$. The observation at the beginning of Section 1.2.5 allows us to identify τ with a homotopy class of trivializations of $TM|_{\partial M}$.

Fix, in this paragraph, a nonsingular section s of $TM|_{\partial M}$. An *Euler structure on M relative to s* is a punctured homotopy class rel ∂M of nonsingular vector fields on M that extend s. We denote by $\mathrm{Eul}(M, s)$ the set of such structures. Obstruction theory says that there is an obstruction $w(M, s) \in H^3(M, \partial M)$ to the existence of such structures and, when the latter happens to vanish, that the set $\mathrm{Eul}(M, s)$ is naturally a $H^2(M, \partial M)$-affine space. (Here, again, we use the given orientation of M to make $\mathbb{Z}$ the coefficients group.) As an application of the Poincaré–Hopf theorem and obstruction calculi on the double $M \cup_{\mathrm{Id}} (-M)$, one obtains that

$$2 \cdot \langle w(M, s), [M, \partial M] \rangle = \langle e(TM|_{\partial M} / \langle s \rangle), [\partial M] \rangle \in \mathbb{Z}. \tag{1.2}$$

The following lemma can be proved formally the same way as Lemma 1.2. The first statement is also a direct consequence of (1.2).

Lemma 1.7. *For any trivialization $t = (t_1, t_2, t_3)$ of $TM|_{\partial M}$ representing τ (which we denote by $t \in \tau$), the nonsingular vector field t_1 can be extended to M. Moreover, for any $t, t' \in \tau$, there exists a canonical $H^2(M, \partial M)$-equivariant bijection*

$$\mathrm{Eul}\,(M, t_1) \xrightarrow{\rho_{t,t'}} \mathrm{Eul}(M, t_1').$$

Lastly, for any $t, t', t'' \in \tau$, we have that $\rho_{t',t''} \circ \rho_{t,t'} = \rho_{t,t''}$.

Definition 1.5. An *Euler structure on M relative to τ* is a pair (v, t) where $t \in \tau$ and $v \in \mathrm{Eul}(M, t_1)$, two such pairs (v, t) and (v', t') being considered as equivalent when $v' = \rho_{t,t'}(v)$. The set of such structures is denoted by $\mathrm{Eul}(M, \tau)$ and can naturally be given the structure of a $H^2(M, \partial M)$-affine space.

Remark 1.5. Following Turaev, one can describe concretely how a $x \in H^2(M, \partial M)$ acts on a $[(v, t)] \in \mathrm{Eul}(M, \tau)$. Let $P^{-1}x \in H_1(M)$ be represented by a smooth oriented knot $K \subset \mathrm{int}(M)$, and let v' be the vector field obtained from v by "Reeb turbulentization" along K (see [28, Section 5.2]). Then, (v', t) represents $[(v, t)] + x$.

The following relative version of Lemma 1.4 can be proved similarly.

Lemma 1.8. *There exists a canonical $H^2(M, \partial M)$-equivariant bijection*

$$\mathrm{Eul}(M, \tau) \xrightarrow{\mu} \mathrm{Spin}^c\,(M, \mu(\tau))\,.$$

1.3.4. Relative Chern classes

Let M be a 3-manifold with boundary and let σ be a Spin-structure on ∂M. In the relative case too, there is a *Chern class* map

$$\mathrm{Spin}^c(M, \sigma) \xrightarrow{c} H^2(M, \partial M),$$

which is affine over the doubling map. It can be defined directly (using Remark 1.2), or undirectly regarding relative Spin^c-structures as classes of vector fields (Section 1.3.3). This is done in the next paragraph.

Let τ be the parallelization on $\mathbb{R} \oplus \mathrm{T}\partial M$ corresponding to σ by μ. For any trivialization t of $\mathrm{T}M|_{\partial M}$ representing τ and for any nonsingular vector field v on M extending t_1, we can consider the relative Euler class

$$e(\mathrm{T}M/\langle v\rangle, t_2) \in H^2(M, \partial M),$$

i.e., the obstruction to extend the nonsingular section t_2 of $\mathrm{T}M/\langle v\rangle$ from ∂M to the whole of M. Clearly, this only depends on the equivalence class $[(v, t)]$ of (v, t) in the sense of Definition 1.5. Thus, we get a canonical map

$$\mathrm{Eul}(M, \tau) \longrightarrow H^2(M, \partial M)$$

which can be verified to be affine over the doubling map thanks to Remark 1.5. Its composition with μ^{-1} is defined to be c. (Compare with Remark 1.4.)

Remark 1.6. For any $\alpha \in \mathrm{Spin}^c(M, \sigma)$, the Chern class $c(\alpha)$ vanishes if and only if α comes from the set $\mathrm{Spin}(M, \sigma)$ defined in Remark 1.3.

We now compute the modulo 2 reduction of a relative Chern class. First, recall that the cobordism group Ω_1^{Spin} is isomorphic to $\mathbb{Z}_2$ [16,23]. For a closed surface S, there is the Atiyah–Johnson correspondence

$$\mathrm{Spin}(S) \xrightarrow[\simeq]{q} \mathrm{Quad}(S)$$

between spin structures on S and quadratic functions with the modulo 2 intersection pairing of S as associated bilinear pairing [1,13]. The quadratic function $q_\sigma : H_1(S; \mathbb{Z}_2) \to \mathbb{Z}_2$ corresponding to $\sigma \in \mathrm{Spin}(S)$ is defined by

$$q_\sigma([\gamma]) = [(\gamma, \sigma|_\gamma)] \in \Omega_1^{\mathrm{Spin}} \simeq \mathbb{Z}_2$$

for any oriented simple closed curve γ on S.

Lemma 1.9. *The following identity holds for any* $\alpha \in \mathrm{Spin}^c(M, \sigma)$:

$$\forall y \in H_2(M, \partial M),\ \langle c(\alpha), y\rangle \bmod 2 = q_\sigma(\partial_*(y)).$$

Here, $\partial_* : H_2(M, \partial M) \to H_1(\partial M)$ *denotes the connecting homomorphism of the pair* $(M, \partial M)$ *and is followed by the modulo* 2 *reduction.*

Proof. The modulo 2 reduction of $c(\alpha)$ is

$$w_2(M, \sigma) \in H^2(M, \partial M; \mathbb{Z}_2),$$

i.e., the obstruction to extend σ to the whole manifold M. Let Σ be a connected immersed surface in M such that $\partial\Sigma$ is $\partial M \cap \Sigma$, $\partial\Sigma$ has no singularity and Σ represents the modulo 2 reduction of y. Then, $\langle c(\alpha), y\rangle \bmod 2 = \langle w_2(M, \sigma), [\Sigma]\rangle$ is equal to $\langle w_2(\Sigma, \sigma|_{\partial\Sigma}), [\Sigma]\rangle$ and so is the obstruction to extend the

Spin-structure $\sigma|_{\partial\Sigma}$ to the whole surface Σ. Since Σ is connected, this is the class of $(\partial\Sigma, \sigma|_{\partial\Sigma})$ in Ω_1^{Spin}. Thus, we have that $\langle c(\alpha), y\rangle \bmod 2 = q_\sigma([\partial\Sigma]) = q_\sigma(\partial_*(y))$. □

Example 1.3. Suppose that ∂M is a disjoint union of tori. Let τ^0 be the distinguished parallelization corresponding to the distinguished Spin-structure σ^0 on the 2-torus (see Example 1.1). An Euler structure on M relative to the union of copies of τ^0 is equivalent to a relative Euler structure in the sense of Turaev [28, Section 5.1]. Lemma 1.9 is a generalization of [31, Lemma 1.3].

1.3.5. Spin^c*-structures as stable complex structures*

We conclude this subsection devoted to the dimension 3 by recalling that, in this case, a Spin^c-structure is equivalent to a stable complex structure on the oriented tangent bundle.

Lemma 1.10. *If M is a closed* 3*-manifold, then the canonical map*

$$\mathrm{U}^s(M) \xrightarrow{\omega} \mathrm{Spin}^c(M)$$

introduced in Section 1.2.7 *is bijective.*

Proof. Endow M with a Riemannian metric and consider a nonsingular vector field v on M. Then, $\mathbb{R}\oplus \mathrm{T}M$ splits as $(\mathbb{R}\oplus\langle v\rangle)\oplus\langle v\rangle^\perp$, which is the sum of two oriented 2-dimensional vector bundles. So, via the inclusion of $\mathrm{U}(1)\times\mathrm{U}(1)$ into $\mathrm{U}(2)$ defined by $(A, B)\mapsto (A)\oplus(B)$, v defines a complex structure J_v on $\mathbb{R}\oplus\mathrm{T}M$. Thus, we get a map from $\mathrm{Eul}(M)$ to the set of stable complex structures on $\mathrm{T}M$ up to punctured homotopy. By obstruction theory applied to the fibration $\mathrm{BU}\to\mathrm{BSO}$ with fiber type SO/U, the latter set is a $H^2(M)$-affine space and that map is $H^2(M)$-equivariant. Thus, since $\pi_3(\mathrm{SO}/\mathrm{U})$ is zero, we get a bijective map

$$\mathrm{Eul}(M) \xrightarrow[\simeq]{J} \mathrm{U}^s(M).$$

It can be verified that $\omega\circ J$ is the map μ from Lemma 1.4. (This verification amounts to checking that some two group homomorphisms from $\mathrm{U}(1)$ to Spin$^c(4)$ coincide.) □

1.4. Gluing of complex spin structures

In this subsection, we deal with the technical problem of gluing Spin^c-structures. We formulate the gluing in terms of (rigid) Spin^c-structures, but the reader may easily translate the statement and the proof in terms of vector fields and Euler structures.

Let M be a closed n-manifold obtained by gluing two n-manifolds M_1 and M_2 along their boundaries:

$$M = M_1 \cup_f M_2.$$

This involves a positive diffeomorphism $f : -\partial M_2 \to \partial M_1$ as well as a collar neighborhood of ∂M_i in M_i. The inclusion $M_i \hookrightarrow M$ will be denoted by j_i.

Lemma 1.11. *For $i = 1, 2$, let s_i be a rigid* Spin^c*-structure on $\mathrm{T}M_i|_{\partial M_i}$. Having identified $\mathbb{R}\oplus\mathrm{T}\partial M_i$ with $\mathrm{T}M_i|_{\partial M_i}$ thanks to the collar, we assume that $s_1\circ(-\mathrm{Id}\oplus\mathrm{T}f) = s_2$. If the relative obstructions*

$w(M_i, s_i)$'s vanish, then the absolute obstruction $w(M)$ does too and there is a canonical gluing *map*

$$\mathrm{Spin}^c(M_1, s_1) \times \mathrm{Spin}^c(M_2, s_2) \xrightarrow{\cup_f} \mathrm{Spin}^c(M)$$

which is affine over

$$\begin{array}{ccc} H^2(M_1, \partial M_1) \oplus H^2(M_2, \partial M_2) & \dashrightarrow & H^2(M) \\ \downarrow{\scriptstyle P^{-1}\times P^{-1}} & & \uparrow{\scriptstyle P} \\ H_{n-2}(M_1) \oplus H_{n-2}(M_2) & \xrightarrow{j_{1,*}\oplus j_{2,*}} & H_{n-2}(M). \end{array}$$

Proof. For $i = 1, 2$, let $\alpha_i \in \mathrm{Spin}^c(M_i, s_i)$ be represented by a rigid structure a_i. The structures a_1 and a_2 can be glued together by means of T*f*: we obtain a rigid Spin^c-structure on M whose homotopy class does not depend on the choices of a_1 and a_2 in α_1 and α_2, respectively. We denote it by $\alpha_1 \cup_f \alpha_2 \in \mathrm{Spin}^c(M)$.

Let us prove that this map $\cup_f$ is affine. For $i = 1, 2$, let $\mathscr{C}_i$ be a smooth triangulation of M_i such that $\mathscr{C}_1|_{\partial M_1}$ corresponds to $\mathscr{C}_2|_{\partial M_2}$ by *f*. We denote by $\mathscr{C}_i^*$ the cellular decomposition of M_i dual to the triangulation $\mathscr{C}_i$.

On the one hand, we consider the *union* $\mathscr{C}$ of the triangulations $\mathscr{C}_1$ and $\mathscr{C}_2$: a simplex of $\mathscr{C}$ is a simplex of $\mathscr{C}_i$ for $i = 1$ or 2, and simplices of ∂M_1 are identified with simplices of ∂M_2 by *f*. On the other hand, we consider the *gluing* $\mathscr{C}^*$ of the cellular decompositions $\mathscr{C}_1^*$ and $\mathscr{C}_2^*$: a cell of $\mathscr{C}^*$ either is a cell of $\mathscr{C}_i^*$ which does not intersect ∂M_i, either is the gluing by *f* of a cell belonging to $\mathscr{C}_1^*$ with a cell of $\mathscr{C}_2^*$ along a face lying in $\partial M_1 \cong -\partial M_2$. Then, $\mathscr{C}$ is a smooth triangulation of M and $\mathscr{C}^*$ is its dual cellular decomposition. Cohomology will be calculated with $\mathscr{C}$ while homology will be computed with $\mathscr{C}^*$.

For $i = 1, 2$, consider some $\alpha_i, \alpha_i' \in \mathrm{Spin}^c(M_i, s_i)$ and set $\alpha = \alpha_1 \cup_f \alpha_2$ and $\alpha' = \alpha_1' \cup_f \alpha_2'$. We want to prove the following equality:

$$j_{1,*}P^{-1}(\alpha_1 - \alpha_1') + j_{2,*}P^{-1}(\alpha_2 - \alpha_2') = P^{-1}(\alpha - \alpha') \in H_{n-2}(M). \tag{1.3}$$

For $i = 1, 2$, let $a_i, a_i' \in \mathrm{Spin}_r^c(M_i)$ represent α_i and α_i' respectively and coincide on the 1-skeleton of $\mathscr{C}_i$ (and, of course, on ∂M_i). Suppose that we have fixed a morphism of oriented vector bundles $\mathrm{T}M_i \to \gamma_{\mathrm{SO}(n)}$: then, the rigid structures a_i and a_i' can be identified with lifts $M_i \to \mathrm{BSpin}^c(n)$ by $\mathrm{B}\pi$ of the base maps $M_i \to \mathrm{BSO}(n)$. The obstruction $\alpha_i - \alpha_i' \in H^2(M_i, \partial M_i)$ is the class of the 2-cocycle which assigns to each 2-simplex e_k^i of $\mathscr{C}_i$ outside ∂M_i, this element z_k^i of $\pi_2(\mathrm{BU}(1)) \simeq \pi_2(K(\mathbb{Z}, 2)) \simeq \mathbb{Z}$ obtained by gluing $a_i|_{e_k^i}$ and $a_i'|_{e_k^i}$ along ∂e_k^i. So, we have that $P^{-1}(\alpha_i - \alpha_i') = [\sum_k z_k^i \cdot e_k^{*,i}]$ if $e_k^{*,i}$ denotes the $(n-2)$-cell dual to e_k^i.

Moreover, $a := a_1 \cup_f a_2$ and $a' := a_1' \cup_f a_2'$ represent α and α' respectively. Using these rigid structures, we can describe explicitely a 2-cocycle representing $\alpha - \alpha'$ as well. This 2-cocycle sends any 2-simplex of $\mathscr{C}_1 \cup_f \mathscr{C}_2$ contained in $\partial M_1 \cong -\partial M_2$ to $0 \in \mathbb{Z}$ so that $P^{-1}(\alpha - \alpha')$ is represented by $\sum_k z_k^1 \cdot e_k^{*,1} + \sum_k z_k^2 \cdot e_k^{*,2}$. □

Suppose now that the manifolds have dimension $n = 3$. This is the gluing lemma that we will use in the next sections.

Lemma 1.12. *Let $\sigma_1 \in \mathrm{Spin}(\partial M_1)$ and $\sigma_2 \in \mathrm{Spin}(\partial M_2)$ be such that $f^*(\sigma_1) = -\sigma_2$. Then, there is a canonical* gluing *map*

$$\mathrm{Spin}^c(M_1, \sigma_1) \times \mathrm{Spin}^c(M_2, \sigma_2) \xrightarrow{\cup_f} \mathrm{Spin}^c(M)$$

which is affine over

$$\begin{array}{ccc} H^2(M_1, \partial M_1) \oplus H^2(M_2, \partial M_2) & \dashrightarrow & H^2(M) \\ \downarrow{\scriptstyle P^{-1}\times P^{-1}} & & \uparrow{\scriptstyle P} \\ H_1(M_1) \oplus H_1(M_2) & \xrightarrow{j_{1,*}\oplus j_{2,*}} & H_1(M). \end{array}$$

Moreover, for any $\alpha_1 \in \mathrm{Spin}^c(M_1, \sigma_1)$ and $\alpha_2 \in \mathrm{Spin}^c(M_2, \sigma_2)$, the following identity between Chern classes holds:

$$P^{-1}c\left(\alpha_1 \cup_f \alpha_2\right) = j_{1,*}P^{-1}c(\alpha_1) + j_{2,*}P^{-1}c(\alpha_2) \in H_1(M).$$

Proof. Choose a rigid Spin-structure s_1 on $\mathrm{T}M_1|_{\partial M_1}$ representing σ_1, which we denoted by $s_1 \in \sigma_1$. This induces a $s_2 \in \sigma_2$ by setting $s_2 = s_1 \circ (-\mathrm{Id} \oplus \mathrm{T}f)$. By Lemma 1.2, the obstructions $w(M_i, \beta s_i)$'s vanish and so, by Lemma 1.11, there is a gluing map with domain $\mathrm{Spin}^c(M_1, \beta s_1) \times \mathrm{Spin}^c(M_2, \beta s_2)$.

Another choice $s_1' \in \sigma_1$ would induce another $s_2' \in \sigma_2$ and would lead to another gluing map this time with domain $\mathrm{Spin}^c(M_1, \beta s_1') \times \mathrm{Spin}^c(M_2, \beta s_2')$. Nevertheless, using the "double collar" of $\partial M_1 \cong -\partial M_2$ in M, one easily sees that the identifications $\rho_{s_1,s_1'}$ and $\rho_{s_2,s_2'}$ from Lemma 1.2 make those two gluing maps agree.

The first assertion of the lemma then follows. The second one is proved with arguments similar to those used in the proof of Lemma 1.11 (gluing of obstructions in compact oriented manifolds using Poincaré duality). □

Remark 1.7. If M is obtained by gluing M_1 and M_2 along only part of their boundaries (so that $\partial M \neq \emptyset$), Lemma 1.12 can easily be generalized to produce Spin^c-structures on M relative to a fixed Spin-structure on its boundary.

2. Linking quadratic function of a three-manifold with complex spin structure

In this section, we define the quadratic function $\phi_{M,\sigma}$ associated to a closed oriented 3-manifold M equipped with a Spin^c-structure σ. We present its elementary properties and connect it to previously known constructions.

2.1. Quadratic functions on torsion Abelian groups

We fix some notations. If A and B are Abelian groups and if $b : A \times A \to B$ is a symmetric bilinear pairing, we denote by $\widehat{b} : A \to \mathrm{Hom}(A, B)$ the adjoint map. The pairing b is said to be *nondegenerate* (respectively *nonsingular*) if $\widehat{b}$ is injective (respectively bijective). We denote by A^* the group $\mathrm{Hom}(A, \mathbb{Z})$ when A is free, the group $\mathrm{Hom}(A, \mathbb{Q})$ when A is a $\mathbb{Q}$-vector space and the group $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ when A is torsion. Lastly, application of the functor $- \otimes \mathbb{Q}$ is indicated by a subscript $\mathbb{Q}$.

2.1.1. Basic notions about quadratic functions

Let G be a torsion Abelian group.

A map $q : G \to \mathbb{Q}/\mathbb{Z}$ is said to be a *quadratic function* on G if

$$b_q(x, y) = q(x + y) - q(x) - q(y)$$

defines a (symmetric) bilinear pairing $b_q : G \times G \to \mathbb{Q}/\mathbb{Z}$. The quadratic function q is said to be *nondegenerate* if b_q is nondegenerate, and *homogeneous* if $q(-x) = q(x)$ for any $x \in G$. Apart from the bilinear pairing b_q, one can associate to q its *radical*

$$\mathrm{Ker}(q) = \mathrm{Ker}\,\widehat{b_q} \subset G,$$

its *homogeneity defect*

$$d_q : G \to \mathbb{Q}/\mathbb{Z}, \quad x \mapsto q(x) - q(-x)$$

and, in case when G happens to be finite, its *Gauss sum*

$$\gamma(q) = \sum_{x \in G} \exp(2i\pi q(x)) \in \mathbb{C}.$$

Given a symmetric bilinear pairing $b : G \times G \to \mathbb{Q}/\mathbb{Z}$, we say that $q : G \to \mathbb{Q}/\mathbb{Z}$ is a quadratic function *over* b if $b_q = b$. The group G^* acts freely and transitively on $\mathrm{Quad}(b)$, the set of quadratic functions over b, just as maps $G \to \mathbb{Q}/\mathbb{Z}$ add up. So, $\mathrm{Quad}(b)$ is a G^*-affine space.

There is a procedure to produce quadratic functions on torsion Abelian groups, known as the "discriminant" construction.

2.1.2. The discriminant construction

In the literature, the discriminant construction is usually restricted to nondegenerate bilinear lattices and produces quadratic functions on finite Abelian groups. The general case has been considered in [5], to which we refer for details and proofs. Here, we briefly review the construction.

A *lattice* H is a free finitely generated Abelian group. A *bilinear lattice* (H, f) is a symmetric bilinear pairing $f : H \times H \to \mathbb{Z}$ on a lattice H. Let also

$$H^\sharp = \{x \in H_{\mathbb{Q}} : f_{\mathbb{Q}}(x, H) \subset \mathbb{Z}\}$$

be the dual lattice. A *Wu class* for (H, f) is an element $w \in H$ such that

$$\forall x \in H, \ \ f(x, x) - f(w, x) \in 2\mathbb{Z}.$$

A *characteristic form* for (H, f) is an element $c \in H^* = \mathrm{Hom}(H, \mathbb{Z})$ satisfying

$$\forall x \in H, \ \ f(x, x) - c(x) \in 2\mathbb{Z}.$$

The sets of characteristic forms and Wu classes for (H, f) are denoted by $\mathrm{Char}(f)$ and $\mathrm{Wu}(f)$ respectively. Those sets are not empty and are related by the map $w \mapsto \widehat{f}(w)$, $\mathrm{Wu}(f) \to \mathrm{Char}(f)$.

Let (H, f) be a bilinear lattice. Consider the torsion Abelian group $G_f = H^\sharp/H$ and the map

$$L_f : G_f \times G_f \to \mathbb{Q}/\mathbb{Z}, \ \ ([x], [y]) \mapsto f_{\mathbb{Q}}(x, y) \bmod 1.$$

The pairing L_f is symmetric and bilinear, with radical $\mathrm{Ker}\,\widehat{L_f} \simeq (\mathrm{Ker}\,\widehat{f}) \otimes \mathbb{Q}/\mathbb{Z}$.

Observe that the adjoint map $\widehat{f_{\mathbb{Q}}} : H_{\mathbb{Q}} \to H^*_{\mathbb{Q}}$ restricted to $H^\sharp$ induces an epimorphism $G_f \to$ Tors Coker $\widehat{f}$. Hence the short exact sequence

$$0 \to \text{Ker } \widehat{L_f} \to G_f \to \text{Tors Coker } \widehat{f} \to 0, \tag{2.1}$$

which can be verified to split (non-canonically). Therefore, G_f is the direct sum of a finite Abelian group with as many copies of $\mathbb{Q}/\mathbb{Z}$ as the rank of Ker $\widehat{f}$. It follows also from (2.1) that the pairing L_f factors to a nondegenerate symmetric bilinear pairing

$$\text{Tors Coker } \widehat{f} \times \text{Tors Coker } \widehat{f} \xrightarrow{\lambda_f} \mathbb{Q}/\mathbb{Z}.$$

The bilinear map $H^* \times H^\sharp \to \mathbb{Q}$ defined by $(\alpha, x) \mapsto \alpha_{\mathbb{Q}}(x)$ induces a bilinear pairing

$$\text{Coker } \widehat{f} \times G_f \xrightarrow{\langle -,- \rangle} \mathbb{Q}/\mathbb{Z} \tag{2.2}$$

which is left nondegenerate and right nonsingular. It is left nonsingular if and only if f is nondegenerate.

Let now (H, f, c) be a bilinear lattice equipped with a characteristic form $c \in H^*$. One can associate to this triple a quadratic function over L_f, namely

$$\phi_{f,c} : G_f \to \mathbb{Q}/\mathbb{Z}, \quad [x] \mapsto \frac{1}{2}(f_{\mathbb{Q}}(x,x) - c_{\mathbb{Q}}(x)) \bmod 1.$$

Definition 2.1. The assignation $(H, f, c) \mapsto (G_f, \phi_{f,c})$ is called the *discriminant* construction.

Let us make a few observations about this construction. First, note that $\phi_{f,c}$ depends on c only mod $2\widehat{f}(H)$. Second, the Abelian group $H^*/\widehat{f}(H)$=Coker $\widehat{f}$ acts freely and transitively on Char$(f)/2\widehat{f}(H)$ by setting

$$\forall [\alpha] \in \text{Coker } \widehat{f}, \quad \forall [c] \in \text{Char}(f)/2\widehat{f}(H), \quad [c] + [\alpha] = [c + 2\alpha] \in \text{Char}(f)/2\widehat{f}(H).$$

Third, since Ker $\widehat{L_f}$ is canonically isomorphic to (Ker $\widehat{f}$) $\otimes \mathbb{Q}/\mathbb{Z}$, any form Ker $\widehat{f} \to \mathbb{Z}$ induces a homomorphism Ker $\widehat{L_f} \to \mathbb{Q}/\mathbb{Z}$. Thus, we get a homomorphism $j_f : (\text{Ker } \widehat{f})^* \to (\text{Ker } \widehat{L_f})^*$.

Theorem 2.1 ([5]). *The assignation $c \mapsto \phi_{f,c}$ induces an embedding*

$$\text{Char}(f)/2\widehat{f}(H) \overset{\phi_f}{\hookrightarrow} \text{Quad}(L_f)$$

which is affine over the opposite of the left adjoint of the pairing (2.2). *Moreover, a function $q \in \text{Quad}(L_f)$ belongs to* Im ϕ_f *if and only if $q|_{\text{Ker } \widehat{L_f}}$ belongs to* Im j_f.

Remark 2.1. The map ϕ_f is bijective if and only if f is nondegenerate.

We now use the algebraic notions above as combinatorial descriptions of topological notions.

2.2. Combinatorial descriptions associated to a surgery presentation

In this subsection, we fix an ordered oriented framed n-component link L in $\mathbf{S}^3$.

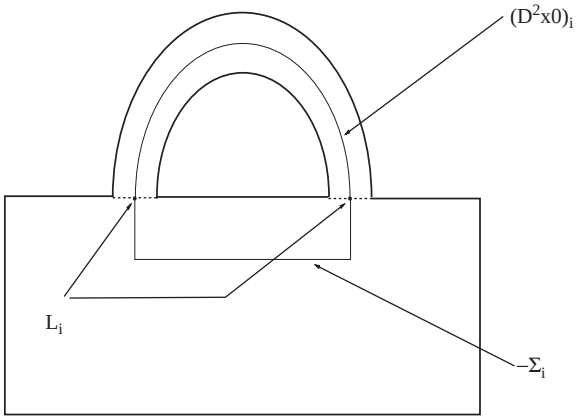


Fig. 1. The preferred basis of $H_2(W_L)$.

We call V_L the 3-manifold obtained from $\mathbf{S}^3$ by surgery along L and we denote by W_L the *trace* of the surgery:

$$V_L = \partial W_L \quad \text{with } W_L = \mathbf{D}^4 \cup \bigcup_{i=1}^{n} (\mathbf{D}^2 \times \mathbf{D}^2)_i,$$

where the 2-handle $(\mathbf{D}^2 \times \mathbf{D}^2)_i$ is attached by embedding $-(\mathbf{S}^1 \times \mathbf{D}^2)_i$ into $\mathbf{S}^3 = \partial \mathbf{D}^4$ in accordance with the specified framing and orientation of L_i.

The group $H_2(W_L)$ is free Abelian of rank n, and is given the *preferred* basis $([S_1], \ldots, [S_n])$ defined as follows. The closed surface S_i is taken to be $(\mathbf{D}^2 \times 0)_i \cup (-\Sigma_i)$, where Σ_i is a Seifert surface for L_i in $\mathbf{S}^3$ which has been pushed off into the interior of $\mathbf{D}^4$ as shown in Fig. 1. The group $H^2(W_L)$ is identified with $\mathrm{Hom}(H_2(W_L), \mathbb{Z})$ by Kronecker evaluation, and is given the dual basis. In the sequel, we simplify the notations by setting $H = H_2(W_L)$ (so that $H^2(W_L)$ is identified with H^*) and by denoting by $f : H \times H \to \mathbb{Z}$ the intersection pairing of W_L. The matrix of f relatively to the preferred basis of H is the linking matrix

$$B_L = (b_{ij})_{i,j=1}^{n}$$

of L. Since (H, f) is a bilinear lattice, the constructions of Section 2.1 apply.

2.2.1. Combinatorial description of Spin-structures

We recall a combinatorial description of $\mathrm{Spin}(V_L)$ due to Blanchet [2]. Define the set

$$\mathscr{S}_L = \left\{ [r] = ([r_i])_{i=1}^{n} \in (\mathbb{Z}_2)^n \ : \ \forall i = 1, \ldots, n, \ \sum_{j=1}^{n} b_{ij} r_j \equiv b_{ii} \bmod 2 \right\}.$$

The elements of $\mathscr{S}_L$ are called *characteristic solutions* of B_L.

Lemma 2.1. *There are canonical bijections*

$$\mathrm{Spin}(V_L) \underset{\simeq}{\longrightarrow} \mathrm{Wu}(f)/2H \underset{\simeq}{\longrightarrow} \mathscr{S}_L.$$

Thus, $\mathscr{S}_L$ shall be referred to as the *combinatorial description of* $\mathrm{Spin}(V_L)$. A refined Kirby's theorem dealing with surgery presentations of closed 3-dimensional Spin-manifolds can be derived from this lemma [2, Theorem (I.1)].

Proof of Lemma 2.1. The preferred basis of H induces an isomorphism $H/2H \simeq (\mathbb{Z}_2)^n$: the bijection between $\mathrm{Wu}(f)/2H$ and $\mathscr{S}_L$ is obtained this way. We now describe a bijection between $\mathrm{Spin}(V_L)$ and $\mathrm{Wu}(f)/2H$. Let σ be a Spin-structure on V_L. The obstruction $w_2(W_L, \sigma)$ to extend σ to W_L belongs to the group $H^2(W_L, V_L; \mathbb{Z}_2) \simeq H_2(W_L; \mathbb{Z}_2) \simeq H/2H$. Since $w_2(W_L, \sigma)$ is sent to $w_2(W_L)$ by the restriction map $H^2(W_L, V_L; \mathbb{Z}_2) \to H^2(W_L; \mathbb{Z}_2)$, a representative for $w_2(W_L, \sigma)$ in H has to be a Wu class for f. □

2.2.2. Combinatorial description of Spin^c*-structures*

Define the set

$$\mathscr{V}_L = \frac{\{s = (s_i)_{i=1}^n \in \mathbb{Z}^n \ : \ \forall i = 1, \ldots, n, \ s_i \equiv b_{ii} \bmod 2\}}{2 \cdot \mathrm{Im}\ B_L},$$

the elements of which are called *Chern vectors* of B_L. According to the following lemma, this set shall be referred to as the *combinatorial description of* $\mathrm{Spin}^c(V_L)$.

Lemma 2.2. *There are canonical bijections*

$$\mathrm{Spin}^c(V_L) \underset{\simeq}{\longrightarrow} \mathrm{Char}(f)/2\widehat{f}(H) \underset{\simeq}{\longrightarrow} \mathscr{V}_L.$$

Proof. The preferred basis of H defines an isomorphism $H^* \simeq \mathbb{Z}^n$, which induces a bijection between $\mathrm{Char}(f)/2\widehat{f}(H)$ and $\mathscr{V}_L$. The restriction map $\mathrm{Spin}^c(W_L) \to \mathrm{Spin}^c(V_L)$ is affine over the map $H^2(W_L) \to H^2(V_L)$ induced by inclusion. By exactness of the pair (W_L, V_L), the latter is surjective and its kernel coincides with the image of $\widehat{f} : H \to H^*$ (by Poincaré duality). Moreover, since $H^2(W_L)$ is free Abelian, a Spin^c-structure on W_L is determined by its Chern class in $H^2(W_L) \simeq H^*$. Such a class has to be a characteristic form for f since its modulo 2 reduction coincides with the second Stiefel–Whitney class $w_2(W_L) \in H^2(W_L; \mathbb{Z}_2) \simeq \mathrm{Hom}(H, \mathbb{Z}_2)$. Therefore, there is a bijection between $\mathrm{Spin}^c(V_L)$ and $\mathrm{Char}(f)/2\widehat{f}(H)$ defined by $\sigma \mapsto [c(\tilde{\sigma})]$ where $\tilde{\sigma}$ is an extension of σ to W_L. (This extension exists since $w(W_L, \sigma)$ lives in $H^3(W_L, V_L) = 0$, see Proposition 1.2.) □

If the Chern vector $[s]$ corresponds to the Spin^c-structure σ, we say that $(L, [s])$ is a *surgery presentation* of the closed 3-dimensional Spin^c-manifold (V_L, σ). On a diagram, we draw the framed link L using the blackboard framing convention, indicate its orientation and decorate each of its components L_i with the integer s_i.

Next, Kirby's theorem [15] can easily be extended to deal with surgery presentations of Spin^c-manifolds. This Spin^c version of Kirby's calculus will be used in the next section.

Theorem 2.2. *Let L and L' be ordered oriented framed links in $\mathbf{S}^3$. Equip them with Chern vectors $[s]$ and $[s']$, which correspond to* Spin^c*-structures σ and σ' on V_L and $V_{L'}$ respectively. Then, the* Spin^c*-manifolds (V_L, σ) and $(V_{L'}, \sigma')$ are* Spin^c*-diffeomorphic if and only if the pairs $(L, [s])$ and $(L', [s'])$ are, up to re-ordering and up to isotopy, related one to the other by a finite sequence of the moves drawn on Fig. 2.*

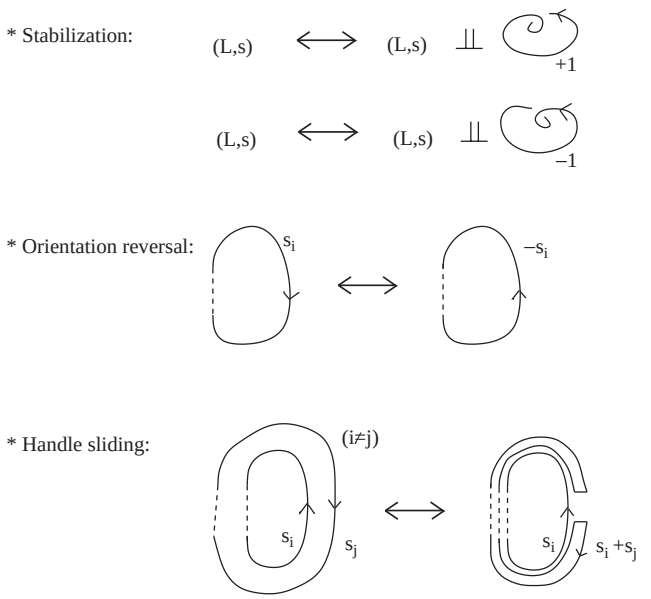


Fig. 2. Spin^c Kirby's moves. (Recall that the blackboard framing convention is used, and that labels refer to Chern vectors.)

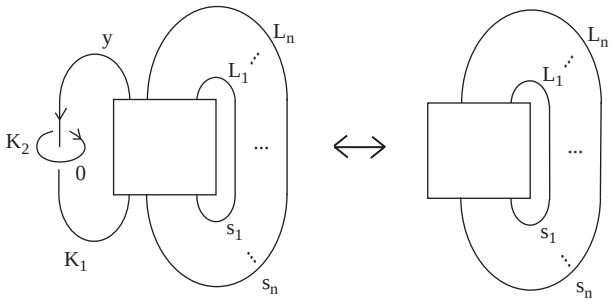


Fig. 3. Spin^c slam dunk move.

Proof. This follows from the usual Kirby's theorem. It suffices to show that, for each Kirby's move $L_1 \to L_2$, the corresponding canonical diffeomorphism $V_{L_1} \to V_{L_2}$ acts at the level of Spin^c-structures as combinatorially described on Fig. 2. This is a straightforward verification. □

Example 2.1. Look at the *slam dunk* move depicted in Fig. 3. Here, we are considering the ordered union $L \cup (K_1, K_2)$ of an n-component ordered oriented framed link L with an oriented framed knot K_1 together with its oriented meridian K_2. The move is

$$(L \cup (K_1, K_2), [(s_1, \ldots, s_n, y, 0)]) \longleftrightarrow (L, [(s_1, \ldots, s_n)]),$$

where y is the framing number of K_1. It relates two closed Spin^c-manifolds which are Spin^c-diffeomorphic, as can be shown by re-writing the proof of [7, Lemma 5] with Spin^c Kirby's calculi.

Remark 2.2. There exists a canonical isomorphism $\varrho : \text{Coker } \widehat{f} \to H^2(V_L)$, as defined by the following commutative diagram:

$$\begin{array}{ccccccc} H^2(W_L, V_L) & \longrightarrow & H^2(W_L) & \longrightarrow & H^2(V_L) & \longrightarrow & 0 \\ \simeq\uparrow & P & \simeq\uparrow & & \simeq\uparrow & \varrho & \\ H & \xrightarrow{\widehat{f}} & H^* & \longrightarrow & \text{Coker } \widehat{f} & \longrightarrow & 0. \end{array}$$

Then, the affine action of $H^2(V_L)$ on $\text{Spin}^c(V_L)$ writes combinatorially:

$$\forall [x] \in \text{Coker } \widehat{f},\ \forall [c] \in \text{Char}(f)/2\widehat{f}(H),\ [c] + [x] = [c + 2x].$$

The Chern class map $c : \text{Spin}^c(V_L) \to H^2(V_L)$ is combinatorially described by the map $c : \text{Char}(f)/2\widehat{f}(H) \to \text{Coker } \widehat{f},\ [c] \mapsto [c]$.

2.2.3. From Spin *to* Spin^c *in a combinatorial way*

We now relate the combinatorial description of $\text{Spin}(V_L)$ to that of $\text{Spin}^c(V_L)$.

Lemma 2.3. *The canonical map $\beta : \text{Spin}(V_L) \to \text{Spin}^c(V_L)$ corresponds to the map $\beta : \text{Wu}(f)/2H \to \text{Char}(f)/2\widehat{f}(H)$ defined by $\beta([w]) = [\widehat{f}(w)]$ or, equivalently, to the map $\beta : \mathscr{S}_L \to \mathscr{V}_L$ defined by $\beta([r]) = [B_L \cdot r]$.*

Proof. Take $\sigma \in \text{Spin}(V_L)$ and let $r_\sigma \in H^2(W_L, V_L) \simeq \mathbb{Z}^n$ be an integral representative for the obstruction $w_2(W_L, \sigma) \in H^2(W_L, V_L; \mathbb{Z}_2) \simeq (\mathbb{Z}_2)^n$ to extend σ to W_L. Let also $\tilde{\sigma} \in \text{Spin}^c(W_L)$ be an extension of $\beta(\sigma) \in \text{Spin}^c(V_L)$. Then, the lemma will follow from the fact that r_σ goes to $c(\tilde{\sigma})$ by the natural map $H^2(W_L, V_L) \to H^2(W_L)$ provided $\tilde{\sigma}$ is appropriately choosen with respect to r_σ. This can be proved undirectly as follows. In case when σ can be extended to W_L, this is certainly true: indeed, we can take $r_\sigma = 0$ and choose as $\tilde{\sigma}$ the image by β of the unique extension of σ to W_L, so that $c(\tilde{\sigma})$ vanishes. The general case can be reduced to this particular one for the following two reasons. First, it is easily verified that for each Kirby's move $L_1 \to L_2$ between ordered oriented framed links, the induced bijections $\mathscr{S}_{L_1} \to \mathscr{S}_{L_2}$ and $\mathscr{V}_{L_1} \to \mathscr{V}_{L_2}$, which are respectively described in [2, Theorem (I.1)] and Theorem 2.2, are compatible with the maps $\beta : \mathscr{S}_{L_k} \to \mathscr{V}_{L_k}$ $(k = 1, 2)$ defined by $\beta([r]) = [B_{L_k} \cdot r]$. Second, according to a theorem of Kaplan [14], there exists an oriented framed link L' in $\mathbf{S}^3$ related to L by a finite sequence of Kirby's moves, and through which $\sigma \in \text{Spin}(V_L)$ goes to $\sigma' \in \text{Spin}(V_{L'})$ with the property that σ' can be extended to $W_{L'}$. □

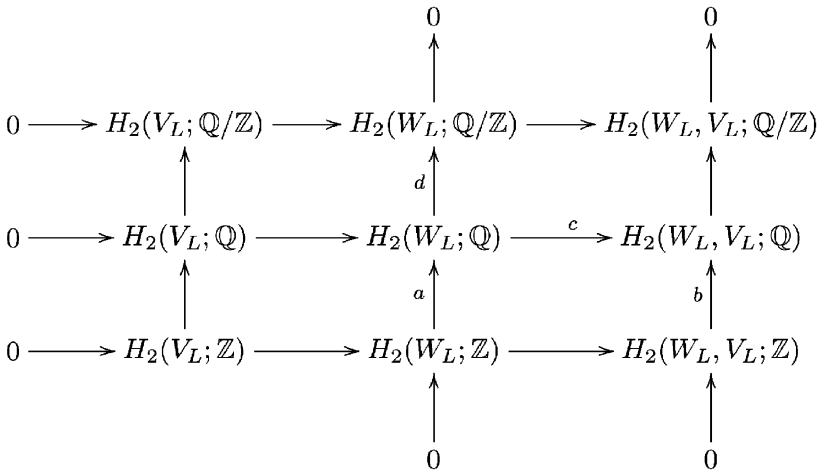
2.2.4. A combinatorial description of $H_2(V_L; \mathbb{Q}/\mathbb{Z})$

We maintain the notations used in Section 2.1.

Lemma 2.4. *There exists a canonical isomorphism*

$$\frac{H^\sharp}{H} \xrightarrow[\simeq]{\kappa} H_2(V_L; \mathbb{Q}/\mathbb{Z}).$$

Proof. Consider the following commutative diagram with exact rows and columns:



The group $H^\sharp$ is the subgroup of $H \otimes \mathbb{Q} = H_2(W_L; \mathbb{Q})$ comprising those $x \in H_2(W_L; \mathbb{Q})$ such that $c(x) \in H_2(W_L, V_L; \mathbb{Q})$ satisfies $c(x) \bullet a(y) \in \mathbb{Z}$ for all $y \in H_2(W_L; \mathbb{Z})$, where $\bullet$ is the rational intersection pairing in W_L. So, we have that

$$H^\sharp = c^{-1}b(H_2(W_L, V_L; \mathbb{Z})).$$

Seeing $H_2(V_L; \mathbb{Q}/\mathbb{Z})$ as a subgroup of $H_2(W_L; \mathbb{Q}/\mathbb{Z})$, we deduce the announced isomorphism from the map d. □

Recall that the quotient group $H^\sharp/H$, which is denoted by G_f in Section 2.1, appears in the short exact sequence (2.1). We now interpret this sequence as an application of the universal coefficients theorem to V_L. We denote by B the Bockstein homomorphism associated to the short exact sequence of coefficients

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Lemma 2.5. *The following diagram is commutative*:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} \widehat{L_f} & \longrightarrow & G_f & \longrightarrow & \operatorname{Tors} \operatorname{Coker} \widehat{f} & \longrightarrow & 0 \\
 & & \kappa\downarrow \simeq & & \kappa\downarrow \simeq & & \varrho\downarrow \simeq & & \\
0 & \longrightarrow & H_2(V_L) \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & H_2\big(V_L; \mathbb{Q}/\mathbb{Z}\big) & \xrightarrow[-P\circ B]{} & \operatorname{Tors} H^2(V_L) & \longrightarrow & 0.
\end{array}$$

Proof. It is enough to prove the commutativity of the right square. Start with a class $m \in H_2(V_L; \mathbb{Q}/\mathbb{Z})$. It can be written as $m = [S \otimes [\frac{1}{n}]]$ where n is a positive integer, S is a 2-chain in V_L with boundary $\partial S = n \cdot X$ and X is a 1-cycle. Then, we have that $B(m) = x \in H_1(V_L)$ if x denotes $[X]$. Let also Y be a relative 2-cycle in (W_L, V_L) with boundary $\partial Y = X$ and set $y = [Y] \in H_2(W_L, V_L)$. Lastly, consider the 2-cycle $U = n \cdot Y - S$ in W_L and set $u = [U] \in H = H_2(W_L)$.

Note that $u\otimes\frac{1}{n}\in H\otimes\mathbb{Q}$ belongs to the dual lattice $H^\sharp$: indeed, $P^{-1}\widehat{f}(u)=i_*(u)\in H_2(W_L,V_L)$ equals $n\cdot y$ so that $\widehat{f}(u)=n\cdot P(y)$. This also shows that $\widehat{f_{\mathbb{Q}}}(u\otimes\frac{1}{n})|_H=P(y)$. So, the map $G_f\to$ Tors Coker $\widehat{f}$ that is featured by the short exact sequence (2.1) sends $[u\otimes\frac{1}{n}]$ to $[P(y)]$.

The canonical map $H\otimes\mathbb{Q}\simeq H_2(W_L;\mathbb{Q})\to H_2(W_L;\mathbb{Q}/\mathbb{Z})$ sends $u\otimes\frac{1}{n}$ to $[(n\cdot Y-S)\otimes[\frac{1}{n}]]=[-S\otimes[\frac{1}{n}]]$. Consequently, we get that $\kappa([u\otimes\frac{1}{n}])=-m$.

The conclusion then follows from the commutativity of the diagram

$$\begin{array}{ccc} H_2(W_L,V_L) & \xrightarrow{\partial_*} & H_1(V_L) \\ \downarrow P & & \downarrow P \\ H^2(W_L) & \xrightarrow{i^*} & H^2(V_L), \end{array}$$

which implies that $\varrho([P(y)])=P(x)$. □

Remark 2.3. Similarly, the pairing (2.2) can easily be interpreted as the intersection pairing of V_L

$$H_1(V_L)\times H_2(V_L;\mathbb{Q}/\mathbb{Z})\xrightarrow{\bullet}\mathbb{Q}/\mathbb{Z}$$

via the isomorphisms $P^{-1}\varrho$: Coker $\widehat{f}\to H_1(V_L)$ and $\kappa: G_f\to H_2(V_L;\mathbb{Q}/\mathbb{Z})$.

2.3. A 4-dimensional definition of the linking quadratic function

Let M be a closed connected oriented 3-manifold equipped with a Spin^c-structure σ. In this subsection, we construct the quadratic function $\phi_{M,\sigma}$ announced in the introduction.

Lemma 2.6. *Fix a homology class $m\in H_2(M;\mathbb{Q}/\mathbb{Z})$. Consider a quadruplet (W,ψ,α,w) formed by a compact oriented 4-manifold W, a positive diffeomorphism $\psi:\partial W\to M$, a* Spin^c*-structure α on W which restricts to $\psi^*(\sigma)$ on the boundary and a class $w\in H_2(W;\mathbb{Q})$, the reduction of which in $H_2(W;\mathbb{Q}/\mathbb{Z})$ coincides with the image of m. Then, the quantity*

$$\phi(M,\sigma,m)=\left[\frac{1}{2}\left(\langle c(\alpha),w\rangle-w\bullet w\right)\right]\in\mathbb{Q}/\mathbb{Z}$$

does not depend on the choice of such a quadruplet.

Remark 2.4. If W is a compact oriented 4-manifold such that $H_1(W)=0$ and there exists a positive diffeomorphism $\psi:\partial W\to M$, then the pair (W,ψ) can be completed to a quadruplet (W,ψ,α,w) with the above property. In particular, such quadruplets do exist since M possesses surgery presentations.

Proof. Let (W',ψ',α',w') be another such quadruplet. We wish to compare the rational numbers $A:=w\bullet w-\langle c(\alpha),w\rangle$ and $A':=w'\bullet w'-\langle c(\alpha'),w'\rangle$.

The homology class m of M can be written as $m=[S\otimes[\frac{1}{n}]]$, where n is a positive integer, S is a 2-chain with boundary $\partial S=n\cdot X$ and X is a 1-cycle. Then, we have that $B(m)=[X]$. Since the image of m in $H_2(W;\mathbb{Q}/\mathbb{Z})$ belongs to the image of $H_2(W;\mathbb{Q})$, the image of $[X]\in H_1(M)$ in $H_1(W)$ is zero. So, one can find a relative 2-cycle Y in $(W,\partial W)$ with boundary $\partial Y=\psi^{-1}(X)$. Consider the 2-cycle

$U = n \cdot Y - \psi^{-1}(S)$ in W. Then, by assumption, w can be written as $w = [-U \otimes \frac{1}{n}] + w_0 \in H_2(W; \mathbb{Q})$, where $w_0 \in H_2(W; \mathbb{Q})$ belongs to the image of $H_2(W; \mathbb{Z})$. We do the same for w' in W' (getting thus some Y', U', w_0').

Next, we consider the closed oriented 4-manifold

$$\overline{W} := W \cup_{\psi^{-1} \circ \psi'} (-W').$$

Gluing rigid Spin^c-structures, it is easy to find a Spin^c-structure $\overline{\alpha}$ on $\overline{W}$ which restricts to α and $-\alpha'$ on W and $-W'$, respectively.

Set $\overline{Y} = i(Y) - i'(Y')$, where i and i' denote the inclusions of W and W' respectively. This is a 2-cycle in $\overline{W}$ with the property that the identity

$$[\overline{Y} \otimes 1] = [i(U) \otimes 1/n - i'(U') \otimes 1/n] = (-i_*(w) + i_*(w_0)) + (i'_*(w') - i'_*(w'_0))$$

holds in $H_2(\overline{W}; \mathbb{Q})$. It follows from this identity that

$$\begin{aligned}[\overline{Y}] \bullet [\overline{Y}] = {} & (w \bullet w + w_0 \bullet w_0 - 2 \cdot w \bullet w_0) \\ & + (-w' \bullet w' - w'_0 \bullet w'_0 + 2 \cdot w' \bullet w'_0),\end{aligned} \tag{2.3}$$

and that

$$\langle c(\overline{\alpha}), [\overline{Y}] \rangle = (-\langle c(\alpha), w \rangle + \langle c(\alpha), w_0 \rangle) + (\langle c(\alpha'), w' \rangle - \langle c(\alpha'), w'_0 \rangle). \tag{2.4}$$

Recall that $w_0 \in H_2(W; \mathbb{Q})$ and $w'_0 \in H_2(W'; \mathbb{Q})$ come from integral classes. Then, by the Wu formula and the fact that a Chern class reduces modulo 2 to the second Stiefel–Whitney class, the integers $[\overline{Y}] \bullet [\overline{Y}]$, $w_0 \bullet w_0$ and $w'_0 \bullet w'_0$ are congruent modulo 2 to $\langle c(\overline{\alpha}), [\overline{Y}] \rangle$, $\langle c(\alpha), w_0 \rangle$ and $\langle c(\alpha'), w'_0 \rangle$, respectively. Adding (2.3) to (2.4), we find that

$$A - A' - 2 \cdot w \bullet w_0 + 2 \cdot w' \bullet w'_0 \equiv 0 \bmod 2.$$

Because the image of $w \in H_2(W; \mathbb{Q})$ in $H_2(W; \mathbb{Q}/\mathbb{Z})$ comes from $H_2(M; \mathbb{Q}/\mathbb{Z})$ and because $w_0 \in H_2(W; \mathbb{Q})$ comes from $H_2(W; \mathbb{Z})$, the rational number $w \bullet w_0$ belongs to $\mathbb{Z}$. The same holds for $w' \bullet w'_0$. We conclude that the rational number $A - A'$ belongs to $2 \cdot \mathbb{Z}$. □

Remark 2.5. A universal class $u \in H^1(K(\mathbb{Q}/\mathbb{Z}, 1); \mathbb{Q}/\mathbb{Z})$ induces a homomorphism

$$\Omega_3^{\text{Spin}^c}(K(\mathbb{Q}/\mathbb{Z}, 1)) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

defined by $[(M, \sigma, f)] \mapsto \phi(M, \sigma, P^{-1} f^*(u))$. This follows from the definition of ϕ in Lemma 2.6.

Consider the linking pairing $\lambda_M : \text{Tors}\, H_1(M) \times \text{Tors}\, H_1(M) \to \mathbb{Q}/\mathbb{Z}$. Composing this with the Bockstein B, one gets a symmetric bilinear pairing

$$H_2(M; \mathbb{Q}/\mathbb{Z}) \times H_2(M; \mathbb{Q}/\mathbb{Z}) \xrightarrow{L_M} \mathbb{Q}/\mathbb{Z}$$

with radical $H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$. Using a cobordism W as in Remark 2.4, one easily proves, for any $m, m' \in H_2(M; \mathbb{Q}/\mathbb{Z})$, the following identity:

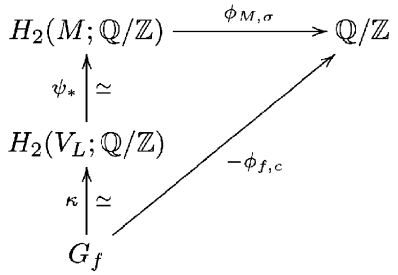
$$\phi(M, \sigma, m + m') - \phi(M, \sigma, m) - \phi(M, \sigma, m') = m \bullet B(m') = L_M(m, m').$$

Definition 2.2. The *linking quadratic function* of the Spin^c-manifold (M, σ) is the map denoted by

$$H_2(M; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\phi_{M,\sigma}} \mathbb{Q}/\mathbb{Z}$$

and defined by $m \mapsto \phi(M, \sigma, m)$.

The discriminant construction allows us to compute combinatorially the quadratic function $\phi_{M,\sigma}$, as soon as a surgery presentation of the Spin^c-manifold (M, σ) is given. Indeed, let L be an ordered oriented framed link in $\mathbf{S}^3$ together with a positive diffeomorphism $\psi : V_L \to M$. With the notations from Section 2.2, (H, f) still denotes the bilinear lattice $(H_2(W_L)$, intersection pairing of $W_L)$, to which the constructions from Section 2.1 apply. Let also $c \in \mathrm{Char}(f)$ represent $\psi^*(\sigma) \in \mathrm{Spin}^c(V_L)$ (in the sense of Lemma 2.2). Then, as can be verified from the definitions, the following diagram commutes:



(2.5)

Note that, in this context, the pairings λ_f and L_f are topologically interpreted as $-\lambda_M$ and $-L_M$, respectively.

2.4. Properties of the linking quadratic function

In this subsection, we fix a closed connected oriented 3-manifold M and prove properties of the map $\phi_M : \mathrm{Spin}^c(M) \to \mathrm{Quad}(L_M)$ defined by $\sigma \mapsto \phi_{M,\sigma}$. Those properties are proved "combinatorially" using (2.5), but may also be proved directly from the very definition of $\phi_{M,\sigma}$.

Next lemma says that $\phi_{M,\sigma}$ is determined on $H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$ by the Chern class $c(\sigma)$. Recall that the modulo 2 reduction of $c(\sigma)$ is $w_2(M) = 0$.

Lemma 2.7. *For any $\sigma \in \mathrm{Spin}^c(M)$, the function $\phi_{M,\sigma}$ is linear on $H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$:*

$$\forall x \otimes [r] \in H_2(M) \otimes \mathbb{Q}/\mathbb{Z}, \quad \phi_{M,\sigma}(x \otimes [r]) = \frac{\langle c(\sigma), x\rangle}{2} \cdot [r] \in \mathbb{Q}/\mathbb{Z}.$$

Proof. The first statement follows from the fact that Ker $\widehat{L_M} = H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$. As for the second statement, it suffices to prove it when $M = V_L$. Suppose that σ is represented by the characteristic form $c \in \mathrm{Char}(f)$ and that $x \in H_2(V_L)$ goes to y in $H = H_2(W_L)$. Then, $x \otimes [r]$ as an element of $H_2(V_L; \mathbb{Q}/\mathbb{Z})$ corresponds to $[y \otimes r]$ in $H^\sharp/H$. Consequently, we have that $\phi_{M,\sigma}(x \otimes [r]) = -\phi_{f,c}([y \otimes r]) = -\frac{1}{2}(r^2 f(y, y) - r \cdot c(y))$ mod 1. Since y belongs to Ker $\widehat{f}$, we obtain that $\phi_{M,\sigma}(x \otimes [r]) = \frac{1}{2} r \cdot c(y) \bmod 1 = \frac{1}{2} r \cdot \langle c(\sigma), x\rangle \bmod 1$, by Remark 2.2. □

Let us consider, for a while, the case when $\sigma \in \mathrm{Spin}^c(M)$ is torsion. Then, Lemma 2.7 implies that $\phi_{M,\sigma}$ vanishes on $H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$: Consequently, $\phi_{M,\sigma}$ factors to a quadratic function over λ_M. In this torsion case, our linking quadratic function is readily seen to agree with that of [4] and, up to a minus sign, with that of [9]. In the next subsection, it is also shown to coincide with that of [19].

In particular, σ may arise from a Spin-structure on M, which happens if and only if $c(\sigma)$ vanishes. Then, the factorization of $\phi_{M,\sigma}$ to Tors $H_1(M)$ coincides with the linking quadratic form defined in [18,24] or [27]. In [21], this quadratic form is used to classify degree 0 invariants in the Spin-refinement of the Goussarov–Habiro theory.

In the sequel, we will use the homomorphism

$$H^2(M) \xrightarrow{\mu_M} \mathrm{Hom}\,(H_2(M;\mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$$

defined by $\mu_M(y) = \langle y, - \rangle$.

Lemma 2.8. *For any $\sigma \in \mathrm{Spin}^c(M)$, the Chern class $c(\sigma)$ is sent by μ_M to the homogeneity defect $d_{\phi_{M,\sigma}} : H_2(M;\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ of the quadratic function $\phi_{M,\sigma}$.*

Proof. Again suppose that $M = V_L$ and that σ is represented by $c \in \mathrm{Char}(f)$. Take $x \in H_2(V_L;\mathbb{Q}/\mathbb{Z})$ represented by $y \in H^\sharp$. One computes that $\phi_{M,\sigma}(x) - \phi_{M,\sigma}(-x) = -\phi_{f,c}([y]) + \phi_{f,c}(-[y]) = c_{\mathbb{Q}}(y) \bmod 1 = \langle c(\sigma), x \rangle$, by Remark 2.2. □

Recall that $\mathrm{Spin}^c(M)$ is an affine space over $H^2(M)$ and that $\mathrm{Quad}(L_M)$ is an affine space over $\mathrm{Hom}(H_2(M;\mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$. Let

$$\mathrm{Hom}\,(H_2(M), \mathbb{Z}) \xrightarrow{j_M} \mathrm{Hom}\,(H_2(M) \otimes \mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$$

be the homomorphism defined by $j_M(l) = l \otimes \mathbb{Q}/\mathbb{Z}$. Next result, which contains Theorem 1, is a direct application of Theorem 2.1 and Remark 2.3.

Theorem 2.3. *The map $\phi_M : \mathrm{Spin}^c(M) \to \mathrm{Quad}(L_M)$ is an affine embedding over the group monomorphism μ_M. Moreover, a function $q \in \mathrm{Quad}(L_M)$ belongs to* Im ϕ_M *if and only if $q|_{H_2(M)\otimes\mathbb{Q}/\mathbb{Z}}$ belongs to* Im j_M.

Remark 2.6. The map ϕ_M is bijective if and only if M is a rational homology 3-sphere.

2.5. An intrinsic definition of the linking quadratic function

Let M be a closed connected oriented 3-manifold equipped with a Spin^c-structure σ. In this subsection, we give for the quadratic function $\phi_{M,\sigma}$ an intrinsic formula which does not refer to 4-dimensional cobordisms.

Here is the idea. Take a $x \in H_2(M;\mathbb{Q}/\mathbb{Z})$. It follows from Lemma 2.8 that

$$2 \cdot \phi_{M,\sigma}(x) = L_M(x,x) + \langle c(\sigma), x \rangle \in \mathbb{Q}/\mathbb{Z}.$$

For any $y \in \mathbb{Q}/\mathbb{Z}$, we denote by $\frac{1}{2} \cdot y$ the set of elements z of $\mathbb{Q}/\mathbb{Z}$ such that $z + z = y$. We are going to select, *correlatively*, an element z_1 in $\frac{1}{2} \cdot L_M(x,x)$ *and* an element z_2 in $\frac{1}{2} \cdot \langle c(\sigma), x \rangle$ such that $\phi_{M,\sigma}(x) = z_1 + z_2$.

Write $x \in H_2(M;\mathbb{Q}/\mathbb{Z})$ as $x = [S \otimes [1/n]]$, where n is a positive integer and S is an oriented immersed surface in M with boundary $n \cdot K$, a bunch of n parallel copies of an oriented knot K in M. Apply now the following stepwise procedure:

- *Step* 1: Choose a nonsingular vector field v on M representing σ as an Euler structure, and which is transverse to K (we claim that it is possible to find such v).
- *Step* 2: Let V be a sufficiently small regular neighborhood of K in M and let K_v be the parallel of K, lying on ∂V, obtained by pushing K along the trajectories of v. By an isotopy, ensure that S is in transverse position with respect to K_v with boundary contained in the interior of V.
- *Step* 3: Define a Spin-structure α_v on $\partial(M\backslash\mathrm{int}(V))$ by requiring its Atiyah–Johnson quadratic form q_{α_v} (Section 1.3.4) to be such that

$$q_{\alpha_v}([\text{meridian of } K]) = 0 \quad \text{and} \quad q_{\alpha_v}([K_v]) = 1.$$

- *Step* 4: Together with the vector field tangent to K_v, v represents a Spin^c-structure σ_v on $M\backslash\mathrm{int}(V)$ relative to the Spin-structure α_v (we claim this). Consider the Chern class $c(\sigma_v) \in H^2(M\backslash\mathrm{int}(V), \partial(M\backslash\mathrm{int}(V)))$.

Proposition 2.1. *By applying the above procedure, we get*

$$\phi_{M,\sigma}(x) = \underbrace{\left[\frac{1}{2n} \cdot K_v \bullet S\right]}_{\in \frac{1}{2}\cdot L_M(x,x)} + \underbrace{\left[\frac{1}{2n} \cdot \langle c(\sigma_v), [S \cap (M\backslash\mathrm{int}(V))]\rangle + \frac{1}{2}\right]}_{\in \frac{1}{2}\cdot\langle c(\sigma),x\rangle} \in \mathbb{Q}/\mathbb{Z}. \tag{2.6}$$

In [19], Looijenga and Wahl associate a quadratic function over λ_M to each pair $(M, \mathcal{J})$ formed by

- a closed connected oriented 3-manifold M,
- a homotopy class of complex structures $\mathcal{J}$ on $\mathbb{R} \oplus TM$ whose first Chern class is torsion.

There is a Spin^c-structure $\omega(\mathcal{J})$ associated to $\mathcal{J}$ (see Section 1.2.7). By assumption, its Chern class is torsion so that $\phi_{M,\omega(\mathcal{J})}$ factors to a quadratic function over λ_M. One can verify, using the inverse of ω described in the proof of Lemma 1.10, that formula (2.6) is equivalent in this case to formula (3.4.1) in [19].

Proof of Proposition 2.1. First of all, we have to justify that the above procedure can actually be carried out.

We begin by proving the claim of Step 1. Let v be an arbitrary nonsingular vector field on M representing σ. It suffices to prove the following claim.

Claim 2.1. Let w be an arbitrary nonsingular vector field tangent to M defined on K. Then, v can be homotoped so as to coincide with w on K.

Proof. Choose a tubular neighborhood W of K, plus an identification $W = (2\mathbf{D}^2) \times \mathbf{S}^1$ such that K corresponds to $0 \times \mathbf{S}^1$. We denote by (e_1, e_2) the standard basis of $\mathbb{R}^2 \supset 2\mathbf{D}^2$. We define $\pi : W \to K$ to be the projection on the core. The solid torus W is parametrized by the cylindric coordinates

$$((r \in [0,2], \theta \in \mathbb{R}/2\pi\mathbb{Z}), \phi \in \mathbb{R}/2\pi\mathbb{Z}).$$

If $p, q \in W$ are such that $\pi(p) = \pi(q)$ (i.e., they belong to the same meridional disk $2\mathbf{D}^2 \times *$), we define the transport map $t_{p,q} : \mathrm{T}_p W \to \mathrm{T}_q W$ as the unique linear map fixing the basis $(e_1, e_2, \frac{\partial}{\partial\phi})$. Deform the vector field v through the homotopy $(v^{(t)})_{t\in[0,1]}$ given at time t and point $p \in W$ by

$$v_p^{(t)} = \begin{cases} t_{\pi(p),p}(v_{\pi(p)}) & \text{if } r(p) \in [0,t] \\ t_{q(p,t),p}(v_{q(p,t)}) & \text{if } r(p) \in [t,2], \text{ with } q(p,t) = \left(\frac{r(p)-t}{1-t/2}, \theta(p), \phi(p)\right) \end{cases}$$

and at time t and point $p \notin W$ by $v_p^{(t)} = v_p$. After such a deformation, the vector field v satisfies the following property: $\forall p \in \mathbf{D}^2 \times \mathbf{S}^1,\ t_{p,\pi(p)}(v_p) = v_{\pi(p)}$. Now, since $\pi_1(\mathbf{S}^2)$ is trivial, $v|_K$ and w have to be homotopic; let $(w^{(t)})_{t\in[0,1]}$ be such a homotopy, beginning at $w^{(0)} = v|_K$ and ending at $w^{(1)} = w$. The homotopy given by

$$v_p^{(t)} = \begin{cases} t_{\pi(p),p}(w_{\pi(p)}^{(t-r(p))}) & \text{if } r(p) \in [0,t] \\ v_p & \text{if } r(p) \in [t,2] \end{cases}$$

if $p \in W$ and by $v_p^{(t)} = v_p$ if $p \notin W$, allows us to deform v to a nonsingular vector field which coincides with w on K. □

Since v is now transverse to K, we can find a regular neighborhood V of K in M plus an identification $V = \mathbf{D}^2 \times \mathbf{S}^1$, such that K corresponds to $0 \times \mathbf{S}^1$ and such that $v|_V$ corresponds to e_1 (recall that (e_1, e_2) denotes the standard basis of $\mathbb{R}^2 \supset \mathbf{D}^2$). We apply steps 2 and 3 (note that K_v then corresponds to $1 \times \mathbf{S}^1$) and we now prove the claim of Step 4. Let $\tau_v \in \mathrm{Spin}(V)$ be defined by the trivialization $(e_1, e_2, \frac{\partial}{\partial\phi})$ of $\mathrm{T}V$. Since $(\tau_v|_{\partial V})|_{1\times\mathbf{S}^1}$ is the non-bounding Spin-structure and since $(\tau_v|_{\partial V})|_{\partial\mathbf{D}^2\times 1}$ spin bounds, we have that $\tau_v|_{\partial V} = -\alpha_v$, i.e. τ_v belongs to $\mathrm{Spin}(V, -\alpha_v)$ with the notation of Remark 1.3. Thus, $v|_{M\setminus\mathrm{int}(V)}$ together with the trivialization $(e_1, e_2, \frac{\partial}{\partial\phi})|_{\partial V}$ of $\mathrm{T}(M\setminus\mathrm{int}(V))|_{\partial V}$ define a $\sigma_v \in \mathrm{Spin}^c(M\setminus\mathrm{int}(V), \alpha_v)$, as claimed in Step 4. For further use, note that σ is the gluing $\sigma_v \cup \beta(\tau_v)$, where $\beta : \mathrm{Spin}(V, -\alpha_v) \to \mathrm{Spin}^c(V, -\alpha_v)$ has been defined in Remark 1.3.

Set $z_1 = [1/2n \cdot K_v \bullet S] \in \mathbb{Q}/\mathbb{Z}$ and $z_2 = [1/2n \cdot \langle c(\sigma_v), [S']\rangle + 1/2] \in \mathbb{Q}/\mathbb{Z}$, where $S' = S \cap (M\setminus\mathrm{int}(V))$. We have that

$$2 \cdot z_1 = [1/n \cdot K_v \bullet S] = [\lambda_M(B(x), B(x))] = L_M(x,x).$$

Moreover, we have that

$$\begin{aligned} 2 \cdot z_2 &= [1/n \cdot \langle c(\sigma_v), [S']\rangle] \\ &= [1/n \cdot P^{-1}(c(\sigma_v)) \bullet [S']] \text{ (intersection in } M\setminus\mathrm{int}(V)) \\ &= P^{-1}(c(\sigma)) \bullet x \text{ (intersection in } M) \\ &= \langle c(\sigma), x\rangle \end{aligned}$$

where the third equality follows from the facts that $x = [S \otimes [1/n]]$, $P^{-1}(c(\sigma)) = i_* P^{-1}(c(\sigma_v)) + i_* P^{-1}(c(\beta(\tau_v))) \in H_1(M)$ (since $\sigma = \sigma_v \cup \beta(\tau_v)$) and $c(\beta(\tau_v)) = 0$ (by Remark 1.6).

We now prove formula (2.6), i.e., the equality $\phi_{M,\sigma}(x) = z_1 + z_2$. Let us work with surgery presentations (even if we could use more general cobordisms as well). Let M' be the 3-manifold obtained from M by doing surgery along the framed knot $(K, (e_1, e_2))$. Conversely, M is the result of the surgery on M' along the dual knot K' of K. Pick a surgery presentation $V_{L'}$ of M'; up to isotopy, the knot $K' \subset M'$ is in

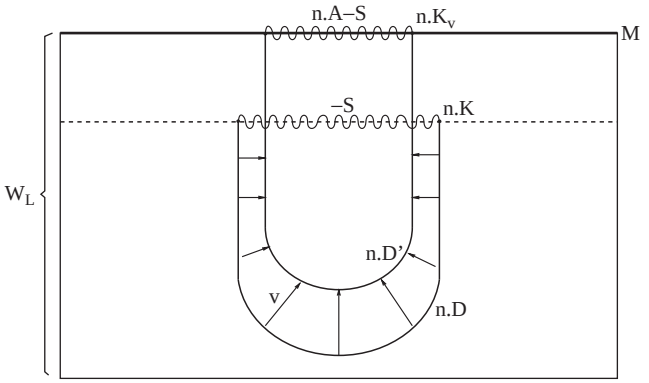


Fig. 4. Two representants of u in transverse position.

$\mathbf{S}^3\backslash L'$. We then find a surgery presentation V_L of M by setting L to be L' union K' with the appropriate framing. This surgery presentation of M has the following advantage: K bounds in the trace W_L of the surgery a disk D whose normal bundle is trivialized by some extension of the trivialization (e_1, e_2) of the normal bundle of K in M. We use the notations fixed in Section 2.2. In particular, $H = H_2(W_L)$ and $f : H \times H \to \mathbb{Z}$ is the intersection pairing of W_L. We define the 2-cycle $U = n \cdot D - S$ where $n \cdot D$ is a bunch of n parallel copies of the disk D with boundary $n \cdot K$; we also set $u = [U] \in H$. Then $u \otimes \frac{1}{n}$ belongs to $H^\sharp$ and the isomorphism $\kappa : H^\sharp/H \to H_2(M; \mathbb{Q}/\mathbb{Z})$ sends$[u \otimes \frac{1}{n}]$ to $-x = -[S \otimes [\frac{1}{n}]]$ (see the proof of Lemma 2.5). So, by diagram (2.5), we obtain that

$$\phi_{M,\sigma}(x) = -\phi_{f,c}\left(-\left[u \otimes \frac{1}{n}\right]\right) = -\frac{1}{2}\left(\frac{1}{n^2} f(u,u) + \frac{1}{n} c(u)\right) \bmod 1,$$

where c is a characteristic form representative for σ.

We calculate the quantity $f(u,u)$. The 2-cycle U is a representant of u. Let D' be a push-off of D by the extension of $e_1 = v|_V$ in such a way that $\partial D'$ is K_v. Let also A be the annulus of an isotopy from $-K_v$ to K in V (e.g., $A = -[0,1] \times \mathbf{S}^1$ in $V = \mathbf{D}^2 \times \mathbf{S}^1$). A second representative for u is $U' = n \cdot D' + n \cdot A - S$. By adding a collar to W_L and stretching the top of U', we can make U in transverse position with U' (see Fig. 4). So, we have that $f(u,u) = U \bullet U' = -nS \bullet K_v$ where the first intersection is calculated in W_L and the second one in M; we are led to

$$\phi_{M,\sigma}(x) = \frac{1}{2n} S \bullet K_v - \frac{1}{2n} c(u) \bmod 1. \tag{2.7}$$

We are now interested in the quantity $c(u)$. Let $\tilde{\sigma}$ be an extension of σ to the manifold W_L and let ξ be the isomorphism class of principal U(1)-bundles on W_L defined by $\tilde{\sigma}$; then c can be choosen to be $c_1(\xi)$. Let p be a representant of ξ and let tr be a trivialization of p on ∂V. Decompose the singular surface U' as $U' = U'_1 \cup U'_2 \cup U'_3$, where $U'_1 = n \cdot D'$, $U'_2 = n \cdot A \cup (-S \cap V)$ and $U'_3 = -S'$. By desingularizing U' so as to be reduced to a calculus of obstructions in an oriented manifold, we obtain that

$$c(u) = \langle c_1(p|_{U'}), [U'] \rangle = \sum_{i=1}^{3} \langle c_1(p|_{U'_i}, \mathrm{tr}\,|_{\partial U'_i}), [U'_i] \rangle \in \mathbb{Z}, \tag{2.8}$$

where $c_1(p|_{U_i'}, \mathrm{tr}\,|_{\partial U_i'}) \in H^2(U_i', \partial U_i')$ is the obstruction to extend the trivialization $\mathrm{tr}\,|_{\partial U_i'}$ of $p|_{U_i'}$ on $\partial U_i'$ to the whole of U_i'. Let $V' \subset W_L$ be the solid torus such that $M' = M\backslash\mathrm{int}(V) \cup V'$. For an appropriate choice of $\tilde{\sigma}$, there exists a Spin^c-structure $\sigma_1 \in \mathrm{Spin}^c(V', -\alpha_v)$ such that $\sigma_v \cup \sigma_1 = \tilde{\sigma}|_{M'}$. Also, for some appropriate choices of p in the class ξ and tr, we have

$$c_1(p|_{V'}, \mathrm{tr}) = c(\sigma_1) \in H^2(V', \partial V'),$$
$$c_1(p|_{V}, \mathrm{tr}) = c(\beta(\tau_v)) \in H^2(V, \partial V),$$
$$c_1(p|_{M\backslash\mathrm{int}(V)}, \mathrm{tr}) = c(\sigma_v) \in H^2(M\backslash\mathrm{int}(V), \partial(M\backslash\mathrm{int}(V))).$$

Then, Eq. (2.8) becomes

$$c(u) = n \cdot \langle c(\sigma_1), [D']\rangle + \langle c(\beta(\tau_v)), [U_2']\rangle - \langle c(\sigma_v), [S']\rangle \in \mathbb{Z}.$$

From the fact that $c(\beta(\tau_v)) = 0$, we deduce that

$$\frac{1}{2n} \cdot c(u) = -\frac{1}{2n} \cdot \langle c(\sigma_v), [S']\rangle + \frac{1}{2} \cdot \langle c(\sigma_1), [D']\rangle \in \mathbb{Q}.$$

Then, showing that $\langle c(\sigma_1), [D']\rangle$ is an odd integer together with (2.7) will end the proof of the proposition. Since $\langle c(\sigma_1), [D']\rangle = q_{-\alpha_v}(\partial_*[D']) = q_{\alpha_v}([K_v]) = 1 \bmod 2$ (by Lemma 1.9), we are done. □

3. Goussarov–Habiro theory for three-manifolds with complex spin structure

In this section, we explain how the Goussarov–Habiro theory can be extended to the context of 3-manifolds equipped with a Spin^c-structure. Then, using the linking quadratic function, we prove Theorem 2 stated in the introduction. This amounts to identifying the degree 0 invariants in the generalized theory.

3.1. Review of the Y-equivalence relation

Recall that the Goussarov–Habiro theory is a theory of finite type invariants for compact oriented 3-manifolds [8,11,12] and is based on the *Y*-surgery as elementary move. In this subsection, we just recall how this surgery move is defined.

Suppose that M is a compact oriented 3-manifold. Let $j : H_3 \hookrightarrow M$ be a positive embedding of the genus 3 handlebody into the interior of M. Set

$$M_j = M\backslash\mathrm{int}(\mathrm{Im}(j)) \cup_{j|_{\partial H_3}} (H_3)_B.$$

Here, $(H_3)_B$ is the surgered handlebody along the six-component framed link B shown in Fig. 5 with the blackboard framing convention.

Remark 3.1. Observe that there is a canonical inclusion $M\backslash\mathrm{int}(\mathrm{Im}(j)) \hookrightarrow M_j$. One can define a self-diffeomorphism h of ∂H_3 (explicitely, as the composition of 6 Dehn twists) such that there exists a diffeomorphism

$$M_j \cong M\backslash\mathrm{int}(\mathrm{Im}(j)) \cup_{j|_{\partial H_3} \circ h} H_3 \tag{3.1}$$

restricting to the identity on $M\backslash\mathrm{int}(\mathrm{Im}(j))$. Moreover, h can be verified to act trivially in homology.

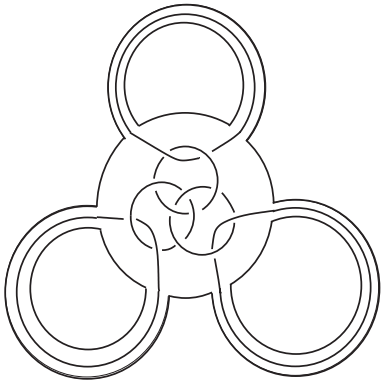


Fig. 5. The framed link *B*.

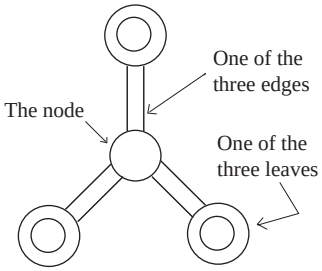


Fig. 6. A *Y*-graph.

A *Y-graph* G in M is an embedding of the surface drawn in Fig. 6 into the interior of M. This surface, of genus 0 with 4 boundary components, is decomposed between *leaves*, *edges* and *node*. Let $j : H_3 \hookrightarrow M$ be a trivialization of a regular neighborhood of G in M. The embedding j is unique, up to ambient isotopy.

Definition 3.1. The manifold obtained from M by *Y-surgery along* G, denoted by M_G, is the positive diffeomorphism class of the manifold M_j. The *Y-equivalence* is the equivalence relation among compact oriented 3-manifolds generated by *Y*-surgeries and positive diffeomorphisms.

Remark 3.2. The *Y*-surgery move has been introduced by Goussarov [11] and is equivalent to Habiro's "A_1-move" [12]. It is equivalent to Matveev's "Borromean surgery" as well, hence the *Y*-equivalence relation is characterized in [22].

3.2. The Y^c-equivalence relation

We define the Y^c-surgery move announced in the introduction, and we outline how this suffices to extend the Goussarov–Habiro theory to manifolds equipped with a Spin^c-structure.

3.2.1. Twist and Spin^c*-structures*

As in Section 1.4, we consider a closed oriented 3-manifold

$$M = M_1 \cup_f M_2$$

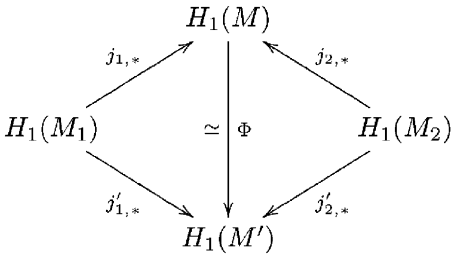
obtained by gluing two compact oriented 3-manifolds M_1 and M_2 with a positive diffeomorphism $f : -\partial M_2 \to \partial M_1$. We add the assumption that ∂M_2 *is connected.*

Let $h : \partial M_2 \to \partial M_2$ be a diffeomorphism *which acts trivially in homology* and consider the manifold

$$M' = M_1 \cup_{f\circ h} M_2.$$

The manifold M' is said to be obtained from M by a *twist*. By Remark 3.1, the Y-surgery move is an instance of a twist move.

By a Mayer–Vietoris argument, there is an isomorphism $\Phi : H_1(M) \to H_1(M')$ which is unambiguously defined by the commutative diagram



where j_1, j_2, j'_1 and j'_2 denote inclusions.

Proposition 3.1. *The twist from M to* M' *induces a canonical bijection*

$$\mathrm{Spin}^c(M) \xrightarrow[\simeq]{\Omega} \mathrm{Spin}^c(M')$$

which is affine over $P\Phi P^{-1} : H^2(M) \to H^2(M')$. *Moreover, the diagram*

$$\begin{array}{ccc} \mathrm{Spin}^c(M) & \xrightarrow{\Omega} & \mathrm{Spin}^c(M') \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} \\ H^2(M) & \xrightarrow[P\Phi P^{-1}]{} & H^2(M') \end{array}$$

is commutative.

Proof. For any $\alpha \in \mathrm{Spin}^c(M)$, we define $\Omega(\alpha)$ as follows. *Choose* $\sigma_2 \in \mathrm{Spin}(\partial M_2)$ and set $\sigma_1 = f_*(-\sigma_2) \in \mathrm{Spin}(\partial M_1)$. Since $h_* : H_1(\partial M_2; \mathbb{Z}_2) \to H_1(\partial M_2; \mathbb{Z}_2)$ is the identity, h acts trivially on $\mathrm{Spin}(\partial M_2)$: this follows from the naturality of the Atiyah–Johnson correspondence $\mathrm{Spin}(\partial M_2) \to \mathrm{Quad}(\partial \mathrm{M}_2)$

(see Section 1.3.4). According to Lemma 1.12, there are two gluing maps

$$\mathrm{Spin}^c(M_1,\sigma_1)\times\mathrm{Spin}^c(M_2,\sigma_2)\xrightarrow{\cup_f}\mathrm{Spin}^c(M)$$

$$\mathrm{Spin}^c(M_1,\sigma_1)\times\mathrm{Spin}^c(M_2,\sigma_2)\xrightarrow{\cup_{f\circ h}}\mathrm{Spin}^c(M')$$

which are affine, via Poincaré duality, over $j_{1,*}\oplus j_{2,*}$ and $j'_{1,*}\oplus j'_{2,*}$ respectively. Since ∂M_2 is connected, the map $\cup_f$ is surjective. *Choose* $\alpha_1\in\mathrm{Spin}^c(M_1,\sigma_1)$ and $\alpha_2\in\mathrm{Spin}^c(M_2,\sigma_2)$ such that $\alpha=\alpha_1\cup_f\alpha_2$, next set

$$\alpha'=\alpha_1\cup_{f\circ h}\alpha_2\in\mathrm{Spin}^c(M')$$

and define $\Omega(\alpha)$ to be α'.

We have to verify that $\Omega(\alpha)$ is well-defined by that procedure. Assume other intermediate choices $\tilde\sigma_2$, $\tilde\alpha_1$ and $\tilde\alpha_2$ instead of σ_2, α_1 and α_2 respectively, leading to $\tilde\alpha':=\tilde\alpha_1\cup_{f\circ h}\tilde\alpha_2$. We claim that $\alpha'=\tilde\alpha'$.

Consider first the particular case when $\tilde\sigma_2=\sigma_2\in\mathrm{Spin}(\partial M_2)$. Since $\alpha_1\cup_f\alpha_2=\alpha=\tilde\alpha_1\cup_f\tilde\alpha_2$, we have that

$$j_{1,*}P^{-1}(\alpha_1-\tilde\alpha_1)+j_{2,*}P^{-1}(\alpha_2-\tilde\alpha_2)=P^{-1}(\alpha-\alpha)=0\in H_1(M).$$

Applying Φ to that identity, we obtain the equation

$$j'_{1,*}P^{-1}(\alpha_1-\tilde\alpha_1)+j'_{2,*}P^{-1}(\alpha_2-\tilde\alpha_2)=0\in H_1(M')$$

whose left term equals $P^{-1}(\alpha'-\tilde\alpha')$. We conclude that $\alpha'=\tilde\alpha'$.

We now turn to the general case. For this, choose an arbitrary element

$$\tau_2\in\mathrm{Spin}^c([0,1]\times\partial M_2,\,0\times(-\sigma_2)\cup 1\times\tilde\sigma_2).$$

Having set $\tilde\sigma_1=f_*(-\tilde\sigma_2)$, define

$$\tau_1=(\mathrm{Id}\times f)_*(-\tau_2)\in\mathrm{Spin}^c([0,1]\times\partial M_1,\,0\times(-\sigma_1)\cup 1\times\tilde\sigma_1).$$

Here, $-\tau_2\in\mathrm{Spin}^c(-[0,1]\times\partial M_2,\,0\times\sigma_2\cup 1\times(-\tilde\sigma_2))$ is obtained from τ_2 by time-reversing. For $i=1,2$, the collar of ∂M_i in M_i and Lemma 1.12 give a map

$$\mathrm{Spin}^c(M_i,\sigma_i)\times\mathrm{Spin}^c\left([0,1]\times\partial M_i,\,0\times(-\sigma_i)\cup 1\times\tilde\sigma_i\right)\xrightarrow{\cup_{\mathrm{col}}}\mathrm{Spin}^c(M_i,\tilde\sigma_i).$$

From the definition of the gluing map $\cup_f$ and by using the "double collar" of $\partial M_1\cong-\partial M_2$ in M, one sees that $\alpha=\alpha_1\cup_f\alpha_2$ may also be written as

$$\alpha=(\alpha_1\cup_{\mathrm{col}}\tau_1)\cup_f(\alpha_2\cup_{\mathrm{col}}\tau_2)\,.$$

It follows from the special case treated previously that, whatever the choices of $\tilde\alpha_1$ and $\tilde\alpha_2$ have been,

$$\tilde\alpha'=(\alpha_1\cup_{\mathrm{col}}\tau_1)\cup_{f\circ h}(\alpha_2\cup_{\mathrm{col}}\tau_2)\,.$$

On the other hand, having set

$$\tau'_1=(\mathrm{Id}\times(f\circ h))_*(-\tau_2)\in\mathrm{Spin}^c([0,1]\times\partial M_1,\,0\times(-\sigma_1)\cup 1\times\tilde\sigma_1),$$

one sees that $\alpha' = \alpha_1 \cup_{f\circ h} \alpha_2$ may also be written as

$$\alpha' = \left(\alpha_1 \cup_{\mathrm{col}} \tau_1'\right) \cup_{f\circ h} \left(\alpha_2 \cup_{\mathrm{col}} \tau_2\right).$$

Consequently, it is enough to prove that

$$\tau_1 = \tau_1' \in \mathrm{Spin}^c([0,1] \times \partial M_1, 0 \times (-\sigma_1) \cup 1 \times \tilde{\sigma}_1). \tag{3.2}$$

The latter space of relative Spin^c-structures is classified by the Chern class map since $H^2([0,1] \times \partial M_1, \partial[0,1] \times \partial M_1)$ has no 2-torsion. Moreover, the naturality of the Chern class and the fact that h preserves the homology imply that

$$c(\tau_1) = (\mathrm{Id} \times f)_*(c(-\tau_2)) = (\mathrm{Id} \times (f \circ h))_*(c(-\tau_2)) = c(\tau_1').$$

We conclude that identity (3.2) holds and that the map Ω is well-defined.

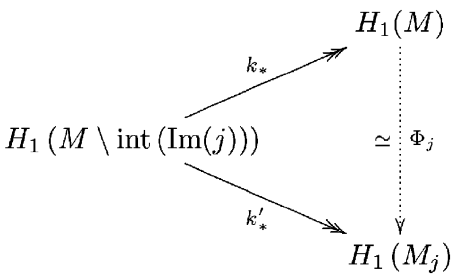
The fact that Ω is affine and the last statement of the proposition are readily derived from the properties of the gluing maps $\cup_f$ and $\cup_{f\circ h}$ stated in Lemma 1.12, and from the definition of the isomorphism Φ. □

Remark 3.3. We could have considered as well the case when M_1 and M_2 have disconnected boundary, but are glued together along a connected component of their boundary to give M (so that $\partial M \cong \partial M' \neq \emptyset$). Then, in view of Remark 1.7, Proposition 3.1 can easily be generalized to involve Spin^c-structures on M and M' relative to a fixed Spin-structure on their identified boundaries.

3.2.2. Definition of the Y^c-surgery move

We explain how Y-surgery makes sense in the setting of Spin^c-manifolds. For simplicity, we consider only the case of a closed oriented 3-manifold M.

Let $j : H_3 \hookrightarrow M$ be an embedding. We denote by $\Phi_j : H_1(M) \to H_1(M_j)$ the isomorphism defined by the commutative diagram



where $k : M\backslash\mathrm{int}(\mathrm{Im}(j)) \hookrightarrow M$ and $k' : M\backslash\mathrm{int}(\mathrm{Im}(j)) \hookrightarrow M_j$ denote inclusions.

Lemma 3.1. *There exists a canonical bijection*

$$\mathrm{Spin}^c(M) \xrightarrow[\simeq]{\Omega_j} \mathrm{Spin}^c\left(M_j\right), \quad \alpha \longmapsto \alpha_j$$

which is affine over $P\Phi_j P^{-1}$. *Moreover, the diagram*

$$\begin{array}{ccc} \mathrm{Spin}^c(M) & \xrightarrow{\Omega_j} & \mathrm{Spin}^c(M_j) \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} \\ H^2(M) & \xrightarrow[P\Phi_j P^{-1}]{} & H^2(M_j) \end{array}$$

is commutative.

Proof. By Remark 3.1, one can define a self-diffeomorphism h of ∂H_3 acting trivially in homology and such that there exists a diffeomorphism

$$M_j = M\backslash \mathrm{int}\,(\mathrm{Im}(j)) \cup_{j|_{\partial H_3}} (H_3)_B \xrightarrow[\cong]{f} M\backslash \mathrm{int}\,(\mathrm{Im}(j)) \cup_{j|_{\partial H_3}\circ h} H_3$$

which restricts to the identity on $M\backslash\mathrm{int}(\mathrm{Im}(j))$. This diffeomorphism induces a bijection

$$\mathrm{Spin}^c\left(M_j\right) \xrightarrow[\simeq]{f_*} \mathrm{Spin}^c\left(M\backslash \mathrm{int}\,(\mathrm{Im}(j)) \cup_{j|_{\partial H_3}\circ h} H_3\right).$$

Also, by Section 3.2.1, there is a canonical bijection

$$\mathrm{Spin}^c\,(M) \xrightarrow[\simeq]{\Omega} \mathrm{Spin}^c\left(M\backslash \mathrm{int}\,(\mathrm{Im}(j)) \cup_{j|_{\partial H_3}\circ h} H_3\right).$$

We define Ω_j to be the composite $f_*^{-1}\Omega$. This composite is easily verified to be independent of the pair (h, f) with the above property. □

Let G be a Y-graph in M. Let also $j : H_3 \hookrightarrow M$ and $j' : H_3 \hookrightarrow M$ be some trivializations of regular neighborhoods of G in M. There exists an ambiant isotopy $(q_t : M \to M)_{t\in[0,1]}$ between j and j': $q_0 = \mathrm{Id}_M$ and $q_1 \circ j = j'$. Let $q : M_j \to M_{j'}$ be the positive diffeomorphism induced by q_1 in the obvious way. One can verify that $q_* \circ \Omega_j = \Omega_{j'}$. Thus, for any Spin^c-structure α on M, the Spin^c-manifolds (M_j, α_j) and $(M_{j'}, \alpha_{j'})$ are Spin^c-diffeomorphic.

Definition 3.2. The Spin^c-manifold obtained from (M, α) by Y^c*-surgery along* G, denoted by (M_G, α_G), is the Spin^c-diffeomorphism class of the manifold (M_j, α_j). We call Y^c*-equivalence* the equivalence relation among closed 3-dimensional Spin^c-manifolds generated by Y^c-surgeries and Spin^c-diffeomorphisms.

In the sequel, the notation M_G will sometimes refer to a representative M_j obtained by fixing a trivialization j of a regular neighborhood of G in M. Similarly, α_G, Ω_G and Φ_G will stand for α_j, Ω_j and Φ_j, respectively.

Remark 3.4. In the case of compact oriented 3-manifolds with boundary, the Y^c-surgery move is defined similarly using Spin^c-structures relative to Spin-structures. (See Remark 3.3.)

It follows from the definition that, for any two disjoint Y-graphs G_1 and G_2 in M, the Spin^c-manifolds $((M_{G_1})_{G_2}, (\alpha_{G_1})_{G_2})$ and $((M_{G_2})_{G_1}, (\alpha_{G_2})_{G_1})$ are Spin^c-diffeomorphic. So, the Y^c-surgery along a family of disjoint Y-graphs makes sense.

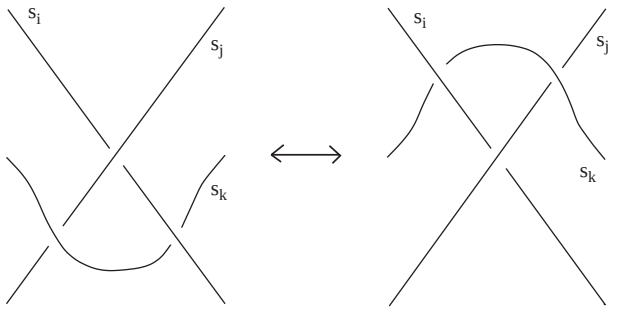


Fig. 7. A Δ^c-move.

Definition 3.3. Let I be an invariant of 3-dimensional Spin^c-manifolds with values in an Abelian group A. The invariant I is said to be *of degree at most d* if, for any 3-dimensional Spin^c-manifold (N, σ) and for any family S of at least $d+1$ pairwise disjoint Y-graphs in N, the identity

$$\sum_{S' \subset S} (-1)^{|S'|} \cdot I(N_{S'}, \sigma_{S'}) = 0 \in A \tag{3.3}$$

holds. Here, the sum is taken over all sub-families S' of S.

Thus, the Y^c-surgery move is the elementary move of a Spin^c-refinement of the Goussarov–Habiro theory of finite type invariants. In particular, two 3-dimensional Spin^c-manifolds are Y^c-equivalent if and only if they are not distinguished by degree 0 invariants. It can be shown that the "calculus of clovers" from [8], which is equivalent to the "calculus of claspers" from [12], extends to Spin^c-manifolds.

Remark 3.5. A Spin-refinement of the Goussarov–Habiro theory has been considered in [21]. In particular, it is shown that the Y-surgery along G induces a canonical bijection $\Theta_G : \mathrm{Spin}(M) \to \mathrm{Spin}(M_G)$. Both refinements of the theory are compatible, in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Spin}(M) & \xrightarrow[\simeq]{\Theta_G} & \mathrm{Spin}(M_G) \\ \beta \downarrow & & \downarrow \beta \\ \mathrm{Spin}^c(M) & \xrightarrow[\Omega_G]{\simeq} & \mathrm{Spin}^c(M_G). \end{array}$$

3.2.3. A combinatorial description of the Y^c-equivalence relation

A given equivalence relation among closed oriented 3-manifolds can sometimes be derived from an unknotting operation via surgery presentations in $\mathbf{S}^3$. It is well-known that the Y-equivalence relation can be formulated that way with the *Δ-move* of [25] as unknotting operation. We refine this to the context of Spin^c-manifolds.

Lemma 3.2. *The Y^c-equivalence relation is generated by* Spin^c*-diffeomorphisms and Δ^c-moves, if the Δ^c-move is defined to be the move depicted on* Fig. 7 *between surgery presentations of closed* 3*-dimensional* Spin^c*-manifolds* (*see* Section 2.2.2).

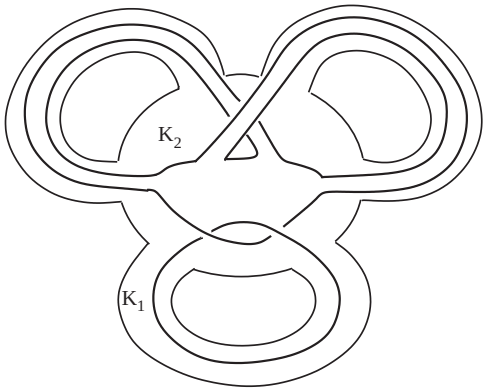


Fig. 8. *Y*-surgery as surgery along a 2-component link.

Proof. Let M be a closed connected oriented 3-manifold and let G be a Y-graph in M. Let $\psi : M \to V_L$ be a surgery presentation of M, where L is an n-component ordered oriented framed link in $\mathbf{S}^3$. Isotope G in M so that $\psi(G)$ becomes disjoint from the link dual to L, then $\psi(G)$ can be regarded as a subset of $\mathbf{S}^3\backslash L$. In the image by ψ of the regular neighborhood of G in M, put the 2-component framed link K depicted on Fig. 8. The link K can be obtained from the link B of Fig. 5 by some slam dunks (see Example 2.1) and handle slidings in H_3. In particular, there is an obvious surgery presentation $\psi' : M_G \to V_{L\cup K}$ induced by ψ. With the viewpoint from Section 2.2.2, we want to identify the combinatorial analog of the bijection Ω_G. In other words, we look forthe map O_G making the diagram

$$\begin{array}{ccc}
\mathcal{V}_L & \overset{O_G}{\dashrightarrow} & \mathcal{V}_{L\cup K} \\
\simeq \uparrow & & \uparrow \simeq \\
\mathrm{Spin}^c(V_L) & & \mathrm{Spin}^c(V_{L\cup K}) \\
\simeq \uparrow \psi_* & & \psi'_* \uparrow \simeq \\
\mathrm{Spin}^c(M) & \xrightarrow[\Omega_G]{} & \mathrm{Spin}^c(M_G)
\end{array}$$

commute. This is contained in the next claim, which will allow us to prove that the Δ^c-move and the Y^c-surgery move are equivalent.

Claim 3.1. Let B_L denote the linking matrix of L and let K be appropriately oriented so that the ordered union of ordered oriented framed links $L \cup K$ has its linking matrix of the form

$$B_{L\cup K} = \left(\begin{array}{ccc|cc}
 & & & x_1 & 0 \\
 & B_L & & \vdots & \vdots \\
 & & & x_l & 0 \\
\hline
x_1 & \cdots & x_l & x & 1 \\
0 & \cdots & 0 & 1 & 0
\end{array}\right).$$

Then, the map O_G sends a Chern vector $[s]$ to the Chern vector $[(s, x, 0)]$.

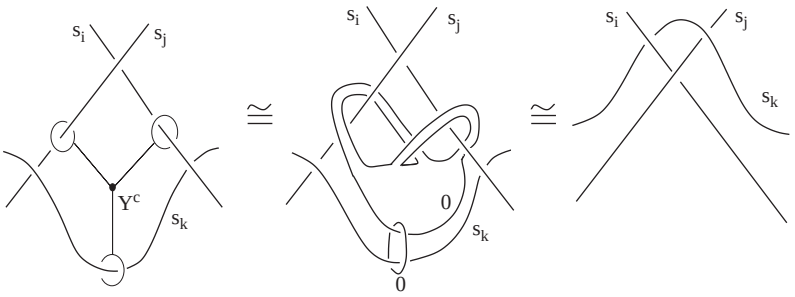


Fig. 9. A Δ^c-move can be realized by a Y^c-surgery.

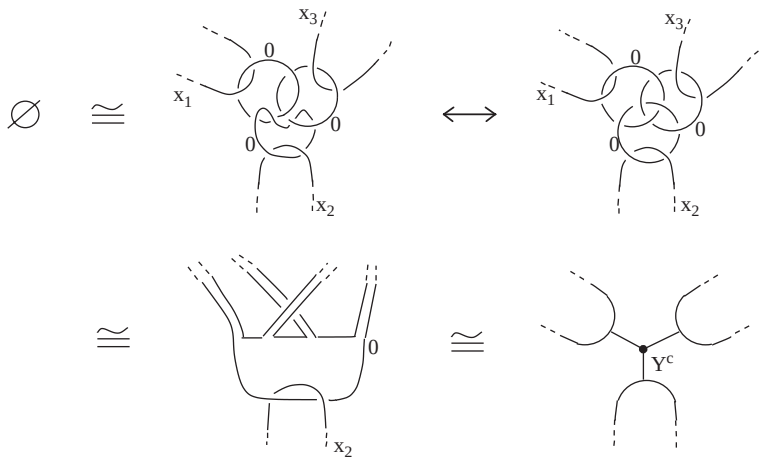


Fig. 10. A Y^c-surgery can be realized by a Δ^c-move.

Proof. As pointed out in Remark 3.5, a Y-surgery along G induces a bijection $\Theta_G : \mathrm{Spin}(M) \to \mathrm{Spin}(M_G)$, a combinatorial analog of which is given in [21]. Using the compatibility between Θ_G and Ω_G together with Section 2.2.3, we see that the claim holds at least for those Chern vectors that come from $\mathscr{S}_L$.

Denote by (H, f) the lattice corresponding to the intersection pairing of W_L, and by (H', f') that of $W_{L\cup K}$. Recall from Remark 2.2 that there are canonical isomorphisms $H^2(V_L) \simeq \mathrm{Coker}\, \widehat{f}$ and $H^2(V_{L\cup K}) \simeq \mathrm{Coker}\, \widehat{f'}$. The isomorphism $P\Phi_G P^{-1} : H^2(M) \to H^2(M_G)$ corresponds then to the isomorphism $\mathrm{Coker}\, \widehat{f} \to \mathrm{Coker}\, \widehat{f'}$ defined by $[y] \mapsto [(y, 0, 0)]$.

Take now $[s] \in \mathscr{V}_L$ arising from $\mathscr{S}_L$ and let $[y] \in \mathbb{Z}^n/\mathrm{Im}\, B_L \simeq \mathrm{Coker}\, \widehat{f}$. We aim to calculate $O_G([s] + [y]) \in \mathscr{V}_{L\cup K}$. The "+" here corresponds to the action of $H^2(V_L)$ on $\mathrm{Spin}^c(V_L)$ (see Remark 2.2). The map Ω_G being affine over $P\Phi_G P^{-1}$, we have that $O_G([s] + [y]) = O_G([s]) + [(y, 0, 0)] = [(s, x, 0)] + [(y, 0, 0)] = [(s + 2y, x, 0)]$. Therefore, the claim also holds for $[s] + [y] = [s + 2y]$. The transitivity of the action of $H^2(V_L)$ on $\mathrm{Spin}^c(V_L)$ allows us to conclude. □

Figs. 9 and 10 prove that, up to Spin^c-diffeomorphisms, a Δ^c-move can be realized by a Y^c-surgery and vice versa.

In Fig. 9, the first Spin^c-diffeomorphism is obtained by applying Claim 3.1, while the second one is obtained from one handle sliding and one slam dunk.

In Fig. 10, the first Spin^c-diffeomorphism is obtained from three slam dunks. Next, a Δ^c-move is applied. The second Spin^c-diffeomorphism is obtained by Spin^c Kirby's calculi (in particular, two slam dunks have been performed), and the last one is obtained from Claim 3.1. □

3.3. Proof of Theorem 2

In this subsection, we prove the characterization of the Y^c-equivalence relation, as announced in the introduction. We need two results concerning classification of quadratic functions up to isomorphism, proved in [5].

3.3.1. Isomorphism classes of quadratic functions

There is a natural notion of isomorphism among triples (H, f, c) defined by bilinear lattices with characteristic form (see Section 2.1): we say that two triples (H, f, c) and (H', f', c') are *isomorphic* if there is an isomorphism $\psi : H \to H'$ such that $f = f' \circ (\psi \times \psi)$ and $c = c' \circ \psi \bmod 2\widehat{f}(H)$. Such triples form a monoid for the orthogonal sum $\oplus$. Two triples (H, f, c) and (H', f', c') are said to be *stably equivalent* if they become isomorphic after stabilizations with some copies of $(\mathbb{Z}, \pm 1, \mathrm{Id})$, which denotes the bilinear lattice defined on $\mathbb{Z}$ by $(1, 1) \mapsto \pm 1$ and equipped with the characteristic form $\mathrm{Id} = \mathrm{Id}_{\mathbb{Z}}$. Note that, for any bilinear lattices (H, f) and (H', f'), there is a map

$$\psi \mapsto \psi^\sharp, \ \mathrm{Iso}(\mathrm{Coker}\,\widehat{f}, \ \mathrm{Coker}\,\widehat{f'}) \to \mathrm{Iso}(G_{f'}, G_f)$$

since the pairing (2.2) is right nonsingular.

Theorem 3.1 (*Deloup and Massuyeau [5]*). *Two bilinear lattices with characteristic form (H, f, c) and (H', f', c') are stably equivalent if, and only if, there exists an element*

$$\psi^\sharp \in \mathrm{Im}(\mathrm{Iso}(\mathrm{Coker}\,\widehat{f}, \mathrm{Coker}\,\widehat{f'}) \to \mathrm{Iso}(G_{f'}, G_f))$$

such that the associated quadratic functions $(G_f, \phi_{f,c})$ and $(G_{f'}, \phi_{f',c'})$ are isomorphic via $\psi^\sharp$. Furthermore, any such isomorphism between $(G_{f'}, \phi_{f',c'})$ and $(G_f, \phi_{f,c})$ lifts to a stable equivalence between (H, f, c) and (H', f', c').

Remark 3.6. Let Ψ be an isomorphism between $(G_{f'}, \phi_{f',c'})$ and $(G_f, \phi_{f,c})$ and suppose that f and f' are degenerate. Then, Ψ does not necessarily arise from an isomorphism $\psi : \mathrm{Coker}\,\widehat{f} \to \mathrm{Coker}\,\widehat{f'}$. In fact, it does if and only if $\Psi|_{\mathrm{Ker}\,\widehat{L_{f'}}} : \mathrm{Ker}\,\widehat{L_{f'}} \to \mathrm{Ker}\,\widehat{L_f}$ lifts to an isomorphism $\mathrm{Ker}\,\widehat{f'} \to \mathrm{Ker}\,\widehat{f}$. (See [5] for details.)

Let now $q : G \to \mathbb{Q}/\mathbb{Z}$ be a quadratic function on an Abelian group G. We shall say that q meets the *finiteness condition* if

- $G/\mathrm{Ker}\,\widehat{b_q}$ is finite,
- the extension G of $\mathrm{Ker}\,\widehat{b_q}$ by $G/\mathrm{Ker}\,\widehat{b_q}$ is split.

We shall also denote by r_q the homomorphism obtained by restricting q to $\mathrm{Ker}\,\widehat{b_q}$.

Theorem 3.2 (*Deloup and Massuyeau [5]*). *Two quadratic functions $q : G \to \mathbb{Q}/\mathbb{Z}$ and $q' : G' \to \mathbb{Q}/\mathbb{Z}$ satisfying the finiteness condition are isomorphic if, and only if, there is an isomorphism $\Psi : G' \to G$ such that $b_{q'} = b_q \circ (\Psi \times \Psi)$, $d_{q'} = d_q \circ \Psi$, $r_{q'} = r_q \circ \Psi|$ and $\gamma(q' \circ s') = \gamma(q \circ s)$ for some Ψ-compatible sections s and s' of the canonical epimorphisms $G \to G/\mathrm{Ker}\,\widehat{b_q}$ and $G' \to G'/\mathrm{Ker}\,\widehat{b_{q'}}$.*

Here, the Ψ-compatibility condition refers to the commutativity of the diagram

$$\begin{array}{ccc} G' & \xleftarrow{s'} & G'/\mathrm{Ker}\,\widehat{b_{q'}} \\ {\scriptstyle\Psi}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\downarrow{\scriptstyle[\Psi]} \\ G & \xleftarrow[s]{} & G/\mathrm{Ker}\,\widehat{b_q} \end{array}$$

where $[\Psi]$ is the isomorphism induced by Ψ.

Remark 3.7. Theorem 3.2 does not claim that $q' = q \circ \Psi$ if the four conditions hold. Nevertheless, as follows from the proof in [5], it is true that there exists an isomorphism $\varphi : G' \to G$ such that $q' = q \circ \varphi$ and $\varphi|_{\mathrm{Ker}\,\widehat{b_{q'}}} = \Psi|_{\mathrm{Ker}\,\widehat{b_{q'}}}$.

We now go into the proof of Theorem 2. In the sequel, we consider two closed connected 3-dimensional Spin^c-manifolds, (M, σ) and (M', σ').

3.3.2. Proof of the equivalence (2) $\Longleftrightarrow$ (3) *of Theorem 2*

Next lemma is easily proved from the definitions.

Lemma 3.3. *Let $\psi : H_1(M) \to H_1(M')$ be an isomorphism, which induces a dual isomorphism $\psi^\sharp : H_2(M'; \mathbb{Q}/\mathbb{Z}) \to H_2(M; \mathbb{Q}/\mathbb{Z})$ with respect to the intersection pairings. The following assertions are equivalent*:

(a) $L_{M'} = L_M \circ (\psi^\sharp \times \psi^\sharp)$,
(b) $\lambda_M = \lambda_{M'} \circ (\psi| \times \psi|)$,
(c) *The following diagram is commutative*:

$$\begin{array}{ccc} H_2(M'; \mathbb{Q}/\mathbb{Z}) & \xrightarrow{B} & \mathrm{Tors}\ H_1(M') \\ {\scriptstyle\psi^\sharp}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\psi|}\uparrow{\scriptstyle\simeq} \\ H_2(M; \mathbb{Q}/\mathbb{Z}) & \xrightarrow{B} & \mathrm{Tors}\ H_1(M). \end{array}$$

Suppose that condition (2) of Theorem 2 is satisfied. This implies that $L_{M'} = L_M \circ (\psi^\sharp \times \psi^\sharp)$ and so that $\lambda_M = \lambda_{M'} \circ (\psi| \times \psi|)$ by Lemma 3.3.

Condition (2) also implies the relation $d_{\phi_{M',\sigma'}} = d_{\phi_{M,\sigma}} \circ \psi^\sharp$ between homogeneity defects of quadratic functions. So, by Lemma 2.8, we have $\langle c(\sigma'), x' \rangle = \langle c(\sigma), \psi^\sharp(x') \rangle$ for all $x' \in H_2(M'; \mathbb{Q}/\mathbb{Z})$. By left nondegeneracy of the pairing $\bullet : H_1(M') \times H_2(M'; \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$, we conclude that $P^{-1}c(\sigma') = \psi(P^{-1}c(\sigma))$.

Last, the quadratic function

$$\phi_{M,\sigma} \circ s = \phi_{M,\sigma} \circ \psi^\sharp \circ s' \circ \psi| = \phi_{M',\sigma'} \circ s' \circ \psi|$$

is isomorphic to $\phi_{M',\sigma'} \circ s'$: hence, these two quadratic functions have identical Gauss sums. Therefore condition (3) holds.

Conversely, suppose that the condition (3) of Theorem 2 is satisfied. The short exact sequence

$$0 \longrightarrow H_2(M) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow H_2(M; \mathbb{Q}/\mathbb{Z}) \xrightarrow{B} \text{Tors } H_1(M) \longrightarrow 0$$

is split, we have that $H_2(M) \otimes \mathbb{Q}/\mathbb{Z} = \text{Ker } \widehat{L_M}$ and Tors $H_1(M)$ is finite: thus, $\phi_{M,\sigma}$ meets the finiteness condition of Section 3.3.1. Since $\lambda_M = \lambda_{M'} \circ (\psi| \times \psi|)$, we obtain by Lemma 3.3 that $L_{M'} = L_M \circ (\psi^\sharp \times \psi^\sharp)$. Since $\psi(P^{-1}c(\sigma)) = P^{-1}c(\sigma')$, we deduce from Lemmas 2.7 and 2.8 that $r_{\phi_{M',\sigma'}} = r_{\phi_{M,\sigma}} \circ \psi^\sharp|$ and that $d_{\phi_{M',\sigma'}} = d_{\phi_{M,\sigma}} \circ \psi^\sharp$, respectively. Also, since $\psi| \circ B \circ \psi^\sharp = B$ (by Lemma 3.3),the ψ-compatibility condition between s and s' required by condition (3) of Theorem 2 coincides with the $\psi^\sharp$-compatibility in the sense of Section 3.3.1. Therefore, by Theorem 3.2, the quadratic functions $\phi_{M,\sigma}$ and $\phi_{M',\sigma'}$ are isomorphic. More precisely, according to Remark 3.7, there exists an isomorphism $\varphi : H_2(M'; \mathbb{Q}/\mathbb{Z}) \to H_2(M; \mathbb{Q}/\mathbb{Z})$ such that $\phi_{M',\sigma'} = \phi_{M,\sigma} \circ \varphi$ and $\varphi|_{H_2(M') \otimes \mathbb{Q}/\mathbb{Z}}$ coincides with $\psi^\sharp|_{H_2(M') \otimes \mathbb{Q}/\mathbb{Z}} = \psi^\sharp \otimes \mathbb{Q}/\mathbb{Z}$. This latter fact, together with Remark 3.6, allows us to precise that φ equals $\eta^\sharp$ for a certain isomorphism $\eta : H_1(M) \to H_1(M')$. Consequently, $\phi_{M',\sigma'} = \phi_{M,\sigma} \circ \eta^\sharp$.

3.3.3. Proof of the equivalence (1) $\Longleftrightarrow$ (2) *of Theorem 2*

We prove implication (1) $\Longrightarrow$ (2) first. By Lemma 3.2, it suffices to prove it when (M, σ) and (M', σ') are related by one Spin^c-diffeomorphism or, for some fixed surgery presentations, by one Δ^c-move. The first case follows immediately from the definition of the linking quadratic function. The second case is deduced from the combinatorial formula for the latter given at the end of Section 2.3, and from the fact that a Δ-move between ordered oriented framed links preserve the linking matrices.

Suppose now that condition (2) is satisfied. We can assume that $M = V_L$ and $M' = V_{L'}$, where L and L' are ordered oriented framed links in $\mathbf{S}^3$. As in Section 2.2, we denote by (H, f) and (H', f') the intersection pairings of W_L and $W_{L'}$, respectively. Let also $c \in \text{Char}(f)$ and $c' \in \text{Char}(f')$ represent σ and σ', respectively. By hypothesis, the quadratic functions $\phi_{f,c} : G_f \to \mathbb{Q}/\mathbb{Z}$ and $\phi_{f',c'} : G_{f'} \to \mathbb{Q}/\mathbb{Z}$ are isomorphic via an isomorphism which is induced by an isomorphism $\text{Coker } \widehat{f} \to \text{Coker } \widehat{f'}$. So, by Theorem 3.1, the bilinear lattices with characteristic form (H, f, c) and (H', f', c') are stably equivalent.

An isomorphism of bilinear lattices with characteristic form can be topologically realized by a finite sequence of Spin^c Kirby's moves (see Theorem 2.2): handle slidings and reversings of orientation. Similarly, a stabilization by $(\mathbb{Z}, \pm 1, \text{Id})$ corresponds to a stabilization by the unknot. Therefore, we can suppose, without loss of generality, that $(H, f, c) \simeq (H', f', c')$ through the isomorphism that identifies the preferred basis of H with that of H'. Concretely, this means that the linking matrices B_L and $B_{L'}$ are equal and that there is a multi-integer s such that the Chern vectors $[s] \in \mathscr{V}_L$ and $[s] \in \mathscr{V}_{L'}$ represent σ and σ', respectively.

A theorem[2] of Murakami and Nakanishi [25, Theorem 1.1] states that two ordered oriented framed links have identical linking matrices if, and only if, they are Δ-equivalent. Then, the "decorated" links

[2] In fact, the first reference is [22] but the proof there is not detailed.

(L, s) and (L', s) are Δ-equivalent: therefore, by Lemma 3.2, the Spin^c-manifolds (M, σ) and (M', σ') are Y^c-equivalent.

Remark 3.8. Observe that the present proof allows for a more precise statement of the equivalence (1) $\Longleftrightarrow$ (2) of Theorem 2. Any finite sequence of Spin^c-diffeomorphisms and Y^c-surgeries

$$(M, \sigma) = (M_0, \sigma_0) \rightsquigarrow (M_1, \sigma_1) \rightsquigarrow (M_2, \sigma_2) \rightsquigarrow \cdots \rightsquigarrow (M_n, \sigma_n) = (M', \sigma')$$

yields an isomorphism $\psi : H_1(M) \to H_1(M')$. This is the composite of the isomorphisms $H_1(M_i) \to H_1(M_{i+1})$, which is taken to be either g_* if the step $(M_i, \sigma_i) \rightsquigarrow (M_{i+1}, \sigma_{i+1})$ is a Spin^c-diffeomorphism g, either the isomorphism Φ_G if the step is the Y^c-surgery along a Y-graph $G \subset M_i$ (Section 3.2.2). This isomorphism ψ satisfies $\phi_{M',\sigma'} = \phi_{M,\sigma} \circ \psi^\sharp$. Conversely, given an isomorphism $\psi : H_1(M) \to H_1(M')$ with this property, one can find a finite sequence of Spin^c-diffeomorphisms and Y^c-surgeries from (M, σ) to (M', σ') inducing ψ at the level of $H_1(-)$. Here, we use the second statement of Theorem 3.1.

3.4. Applications and problems

We conclude this paper with some applications of our results illustrated by a few examples. We also state a few problems.

3.4.1. The quotient set Spin$^c(M)/Y^c$

Given a closed oriented 3-manifold M, one may consider the quotient set

$$\mathrm{Spin}^c(M)/Y^c$$

of Spin^c-structures on M modulo the Y^c-equivalence relation. Let us consider a few examples.

Example 3.1. Take $M = \mathbb{R}\mathrm{P}^3$. This manifold has two distinct Spin^c-structures σ_0 and σ_1, both arising from Spin-structures. The quadratic functions ϕ_{M,σ_0} and ϕ_{M,σ_1} have different Gauss sums (which are $\exp(2i\pi/8)$ and $\exp(-2i\pi/8) \in \mathbb{C}$). Therefore, by Corollary 2, σ_0 is not Y^c-equivalent to σ_1.

Example 3.2. Take M such that $H_1(M) \simeq \mathbb{Z}^n$. According to Corollary 1, the set Spin$^c(M)/Y^c$ can be identified with $(2\mathbb{Z}^n)/\mathrm{GL}(n;\mathbb{Z})$ by the Chern class map.

In particular, if $M = \mathbf{S}^2 \times \mathbf{S}^1$ and if an isomorphism $H_1(M) \simeq \mathbb{Z}$ is fixed, we denote by α_k the unique element of Spin$^c(M)$ such that $c(\alpha_k) = 2k \in \mathbb{Z}$, with $k \in \mathbb{Z}$. Then, the Y^c-equivalence classes are $\{\alpha_0\}$ and $\{\alpha_k, \alpha_{-k}\}$ with $k > 0$. Observe from Theorem 2.2, that these classes coincide with the diffeomorphism classes.

Example 3.3. Take $M = (\mathbf{S}^2 \times \mathbf{S}^1)\sharp\mathbb{R}\mathrm{P}^3$. By applying equivalence (1) $\Leftrightarrow$ (2) of Theorem 2, the Y^c-equivalence classes are seen to be $\{\alpha_0\sharp\sigma_0\}$, $\{\alpha_0\sharp\sigma_1\}$, $\{\alpha_k\sharp\sigma_0,\ \alpha_k\sharp\sigma_1,\ \alpha_{-k}\sharp\sigma_0,\ \alpha_{-k}\sharp\sigma_1\}$ with $k > 0$ odd,$\{\alpha_k\sharp\sigma_0, \alpha_{-k}\sharp\sigma_0\}$ and $\{\alpha_k\sharp\sigma_1, \alpha_{-k}\sharp\sigma_1\}$ with $k > 0$ even. Again, observe from Theorem 2.2, that these classes coincide with the diffeomorphism classes.

In light of the previous examples, it is natural to ask whether the diffeomorphism classes of Spin^c-structures of a given closed oriented 3-manifold M coincide with the Y^c-equivalence classes. To answer this question by the negative, let us consider a class of manifolds for which the Spin^c-structures have

been classified: the family of lens spaces. Let $p \geqslant 2$ be an integer, let q_1, q_2 be some invertible elements of $\mathbb{Z}_p$ and let $L(p; q_1, q_2)$ be the corresponding lens space with the orientation induced from the canonical orientation of $\mathbf{S}^3$.

Theorem 3.3 (*Turaev [28]*). *The number of orbits of* Spin^c*-structures under the action of the group of positive self-diffeomorphisms of* $L(p; q_1, q_2)$ *is*

- $[p/2] + 1$, *if* $q_1^2 \neq q_2^2$ *or* $q_1 = \pm q_2$,
- $p/2 - b(p; q_1, q_2)/4 + c(p; q_1, q_2)/2$, *if* $q_1^2 = q_2^2$ *and* $q_1 \neq \pm q_2$.

Here, for $x \in \mathbb{Q}$, $[x]$ *denotes the greatest integer less or equal than* x, $b(p; q_1, q_2)$ *is the number of* $i \in \mathbb{Z}_p$ *for which* i, $q_1 + q_2 - i$ *and* $q_2 q_1^{-1} i$ *are pairwise different, and* $c(p; q_1, q_2)$ *is the number of* $i \in \mathbb{Z}_p$ *such that* $i = q_1 + q_2 - i = q_2 q_1^{-1} i$.

Proof. In [28, Section 9.2.1], the Euler structures on $L(p; q_1, q_2)$ are classified up to diffeomorphisms. The same kind of arguments can be used to classify these up to *positive* diffeomorphisms. Details are left to the reader. □

The classification of the Spin^c-structures on $L(p; q_1, q_2)$ up to Y^c-equivalence is easily obtained from Corollary 2. For instance, let us suppose that p is odd. Then, Spin$^c(L(p; q_1, q_2))/Y^c$ can be identified via the Chern class map with the quotient set $\mathbb{Z}_p/\sim$, where

$$\forall i, j \in \mathbb{Z}_p, \quad (i \sim j) \iff (\exists r \in \mathbb{Z}_p,\ r^2 = 1 \text{ and } j = ri).$$

Example 3.4. Let $k \geqslant 4$ be an even integer and let $p = k^2 - 1$. Then, there are some Spin^c-structures on $L(p; 1, 1)$ which are Y^c-equivalent but which are not diffeomorphic. Indeed, according to Theorem 3.3, Spin$^c(L(p; 1, 1))$ contains $(p-1)/2 + 1$ diffeomorphism classes. But, $k^2 = 1 \in \mathbb{Z}_p$ and $k \neq \pm 1 \in \mathbb{Z}_p$, so the cardinality of Spin$^c(L(p; 1, 1))/Y^c$ is strictly less than $(p-1)/2 + 1$.

3.4.2. Reidemeister–Turaev torsions

Let $\tau(M, \sigma)$ denote the maximal Abelian Reidemeister–Turaev torsion of a closed oriented 3-manifold M equipped with an Euler structure or, equivalently, a Spin^c-structure σ [32]. If M is a rational homology sphere, it turns out that $\phi_{M,\sigma}$ can be explicitely computed from $\tau(M, \sigma)$ [6,26]. Thus, according to Corollary 2, part of $\tau(M, \sigma)$ is of degree 0.

Problem 3.1. Derive from Reidemeister–Turaev torsions higher degree finite type invariants of closed 3-dimensional Spin^c-manifolds.

In the last chapter of [20], it is studied how Reidemeister–Turaev torsions vary under those twists defined in Section 3.2.1. This variation is difficult to control for a generic Y-graph. Nevertheless, this variation can be calculated explicitely in case of "looped clovers". It is shown that Reidemeister–Turaev torsions satisfy a certain multiplicative degree 1 relation involving surgeries along looped clovers.

3.4.3. From the Spin-refinement of the theory to its Spin^c*-refinement*

According to Remark 3.5, any Spin^c-invariant of degree d in the Goussarov–Habiro theory induces a Spin-invariant of degree d. The converse is not true.

Example 3.5. The Rochlin invariant $R(M,\sigma) \in \mathbb{Z}_{16}$ of a closed Spin-manifold (M,σ) of dimension 3 is a finite type invariant of degree 1 [21]. But, it does not lift to an invariant of Spin^c-manifolds in general. Indeed, consider the torus $\mathbf{T}^3$ and its canonical Spin-structure σ^0 (induced by its Lie group structure), choose also σ' in Spin($\mathbf{T}^3$) different from σ^0. Then, $\beta(\sigma')$ and $\beta(\sigma^0)$ coincide, but $R(\mathbf{T}^3,\sigma^0) = 8$ is not equal to $R(\mathbf{T}^3,\sigma') = 0$.

On the contrary, we have in degree 0 the following consequence of both Theorem 2 and [21, Theorem 1].

Corollary 3.1. *Let (M,σ) and (M',σ') be closed* 3*-dimensional* Spin*-manifolds. Then, (M,σ) and (M',σ') are distinguished by degree* 0 Spin*-invariants if and only if $(M,\beta(\sigma))$ and $(M',\beta(\sigma'))$ are distinguished by degree* 0 Spin^c*-invariants.*

Problem 3.2. Compare in higher degrees the Spin^c-refinement of the Goussarov–Habiro theory with its Spin-refinement.

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