NON-COMMUTATIVE CHARACTERISTIC POLYNOMIALS AND COHN LOCALIZATION

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Abstract

Almkvist proved that for a commutative ring A the characteristic polynomial of an endomorphism $\alpha : P \longrightarrow P$ of a finitely generated projective A-module determines (P, α) up to extensions. For a non-commutative ring A the generalized characteristic polynomial of an endomorphism $\alpha : P \longrightarrow P$ of a finitely generated projective A-module is defined to be the Whitehead torsion $[1 - x\alpha] \in K_1(A[[x]])$, which is an equivalence class of formal power series with constant coefficient 1.

The paper gives an example of a non-commutative ring A and an endomorphism $\alpha : P \longrightarrow P$ for which the generalized characteristic polynomial does not determine (P, α) up to extensions. The phenomenon is traced back to the non-injectivity of the natural map $\Sigma^{-1}A[x] \longrightarrow A[[x]]$, where $\Sigma^{-1}A[x]$ is the Cohn localization of $A[x]$ inverting the set Σ of matrices in $A[x]$ sent to an invertible matrix by $A[x] \longrightarrow A; x \longmapsto 0$.

1. *Introduction*

We begin by recalling the definition of the characteristic polynomial $\mathrm{ch}_x(\mathbb{C}^n, \alpha)$ of an endomorphism $\alpha : \mathbb{C}^n \longrightarrow \mathbb{C}^n$:

$$\mathrm{ch}_x(\mathbb{C}^n, \alpha) = \det(I - Mx) \in 1 + x\mathbb{C}[x],$$

where M is an $n \times n$ matrix representing α with respect to any choice of basis. (The polynomial defined here can be called the 'reverse characteristic polynomial' to distinguish between $\det(I - xM)$ and $\det(M - xI)$.)

Of course, ch_x is not a complete invariant of the endomorphism; for example the matrices

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

have the same characteristic polynomial although they are not conjugate. On the other hand, if one is given the dimension n and the characteristic polynomial $\mathrm{ch}_x(\mathbb{C}^n, \alpha)$, one can compute all the eigenvalues of α. The Jordan normal form implies that $(\mathbb{C}^n, \alpha)$ is determined uniquely up to choices of extension (cf. [**8**]).

The notion 'unique up to choices of extension' can be made precise without relying on a structure theorem for endomorphisms by introducing the reduced endomorphism class group $\widetilde{\mathrm{End}}_0(A)$ where A denotes any ring [**1, 2, 7**]. $\widetilde{\mathrm{End}}_0(A)$ is the abelian group with

(1) one generator $[A^n, \alpha]$ for each isomorphism class of pairs (A^n, α) where $\alpha : A^n \longrightarrow A^n$;

(2) a relation $[A^n, \alpha] + [A^{n''}, \alpha''] = [A^{n'}, \alpha']$ for each exact sequence

$$0 \longrightarrow A^n \xrightarrow{\theta} A^{n'} \xrightarrow{\theta'} A^{n''} \longrightarrow 0 \tag{1}$$

such that $\theta\alpha = \alpha'\theta$ and $\theta'\alpha' = \alpha''\theta'$;

(3) a relation $[A^n, 0] = 0$ for each n.

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$$\begin{tikzcd} \widetilde{\mathrm{End}}_0(A) \arrow[r, "\widehat{\mathrm{ch}}_x"] \arrow[d, "\mathrm{ch}_x"', "\cong"] & K_1(A[[x]]) \arrow[d, "\det"] \\ \tilde{A}_0 \arrow[r] & A[[x]]^\bullet \end{tikzcd}$$

FIGURE 1.

(Although free modules A^n simplify the presentation, the group $\widetilde{\mathrm{End}}_0(A)$ is unchanged if one substitutes finitely generated projective modules throughout (see section 2.1).) $\widetilde{\mathrm{End}}_0(\mathbb{C})$, for example, is a free abelian group with one generator $[\mathbb{C}, \lambda]$ for each non-zero eigenvalue $\lambda \in \mathbb{C}\backslash 0$.

If A is a commutative ring, Almkvist proved [**2**] that the characteristic polynomial

$$\mathrm{ch}_x(A^n, \alpha) = \det(1 - \alpha x : A[x]^n \longrightarrow A[x]^n) \tag{2}$$

induces an isomorphism

$$\mathrm{ch}_x : \widetilde{\mathrm{End}}_0(A) \longrightarrow \tilde{A}_0 = \left\{ \frac{1 + a_1 x + \ldots + a_n x^n}{1 + b_1 x + \ldots + b_m x^m} \,\middle|\, a_i, b_i \in A \right\},$$

so no further invariants are needed to classify endomorphisms up to extensions.

If we do not assume that A is commutative then the definition (2) above does not apply. However, $1 - \alpha x : A[[x]] \longrightarrow A[[x]]$ is a well-defined automorphism (with inverse $1 + \alpha x + \alpha^2 x^2 + \ldots$) where $A[[x]]$ denotes the ring of formal power series in a central indeterminate x. One can therefore define the *generalized characteristic polynomial* $\widehat{\mathrm{ch}}_x(A^n, \alpha)$ to be the element $[1 - \alpha x]$ of the Whitehead group $K_1(A[[x]])$, inducing a group homomorphism $\widehat{\mathrm{ch}}_x : \widetilde{\mathrm{End}}_0(A) \longrightarrow K_1(A[[x]])$. As Pajitnov observed [**10**, **11**], a Gaussian elimination argument (see section 2.2) yields

$$K_1(A[[x]]) = K_1(A) \oplus W_1(A),$$

where $W_1(A)$ is the image in $K_1(A[[x]])$ of the group $1 + xA[[x]]$ of Witt vectors.

If A is commutative then $\tilde{A}_0$ injects naturally into the group of units $A[[x]]^\bullet$ and the commutative square shown in Figure 1 implies that $\widehat{\mathrm{ch}}_x$ is an injection.

The question arises whether $\widehat{\mathrm{ch}}_x$ is still injective when A is non-commutative. The main result of the present paper is that the answer can be negative.

PROPOSITION 1.1. *The non-commutative ring*

$$S = \mathbb{Z}\langle f, s, g \mid fg, fsg, fs^2g, \ldots \rangle$$

is such that $\widehat{\mathrm{ch}}_x : \widetilde{\mathrm{End}}_0(S) \longrightarrow K_1(S[[x]])$ *is not injective.*

Specifically the two endomorphisms $S \longrightarrow S$ given by $a \longmapsto as$ and $a \longmapsto a(1 - gf)s$ will be shown to have the same image under $\widehat{\mathrm{ch}}_x$ although they represent distinct classes in $\widetilde{\mathrm{End}}_0(S)$. The proof depends on the fact that the functor $A \longmapsto \widetilde{\mathrm{End}}_0(A)$ commutes with direct limits whereas $A \longmapsto A[[x]]$ does not.

To put Proposition 1.1 into context and explain the origins of the ring S, we require a certain universal localization $\Sigma^{-1}A[x]$ [**5**, Chapter 7; **14**, Chapter 4] which P. M. Cohn constructed by adjoining formal inverses to a set Σ of matrices. Here, Σ contains precisely the matrices which become invertible under the augmentation $\epsilon : A[x] \longrightarrow A; x \longmapsto 0$ (or equivalently are invertible in $A[[x]]$).

By the universal property of Cohn localization, the inclusion of $A[x]$ in $A[[x]]$ factors in a unique way through $\Sigma^{-1}A[x]$:

$$A[x] \xrightarrow{i_\Sigma} \Sigma^{-1}A[x] \xrightarrow{\gamma} A[[x]]. \tag{3}$$

In particular $i_\Sigma : A[x] \longrightarrow \Sigma^{-1}A[x]$ is injective for all rings A (which is not true of some Cohn localizations).

If A is commutative then $\Sigma^{-1}A[x]$ is the usual commutative localization, inverting

$$\{\det(\sigma) \mid \sigma \in \Sigma\} = \{p \in A[x] \mid \epsilon(p) \text{ is invertible}\},$$

so $\gamma : \Sigma^{-1}A[x] \longrightarrow A[[x]]$ is also injective. On the other hand in Section 3 we prove the following.

PROPOSITION 1.2. *The non-commutative ring S is such that*

$$\gamma : \Sigma^{-1}S[x] \longrightarrow S[[x]]$$

is not injective.

In fact, Proposition 1.1 is an algebraic K-theory version of Proposition 1.2; for a theorem due to Ranicki [**12**, Proposition 10.16] states that for any ring A

$$K_1(\Sigma^{-1}A[x]) \cong K_1(A) \oplus \widetilde{\mathrm{End}}_0(A), \tag{4}$$

where the split injection $\widetilde{\mathrm{End}}_0(A) \longrightarrow K_1(\Sigma^{-1}A[x])$ is $[A^n, \alpha] \longmapsto [1 - \alpha x]$. One can reinterpret Proposition 1.1 as the statement that the natural homomorphism of groups $K_1(\Sigma^{-1}S[x]) \longrightarrow K_1(S[[x]])$ is not injective; by Proposition 1.2 the phenomenon is not peculiar to algebraic K-theory.

Propositions 1.1 and 1.2 are proved in Sections 2 and 3, respectively. The proofs are independent of each other, and do not assume identity (4).

Section 4 is expository. Firstly we show that, for any ring A, the image of γ is the ring $\mathscr{R}^A$ of rational power series; by definition $\mathscr{R}^A$ is the smallest subring of $A[[x]]$ which contains $A[x]$ and is such that elements of $\mathscr{R}^A$ which are invertible in $A[[x]]$ are invertible in $\mathscr{R}^A$, that is, $\mathscr{R}^A \cap A[[x]]^\bullet = (\mathscr{R}^A)^\bullet$. We work in greater generality replacing the single indeterminate x in (3) by a set $X = \{x_1, \dots, x_\mu\}$ of non-commuting indeterminates

$$A\langle X\rangle \xrightarrow{i_\Sigma} \Sigma^{-1}A\langle X\rangle \xrightarrow{\gamma} A\langle\langle X\rangle\rangle.$$

Secondly we prove that each $\alpha \in \Sigma^{-1}A\langle X\rangle$ can be expressed (non-uniquely) in the form $\alpha = f(1 - s_1x_1 - \ldots - s_\mu x_\mu)^{-1}g$ where $f \in A^n$ is a row vector, $g \in A^n$ is a column vector and $s_1, \dots, s_\mu$ are $n \times n$ matrices with entries in A. This is a version of Schützenberger's theorem ([**15**, **16**], see also [**3**, Chapter 1; **4** §6]). One can think of the elements of $\Sigma^{-1}A\langle X\rangle$ as equivalence classes of finite dimensional linear machines $(f, s_1, \dots, s_\mu, g)$ which generate the power series

$$\gamma(\alpha) = fg + \sum_{i=1}^{\mu} f s_i g x_i + \sum_{i,j=1}^{\mu} f s_i s_j g x_i x_j + \ldots.$$

Cohn wrote [**5**, p. 487] that 'the basic idea … to invert matrices rather than elements was inspired by the rationality criteria of Schützenberger and Nivat …'.

Motivated by the theory of multi-dimensional boundary links, Farber and Vogel proved [**6**] that if A is a (commutative) principal ideal domain then the Cohn localization of the free group ring AF_μ (inverting those matrices which are invertible after augmentation $AF_\mu \longrightarrow A$) is isomorphic to the ring $\mathscr{R}^A$ of rational power series. In Section 5 we show that this localization of the free group ring is isomorphic to $\Sigma^{-1}A\langle X\rangle$ so $\gamma : \Sigma^{-1}A\langle X\rangle \longrightarrow \mathscr{R}^A$ is an isomorphism. By contrast, Proposition 1.2 above says that $\Sigma^{-1}S\langle X\rangle$ is larger than $\mathscr{R}^S$ even when $|X| = 1$; distinct classes of linear machines can generate the same rational power series.

2. *Algebraic K-theory*

2.1. *Definitions*

Let A be a ring, assumed to be associative and to contain a 1. We recall first the definitions of the Grothendieck group $K_0(A)$, the Whitehead group $K_1(A)$ and the less widely known endomorphism class group

$$\mathrm{End}_0(A) = K_0(\text{endomorphism category over}\, A).$$

DEFINITION 2.1. $K_0(A)$ is the abelian group with one generator $[P]$ for each isomorphism class of finitely generated projective A-modules and one relation $[P'] = [P] + [P'']$ for each identity $P' \cong P \oplus P''$.

Let $\mathrm{End}(A)$ denote the category of pairs (P, α) where P is a projective (left) A-module and $\alpha : P \longrightarrow P$ is an A-module endomorphism. A morphism $\theta : (P, \alpha) \longrightarrow (P', \alpha')$ in $\mathrm{End}(A)$ is an A-module map $\theta : P \longrightarrow P'$ such that $\theta\alpha = \alpha'\theta$. A sequence of objects and morphisms

$$0 \longrightarrow (P, \alpha) \xrightarrow{\theta} (P', \alpha') \xrightarrow{\theta'} (P'', \alpha'') \longrightarrow 0 \tag{5}$$

is exact if $0 \longrightarrow P \xrightarrow{\theta} P' \xrightarrow{\theta'} P'' \longrightarrow 0$ is an exact sequence.

Let $\mathrm{Aut}(A) \subset \mathrm{End}(A)$ denote the full subcategory of pairs (P, α) such that $\alpha : P \longrightarrow P$ is an automorphism.

DEFINITION 2.2. The Whitehead group $K_1(A)$ is the abelian group generated by the isomorphism classes $[P, \alpha]$ of $\mathrm{Aut}(A)$ subject to the following relations.

(1) If $0 \longrightarrow (P, \alpha) \longrightarrow (P', \alpha') \longrightarrow (P'', \alpha'') \longrightarrow 0$ is an exact sequence then $[P', \alpha'] = [P'', \alpha''] + [P, \alpha]$.

(2) $[P, \alpha] + [P, \alpha'] = [P, \alpha\alpha']$.

Alternatively, in terms of matrices,

$$K_1(A) = \mathrm{GL}(A)^{\mathrm{ab}} = \frac{\mathrm{GL}(A)}{E(A)} = \frac{\varinjlim \mathrm{GL}_n(A)}{\varinjlim E_n(A)},$$

where $E_n(A)$ is the subgroup of $\mathrm{GL}_n(A)$ generated by elementary matrices $e_{ij}(a)$ which have 1s on the diagonal, a in the ijth position and 0s elsewhere ($a \in A, 1 \leqslant i, j \leqslant n$ and $i \neq j$). See for example [**13**] for further details. If $M, M' \in \mathrm{GL}(A)$ and $[M] = [M'] \in K_1(A)$ then we write $M \sim M'$.

DEFINITION 2.3. The endomorphism class group $\mathrm{End}_0(A) = K_0(\mathrm{End}(A))$ is the abelian group with one generator $[P, \alpha]$ for each isomorphism class in $\mathrm{End}(A)$ and a relation

$$[P', \alpha'] = [P'', \alpha''] + [P, \alpha] \tag{6}$$

corresponding to each exact sequence (5) above.

Since every exact sequence of projective modules splits, we recover $K_0(A)$ by omitting the endomorphisms in Definition 2.3. The forgetful map

$$\mathrm{End}_0(A) \longrightarrow K_0(A); \quad [P,\alpha] \longmapsto [P]$$

is surjective and split by $[P] \longmapsto [P,0]$ so that $\mathrm{End}_0(A) \cong K_0(A) \oplus \widetilde{\mathrm{End}}_0(A)$ with

$$\widetilde{\mathrm{End}}_0(A) = \mathrm{Ker}(\mathrm{End}_0(A) \longrightarrow K_0(A)) \cong \mathrm{Coker}(K_0(A) \longrightarrow \mathrm{End}_0(A)).$$

Note that $\mathrm{End}_0(\_)$ and $\widetilde{\mathrm{End}}_0(\_)$ are functors; a ring homomorphism $p : A \longrightarrow A'$ induces a group homomorphism

$$\begin{aligned} \mathrm{End}_0(A) &\longrightarrow \mathrm{End}_0(A') \\ [P,\alpha] &\longmapsto [A' \otimes_A P, 1 \otimes \alpha]. \end{aligned}$$

The same group $\widetilde{\mathrm{End}}_0(A)$ is obtained if, as in the introduction, one starts with free modules in place of projective modules: let $K_0^h(A)$ denote the Grothendieck group generated by free modules $[A^n]$ subject to relations $[A^{m+n}] = [A^m] + [A^n]$. Nearly all of the rings usually encountered (including the ring S of the present paper) have 'invariant basis number', $A^n \cong A^m \Longrightarrow n = m$, which implies that $K_0^h(A) = \mathbb{Z}$.

Let $\mathrm{End}^h(A) \subset \mathrm{End}(A)$ denote the full subcategory of pairs (A^n, α). Then $\mathrm{End}_0^h(A) = K_0(\mathrm{End}^h(A))$ satisfies $\mathrm{End}_0^h(A) \cong K_0^h(A) \oplus \widetilde{\mathrm{End}}_0^h(A)$, where

$$\widetilde{\mathrm{End}}_0^h(A) = \mathrm{Ker}(\mathrm{End}_0^h(A) \longrightarrow K_0^h(A)) \cong \mathrm{Coker}(K_0^h \longrightarrow \mathrm{End}_0^h(A)).$$

LEMMA 2.4. *There is a natural isomorphism* $\widetilde{\mathrm{End}}_0^h(A) \cong \widetilde{\mathrm{End}}_0(A)$.

Proof. The homomorphism

$$\begin{aligned} \widetilde{\mathrm{End}}_0^h(A) \cong \frac{\mathrm{End}_0^h(A)}{\mathbb{Z}} &\longrightarrow \frac{\mathrm{End}_0(A)}{K_0(A)} \cong \widetilde{\mathrm{End}}_0(A) \\ [A^n,\alpha] &\longrightarrow [A^n,\alpha] \end{aligned}$$

has inverse $[P,\alpha] \longmapsto [P \oplus Q, \alpha \oplus 0]$, where Q is a finitely generated A-module such that $P \oplus Q$ is free. The definition of the inverse does not depend on the choice of Q and plainly $[P,0] \longmapsto 0$ so we need only check that the 'exact sequence relations' (6) are respected. Suppose we are given an exact sequence (5). Choose finitely generated A-modules Q and Q'' such that $P \oplus Q$ and $P'' \oplus Q''$ are free. Then $P' \oplus Q \oplus Q'' \cong P \oplus P'' \oplus Q \oplus Q''$ is free and there is an exact sequence of endomorphisms

$$0 \longrightarrow (P \oplus Q, \alpha \oplus 0) \longrightarrow (P' \oplus Q \oplus Q'', \alpha' \oplus 0 \oplus 0) \longrightarrow (P'' \oplus Q'', \alpha \oplus 0) \longrightarrow 0.$$

□

It follows from Lemma 2.4 that $\widetilde{\mathrm{End}}_0(A)$ has an equivalent definition in terms of matrices: let $M_n(A)$ denote the ring of $n \times n$ matrices with entries in A. Regarding A^n as a module of row vectors, a matrix $M \in M_n(A)$ represents the endomorphism of A^n which multiplies by M on the right. $\widetilde{\mathrm{End}}_0(A)$ is isomorphic to the group generated by $\{[M] \mid M \in \bigcup_{n=1}^{\infty} M_n(A)\}$ subject to the following relations.

(1) If $M \in M_n(A)$ and $M' \in M_{n'}(A)$ then

$$[M] + [M'] = \begin{bmatrix} M & N \\ 0 & M' \end{bmatrix}$$

for all $n \times n'$ matrices N.

(2) If $M, P \in M_n(A)$ and P is invertible then $[M] = [PMP^{-1}]$.
(3) If all the entries in M are zero then $[M] = 0$.

2.2. *Rings of formal power series*

Let $A[[x]]$ be the ring of formal power series in the central indeterminate x.

To define the generalized characteristic polynomial of (P, α) we observe that $1 - \alpha x$ has inverse $1 + \alpha x + \alpha^2 x^2 + \ldots$ when regarded as an endomorphism of $P[[x]] = A[[x]] \otimes_A P$. Thus $1 - \alpha x$ represents an element of $K_1(A[[x]])$. Now

$$\begin{aligned} \widehat{\mathrm{ch}}_x : \widetilde{\mathrm{End}_0}(A) &\longrightarrow K_1(A[[x]]) \\ [P, \alpha] &\longmapsto [1 - \alpha x : P[[x]] \longrightarrow P[[x]]\,] \end{aligned}$$

is well defined because $\widehat{\mathrm{ch}}_x(P, 0) = 0 \in K_1(A[[x]])$ and an exact sequence (5) gives rise to an exact sequence

$$0 \longrightarrow (P[[x]], 1 - \alpha x) \longrightarrow (P'[[x]], 1 - \alpha' x) \longrightarrow (P''[[x]], 1 - \alpha'' x) \longrightarrow 0.$$

LEMMA 2.5. (i) $K_1(A[[x]]) = K_1(A) \oplus W_1(A)$, *where* $W_1(A)$ *is the image in* $K_1(A[[x]])$ *of the group* $W(A) = 1 + xA[[x]]$.

(ii) *If* A *is commutative then* $W_1(A) = W(A) = 1 + xA[[x]]$.

This result and an argument showing that the abelianized group $(1 + xA[[x]])^{\mathrm{ab}}$ is in general larger than $W_1(A)$ can be found in **[11]**.

Proof of Lemma 2.5. (i) Let ϵ denote the augmentation map $A[[x]] \longrightarrow A$; $x \longmapsto 0$. We shall prove that the sequence

$$0 \longrightarrow W_1(A) \longrightarrow K_1(A[[x]]) \xrightarrow{\epsilon} K_1(A) \longrightarrow 0$$

is split exact.

The composite

$$A \longrightarrow A[[x]] \xrightarrow{\epsilon} A$$

is the identity map so $\epsilon : K_1(A[[x]]) \longrightarrow K_1(A)$ is surjective and split.

We have only to show that an element δ of $K_1(A[[x]])$ which becomes zero in $K_1(A)$ can be written $\delta = [1 + x\xi]$ for some $\xi \in A[[x]]$. We may certainly write $\delta = [\delta_0 + x\delta_1 + x^2\delta_2 + \ldots]$ with $\delta_i \in M_n(A)$ for each i and with δ_0 invertible. Now $[\delta_0] = 0 \in K_1(A)$ so $\delta = [1 + \eta]$ where $\eta = \sum_{i=1}^{\infty} \delta_0^{-1}\delta_i x^i$. Since the diagonal entries of $1 + \eta$ are invertible and all other entries are in $xA[[x]]$, we can reduce $1 + \eta$ by elementary row operations to a diagonal matrix with entries in $1 + xA[[x]]$. Thus $\delta = [1 + x\xi]$ where $1 + x\xi$ is the product of the diagonal entries.

(ii) Taking determinants gives a homomorphism to the group of units

$$\det : K_1(A[[x]]) \longrightarrow A[[x]]^{\bullet}.$$

Every element of $W_1(A)$ can be written in the form $[1 + x\xi]$ so the restriction of det to $W_1(A)$ is inverse to the canonical map $1 + xA[[x]] \longrightarrow W_1(A)$. □

2.3. *Proof of Proposition* 1.1

Recall that S denotes the quotient of the free ring $\mathbb{Z}\langle f, s, g \rangle$ by the two-sided ideal generated by the set $\{fs^ig \mid i = 0, 1, 2, \ldots\}$. There are two statements to prove as follows.

LEMMA 2.6. *$[S, s]$ and $[S, (1 - gf)s]$ are distinct classes in $\widetilde{\mathrm{End}}_0(S)$.*

LEMMA 2.7. *$\widehat{\mathrm{ch}}_x[S, s] = \widehat{\mathrm{ch}}_x[S, (1 - gf)s]$ in $K_1(S[[x]])$.*

Proof. We aim to show that $[1 - sx] = [1 - (1 - gf)sx] \in K_1(S[[x]])$. In $S[[x]]$ we have

$$f(1 - sx)^{-1}g = fg + (fsg)x + (fs^2g)x^2 + (fs^3g)x^3 + \ldots = 0,$$

so

$$\begin{aligned}
1 - sx &\sim \begin{pmatrix} 1 & 0 \\ 0 & 1 - sx \end{pmatrix}\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 + f(1 - sx)^{-1}g & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -(1 - sx)^{-1}g & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & f \\ -g & 1 - sx \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix}\begin{pmatrix} 1 & -f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & f \\ -g & 1 - sx \end{pmatrix}\begin{pmatrix} 1 & -f \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 - (1 - gf)sx \end{pmatrix} \quad \text{since } fg = 0 \\
&\sim 1 - (1 - gf)sx.
\end{aligned}$$

□

Before we prove Lemma 2.6, it is convenient to define a second invariant χ. In terms of matrices,

$$\begin{aligned}
\chi : \widetilde{\mathrm{End}}_0(A) &\longrightarrow \overline{A}[[x]] \\
[M] &\longmapsto \sum_{i=1}^{\infty} \mathrm{Trace}(M^i)x^i,
\end{aligned}$$

where $\overline{A}$ denotes the quotient of A by the abelian group generated by commutators (cf. [**9**])

$$\overline{A} = \frac{A}{\mathbb{Z}\{ab - ba \mid a, b \in A\}}.$$

EXAMPLE 2.8. Let X be a set and suppose that A is the free ring $\mathbb{Z}\langle X\rangle$ generated by X. The free monoid X^* of words in the alphabet X is a basis for $\mathbb{Z}\langle X\rangle$ as a $\mathbb{Z}$-module. Each commutator $ab - ba$ with $a, b \in \mathbb{Z}\langle X\rangle$ is a linear combination of 'basic' commutators $\sum_i \lambda_i(u_iv_i - v_iu_i)$ where $\lambda_i \in \mathbb{Z}$ and $u_i, v_i \in X^*$ so the commutator submodule $\mathbb{Z}\{ab - ba \mid a, b \in \mathbb{Z}\langle X\rangle\} \subset \mathbb{Z}\langle X\rangle$ is spanned by elements $w - w'$ with $w, w' \in X^*$ and w' a cyclic permutation of w (written $w \sim w'$). It follows that $\overline{\mathbb{Z}\langle X\rangle} = \mathbb{Z}\{X^*/\sim\}$.

We emphasize that the abelian group $\overline{A}$ is in general larger than the commutative ring A^{ab}, the latter being the quotient of A by the two-sided ideal generated by $\{ab - ba \mid a, b \in A\}$. Nevertheless, if M and N are $n \times n$ matrices with entries in A then $\mathrm{Trace}(MN) = \mathrm{Trace}(NM) \in \overline{A}$ and it follows that χ is well-defined on $\widetilde{\mathrm{End}}_0(A)$.

$$\begin{tikzcd}
\widetilde{\mathrm{End}}_0(A) \arrow[r, "\widehat{\mathrm{ch}}_x"] \arrow[dr, "\chi"'] & K_1(A[[x]]) \arrow[d, "T"] \\
& \overline{A}[[x]]
\end{tikzcd}$$

FIGURE 2.

REMARK 2.9. χ is in general a weaker invariant then $\widehat{\mathrm{ch}}_x$. There is a commutative triangle as shown in Figure 2, where

$$T[M] = -\mathrm{Trace}\left(\left(x\frac{d}{dx}M\right)M^{-1}\right)$$

for $M \in \mathrm{GL}(A[[x]])$. Differentiation is defined formally by

$$\frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Proof of Lemma 2.6. We define a family of rings

$$S_m := \mathbb{Z}\langle f, s, g \mid fg, fsg, \ldots, fs^m g\rangle.$$

There is an obvious surjection $p_m : S_m \longmapsto S_{m+1}$ for each $m \in \mathbb{N}$ and S is the direct limit of the system $S = \varinjlim S_m$.

By Lemma A.2 of Appendix A we have $\widetilde{\mathrm{End}}_0(S) = \varinjlim \widetilde{\mathrm{End}}_0(S_m)$ so it suffices to prove that for each $m \in \mathbb{N}$

$$[S_m, s] \neq [S_m, (1 - gf)s] \in \widetilde{\mathrm{End}}_0(S_m).$$

We shall see that χ is sensitive enough to distinguish these two endomorphism classes. Indeed, $\chi[S_m, (1-gf)s] = \sum_{i=1}^{\infty}((1-gf)s)^i x^i$ and in particular the coefficient of x^{m+1} is

$$((1-gf)s)^{m+1} = s^{m+1} - (gfs^{m+1} + sgfs^m + \ldots + s^m gfs) + \text{other terms},$$

where in each of the 'other terms' two or more occurrences of gf intersperse $m+1$ copies of s. Since $ab = ba \in \overline{S_m}$ for all $a, b \in S_m$, one may perform a cyclic permutation of the letters in each term to obtain

$$((1-gf)s)^{m+1} = s^{m+1} - (m+1)fs^{m+1}g,$$

the 'other terms' disappearing by the defining relations $fg = \ldots = fs^m g = 0$ of S_m. Now the coefficient of x^{m+1} in $\chi[S, s]$ is s^{m+1} so it remains to prove that $(m+1)fs^{m+1}g \neq 0$ in $\overline{S_m}$. We shall argue by contradiction.

Let X denote the alphabet $\{f, s, g\}$. If $(m+1)fs^{m+1}g = 0 \in \overline{S_m}$ then there is an equation in $\mathbb{Z}\langle X\rangle$

$$(m+1)fs^{m+1}g = \sum_{i=1}^{l}(w_i - w_i') + r_0 fgr_0' + r_1 fsgr_1' + \ldots + r_m fs^m gr_m', \tag{7}$$

where $r_j, r_j' \in \mathbb{Z}\langle X\rangle$ for $1 \leqslant j \leqslant m$ and $w_i, w_i' \in X^*$ are such that $w_i \sim w_i'$ for $1 \leqslant i \leqslant l$ as in Example 2.8.

Let V denote the $\mathbb{Z}$-module generated by the cyclic permutations of $fs^{m+1}g$ and let W be the $\mathbb{Z}$-module generated by all other words in X^*

$$\mathbb{Z}\langle X\rangle = V \oplus W = \mathbb{Z}\{w \in X^* \mid w \sim fs^{m+1}g\} \oplus \mathbb{Z}\{w \in X^* \mid w \nsim fs^{m+1}g\}.$$

Each basic commutator $w - w'$ is either in V or in W and

$$r_0 f g r_0' + r_1 f s g r_1' + \ldots + r_m f s^m g r_m' \in W,$$

so by equation (7)

$$(m+1) f s^{m+1} g = \sum_{i \in I} w_i - w_i',$$

where $I = \{i \mid w_i \sim f s^{m+1} g\} \subset \{1, \ldots, l\}$. We have reached a contradiction (for example, put $f = g = s = 1$) and the proof of Proposition 1.1 is complete. $\square$

3. *Cohn localization*

In this section, we briefly review Cohn localization before proving Proposition 1.2.

3.1. *Definitions*

If A is a ring and Σ is any set of matrices with entries in A then a ring homomorphism $A \longrightarrow B$ is said to be Σ-inverting if every matrix in Σ is mapped to an invertible matrix over B. The Cohn localization $i_\Sigma : A \longrightarrow \Sigma^{-1}A$ is the (unique) ring homomorphism with the universal property that every Σ-inverting homomorphism $A \longrightarrow B$ factors uniquely through i_Σ. Note that i_Σ is not in general an injection; it may even be the case that $\Sigma^{-1}A = 0$.

If A is commutative, then $\Sigma^{-1}A$ coincides with the commutative ring of quotients $S^{-1}R$ with $S = \{\det(M) \mid M \in \Sigma\}$.

For non-commutative A, Cohn constructed $\Sigma^{-1}A$ by generators and relations as follows [**5**, p. 390]. For each $m \times n$ matrix $M \in \Sigma$ take a set of mn symbols arranged as an $n \times m$ matrix M'. $\Sigma^{-1}A$ is generated by the elements of A together with all the symbols in the matrices M', subject to the relations holding in A and the equations $MM' = I$ and $M'M = I$. Schofield [**14**, Chapter 4] gave a slightly more general construction, inverting a set Σ of homomorphisms between finitely generated projective A-modules.

Given any ring homomorphism $A \longrightarrow B$, we may define Σ to be the set of matrices in A which are invertible in B, obtaining

$$A \xrightarrow{i_\Sigma} \Sigma^{-1}A \xrightarrow{\gamma} B.$$

Every matrix with entries in $\Sigma^{-1}A$ can be expressed (non-uniquely) in the form $f\sigma^{-1}g$ where f, σ and g are matrices with entries in A and $\sigma \in \Sigma$ (see for example [**14**, p. 52]).

We shall also need the following lemma in Section 4.

LEMMA 3.1. *A matrix α with entries in $\Sigma^{-1}A$ is invertible if and only if its image $\gamma(\alpha)$ is invertible. In particular,* $\mathrm{Im}(\gamma)^\bullet = B^\bullet \cap \mathrm{Im}(\gamma)$.

Proof. The 'only if' part is easy. Conversely, suppose that $\gamma(\alpha)$ is invertible and $\alpha = f\sigma^{-1}g$ as above. The equation

$$\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f\sigma^{-1}g & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\sigma^{-1}g & 1 \end{pmatrix} = \begin{pmatrix} 0 & f \\ -g & \sigma \end{pmatrix} \tag{8}$$

implies that α is invertible if and only if

$$\begin{pmatrix} 0 & f \\ -g & \sigma \end{pmatrix}$$

is invertible, but applying γ to equation (8) we learn that

$$\gamma\begin{pmatrix} 0 & f \\ -g & \sigma \end{pmatrix}$$

is invertible and hence that

$$\begin{pmatrix} 0 & f \\ -g & \sigma \end{pmatrix} \in \Sigma.$$

Thus

$$\begin{pmatrix} 0 & f \\ -g & \sigma \end{pmatrix}$$

and α are invertible over $\Sigma^{-1}A$. $\square$

3.2. *Proof of Proposition* 1.2

We recall that S denotes the ring

$$\mathbb{Z}\langle f, s, g \mid fg, fsg, fs^2g, \ldots\rangle;$$

let Σ be the set of matrices $\sigma = \sigma_0 + \sigma_1 x + \ldots + \sigma_n x^n$ with entries in $S[x]$ such that σ_0 is invertible (so σ is invertible in $S[[x]]$).

We will prove the following two statements:

(a) The element $f(1-sx)^{-1}g$ is non-zero in $\Sigma^{-1}S[x]$.

(b) $f(1-sx)^{-1}g$ lies in the kernel of the natural map $\gamma : \Sigma^{-1}S[x] \longrightarrow S[[x]]$.

Statement (b) follows directly from the definition of S

$$\gamma(f(1-sx)^{-1}g) = fg + (fsg)x + (fs^2g)x^2 + \ldots = 0 \in S[[x]].$$

To prove (a) we express S once again as the direct limit $\varinjlim S_m$ with

$$S_m := \mathbb{Z}\langle f, s, g \mid fg, fsg, \ldots, fs^m g\rangle$$

and the augmentations $\epsilon : S_m[x] \longrightarrow S_m; x \longrightarrow 0$ fit into a commutative diagram, Figure 3.

Let Σ_m denote the set of matrices in $S_m[x]$ which become invertible under ϵ, so that $p_m(\Sigma_m) \subset \Sigma_{m+1}$ and $\Sigma = \varinjlim \Sigma_m$. By Lemma A.1 of *Appendix* A,

$$\Sigma^{-1}S[x] = \varinjlim \Sigma_m^{-1} S_m[x],$$

so it suffices to show that $f(1-sx)^{-1}g \neq 0 \in \Sigma_m^{-1}S_m[x]$ for each $m \in \mathbb{N}$. However $\gamma(f(1-sx)^{-1}g) = \sum_{n=0}^{\infty}(fs^n g)x^n$ which is non-zero in $S_m[[x]]$ because there does not

$$\begin{tikzcd}
\cdots \arrow[r] & S_m[x] \arrow[r, "p_m"] \arrow[d, "\epsilon"] & S_{m+1}[x] \arrow[r] \arrow[d, "\epsilon"] & \cdots \\
\cdots \arrow[r] & S_m \arrow[r, "p_m"] & S_{m+1} \arrow[r] & \cdots
\end{tikzcd}$$

FIGURE 3.

exist an equation

$$fs^n g = r_0 f g r_0' + r_1 f s g r_1' + \ldots + r_m f s^m g r_m' \in \mathbb{Z}\langle f, s, g\rangle$$

with $n > m$ and $r_i, r_i' \in \mathbb{Z}\langle f, s, g\rangle$ for $1 \leqslant i \leqslant m$. Thus $f(1 - sx)^{-1} g \neq 0 \in \Sigma_m^{-1} S_m[x]$ and the proof of Proposition 1.2 is complete. □

4. *Many indeterminates*

Let A be any ring, let $X = \{x_1, \ldots, x_\mu\}$ be a finite set, and let X^* be the free monoid of words in the alphabet X. The free A-algebra

$$A\langle X\rangle = A \otimes_{\mathbb{Z}} \mathbb{Z}\langle X\rangle$$

is graded by word length in X^* and is therefore a subring of its completion $A\langle\langle X\rangle\rangle$ the elements of which are formal power series $p = \sum_w p_w w$ with $p_w \in A$ for each $w \in X^*$.

Let Σ denote the set of matrices in $A\langle X\rangle$ which are sent to an invertible matrix by the augmentation $\epsilon : A\langle X\rangle \longrightarrow A_i;\ x_i \longmapsto 0$ for all i. Σ is precisely the set of matrices which are invertible over $A\langle\langle X\rangle\rangle$, so the inclusion of $A\langle X\rangle$ in $A\langle\langle X\rangle\rangle$ factors uniquely through $\Sigma^{-1}A\langle X\rangle$:

$$A\langle X\rangle \xrightarrow{i_\Sigma} \Sigma^{-1}A\langle X\rangle \xrightarrow{\gamma} A\langle\langle X\rangle\rangle.$$

4.1. *Rational power series*

In this section we describe the image of γ.

Definition 4.1. Let $\mathscr{R}^A$ denote the rational closure of $A\langle X\rangle$. In other words $\mathscr{R}^A$ is the intersection of all the rings R such that $A\langle X\rangle \subset R \subset A\langle\langle X\rangle\rangle$ and $R^\bullet = R \cap A\langle\langle X\rangle\rangle^\bullet$. A power series $p \in \mathscr{R}^A$ is said to be *rational.*

Proposition 4.2. $\gamma(\Sigma^{-1}A\langle X\rangle) = \mathscr{R}^A$.

Proof. To prove that $\mathscr{R}^A \subset \mathrm{Im}\,(\gamma)$, we note that $\mathrm{Im}\,(\gamma)^\bullet = \mathrm{Im}\,(\gamma) \cap A\langle\langle X\rangle\rangle^\bullet$ by Lemma 3.1 above.

Conversely, to prove that $\mathrm{Im}\,(\gamma) \subset \mathscr{R}^A$ it suffices to show that every matrix $\sigma \in \Sigma$ has an inverse with entries in $\mathscr{R}^A$ so that there is a commutative diagram, Figure 4.

Recall that $\epsilon : A\langle X\rangle \longrightarrow A$ is the augmentation given by $\epsilon(x_i) = 0$ for all i. Multiplying σ by $\epsilon(\sigma)^{-1}$ if necessary we can assume that $\epsilon(\sigma) = I$. Each diagonal entry of σ has an inverse in $\mathscr{R}^A$ so, after elementary row operations (which are of course invertible), σ becomes a diagonal matrix where each diagonal entry σ_{ii} has $\epsilon(\sigma_{ii}) = 1$ (cf. the proof of Lemma 2.5(i)). By the definition of $\mathscr{R}^A$, each σ_{ii} has an inverse in $\mathscr{R}^A$. □

$$\begin{tikzcd}
A\langle X\rangle \arrow[d, "i_\Sigma"'] \arrow[dr] \arrow[drr] & & \\
\Sigma^{-1}A\langle X\rangle \arrow[r] & \mathscr{R}^A \arrow[r] & A\langle\langle X\rangle\rangle
\end{tikzcd}$$

Figure 4.

4.2. *Schützenberger's theorem*

PROPOSITION 4.3. *Every matrix α with entries in $\Sigma^{-1}A\langle X\rangle$ can be expressed (non-uniquely) in the form*

$$\alpha = f(1 - s_1x_1 - \ldots - s_\mu x_\mu)^{-1}g, \tag{9}$$

where $f, s_1, \ldots, s_\mu$ and g are matrices with entries in A.

Proof. It suffices to show that α has the form $f\sigma^{-1}g$ where f and g have entries in A and $\sigma = \sigma_0 + \sum_{i=1}^{\mu}\sigma_i x_i$ is linear with σ_0 invertible, for then $\alpha = (f\sigma_0)(\sigma_0^{-1}\sigma)g$.

Note first that if $\alpha_1 = f_1\sigma_1^{-1}g_1$ and $\alpha_2 = f_2\sigma_2^{-1}g_2$ then

$$\alpha_1 - \alpha_2 = \begin{pmatrix} f_1 & -f_2 \end{pmatrix}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}^{-1}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \tag{10}$$

and

$$\alpha_1\alpha_2 = \begin{pmatrix} f_1 & 0 \end{pmatrix}\begin{pmatrix} \sigma_1 & -g_1f_2 \\ 0 & \sigma_2 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ g_2 \end{pmatrix} \tag{11}$$

whenever the left-hand sides make sense (cf. [**14**, p. 52]. Hence we need only treat the cases where (i) α has entries in $A\langle X\rangle$ and (ii) $\alpha = \sigma^{-1}$ with $\sigma \in \Sigma$.

If α has entries in $A\langle X\rangle$, then, by repeated application of the equation

$$\begin{pmatrix} a+bc & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ -c & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \tag{12}$$

in which a, b, c and 1 denote matrices, some stabilization

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$$

can be expressed as a product of linear matrices. Each linear matrix $a_0 + a_1x_1 + \ldots + a_\mu x_\mu$ can be written

$$(1 \quad 0)\begin{pmatrix} 1 & -a_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{i=1}^{\mu}(1 \quad 0)\begin{pmatrix} 1 & -a_ix_i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and equations (10) and (11) imply that

$$\alpha = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is of the required form $f\sigma^{-1}g$.

The case $\alpha = \sigma^{-1}$ is similar (but slightly easier); we repeatedly apply equation (12) to express (a stabilization of) σ^{-1} as a product of inverses of linear matrices in Σ and then apply equation (11). □

A power series $p \in A\langle\langle X\rangle\rangle$ is said to be *recognizable* if it is of the form

$$p = fg + \sum_{i=1}^{\mu} fs_igx_i + \sum_{i,j=1}^{\mu} fs_is_jgx_ix_j + \ldots,$$

where $f \in A^n$ is a row vector, $g \in A^n$ is a column vector and each s_i is an $n \times n$ matrix in A. Propositions 4.3 and 4.2 imply the following.

COROLLARY 4.4 (Schützenberger's theorem). *A power series $p \in A\langle\langle X\rangle\rangle$ is rational if and only if it is recognizable.*

5. *Localization of the free group ring*

We identify the localization of the group ring of the free group studied in [**6**] with the localization $\Sigma^{-1}A\langle X\rangle$ of the present paper.

Let F_μ denote the free group on generators $z_1, \ldots, z_\mu$, and as usual let A be a (not necessarily commutative) ring. AF_μ will denote the group ring, in which the elements of the group F_μ are assumed to commute with elements of A. Let $\epsilon : AF_\mu \longrightarrow A$; $z_i \longmapsto 1$ for all i and let Ψ denote the set of square matrices M in AF_μ such that $\epsilon(M)$ is invertible. Ψ is denoted Σ in [**6**].

All the matrices in Ψ become invertible under the Magnus embedding of the group ring

$$\begin{aligned} AF_\mu &\longrightarrow A\langle\langle X\rangle\rangle, \\ z_i &\longmapsto 1 + x_i, \\ z_i^{-1} &\longmapsto 1 - x_i + x_i^2 - x_i^3 + \ldots , \end{aligned}$$

so the embedding factors through $\Psi^{-1}AF_\mu$:

$$AF_\mu \xrightarrow{i_\Psi} \Psi^{-1}AF_\mu \xrightarrow{\gamma} A\langle\langle X\rangle\rangle.$$

Farber and Vogel proved that if A is a (commutative) principle ideal domain then γ is an injection and the image of γ is the ring $\mathscr{R}^A$ of rational power series.

Now, for general A, let $m : A\langle X\rangle \longrightarrow AF_\mu$ be the ring homomorphism defined by $x_i \longmapsto z_i - 1$ for all i. There is a commutative diagram, Figure 5, so $m(\Sigma) \subset \Phi$ and m induces a homomorphism $m : \Sigma^{-1}A\langle X\rangle \longrightarrow \Psi^{-1}AF_\mu$ which fits into a commutative diagram, Figure 6.

PROPOSITION 5.1. *$m : \Sigma^{-1}A\langle X\rangle \longrightarrow \Psi^{-1}AF_\mu$ is an isomorphism.*

Proof. The map $l : AF_\mu \longrightarrow \Sigma^{-1}A\langle X\rangle$; $z_i \longmapsto x_i + 1$, indicated by the broken arrow, fits into the commutative diagram of Figure 6. l is Ψ-inverting and therefore induces a map $\Psi^{-1}AF_\mu \longrightarrow \Sigma^{-1}A\langle X\rangle$ which, by the universal properties of i_Σ and i_Ψ, is inverse to m. □

$$\begin{tikzcd} A\langle X\rangle \arrow[r, "\epsilon"] \arrow[d, "l"'] & A \\ AF_\mu \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

FIGURE 5.

$$\begin{tikzcd} A\langle X\rangle \arrow[r, "i_\Sigma"] \arrow[d, "m"'] & \Sigma^{-1}A\langle X\rangle \arrow[r, "\gamma"] \arrow[d, "m"] & A\langle\langle X\rangle\rangle \\ AF_\mu \arrow[r, "i_\Psi"'] \arrow[ur, dashed, "l"] & \Psi^{-1}AF_\mu \arrow[ur, "\gamma"'] & \end{tikzcd}$$

FIGURE 6.

Appendix A. *Direct limits*

In this appendix we prove that Cohn localization and the functor $\widetilde{\mathrm{End}}(\_)$ commute with direct limits.

A.1. *Cohn localization*

First we make the former claim more precise. Suppose that I is a directed set and $(\{A_m\}_{m\in I}, \{f^l_m : A_m \longrightarrow A_l\}_{m\leqslant l})$ is a direct system of rings. Suppose further that for each $m \in I$ we have a set of matrices Σ_m with entries in A_m such that $f^l_m(\Sigma_m) \subset \Sigma_l$ whenever $m \leqslant l$. If $i_m : A_m \longrightarrow \Sigma_m^{-1}A_m$ is the universal Σ_m-inverting ring homomorphism for each m, then when $m \leqslant l$ the composite

$$A_m \xrightarrow{f^l_m} A_l \xrightarrow{i_l} \Sigma_l^{-1}A_l$$

is Σ_m-inverting and therefore factors through a map $\Sigma^{-1}f^l_m : \Sigma_m^{-1}A_m \longrightarrow \Sigma_l^{-1}A_l$. It is easy to see that $\Sigma^{-1}f^k_l \circ \Sigma^{-1}f^l_m = \Sigma^{-1}f^k_m$ when $m \leqslant l \leqslant k$.

For any ring A let $\mathscr{M}(A)$ denote the set of matrices (of any size and shape) with entries in A. The inclusions $\Sigma_m \subset \mathscr{M}(A_m)$ induce an injection

$$\varinjlim \Sigma_m \longrightarrow \varinjlim \mathscr{M}(A_m) = \mathscr{M}(\varinjlim A_m).$$

Lemma A.1. *There is a natural isomorphism*

$$(\varinjlim \Sigma_m)^{-1}(\varinjlim A_m) \cong \varinjlim (\Sigma_m^{-1}A_m).$$

Proof. One can check that the canonical map $\varinjlim i_m : \varinjlim A_m \longrightarrow \varinjlim \Sigma_m^{-1}A_m$ is universal among $(\varinjlim \Sigma_m)^{-1}$-inverting homomorphisms. The details are left to the reader. $\square$

A.2. *The endomorphism class group*

We consider next the functor $\widetilde{\mathrm{End}}_0(\_)$.

Lemma A.2. *There is a natural isomorphism*

$$\varinjlim \widetilde{\mathrm{End}}_0(A_m) \cong \widetilde{\mathrm{End}}_0(\varinjlim A_m).$$

Proof. The canonical maps $f_m : A_m \longrightarrow \varinjlim A_m$ induce maps $f_m : \widetilde{\mathrm{End}}_0(A_m) \longrightarrow \widetilde{\mathrm{End}}_0(\varinjlim A_m)$ satisfying $f_l f^l_m = f_m$ for $m \leqslant l$. We aim to prove that any other system of maps $g_m : \widetilde{\mathrm{End}}_0(A_m) \longrightarrow T$ with $g_l f^l_m = g_m$ for $m \leqslant l$ factors uniquely through $\widetilde{\mathrm{End}}_0(\varinjlim A_m)$; see Figure 7.

Suppose that $[M]$ is a generator of $\widetilde{\mathrm{End}}_0(\varinjlim A_m)$ where $M \in M_n(\varinjlim A_m)$. M is

$$\begin{tikzcd}
\{\widetilde{\mathrm{End}}_0(A_m)\} \arrow[r, "\{g_m\}"] \arrow[d, "\{f_m\}"] & T \\
\widetilde{\mathrm{End}}_0(\varinjlim A_m) \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

Figure 7.

the image $f_m(M_m)$ of some matrix $M_m \in M_n(A_m)$, so we can define $g[M] = g_m[M_m]$. To show that g is well-defined there are two things to check:

(i) If $M_l \in M_n(A_l)$ is an alternative choice with $f_l(M_l) = M$ then we require $g_m[M_m] = g_l[M_l]$. Indeed, there exists k such that $l \leqslant k$, $m \leqslant k$ and $f_l^k(M_k) = f_m^k(M_m) \in M_n(A_k)$. Hence $g_m[M_m] = g_k f_m^k[M_m] = g_k f_l^k[M_l] = g_l[M_l]$.

(ii) We must check that g respects the defining relations of $\widetilde{\mathrm{End}}_0(\varinjlim A_m)$.

(1) A matrix

$$\begin{pmatrix} M & N \\ 0 & M' \end{pmatrix}$$

is the image of some matrix

$$\begin{pmatrix} M_m & N_m \\ 0 & M'_m \end{pmatrix},$$

so

$$g\begin{bmatrix} M & N \\ 0 & M' \end{bmatrix} = g_m\begin{bmatrix} M_m & N_m \\ 0 & M'_m \end{bmatrix} \neq g_m([M_m] + [M'_m]) = g[M] + g[M'].$$

(2) Suppose that $M' = PMP^{-1}$ for some invertible matrix P. For large enough m we can choose $P_m, Q_m \in M_n(A_m)$ to represent P and P^{-1} respectively. Since $I = f_m(P_m)f_m(Q_m)$ there exists $k \geqslant m$ such that $P_kQ_k = I \in M_n(A_k)$ where $P_k = f_m^k P_m$ and $Q_k = f_m^k Q_m$. Thus $M' = f_k(P_k M_k P_k^{-1})$ and $g[M'] = g_k[P_k M_k P_k^{-1}] = g_k[M_k] = g[M]$.

(3) If M is the zero matrix, $g[M] = 0$.

Uniqueness of g follows from the fact that every class $[M]$ in $\widetilde{\mathrm{End}}_0(\varinjlim A_m)$ is an image of a class $[M_m] \in \widetilde{\mathrm{End}}_0(A_m)$. □

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