

UNIVERSAL LOCALIZATION OF TRIANGULAR MATRIX RINGS

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ABSTRACT. If R is a triangular 2×2 matrix ring, the columns P and Q are f.g. projective R -modules. We describe the universal localization of R which makes invertible an R -module morphism $\sigma : P \rightarrow Q$, generalizing a theorem of A. Schofield. We also describe the universal localization of R -modules.

1. INTRODUCTION

Suppose R is an associative ring (with 1) and $\sigma : P \rightarrow Q$ is a morphism between finitely generated projective R -modules. There is a universal way to localize R in such a way that σ becomes an isomorphism. More precisely, there is a ring morphism $R \rightarrow \sigma^{-1}R$ which is universal for the property that

$$\sigma^{-1}R \otimes_R P \xrightarrow{1 \otimes \sigma} \sigma^{-1}R \otimes_R Q$$

is an isomorphism (Cohn [7, 9, 8, 6], Bergman [4, 5], Schofield [17]). Although it is often difficult to understand universal localizations when R is non-commutative¹ there are examples where elegant descriptions of $\sigma^{-1}R$ have been possible (e.g. Cohn and Dicks [10], Dicks and Sontag [11, Thm. 24], Farber and Vogel [12] Ara, González-Barroso, Goodearl and Pardo [1, Example 2.5]). The purpose of this paper is to describe and to generalize some particularly interesting examples due to A. Schofield [17, Thm. 13.1] which have application in topology (e.g., Ranicki [16, Part 2]).

We consider a triangular matrix ring $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ where A and B are associative rings (with 1) and M is an (A, B) -bimodule. Multiplication in R is given by

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix} = \begin{pmatrix} aa' & am' + mb' \\ 0 & bb' \end{pmatrix}$$

for all $a, a' \in A$, $m, m' \in M$ and $b, b' \in B$. The columns $P = \begin{pmatrix} A \\ 0 \end{pmatrix}$ and $Q = \begin{pmatrix} M \\ B \end{pmatrix}$ are f.g. projective left R -modules with

$$P \oplus Q \cong R.$$

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¹If R is commutative one obtains a ring of fractions; see Bergman [4, p. 68].

For the general theory of triangular matrix rings see Haghany and Varadarajan [13, 14].

We shall describe in Theorem 2.4 the universal localization $R \rightarrow \sigma^{-1}R$ which makes invertible a morphism $\sigma : P \rightarrow Q$. Such a morphism can be written $\sigma = \begin{pmatrix} j \\ 0 \end{pmatrix}$ where $j : A \rightarrow M$ is a morphism of left A -modules. Examples follow, in which restrictions are placed on A, B, M and σ . In particular, Example 2.8 recovers Theorem 13.1 of Schofield [17]. We proceed to describe the universal localization $\sigma^{-1}N = \sigma^{-1}R \otimes_R N$ of an arbitrary left module N for the triangular matrix ring R (see Theorem 2).

The structure of this paper is as follows: definitions, statements of results and examples are given in Section 2 and the proofs are collected in Section 3.

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2. STATEMENTS AND EXAMPLES

Let us first make more explicit the universal property of localization:

Definition 2.1. A ring morphism $R \rightarrow R'$ is called σ -invertible if

$$\text{id} \otimes \sigma : R' \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix} \rightarrow R' \otimes_R \begin{pmatrix} M \\ B \end{pmatrix}$$

is an isomorphism. The universal localization $i_\sigma : R \rightarrow \sigma^{-1}R$ is the initial object in the category of σ -invertible ring morphisms $R \rightarrow R'$. In other words, every σ -invertible ring morphism $R \rightarrow R'$ factors uniquely as a composite $R \rightarrow \sigma^{-1}R \rightarrow R'$.

Definition 2.2. An (A, M, B) -ring (S, f_A, f_M, f_B) is a ring S together with ring morphisms $f_A : A \rightarrow S$ and $f_B : B \rightarrow S$ and an (A, B) -bimodule morphism $f_M : M \rightarrow S$.

$$\begin{array}{ccc} A & \xrightarrow{f_A} & S & \xleftarrow{f_B} & B \\ & & f_M \uparrow & & \\ & & M & & \end{array}$$

It is understood that the (A, B) -bimodule structure on S is induced by f_A and f_B , so that $f_A(a)f_M(m) = f_M(am)$ and $f_M(m)f_B(b) = f_M(mb)$ for all $a \in A, b \in B$ and $m \in M$.

A morphism $(S, f_A, f_M, f_B) \rightarrow (S', f'_A, f'_M, f'_B)$ of (A, M, B) -rings is a ring morphism $\theta : S \rightarrow S'$ such that i) $\theta f_A = f'_A$, ii) $\theta f_M = f'_M$ and iii) $\theta f_B = f'_B$.

Definition 2.3. Suppose $p \in M$. Let $(T(M, p), \rho_A, \rho_M, \rho_B)$ denote the initial object in the subcategory of (A, M, B) -rings with the property $\rho_M(p) = 1$. For brevity we often write $T = T(M, p)$.

The ring T can be explicitly described in terms of generators and relations as follows. We have one generator x_m for each element $m \in M$ and relations:

- (+) $x_m + x_{m'} = x_{m+m'}$,
- (a) $x_{ap}x_m = x_{am}$,
- (b) $x_mx_{pb} = x_{mb}$,
- (id) $x_p = 1$,

for all $m, m' \in M$, $a \in A$ and $b \in B$. The morphisms ρ_A, ρ_M, ρ_B are

$$\begin{aligned} \rho_A &: A \rightarrow T; a \mapsto x_{ap}, \\ \rho_B &: B \rightarrow T; b \mapsto x_{pb}, \\ \rho_M &: M \rightarrow T; m \mapsto x_m. \end{aligned}$$

Suppose $\sigma : \begin{pmatrix} A \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} M \\ B \end{pmatrix}$ is a morphism of left R -modules. We may write $\sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix}$ for some $p \in M$. Let $T = T(M, p)$.

Theorem 2.4. *The universal localization $R \rightarrow \sigma^{-1}R$ is (isomorphic to)*

$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \xrightarrow{\begin{pmatrix} \rho_A & \rho_M \\ 0 & \rho_B \end{pmatrix}} \begin{pmatrix} T & T \\ T & T \end{pmatrix}.$$

- Example 2.5.** (1) Suppose $A = B = M$ and multiplication in A defines the (A, A) -bimodule structure on M . If $p = 1$, then $T = A$ and $\rho_A = \rho_M = \rho_B = \text{id}_A$.
 (2) Suppose $A = B$ and $M = A \oplus A$ with the obvious bimodule structure. If $p = (1, 0)$, then T is the polynomial ring $A[x]$ in a central indeterminate x . The map $\rho_A = \rho_B$ is the inclusion of A in $A[x]$ while $\rho_M(1, 0) = 1$ and $\rho_M(0, 1) = x$.

The universal localizations corresponding to Example 2.5 are

- (1) $\begin{pmatrix} A & A \\ 0 & A \end{pmatrix} \rightarrow \begin{pmatrix} A & A \\ A & A \end{pmatrix}$;
 (2) $\begin{pmatrix} A & A \oplus A \\ 0 & A \end{pmatrix} \rightarrow \begin{pmatrix} A[x] & A[x] \\ A[x] & A[x] \end{pmatrix}$.

Remark 2.6. One can regard the triangular matrix rings in these examples as path algebras over A for the quivers



The universal localizations $R \rightarrow \sigma^{-1}R$ are obtained by introducing an inverse to the arrow in (1) and by introducing an inverse to one of the arrows in (2). See for example Benson [2, p. 99] for an introduction to quivers.

The following examples subsume these.

- Example 2.7.** (1) (Amalgamated free product; Schofield [17, Thm. 4.10])
 Suppose $i_A : C \rightarrow A$ and $i_B : C \rightarrow B$ are ring morphisms and $M = A \otimes_C B$. If $p = 1 \otimes 1$, then T is the amalgamated free product $A \sqcup_C B$ and appears in the pushout square

$$\begin{array}{ccc} C & \xrightarrow{i_A} & A \\ i_B \downarrow & & \downarrow \rho_A \\ B & \xrightarrow{\rho_B} & T \end{array}$$

The map ρ_M is given by $\rho_M(a \otimes b) = \rho_A(a)\rho_B(b)$ for all $a \in A$ and $b \in B$. We recover part (1) of Example 2.5 by setting $A = B = C$ and $i_A = i_B = \text{id}$.

- (2) (HNN extension) Suppose $A = B$ and $i_1, i_2 : C \rightarrow A$ are ring morphisms. Let $A \otimes_C A$ denote the tensor product with C acting via i_1 on the first copy of A and by i_2 on the second copy. Let $M = A \oplus (A \otimes_C A)$ and $p = (1, 0 \otimes 0)$. Now $T = A *_C \mathbb{Z}[x]$ is generated by the elements in A together with an indeterminate x and has the relations in A together with $i_1(c)x = xi_2(c)$ for each $c \in C$. We have $\rho_A(a) = \rho_B(a) = a$ for all $a \in A$ while $\rho_M(1, 0 \otimes 0) = 1$ and $\rho_M(0, a_1 \otimes a_2) = a_1xa_2$. If $C = A$ and $i_1 = i_2 = \text{id}_A$, we recover part (2) of Example 2.5.

The following example is Theorem 13.1 of Schofield [17] and generalizes Example 2.7.

- Example 2.8.** (1) Suppose p generates M as a bimodule, i.e., $M = ApB$. Now T is generated by the elements of A and the elements of B subject to the relation $\sum_{i=1}^n a_i b_i = 0$ if $\sum_{i=1}^n a_i p b_i = 0$ (with $a_i \in A$ and $b_i \in B$). This ring T is denoted $A \sqcup_{(M,p)} B$ in [17, Ch. 13]. The maps ρ_A and ρ_B are obvious and ρ_M sends $\sum_i a_i p b_i$ to $\sum_i a_i b_i$.
- (2) Suppose $M = ApB \oplus N$ for some (A, B) -bimodule N . Now T is the tensor ring over $A \sqcup_{(M,p)} B$ of

$$(A \sqcup_{(M,p)} B) \otimes_A N \otimes_B (A \sqcup_{(M,p)} B).$$

We may vary the choice of p as the following example illustrates:

- Example 2.9.** Suppose $A = B = M = \mathbb{Z}$ and $p = 2$. In this case $T = \mathbb{Z}[\frac{1}{2}]$ and $\rho_A = \rho_B$ is the inclusion of \mathbb{Z} in $\mathbb{Z}[\frac{1}{2}]$ while $\rho_M(n) = n/2$ for all $n \in \mathbb{Z}$.

Example 2.9 can be verified by direct calculation using Theorem 2.4 or deduced from part (1) of Example 2.5 by setting $a_0 = b_0 = 2$ in the following more general proposition. Before stating it, let us remark that the universal property of $T = T(M, p)$ implies that $T(M, p)$ is functorial in (M, p) . An (A, B) -bimodule morphism $\phi : M \rightarrow M'$ with $\phi(p) = p'$ induces a ring morphism $T(M, p) \rightarrow T(M', p')$.

Proposition 2.10. *Suppose A and B are rings, M is an (A, B) -bimodule and $p \in M$. If $a_0 \in A$ and $b_0 \in B$ satisfy $a_0 m = m b_0$ for all $m \in M$, then:*

- (1) *The element $\rho_M(a_0 p) = x_{a_0 p} = x_{p b_0}$ is central in $T(M, p)$.*
- (2) *The ring morphism $\phi : T(M, p) \rightarrow T(M, a_0 p) = T(M, p b_0)$ induced by the bimodule morphism $\phi : M \rightarrow M; m \mapsto a_0 m = m b_0$ is the universal localization of $T(M, p)$ making invertible the element $x_{a_0 p}$.*

Since $x_{a_0 p}$ is central, each element in $T(M, a_0 p)$ can be written as a fraction α/β with numerator $\alpha \in T(M, p)$ and denominator $\beta = x_{a_0 p}^r$ for some nonnegative integer r .

Having described universal localization of the ring R in Theorem 2.4 we may also describe the universal localization $\sigma^{-1}R \otimes_R N$ of a left R -module N .

For the convenience of the reader let us first recall the structure of modules over a triangular matrix ring.

Lemma 2.11. *Every left R -module N can be written canonically as a triple*

$$(N_A, N_B, f : M \otimes_B N_B \rightarrow N_A)$$

where N_A is a left A -module, N_B is a left B -module and f is a morphism of left A -modules.

A proof of this lemma is included in Section 3 below. Localization of modules can be expressed as follows.

Theorem 2.12.² For any left R -module $N = (N_A, N_B, f)$ the localization left $\sigma^{-1}R$ -module $\sigma^{-1}N = \sigma^{-1}R \otimes_R N$ is isomorphic to $\begin{pmatrix} L \\ L \end{pmatrix}$ with $\sigma^{-1}R = M_2(T)$, $T = T(M, p)$, L the left T -module defined by

$$\begin{aligned} L &= (T \ T) \otimes_R N \\ &= \operatorname{coker} \left(\begin{pmatrix} 1 \otimes f \\ g \otimes 1 \end{pmatrix} : T \otimes_A M \otimes_B N_B \rightarrow (T \otimes_A N_A) \oplus (T \otimes_B N_B) \right) \end{aligned}$$

with g the (T, B) -bimodule morphism

$$g : T \otimes_A M \rightarrow T ; t \otimes m \mapsto -tx_m,$$

and $M_2(T)$ acting on the left of $\begin{pmatrix} L \\ L \end{pmatrix}$ by matrix multiplication.

3. PROOFS

The remainder of this paper is devoted to the proofs of Theorem 2.4, Proposition 2.10 and Theorem 2.

3.1. Localization as pushout. Before proving Theorem 2.4 we show that there is a pushout diagram

$$\begin{array}{ccc} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} & \longrightarrow & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \\ \alpha \downarrow & & \downarrow \\ R & \longrightarrow & \sigma^{-1}R \end{array}$$

where $\alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\alpha \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix}$. Bergman observed [4, p. 71] that more generally, up to Morita equivalence every localization $R \rightarrow \sigma^{-1}R$ appears in such a pushout diagram.

It suffices to check that the lower horizontal arrow in any pushout

$$\begin{array}{ccc} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} & \longrightarrow & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \\ \alpha \downarrow & & \theta \downarrow \\ R & \xrightarrow{i} & S \end{array}$$

is i) σ -inverting and ii) universal among σ -inverting ring morphisms. The universal property of a pushout will be shown to be the universal property of a universal localization, so that such a commutative diagram is a pushout if and only if S is a universal localization $\sigma^{-1}R$.

²This corrects Theorem 2.12 in the preprint version arXiv:math.RA/0407407.

i) The map $\text{id} \otimes \sigma : S \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix} \rightarrow S \otimes_R \begin{pmatrix} M \\ B \end{pmatrix}$ has inverse given by the composite

$$S \otimes_R \begin{pmatrix} M \\ B \end{pmatrix} \subset S \otimes_R R \cong S \xrightarrow{\gamma} S \cong S \otimes_R R \rightarrow S \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix}$$

where γ multiplies on the right by $\theta \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

ii) If $i' : R \rightarrow S'$ is a σ -inverting ring morphism, then there is an inverse $\psi : S' \otimes_R \begin{pmatrix} M \\ B \end{pmatrix} \rightarrow S' \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix}$ to $\text{id} \otimes \sigma$. It is argued shortly below that there is a (unique) diagram

$$(1) \quad \begin{array}{ccc} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} & \longrightarrow & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \\ \alpha \downarrow & & \downarrow \theta' \\ R & \xrightarrow{i} & S \\ & \searrow i' & \swarrow \psi \\ & & S' \end{array}$$

where θ' sends $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ to $\psi \left(1 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \in S' \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix} \subset S'$. Since S is a pushout, there is a unique morphism $S \rightarrow S'$ to complete the diagram and so i' factors uniquely through i .

To show uniqueness of (1), note that in S' multiplication on the right by $\theta' \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ must coincide with the morphism

$$\begin{pmatrix} 0 & 0 \\ \text{id} \otimes \sigma & 0 \end{pmatrix} : S' \otimes \begin{pmatrix} A \\ 0 \end{pmatrix} \oplus S' \otimes \begin{pmatrix} M \\ B \end{pmatrix} \longrightarrow S' \otimes \begin{pmatrix} A \\ 0 \end{pmatrix} \oplus S' \otimes \begin{pmatrix} M \\ B \end{pmatrix},$$

so multiplication on the right by $\theta' \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ coincides with $\begin{pmatrix} 0 & \psi \\ 0 & 0 \end{pmatrix}$. Now $1 \in S'$ may be written

$$\left(1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 1 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \in S' \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix} \oplus S' \otimes_R \begin{pmatrix} M \\ B \end{pmatrix}$$

so $\theta' \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \psi \left(1 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$. The reader may verify that this formula demonstrates the existence of a commutative diagram (1).

3.2. Identifying $\sigma^{-1}R$.

Proof of Theorem 2.4. It suffices to show that the diagram of ring morphisms

$$\begin{array}{ccc} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} & \longrightarrow & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \\ \alpha \downarrow & & \downarrow \\ \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} & \xrightarrow{\rho} & \begin{pmatrix} T & T \\ T & T \end{pmatrix} \end{array}$$

is a pushout, where $T = T(M, p)$, $\rho = \begin{pmatrix} \rho_A & \rho_M \\ 0 & \rho_B \end{pmatrix}$ and α is defined as in Section 3.1. Given a diagram of ring morphisms

$$(2) \quad \begin{array}{ccc} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} & \longrightarrow & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \\ \alpha \downarrow & & \downarrow \\ \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} & \xrightarrow{\rho} & \begin{pmatrix} T & T \\ T & T \end{pmatrix} \\ & \searrow \rho' & \nearrow \theta \\ & & S \end{array}$$

we must show that there is a unique morphism γ to complete the diagram. The map θ induces a decomposition of S as a matrix ring $M_2(S') = \begin{pmatrix} S' & S' \\ S' & S' \end{pmatrix}$ with S' the centralizer of $\theta(M_2(\mathbb{Z})) \subset S$. In particular, $\theta(e_{ij}) = e_{ij}$ for $i, j \in \{1, 2\}$. Any morphism γ which makes the diagram commute must be of the form $\gamma = M_2(\gamma')$ for some ring morphism $\gamma' : T \rightarrow S'$ (e.g. Cohn [9, p. 1] or Lam [15, (17.7)]). Commutativity of the diagram implies that ρ' also respects the 2×2 matrix structure and we may write

$$\rho' = \begin{pmatrix} \rho'_A & \rho'_M \\ 0 & \rho'_B \end{pmatrix} : \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \longrightarrow \begin{pmatrix} S' & S' \\ S' & S' \end{pmatrix}$$

with $\rho'_M(p) = 1$ as one sees by considering the images of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ under the maps in diagram (2) above. Since ρ' is a ring morphism, one finds

$$\begin{pmatrix} \rho'_A(aa') & \rho'_M(am' + mb') \\ 0 & \rho'_B(bb') \end{pmatrix} = \begin{pmatrix} \rho'_A(a)\rho'_A(a') & \rho'_A(a)\rho'_M(m') + \rho'_M(m)\rho'_B(b') \\ 0 & \rho'_B(b)\rho'_B(b') \end{pmatrix}$$

for all $a, a' \in A$, $b, b' \in B$ and $m, m' \in M$. Hence the maps $\rho'_A : A \rightarrow S'$ and $\rho'_B : B \rightarrow S'$ are ring morphisms and ρ'_M is a morphism of (A, B) -bimodules. Thus S' is an (A, M, B) -ring with respect to the maps $\rho'_A, \rho'_M, \rho'_B$ such that $\rho'_M(p) = 1$. By the universal property of T there exists a unique morphism $\gamma' : T \rightarrow S'$ such that $M_2(\gamma') : M_2(T) \rightarrow M_2(S') = S$ completes diagram (2) above. \square

Proof of Proposition 2.10. 1. In $T(M, p)$ we have $x_{a_0p}x_m = x_{a_0m} = x_{mb_0} = x_mx_{pb_0} = x_mx_{a_0p}$ for all $m \in M$.

2. The map $\phi : M \rightarrow M; m \mapsto a_0m$ induces

$$(3) \quad \begin{aligned} \phi : T(M, p) &\rightarrow T(M, a_0p), \\ x_m &\mapsto x_{a_0m}. \end{aligned}$$

In particular, $\phi(x_{a_0p}) = x_{a_0^2p} \in T(M, a_0p)$ and we have

$$x_{a_0^2p}x_p = x_{a_0(a_0p)}x_p = x_{a_0p} = 1 = x_{pb_0} = x_px_{pb_0^2} = x_px_{a_0^2p},$$

so $\phi(x_{a_0p})$ is invertible.

We must check that (3) is universal. If $f : T(M, p) \rightarrow S$ is a ring morphism and $f(x_{a_0p})$ is invertible, we claim that there exists a unique $\tilde{f} : T(M, a_0p) \rightarrow S$ such that $f\phi = \tilde{f}$.

Uniqueness: Suppose $\tilde{f}\phi = f$. For each $m \in M$ we have

$$\tilde{f}(x_{a_0m}) = \tilde{f}\phi(x_m) = f(x_m).$$

Now $f(x_{a_0p})\tilde{f}(x_m) = \tilde{f}\phi(x_{a_0p})\tilde{f}(x_m) = \tilde{f}(x_{a_0(a_0p)x_m}) = \tilde{f}(x_{a_0m}) = f(x_m)$, so

$$(4) \quad \tilde{f}(x_m) = (f(x_{a_0p}))^{-1}f(x_m).$$

Existence: It is straightforward to check that equation (4) provides a definition of \tilde{f} which respects the relations (+), (a), (b) and (id) in $T(M, a_0p)$. Relation (b), for example, is proved by the equations

$$\tilde{f}(x_m)\tilde{f}(x_{a_0pb}) = f(x_{a_0p})^{-1}f(x_m)f(x_{pb}) = f(x_{a_0p})^{-1}f(x_{mb}) = \tilde{f}(x_{mb})$$

and the other relations are left to the reader. □

3.3. Module localization. We turn finally to the universal localization $\sigma^{-1}R \otimes_R N$ of an R -module N .

Proof of Lemma 2.11. If N is a left R -module, set $N_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} N$ and set $N_B = N/N_A$. If $m \in M$ and $n_B \in N_B$ choose a lift $x \in N$ and define the map $f : M \otimes N_B \rightarrow N_A$ by $f(m \otimes n_B) = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} x$. Conversely, given a triple (N_A, N_B, f) one recovers a left R -module $\begin{pmatrix} N_A \\ N_B \end{pmatrix}$ with

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} n_A \\ n_B \end{pmatrix} = \begin{pmatrix} an_A + f(m \otimes n_B) \\ bn_B \end{pmatrix}$$

for all $a \in A, b \in B, m \in M, n_A \in N_A, n_B \in N_B$. □

Proof of Theorem 2. As in the statement, let $T = T(M, p)$ and define the left T -module

$$L = \text{coker} \left(\begin{pmatrix} 1 & f \\ g & 1 \end{pmatrix} : T \otimes_A M \otimes_B N_B \rightarrow (T \otimes_A N_A) \oplus (T \otimes_B N_B) \right).$$

We shall establish an isomorphism of left T -modules

$$(5) \quad (T \quad T) \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix} \cong L$$

and leave to the reader the straightforward deduction that there is an isomorphism of $\sigma^{-1}R$ -modules

$$\sigma^{-1}R \otimes_R N = \begin{pmatrix} T & T \\ T & T \end{pmatrix} \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix} \cong \begin{pmatrix} L \\ L \end{pmatrix}.$$

The left T -module morphism

$$\alpha : L \rightarrow (T \quad T) \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix};$$

$$(t \otimes n_A, t' \otimes n_B) \mapsto (t \quad 0) \otimes_R \begin{pmatrix} n_A \\ 0 \end{pmatrix} + (0 \quad t') \otimes_R \begin{pmatrix} 0 \\ n_B \end{pmatrix}$$

is well defined, since

$$\begin{aligned} &\alpha(t \otimes_A f(m, n_B), g(t, m) \otimes_B n_B) \\ &= \alpha(t \otimes_A f(m, n_B), -tx_m \otimes_B n_B) \\ &= (t \ 0) \otimes_R \begin{pmatrix} f(m, n_B) \\ 0 \end{pmatrix} - (0 \ tx_m) \otimes_R \begin{pmatrix} 0 \\ n_B \end{pmatrix} \\ &= (t \ 0) \otimes_R \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ n_B \end{pmatrix} - (t \ 0) \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \otimes_R \begin{pmatrix} 0 \\ n_B \end{pmatrix} \\ &= 0 \in (T \ T) \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix}. \end{aligned}$$

The left T -module morphism

$$\beta : (T \ T) \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix} \rightarrow L; (t \ t') \otimes_R \begin{pmatrix} n_A \\ n_B \end{pmatrix} \mapsto (t \otimes n_A, t' \otimes n_B)$$

is well defined, since

$$\begin{aligned} &\beta((t \ t') \otimes_R \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} n_A \\ n_B \end{pmatrix} - (t \ t') \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \otimes_R \begin{pmatrix} n_A \\ n_B \end{pmatrix}) \\ &= (t \otimes (an_A + f(m, n_b)), t' \otimes bn_B) - (ta \otimes n_A, (tx_m + t'b) \otimes n_B) \\ &= (t \otimes f(m, n_B), -tx_m \otimes n_B) \\ &= (1 \otimes f, g \otimes 1)(t \otimes m \otimes n_B) = 0 \in L. \end{aligned}$$

It is immediate that $\beta\alpha = \text{id}$. To prove (5) we must check that $\alpha\beta = \text{id}$ or in other words, that

$$(t \ t') \otimes_R \begin{pmatrix} n_A \\ n_B \end{pmatrix} = (t \ 0) \otimes_R \begin{pmatrix} n_A \\ 0 \end{pmatrix} + (0 \ t') \otimes_R \begin{pmatrix} 0 \\ n_B \end{pmatrix}.$$

This equation follows from the next two calculations:

$$\begin{aligned} (t \ 0) \otimes_R \begin{pmatrix} 0 \\ n_B \end{pmatrix} &= (t \ 0) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes_R \begin{pmatrix} 0 \\ n_B \end{pmatrix} = 0; \\ (0 \ t') \otimes_R \begin{pmatrix} n_A \\ 0 \end{pmatrix} &= (0 \ t') \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes_R \begin{pmatrix} n_A \\ 0 \end{pmatrix} = 0. \end{aligned}$$

□

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