Differential structures on a product of spheres: II*

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1. Introduction

In this paper we complete the classification of differential structures on a product of spheres that was begun in [2] (see Theorem 3.3 in § 3). At the same time we show that the pairing

$$\tau_{n,k}\colon \theta_n \otimes \pi_k(SO(n-1)) \longrightarrow \theta_{n+k} \qquad (n \neq 3)$$

of Milnor-Munkres-Novikov is zero if $k \geqq n-3$ (see Theorem 3.1 in § 3). These two results are related through the pairing

$$\rho_{n,k}\colon \theta_n \otimes \pi_{n+k}(S^n) \longrightarrow \theta_{n+k} \qquad \text{for } k < n-1$$

that was defined recently by G. Bredon [1] and which generalized $\tau_{n,k}$ (see Lemma 2.2 below). Here θ_n is the group of **h**-cobordism classes of homotopy n-spheres under the connected sum operation $+$, and S^n denotes the standard unit n-sphere in real $(n+1)$-space R^{n+1}. In particular, we show the following. Let A^n be a homotopy n-sphere, and let $I(S^k \times A^n)$ denote the set of diffeomorphism classes of those homotopy $(n+k)$-spheres V^{n+k} such that $(S^k \times A^n) + V^{n+k}$ is diffeomorphic to $S^k \times A^n$. We assume that $n + k \geqq 5$. Then $I(S^k \times A^n)$ is the cyclic subgroup of θ_{n+k} such that

$$I(S^k \times A^n) = \tau_{n,k}(A^n \otimes \pi_k(SO(n-1))) \,.$$

Thus if $k < n-1$, then $I(S^k \times A^n)$ may be described as the set of elements of the form $\rho_{n,k}(A^n \otimes x)$, where x lies in the image of the Hopf-Whitehead homomorphism $J\colon \pi_k(SO(n)) \to \pi_{n+k}(S^n)$. If either $k \geqq n-3$ or $k \equiv 2,4,5,6 \pmod 8$, then $I(S^k \times A^n) = 0$, and hence the pairing $\tau_{n,k}$ is trivial in these dimensions. On the other hand, the pairing $\tau_{n,k}$ corresponds to composition in the stable homotopy groups of spheres. Thus for example, if A^{14} represents the non-zero element of $\theta_{14} \approx Z_2$, then $I(S^3 \times A^{14})$ is of order two in the group θ_{17} of order sixteen.

This result is of interest in the following connection. Let M be a manifold that is homeomorphic to a product of standard spheres $S^k \times S^n$ such that $2 \leqq k \leqq n$ and $n + k \geqq 6$. In [2] we showed that M is diffeomorphic to

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$(S^k \times A^n) + V^{n+k}$, where A^n and V^{n+k} are homotopy spheres. If $n-3 \leqq k \leqq n$, then M is diffeomorphic to $(S^k \times S^n) + V^{n+k}$ and V^{n+k} is unique (see [2, Th. 1]). More generally, it was shown that if B^n and U^{n+k} are homotopy spheres such that M is also diffeomorphic to $(S^k \times B^n) + U^{n+k}$, then $S^k \times A^n$ and $S^k \times B^n$ are diffeomorphic. Furthermore, if $k \equiv 2, 4, 5, 6 \pmod 8$, then V^{n+k} is diffeomorphic to U^{n+k}. This last conclusion is not true precisely when the pairing $\tau_{n,k}\colon \theta_n \otimes \pi_k(SO(n-1)) \to \theta_{n+k}$ is non-trivial. In fact, it is shown here that if A^n and V^{n+k} are homotopy spheres, then $(S^k \times A^n) + V^{n+k}$ is diffeomorphic to $S^k \times A^n$ if and only if there exists an element $\alpha \in \pi_k(SO(n-1))$ such that

$$V^{n+k} = \tau_{n,k}(A^n \otimes \alpha) .$$

As an example we show that $S^3 \times S^{14}$ has exactly twenty-four differential structures. In general the complete classification of differential structures on $S^k \times S^n$ (up to diffeomorphism) is given by Theorem 3.3 in § 3.

2. The pairings

Differentiable means of class C^∞ here, and diffeomorphisms are orientation preserving. We begin by giving the usual construction of the pairing $\tau_{n,k}\colon \theta_n \otimes \pi_k(SO(n-1)) \to \theta_{n+k}$ for $n \neq 3,4$. Let A^n represent an element of θ_n, and let $\alpha\colon S^k \to SO(n-1)$ be a differentiable map that represents an element of $\pi_k(SO(n-1))$. It is known that A^n may be represented by a diffeomorphism $g'\colon R^{n-1} \to R^{n-1}$ that is the identity outside of some compact set. Define diffeomorphisms F' and G' of $S^k \times R^{n-1}$ by writing

$$F'(u, v) = (u, \alpha(u)\cdot v) \qquad \text{and} \qquad G'(u, v) = (u, g'(v)) .$$

Here $\alpha(u)\cdot v$ denotes the action of the rotation group $SO(n-1)$ on R^{n-1}. Then $F'^{-1}G'F'$ is a diffeomorphism of $S^k \times R^{n-1}$ that is the identity outside of some compact set. Furthermore, $F'^{-1}G'F'$ induces a diffeomorphism of S^{n+k-1} by embedding $S^k \times R^{n-1}$ in S^{n+k-1} in the standard way, sending $(u, v) \in S^k \times R^{n-1}$ into $(u,v)/||(u,v)||$. Then this diffeomorphism of S^{n+k-1} represents $\tau_{n,k}(A^n \otimes \alpha)$. It can be shown by standard arguments that this correspondence defines a pairing. In fact, this follows by application of Lemma 2.2 below and [1, Prop. 1.2].

Now we give an interpretation of $\tau_{n,k}(A^n \otimes \alpha)$ that will be useful here. Up to diffeomorphisms of S^{n-1} that extend to diffeomorphisms of the unit n-disc D^n, the class of A^n in θ_n is completely determined by a diffeomorphism $g\colon S^{n-1} \to S^{n-1}$. Specifically, A^n is diffeomorphic to $D_1^n \cup_g D_2^n$, the disjoint union of two copies of the n-disc with $u \in \partial D_2^n$ and $g(u) \in \partial D_1^n$ identified and with the orientation of D_2^n. In fact, we can take g to be that diffeomorphism induced by the diffeomorphism $g'\colon R^{n-1} \to R^{n-1}$ of the preceding paragraph *via* the standard embedding of R^{n-1} in S^{n-1} (stereographic projection from the south pole of S^{n-1}). We can assume that g is the identity on the southern hemisphere D_-^{n-1}. Now let

$$s\colon SO(n-1) \subset SO(n)$$

denote the natural inclusion, and define diffeomorphisms F and G of $S^k \times S^{n-1}$ by writing

$$F(u, v) = \big(u, s\alpha(u)\cdot v\big) \qquad \text{and} \qquad G(u, v) = \big(u, g(v)\big)\ .$$

Let $v_0 = (0, \cdots, 0, -1)$ denote the south pole of S^{n-1}. Since each element of $SO(n-1)$ leaves v_0 fixed it follows that F is the identity on the k-sphere $S^k \times v_0$. Furthermore, there is a closed $(n-1)$-disc neighborhood B^{n-1} of v_0 in D_-^{n-1} such that

$$F(S^k \times B^{n-1}) \subset S^k \times D_-^{n-1}\ .$$

Since G is the identity on $S^k \times D_-^{n-1}$ it follows that $F^{-1}GF$ is the identity on $S^k \times B^{n-1}$. Thus if B^{n+k} is a small $(n+k)$-disc embedded in the interior of $D^{k+1} \times S^{n-1}$, then it follows that $F^{-1}GF$ may be extended to a diffeomorphism of $D^{k+1} \times S^{n-1} - \operatorname{Int} B^{n+k}$. This gives rise to a diffeomorphism of the $(n+k-1)$-sphere ∂B^{n+k} and hence determines an element of θ_{n+k}. It is not hard to see that this element is $\tau_{n,k}(A^n \otimes \alpha)$. We state this as follows.

LEMMA 2.1. *$\tau_{n,k}(A^n \otimes \alpha)$ is the obstruction to extending the diffeomorphism $F^{-1}GF$ of $S^k \times S^{n-1}$ to one of $D^{k+1} \times S^{n-1}$.*

The pairing $\rho_{n,k}\colon \theta_n \otimes \pi_{n+k}(S^n) \to \theta_{n+k}$ of Bredon [1] is constructed as follows for $n > 4$. Let A^n be a homotopy n-sphere and let $x \in \pi_{n+k}(S^n)$. Represent A^n by a diffeomorphism $g\colon S^{n-1} \to S^{n-1}$. By means of the Pontrjagin-Thom construction we can represent $x \in \pi_{n+k}(S^n)$ by a framed k-manifold X^k in S^{n+k}. Let

$$f\colon X^k \times D^n \longrightarrow S^{n+k}$$

be a corresponding product representation of a tubular neighborhood of X^k in S^{n+k}, and define a diffeomorphism G of $X^k \times S^{n-1}$ by the equation $G(u, v) = \big(u, g(v)\big)$. Now we define a homotopy $(n+k)$-sphere by writing

$$\big(S^{n+k} - f(X^k \times \operatorname{Int} D^n)\big) \cup_{fG} (X^k \times D^n)\ .$$

This represents an element $\rho_{n,k}(A^n \otimes x)$ of θ_{n+k}, and one must show that this element is independent of the choices made. Bredon shows that the function $\rho_{n,k}$ is linear in the second variable for all integers n and k, and that $\rho_{n,k}$ is linear in the first variable provided that x is a suspension element. Thus if

$k < n - 1$, then $\rho_{n,k}$ is bilinear. The reader is referred to [1] for details.

Let $J\colon \pi_k(SO(n)) \to \pi_{n+k}(S^n)$ denote the Hopf-Whitehead homomorphism, and let $s_*\colon \pi_k(SO(n-1)) \to \pi_k(SO(n))$ be induced by the inclusion.

LEMMA 2.2. *The following diagram is commutative* ($n \neq 3$):

$$\begin{array}{ccc} \theta_n \otimes \pi_k(SO(n-1)) & & \\ \Big\downarrow{\scriptstyle 1\otimes(J\circ s_*)} & \searrow^{\pm\tau_{n,k}} & \\ & & \theta_{n+k} \\ & \nearrow_{\rho_{n,k}} & \\ \theta_n \otimes \pi_{n+k}(S^n) & & \end{array} .$$

Remark. The ambiguity $\pm\tau_{n,k}$ in Lemma 2.2 can be removed by establishing orientation conventions in the definition of the Hopf-Whitehead homomorphism J.

PROOF. We write S^{n+k} in the form

(1) $$S^{n+k} = (D^{k+1} \times S^{n-1}) \cup_1 (S^k \times D^n) ,$$

and give it the orientation of $S^k \times D^n$. Let $g\colon S^{n-1} \to S^{n-1}$ be a diffeomorphism that represents an element A^n of θ_n, and let $\alpha\colon S^k \to SO(n-1)$ be a differentiable map. Then $\pm\rho_{n,k}(A^n \otimes Js_*(\alpha))$ is represented by the homotopy $(n+k)$-sphere

(2) $$(D^{k+1} \times S^{n-1}) \cup_{F^{-1}G} (S^k \times D^n) ,$$

where F and G are diffeomorphisms of $S^k \times S^{n-1}$ defined by the equations

(3) $$F(u, v) = (u, s\alpha(u)\cdot v) \quad \text{and} \quad G(u, v) = (u, g(v)) .$$

The homotopy sphere (2) is given the orientation of $S^k \times D^n$. Now we attempt to define a diffeomorphism from (1) to (2). First we map $(u, v) \in S^k \times D^n$ of (1) into $F(u, v) \in S^k \times D^n$ of (2) and then try to extend this to a diffeomorphism of the complements $D^{k+1} \times S^{n-1}$. The diffeomorphism on the boundary of $D^{k+1} \times S^{n-1}$ is clearly $F^{-1}GF$ and by Lemma 1, the obstruction to extending this to a diffeomorphism of $D^{k+1} \times S^{n-1}$ is $\tau_{n,k}(A^n \otimes \alpha)$. It follows that the obstruction to extending our mapping to a diffeomorphism from (1) to (2) is $(-1)^{k+1}\tau_{n,k}(A^n \otimes \alpha)$ since the orientation of $D^{k+1} \times S^{n-1}$ is changed by $(-1)^{k+1}$ in both (1) and (2). This implies that (2) represents $(-1)^k\tau_{n,k}(A^n \otimes \alpha)$, completing the proof.

3. Relationship to the action of θ_{n+k}

In this section we prove the following.

THEOREM 3.1. *Let A^n be a homotopy n-sphere and let V^{n+k} be a homotopy*

$(n+k)$-sphere such that $n+k \geqq 6$ and $k \geqq 1$. Then $(S^k \times A^n) + V^{n+k}$ is diffeomorphic to $S^k \times A^n$ if and only if there exists an element $\alpha \in \pi_k(SO(n-1))$ such that $V^{n+k} = \tau_{n,k}(A^n \otimes \alpha)$. In particular, if $k \geqq n-3$, then $\tau_{n,k} = 0$.

PROOF. Suppose that $(S^k \times A^n) + V^{n+k}$ is diffeomorphic to $S^k \times A^n$. It was proved in [2, Lem. 2] that if $k \geqq n-3$, then V^{n+k} is diffeomorphic to S^{n+k}. Therefore we can assume that $k < n-1$. Since $\theta_6 = 0$, we can further assume that $n > 4$. Now let $g: S^{n-1} \to S^{n-1}$ be a diffeomorphism that represents A^n. It was shown in the proof of [2, Lem. 2] that there is a differentiable map $\alpha: S^k \to SO(n)$ such that V^{n+k} is diffeomorphic to either

$$(D^{k+1} \times S^{n-1}) \cup_{F^{-1}G} (S^k \times D^n)$$

or

$$(D^{k+1} \times S^{n-1}) \cup_{F^{-1}G^{-1}} (S^k \times D^n) ,$$

where F and G are diffeomorphisms of $S^k \times S^{n-1}$ defined by the equations $F(u, v) = (u, \alpha(u) \cdot v)$ and $G(u, v) = (u, g(v))$. Since $k < n-1$, it follows that

$$s_*: \pi_k(SO(n-1)) \longrightarrow \pi_k(SO(n))$$

is surjective, and hence we can assume that α represents an element in $\pi_k(SO(n-1))$. Now it follows from Lemma 2.2 that $V^{n+k} = \pm\tau_{n,k}(A^n \otimes \alpha)$.

Conversely, suppose that $\alpha: S^k \to SO(n-1)$ is a differentiable map such that $V^{n+k} = (-1)^k \tau_{n,k}(A^n \otimes \alpha)$. Now we refer to the proof of Lemma 2.2 above. First we note that V^{n+k} is diffeomorphic to the manifold (2), where F and G are diffeomorphisms of $S^k \times S^{n-1}$ defined by (3). We define a mapping from (1) to (2) by sending $(u, v) \in D^{k+1} \times S^{n-1}$ of (1) into $(u, g(v)) \in D^{k+1} \times S^{n-1}$ of (2) and try to extend this to a diffeomorphism of the complements $S^k \times D^n$. The diffeomorphism on the boundary $(-1)^k(S^k \times S^{n-1})$ of $S^k \times D^n$ is $G^{-1}FG$. Now we can assume that α maps the southern hemisphere D^k_- of S^k into the identity of $SO(n-1)$, and hence it follows that $G^{-1}FG$ is the identity on $D^k_- \times S^{n-1}$. Thus if B^{n+k} is a small $(n+k)$-disc embedded in the interior of $S^k \times D^n$, then it follows that $G^{-1}FG$ can be extended to a diffeomorphism of $S^k \times D^n -$ Interior B^{n+k}. This induces a diffeomorphism of the $(n+k-1)$-sphere ∂B^{n+k} which must represent $(-1)^{k+1}\tau_{n,k}(A^n \otimes \alpha)$. We state this result as follows.

LEMMA 3.2. *The obstruction to extending the diffeomorphism $G^{-1}FG$ of $(-1)^k(S^k \times S^{n-1}) = \partial(S^k \times D^n)$ to one of $S^k \times D^n$ is $(-1)^{k+1}\tau_{n,k}(A^n \otimes \alpha)$.*

Now we can write $S^k \times A^n$ in the form

$$S^k \times D^n_1 \cup_G S^k \times D^n_2 , \tag{4}$$

where D_1^n and D_2^n are two copies of the n-disc, and $(u, v) \in S^k \times \partial D_2^n$ is identified with $G(u, v) \in S^k \times \partial D_1^n$. The correspondence $(u, v) \longrightarrow F(u, v)$ defines a diffeomorphism of $S^k \times D_1^n$ and we try to extend this to a diffeomorphism of the manifold (4). The diffeomorphism induced on the boundary of $S^k \times D_2^n$ is clearly $G^{-1}FG$ and hence we can apply Lemma 3 to extend to a diffeomorphism

$$S^k \times A^n \longrightarrow (S^k \times A^n) + V^{n+k} ,$$

as desired.

The final statement in the theorem now follows from [2, Lem. 2].

We conclude this section with some remarks that relate the above theorem to the classification of differential structures on $S^k \times S^n$ up to diffeomorphism. Let Φ_n^{k+1} be the subgroup of θ_n consisting of those homotopy n-spheres that embed in R^{n+k+1} with a trivial normal bundle. It is known that if $n > 4$ and $k \geqq 1$, then Φ_n^{k+1} contains the subgroup bP_{n+1} of those homotopy n-spheres that bound parallelizable manifolds. If, in addition, $n < 2k + 1$ and $n \neq 2^a - 2$, then we have the exact sequence of Hsiang-Levine-Szczarba:

$$0 \longrightarrow bP_{n+1} \longrightarrow \Phi_n^{k+1} \xrightarrow{\ \psi\ } \frac{\pi_{n+k+1}(S^{k+1})}{J_n^{k+1}} \longrightarrow 0 ,$$

where J_n^{k+1} is the image of the Hopf-Whitehead homomorphism, and ψ is the homomorphism defined *via* the Pontrjagin-Thom construction. Finally, let A_*^n denote the class of A^n in θ_n/Φ_n^{k+1}.

First we consider manifolds of the form $S^k \times A^n$ such that A^n is a homotopy n-sphere, $n \geqq 5$, and $k \geqq 2$. It was shown in [2, Th. 2] that two such manifolds $S^k \times A^n$ and $S^k \times B^n$ are diffeomorphic if and only if $A_*^n = \pm B_*^n$. It was also shown in [2, Lem. 2] that θ_{n+k} acts non-trivially on the usual differential structure of $S^k \times S^n$ and hence it follows from Theorem 3.1 above that the pairing $\tau_{n,k}$ induces a pairing

$$(5) \qquad \tau'_{n,k}\colon (\theta_n/\Phi_n^{k+1}) \otimes \pi_k(SO(n-1)) \longrightarrow \theta_{n+k} \qquad (k \geqq 2) .$$

Now let M be a manifold that is homeomorphic to $S^k \times S^n$ such that $n + k \geqq 6$ and $n \geqq k \geqq 2$. We recall that there are homotopy spheres A^n and V^{n+k} such that M is diffeomorphic to $(S^k \times A^n) + V^{n+k}$ (cf. [2, Th. 1]). If M is also diffeomorphic to $(S^k \times B^n) + U^n$, then $S^k \times A^n$ and $S^k \times B^n$ are diffeomorphic (cf. [2, Lem. 3]). Thus we can state the following classification theorem.

THEOREM 3.3. *Let $S^k \times S^n$ be a product of standard spheres such that $2 \leqq k \leqq n$ and $n + k \geqq 6$. Then, under the relation of orientation preserving diffeomorphism, the diffeomorphism classes of manifolds homeomorphic to $S^k \times S^n$ are in a one-to-one correspondence with the equivalence classes on the set $(\theta_n/\Phi_n^{k+1}) \times \theta_{n+k}$, where (A_*^n, V^{n+k}) and (B_*^n, U^{n+k}) are equivalent* *if and only if* $A_*^n = \pm B_*^n$ *and there exists* $\alpha \in \pi_k(SO(n-1))$ *such that* $\tau'_{n,k}(A_*^n \otimes \alpha) = V^{n+k} - U^{n+k}$.

In the next section we compute the number of differential structures on $S^3 \times S^{14}$ and $S^3 \times S^{10}$; these correspond to the non-trivial pairings $\tau_{14,3}$ and $\tau_{10,3}$. On the other hand, if either $k \equiv 2,4,5,6 \pmod 8$ or $n \geqq k \geqq n-3$, then the equivalence relation on $(\theta_n/\Phi_n^{k+1}) \times \theta_{n+k}$ collapses to (A_*^n, V^{n+k}) is equivalent to (B_*^n, U^{n+k}) if and only if $A_*^n = \pm B_*^n$ and $V^{n+k} = U^{n+k}$.

4. Two calculations in the case where $\tau_{n,k}$ is non-trivial

In this section we indulge in some numbers to show that there are exactly twenty-four distinct (non-diffeomorphic) differential structures on $S^3 \times S^{14}$. We also show that there are $4d$ differential structures on $S^3 \times S^{10}$, where d is ambiguously 1 or 2.

In what follows Π_q denotes the stable q-stem $\pi_{q+m}(S^m)$ for $m > q+1$, and J_q denotes the image of the stable J-homomorphism $J\colon \pi_q(SO(m)) \to \pi_{q+m}(S^m)$ for $m > q+1$. Let

$$p'\colon \theta_n \longrightarrow \Pi_n/J_n$$

be the homomorphism of Kervaire-Milnor defined *via* the Pontrjagin-Thom construction. It is known that p' is surjective if $n \neq 2^a - 2$, and that its kernel bP_{n+1} is zero if n is even. Now if $k < n-1$, then the following diagram is commutative, where the bottom map is induced by the composition $\Pi_n \otimes \Pi_k \to \Pi_{n+k}$:

$$\begin{array}{ccc} \theta_n \otimes \pi_k(SO(n-1)) & \xrightarrow{\pm\tau_{n,k}} & \theta_{n+k} \\ {\scriptstyle p'\otimes(J\circ s_*)}\downarrow & & \downarrow{\scriptstyle p'} \\ (\Pi_n/J_n)\otimes\Pi_k & \longrightarrow & \Pi_{n+k}/J_{n+k} \end{array} \tag{6}$$

We believe that this is originally due to Milnor and follows from [1, Cor. 2.2] and Lemma 2.2 above.

First consider the case where $n = 10$ and $k = 3$. It is known that $\theta_{10} \approx Z_6$, $\pi_3(SO(9)) \approx Z$, $\theta_{13} \approx Z_3$, $\Pi_3 \approx Z_{24}$, and $J \circ s_*\colon \pi_3(SO(9)) \to \Pi_3$ is surjective. Furthermore, the groups bP_{11}, bP_{14}, J_{10} and J_{13} are zero and hence we have isomorphisms

$$p'\colon \theta_{10} \approx \Pi_{10} \qquad \text{and} \qquad p'\colon \theta_{13} \approx \Pi_{13}\,.$$

Thus we have a commutative diagram

$$\begin{array}{ccc} \theta_{10} \otimes \pi_3(SO(9)) & \xrightarrow{\pm\tau_{10,3}} & \theta_{13} \\ {\scriptstyle p'\otimes(J\circ s_*)}\downarrow & & \approx\downarrow{\scriptstyle p'} \\ \Pi_{10}\otimes\Pi_3 & \longrightarrow & \Pi_{13} \end{array}$$

Now according to Toda [4, p. 189] the composition $\Pi_{10} \otimes \Pi_3 \rightarrow \Pi_{13}$ is non-trivial and hence, if Σ^{10} is a generator of $\theta_{10} \approx Z_6$ and α is a generator of $\pi_3(SO(9))$, then $\tau_{10,3}(\Sigma^{10} \otimes \alpha)$ generates $\theta_{13} \approx Z_3$. It follows that Φ^4_{10} is either zero or of order two. If Φ^4_{10} is zero, then it follows that $S^3 \times S^{10}$ has exactly eight differential structures. If Φ^4_{10} is of order two, then θ_{10}/Φ^4_{10} is of order three and $S^3 \times S^{10}$ has four differential structures. The ambiguity exists here because we do not know if $3\Sigma^{10}$ embeds in R^{14} with a trivial normal bundle. In fact, it is not known if $3\Sigma^{10}$ embeds in R^{14} at all (cf. [3, p. 48]) In either case it follows that neither Σ^{10} nor $2\Sigma^{10}$ embeds in R^{14} with a trivial normal bundle. Actually, Levine [3] has shown that these homotopy 10-spheres do not even embed in R^{14}.

Now we consider the case where $n = 14$ and $k = 3$. We note that $\theta_{14} \approx Z_2$ and that θ_{17} has order sixteen. Furthermore, the groups bP_{15} and J_{14} are zero whereas the groups bP_{18} and J_{17} are each of order two. Thus we have a monomorphism $p' \colon \theta_{14} \rightarrow \Pi_{14} \approx Z_2 + Z_2$. According to Toda [4, p. 189] Π_{14} has two generators, a composition $\sigma^2 = \sigma \circ \sigma$ and an element κ, where σ is of order sixteen in $\Pi_7 \approx Z_{240}$. Furthermore, if λ is a generator of $\Pi_3 \approx Z_{24}$, then Toda shows that the composition $\lambda \circ \sigma^2 = 0$ whereas $\lambda \circ \kappa$ is a non-zero element of $\Pi_{17} \approx Z_2 + Z_2 + Z_2 + Z_2$ that does not lie in J_{17}. Now we show that if Σ^{14} represents the non-zero element of $\theta_{14} \approx Z_2$, then $p'(\Sigma^{14}) \neq \sigma^2$. In fact, it is known that σ^2 is represented by the framed manifold $(S^7, \varphi) \times (S^7, \varphi)$, where φ is the framing defined as follows. Consider the standard sphere $S^7 \subset R^{m+7}$ (m large) and let φ denote the framing of the normal bundle of S^7 that corresponds to a generator of $\pi_7(SO(m)) \approx Z$. Since $J \colon \pi_7(SO(m)) \rightarrow \Pi_7 \approx Z_{240}$ is surjective, it follows that (S^7, φ) represents a generator of Π_7 *via* the Pontrjagin-Thom construction. Furthermore, according to Milnor $(S^7, \varphi) \times (S^7, \varphi)$ has *Arf* invariant one and represents σ^2. On the other hand, since $\pi_{14}(SO) = 0$, Σ^{14} has an essentially unique framing ψ such that the framed manifold (Σ^{14}, ψ) represents $p'(\Sigma^{14})$. Moreover, (Σ^{14}, ψ) has *Arf* invariant zero and hence it follows that $p'(\Sigma^{14}) \neq \sigma^2$. Consequently $p'(\Sigma^{14})$ is equal to either κ or $\kappa + \sigma^2$ which implies that the composition $\lambda \circ p'(\Sigma^{14}) = \lambda \circ \kappa$ does not lie in J_{17}. Since (6) is a commutative diagram, it follows that $p'\tau_{14,3}(\Sigma^{14} \otimes \alpha) \neq 0$, where α generates $\pi_3(SO(13)) \approx Z$. Consequently $\tau_{14,3}(\Sigma^{14} \otimes \alpha)$ is of order two in θ_{17} which implies that $\Phi^4_{14} = 0$. It follows that $S^3 \times S^{14}$ has exactly twenty-four differential structures. Moreover, we see that Σ^{14} does not embed in R^{18} with a trivial normal bundle.

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